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SURJECTIVE NONSYMMETRIC NORM PRESERVING MAPS BETWEEN $\mathrm{Lip}(X, d, \tau)$-ALGEBRAS

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Abstract. Let A be a real subalgebra of $C(X, \tau)$ containing $\mathrm{Lip}(X, d, \tau)$ and let B be a real subalgebra of $C(Y, \eta)$ containing $\mathrm{Lip}(Y, \rho, \eta)$, where (X, d) and (Y, ρ) are compact metric spaces and τ and η are Lipschitz involutions on (X, d) and (Y, ρ), respectively. In this paper, we first study surjective multiplicatively uniform norm preserving maps T from A to B and determine their structures. Next, we give a description of surjective maps $T : A \to B$ satisfying the nonsymmetric uniform norm condition $\|T(f)T(g) - 1_Y\|_Y = \|fg - 1_X\|_X$ for all $f, g \in A$. For such a map T, we show that $(T(1_X))^2 = 1_Y$ and T is multiplicatively uniform norm preserving. In the case that $A = \mathrm{Lip}(X, d, \tau)$ and $B = \mathrm{Lip}(Y, \rho, \eta)$, we prove that $T(A_{\mathbb{R}})$ is a subset of $B_{\mathbb{R}}$ and there exists a bijective map $\varphi \colon Y \to X$ with $\varphi \circ \eta = \tau \circ \varphi$ on Y such that φ is continuous at each $y \in Y$ with $\eta(y) = y$, φ^{-1} is continuous at each $x \in X$ with $\tau(x) = x$, and $T(f) = T(1_X) \cdot (f \circ \varphi)$ on Y for all $f \in A_{\mathbb{R}}$, where $A_{\mathbb{R}}$ is the set of all $f \in A$ for which f is real-valued on X.

1. Introduction and preliminaries

The symbol $\mathbb{K}$ denotes the scalar field, which is either $\mathbb{R}$ or $\mathbb{C}$. The study of surjective maps (linear or nonlinear) between commutative Banach algebras over $\mathbb{K}$ of $\mathbb{K}$-valued functions that preserve multiplicatively the spectrum, the range, the norm, the peripherally range or the peripherally spectrum has attracted the attention of several mathematicians in recent years.

Let $(A, \|\cdot\|)$ and $(B, \|\cdot\|)$ be normed algebras over $\mathbb{K}$ and let T be a map from A into B. We say that T is *norm preserving* if $\|T(a)\| = \|a\|$ for all $a \in A$. The

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map T is called an *isometry* if $\|T(a_1)-T(a_2)\| = \|a_! - a_2\|$ for all $a_1, a_2 \in A$. We say that T is a *multiplicatively norm preserving* if $\|T(a_1)T(a_2)\| = \|a_1a_2\|$ for all $a_1, a_2 \in A$.

Let A be an algebra over $\mathbb{K}$ with unit e_A. For $a \in A$, the spectrum of a is denoted by $\sigma_A(a)$. It is known that $\sigma_A(a)$ is a nonempty compact subset of $\mathbb{C}$ whenever A is a Banach algebra with unit over $\mathbb{K}$ (See [8, Theorem 1.2.8] for $\mathbb{K} = \mathbb{C}$ and [10, Corollary 1.1.18 and Theorem 1.1.19] for $\mathbb{K} = \mathbb{R}$.). For $a \in A$, the *peripheral spectrum* of a is denoted by $\sigma_{\pi,A}(a)$ and defined by

$$\sigma_{\pi,A}(a) = \{\lambda \in \sigma_A(a) \colon |\lambda| = \max\{|w| \colon w \in \sigma_A(a)\}\}.$$

Let $(A, \|\cdot\|)$ and $(B, \|\cdot\|)$ be unital Banach algebras over $\mathbb{K}$ and let T be a map from A into B. We say that T is *spectrum preserving* if $\sigma_B(T(a)) = \sigma_A(a)$ for all $a \in A$. The map T is called *peripherally spectrum preserving* if $\sigma_{\pi,B}(T(a)) = \sigma_{\pi,A}(a)$ for all $a \in A$. We say that T is *multiplicatively spectrum preserving* if $\sigma_B(T(a_1f)T(a_2)) = \sigma_A(a_1a_2)$ for all $a_1, a_2 \in A$. The map T is called *multiplicatively peripheral spectrum preserving* if $\sigma_{\pi,B}(T(a_1)T(a_2)) = \sigma_{\pi,A}(a_1a_2)$ for all $a_1, a_2 \in A$.

Let X be a compact Hausdorff space. We denote by $C_{\mathbb{K}}(X)$ the set of all $\mathbb{K}$-valued continuous functions on X. It is known that $(C_{\mathbb{K}}(X), \|\cdot\|_X)$ is a unital commutative Banach algebra over $\mathbb{K}$ with unit 1_X, the constant function on X with value 1, where $\|\cdot\|_X$ is the uniform norm on $C_{\mathbb{K}}(X)$, which is defined by

$$\|f\|_X = \sup\{|f(x)| \colon x \in X\} \qquad (f \in C_{\mathbb{K}}(X)).$$

We write $C(X)$ instead of $C_{\mathbb{C}}(X)$. For $f \in C(X)$, the range of f is denoted by $\mathrm{Ran}_X(f)$ and defined by $\mathrm{Ran}_X(f) = \{f(x) \colon x \in X\}$. The set of all $x \in X$ for which $\|f\|_X = |f(x)|$ is called the *maximizing set* of f and denoted by $M(f)$. The peripheral range of f is denoted by $\mathrm{Ran}_{\pi,X}(f)$ and defined by $\mathrm{Ran}_{\pi,X}(f) = \{f(x) \colon x \in X, \|f\|_X = |f(x)|\}$. A *complex function algebra* on X is a complex subalgebra of $C(X)$ that separates the points of X and contains 1_X. A *complex Banach function algebra* on X is a complex function algebra A on X such that it is a unital Banach algebra under an algebra norm $\|\cdot\|$. If the algebra norm on A is the uniform norm $\|\cdot\|_X$, then A is called a *complex uniform function algebra* on X.

Let X and Y be compact Hausdorff spaces, let A and B be subalgebras of $C(X)$ and $C(Y)$ over $\mathbb{K}$, respectively, and let $T : A \to B$ be a map. We say that T is *range preserving* if $\mathrm{Ran}_Y(T(f)) = \mathrm{Ran}_X(f)$ for all $f \in A$. The map T is called *multiplicatively range preserving* if $\mathrm{Ran}_Y(T(f)T(g)) = \mathrm{Ran}_X(fg)$ for all $f, g \in A$. We say that T is *peripherally range preserving* if $\mathrm{Ran}_{\pi,Y}(T(f)) = \mathrm{Ran}_{\pi,X}(f)$ for all $f \in A$. The map T is called *peripherally multiplicative range preserving* if $\mathrm{Ran}_{\pi,Y}(T(f)T(g)) = \mathrm{Ran}_{\pi,X}(fg)$ for all $f, g \in A$.

Molnár [14] showed that if $T : C_{\mathbb{K}}(X) \to C_{\mathbb{K}}(X)$ is a surjective multiplicatively spectrum preserving map, whenever X is a first countable compact Hausdorff space, then there exists a homeomorphism $\varphi : Y \to X$ such that $T(f) = T(1_X) \cdot (f \circ \varphi)$ for all $f \in C_{\mathbb{K}}(X)$. Rao and Roy [16] obtained a generalization of Molnár's theorem for natural complex uniform function algebras. Lambert, Luttman, and Tonev [11] characterized multiplicatively uniform norm preserving maps between complex uniform function algebras and showed that these maps are essentially weighted composition operators in modulus. Lee [12] generalized their results to complex function algebras. Hatori, Miura, and Takagi [4] generalized the result of Rao and Roy for multiplicatively range preserving maps between complex uniform function algebras. Pazandeh and Sady [15] studied the surjective maps $T : A \to B$ satisfying the nonsymmetric multiplicatively norm condition $\|T(f)T(g) - 1_Y\|_Y = \|fg - 1_X\|_X$ whenever A and B are either certain uniformly closed complex subspaces of $C(X)$ and $C(Y)$, respectively, or their positive cones and characterized the structure of these maps.

For a topological space X, a self-map τ of X is called a *topological involution* on X if τ is continuous and $\tau(\tau(x)) = x$ for all $x \in X$. Let X be a compact Hausdorff space and let τ be a topological involution on X. The map $\tau^* : C(X) \to C(X)$ defined by $\tau^*(f) = \overline{f} \circ \tau$, $f \in C(X)$, is an algebra involution on $C(X)$, which is called the *algebra involution induced by* τ on $C(X)$. Define

$$C(X, \tau) = \{f \in C(X) : \tau^*(f) = f\}.$$

Then $C(X, \tau)$ is a self-adjoint uniformly closed real subalgebra of $C(X)$ that contains 1_X and separates the points of X. In addition, $i1_X \notin C(X, \tau)$ and $C(X) = C(X, \tau) \oplus iC(X, \tau)$. A *real function algebra* on (X, τ) is a real subalgebra of $C(X, \tau)$ that contains 1_X and separates the points of X. A *real uniform function algebra* on (X, τ) is a uniformly closed function algebra on (X, τ). These algebras were first introduced in [9]. We refer the reader to the book by Kulkarni and Limaye [10] for details and background on the real uniform function algebras. It is easy to see that if A is a nonempty self-adjoint subset of $C(X, \tau)$, then $\overline{A}^{\|\cdot\|_X}$ is self-adjoint, where $\overline{A}^{\|\cdot\|_X}$ is the uniform closure of A in $C(X, \tau)$.

Let X be a compact Hausdorff space and let A be a nonempty subset of $C(X)$. A function $f \in A$ is called a *peaking function* in A if $\mathrm{Ran}_{\pi,X}(f) = \{1\}$. Let $\mathcal{P}(A)$ denote the set of all peaking function in A. For $x \in X$, we denote by $\mathcal{P}_x(A)$ the set of all $f \in \mathcal{P}(A)$ for which $f(x) = 1$. For a nonempty subset E of X, let $\mathcal{F}_E(A)$ denote the set of all $f \in A$ for which $\|f\|_X = 1$ and $|f(x)| = 1$ whenever $x \in E$. Note that $\mathcal{F}_E(A)$ contains 1_X. For $x \in X$, we write $\mathcal{F}_x(A)$ instead of $\mathcal{F}_{\{x\}}(A)$. It is easy to see that if τ is a topological involution on X and A is a nonempty subset of $C(X, \tau)$, then $\mathcal{F}_x(A) = \mathcal{F}_{\tau(x)}(A) = \mathcal{F}_{\{x,\tau(x)\}}(A)$.

Let X be a compact Hausdorff space, let τ be a topological involution on X, and let A be a nonempty subset of $C(X, \tau)$. A function $f \in A$ is called an (i)-*peaking function* in A if $\mathrm{Ran}_{\pi,X}(f) = \{-i, i\}$. For $x \in X$, we denote by $i\mathcal{P}_x(A)$ the set of all (i)-peaking functions f in A for which $f(x) = i$. These types of functions were first introduced in [13].

Let (X, d) and (Y, ρ) be metric spaces. A map $\psi : X \to Y$ is called a *Lipschitz mapping* from (X, d) to (Y, ρ) if there exists a positive constant C such that $\rho(\psi(x), \psi(y)) \leq Cd(x, y)$ for all $x, y \in X$. A *Lipschitz homeomorphism* from (X, d) to (Y, ρ) is a bijective map $\psi : X \to Y$ such that ψ is a Lipschitz mapping from (X, d) to (Y, ρ) and ψ^{-1} is a Lipschitz mapping from (Y, ρ) to (X, d).

Let (X, d) be a metric space. For a $\mathbb{K}$-valued function f on X, the *Lipschitz constant* f is denoted by

$$p_{(X,d)}(f) = \sup\left\{\frac{|f(x) - f(y)|}{d(x,y)} : x, y \in X, x \neq y\right\}.$$

A $\mathbb{K}$-valued function f on X is called a *Lipschitz function* on (X, d) if $p_{(X,d)}(f) < \infty$, i.e., f is a Lipschitz mapping from (X, d) to $\mathbb{K}$ with the Euclidean metric.

Let (X, d) be a compact metric space. We denote by $\mathrm{Lip}_{\mathbb{K}}(X, d)$ the set of all $\mathbb{K}$-valued Lipschitz functions f on (X, d). Then $\mathrm{Lip}_{\mathbb{K}}(X, d)$ is a subalgebra of $C_{\mathbb{K}}(X)$ that contains 1_X and separates the points of X. We write $\mathrm{Lip(X, d)}$ instead of $\mathrm{Lip}_{\mathbb{C}}(X, d)$. These algebras are called *Lipschitz algebras* and were first studied by Sherbert [17, 18].

Surjective multiplicatively norm preserving maps and peripherally spectrum preserving maps between Lipschitz algebras studied in [5, 6, 7]. Burgos, Jiménez-Vargas, and Villegas-Vallecillos [3] characterized the structure of surjective maps T between Lipschitz algebras $\mathrm{Lip}(X, d)$ and $\mathrm{Lip}(Y, \rho)$ satisfying the nonsymmetric norm conditions $\|T(f)\overline{T(g)} - 1_Y\|_Y = \|f\overline{g} - 1_X\|_X$.

Let (X, d) be a metric space. A self-map τ of X is called a *Lipschitz involution* on (X, d) if τ is a Lipschitz mapping from (X, d) to (X, d) and $\tau(\tau(x)) = x$ for all $x \in X$.

Let (X, d) be a compact metric space, let τ be a Lipschitz involution on (X, d), and let τ^* be the algebra involution induced by τ on $C(X)$. Then $\tau^*(\mathrm{Lip}(X, d)) = \mathrm{Lip}(X, d)$. Define

$$\mathrm{Lip}(X, d, \tau) = \{f \in \mathrm{Lip}(X, d) : \tau^*(f) = f\}.$$

Then $\mathrm{Lip}(X, d, \tau)$ is a self-adjoint real subalgebra of $C(X, \tau)$ and $\mathrm{Lip}(X, d)$ that contains 1_X and separates the points of X. In addition, $i1_X \notin \mathrm{Lip}(X, d, \tau)$ and

$$\mathrm{Lip}(X, d) = \mathrm{Lip}(X, d, \tau) \oplus i\mathrm{Lip}(X, d, \tau).$$

Note that $\mathrm{Lip}(X, d, \tau) = \mathrm{Lip}_{\mathbb{R}}(X, d)$ if and only if τ is the identity map on X. These algebras are called *real Lipschitz algebras of complex-valued functions with Lipschitz involution* and were first introduced in [2]. Surjective peripherally multiplicative spectrum preserving maps between these algebras were studied and characterized in [1]. Since $\mathrm{Lip}(X, d, \tau)$ is a self-adjoint real function algebra on (X, τ), we deduce that $\overline{\mathrm{Lip}(X, d, \tau)}^{\|\cdot\|_X}$ is a self-adjoint uniform function algebra on (X, τ). Therefore, by [9, Proposition 1.1], we have $\overline{\mathrm{Lip}(X, d, \tau)}^{\|\cdot\|_X} = C(X, \tau)$.

Let (X, d) and (Y, ρ) be compact metric spaces, let τ and η be Lipschitz involutions on (X, d) and (Y, ρ), respectively, let A be a real subalgebra of $C(X, \tau)$ containing $\mathrm{Lip}(X, d, \tau)$, and let B be a real subalgebra of $C(Y, \eta)$ containing $\mathrm{Lip}(Y, \rho, \eta)$. Let $x_\tau = \{x, \tau(x)\}$ for $x \in X$ and $y_\eta = \{y, \eta(y)\}$ for $y \in Y$. We denote by X_τ the set of all x_τ for which $x \in X$ and Y_η the set of all y_η for which $y \in Y$. In section 2, we give a description of surjective multiplicatively uniform norm preserving maps $T\colon A \to B$ and show that for such a map T, there exists a unique bijective map $\Psi\colon X_\tau \to Y_\eta$ such that $|f(x)| = |T(f)(y)|$ for all $f \in A$, $x \in X$ and $y \in \Psi(x_\tau)$. In section 3, we give a description of surjective maps $T\colon A \to B$ satisfying the nonsymmetric uniform norm condition

$\|T(f)T(g) - 1_Y\|_Y = \|fg - 1_X\|_X$ for all $f, g \in A$. For such a map T, we show that $(T(1_X))^2 = 1_Y$ and T is a multiplicatively uniform norm preserving. In the case that $A = \mathrm{Lip}(X, d, \tau)$ and $B = \mathrm{Lip}(Y, \rho, \eta)$, we prove that there exists a bijective map $\varphi\colon Y \to X$ with $\varphi \circ \eta = \tau \circ \varphi$ on Y such that φ is continuous at each $y \in Y$ with $\eta(y) = y$ and φ^{-1} is continuous at each $x \in X$ with $\tau(x) = x$ and $T(f) = T(1_X) \cdot (f \circ \varphi)$ for all $f \in A_{\mathbb{R}}$, where $A_{\mathbb{R}}$ is the set of all real-valued functions in A. Finally, we show that if T is a surjective map from $\mathrm{Lip}_{\mathbb{R}}(X, d)$ to $\mathrm{Lip}_{\mathbb{R}}(Y, \rho)$ satisfying $\|T(f)T(g) - 1_Y\|_Y = \|fg - 1_X\|_X$ for all $f, g \in \mathrm{Lip}_{\mathbb{R}}(X, d)$, then $(T(1_X))^2 = 1_Y$ and there exists a Lipschitz homeomorphism φ from (Y, ρ) to (X, d) such that $T(f) = T(1_X) \cdot (f \circ \varphi)$ for all $f \in \mathrm{Lip}_{\mathbb{R}}(X, d)$.

2. Multiplicatively uniform norm preserving maps

Throughout this section, we assume that (X, d) and (Y, ρ) are compact metric spaces, that τ and η are Lipschitz involutions on X and Y, respectively, that A is a real subalgebra of $C(X, \tau)$ that contains $\mathrm{Lip}(X, d, \tau)$, and that B is a real subalgebra of $C(Y, \eta)$ that contains $\mathrm{Lip}(Y, \rho, \eta)$.

Our purpose in this section is to show that every surjective multiplicatively uniform norm preserving map $T : A \to B$ induces a bijective map $\Psi : X_\tau \to Y_\eta$ in such a way that if $x \in X$ and $y \in \Psi(x_\tau)$, then $|f(x)| = |T(f)(y)|$ for all $f \in A$. To this aim, we need some lemmas.

Lemma 2.1. *Let $T : A \to B$ be a uniform norm preserving map and $|T(f)| \leq |T(g)|$ on Y whenever $f, g \in A$ with $|f| \leq |g|$ on X. Then for every $x \in X$, there exists $y \in Y$ such that $T(\mathcal{F}_x(A)) \subseteq \mathcal{F}_y(B)$.*

Proof. Let $x \in X$. For each $f \in \mathcal{F}_x(A)$, define

$$F_x(f) = \{y \in Y : T(f) \in \mathcal{F}_y(B)\}.$$

Let $f \in \mathcal{F}_x(A)$. Since Y is compact and $T(f) \in C(Y)$, there exists $y_0 \in Y$ such that $\|T(f)\|_Y = |T(f)(y_0)|$. According to $f \in \mathcal{F}_x(A)$ and T is uniform norm preserving, we get $\|T(f)\|_Y = \|f\|_X = 1$. Therefore, $\|T(f)\|_Y = 1 = |T(f)(y_0)|$ and so $y_0 \in F_x(f)$. Thus $F_x(f)$ is nonempty. According to the continuity of $|T(f)|$ on Y and the definition of $\mathcal{F}_w(B)$ for $w \in Y$, we deduce that $F_x(f)$ is a closed subset of Y.

To prove the assertion, it is sufficient to show that $\bigcap_{f \in \mathcal{F}_x(A)} F_x(f)$ is nonempty. To this end, we prove that $\{F_x(f) : f \in \mathcal{F}_x(A)\}$ has the finite intersection property because Y is a compact Hausdorff space. Pick $f_1, \ldots, f_n \in \mathcal{F}_x(A)$ and take $g = \prod_{j=1}^n f_j$. Then $g \in A$ and $|g| \leq |f_j|$ on X for all $j \in \{1, \ldots, n\}$. Therefore, $|T(g)| \leq |T(f_j)|$ on Y for all $j \in \{1, \ldots, n\}$. We claim that

$$F_x(g) \subseteq F_x(f_j) \tag{2.1}$$

for all $j \in \{1, \ldots, n\}$. Let $j \in \{1, \ldots, n\}$ be given. Assume that $y \in F_x(g)$. Then $y \in Y$ and $T(g) \in \mathcal{F}_y(B)$. This implies that $\|T(g)\|_Y = |T(g)(y)| = 1$. According to the fact that T is uniform norm preserving and $\|f_j\|_X = 1$, we get

$$1 = |T(g)(y)| \leq |T(f_j)(y)| \leq \|T(f_j)\|_Y = \|f_j\|_X = 1.$$

Thus $T(f_j) \in \mathcal{F}_y(B)$. Therefore, $y \in F_x(f_j)$ and so (2.1) holds. Since (2.1) holds for all $j \in \{1, \ldots, n\}$, we deduce that $F_x(g) \subseteq \bigcap_{j=1}^n F_x(f_j)$. Therefore, $\bigcap_{j=1}^n F_x(f_j)$ is nonempty since $F_x(g)$ is nonempty. □

Lemma 2.2. *Let $T : A \to B$ be a map and let $|f| \leq |g|$ on X, whenever $f, g \in A$ with $|T(f)| \leq |T(g)|$ on Y. Suppose that $x, z \in X$. Then $x_\tau = z_\tau$ if and only if $T(\mathcal{F}_x(A)) \subseteq T(\mathcal{F}_z(A))$.*

Proof. The necessity part is obvious. To prove the sufficiency part, assume that $T(\mathcal{F}_x(A)) \subseteq T(\mathcal{F}_z(A))$ but $x_\tau \neq z_\tau$. By [1, Lemma 2.1], there exists a function $h \in \mathrm{Lip}(X, d, \tau)$ with $h(x) = h(\tau(x)) = 1$ and $0 \leq h(t) < 1$ for all $t \in X \setminus \{x, \tau(x)\}$. Clearly, $h \in \mathcal{F}_x(\mathrm{Lip}(X, d, \tau))$ and so $h \in \mathcal{F}_x(A)$. Therefore, there exists $f \in \mathcal{F}_z(A)$ such that $T(h) = T(f)$ since $T(\mathcal{F}_x(A)) \subseteq T(\mathcal{F}_z(A))$. Thus $|T(f)| \leq |T(h)|$ on Y. This implies that $|f| \leq |h|$ on X. Therefore, $1 = |f(z)| \leq |h(z)| < 1$, which is impossible. Hence, the sufficiency part holds. □

Lemma 2.3. *Let (X, d) be a compact metric space, let $\tau : X \to X$ be a Lipschitz involution on (X, d), let A be a real subalgebra of $C(X, \tau)$ that contains* $\mathrm{Lip}(X, d, \tau)$, *and let $x, z \in X$. Then $x_\tau = z_\tau$ if and only if $\mathcal{F}_x(A) \subseteq \mathcal{F}_z(A)$.*

Note that Lemma 2.3 is an immediate consequence of [1, Lemma 2.8] with exactly the same proof since $\mathrm{Lip}(X, d, \tau)$ is a subset of A.

Lemma 2.4. *Let $T : A \to B$ be a surjective map satisfying the following conditions:*

(a) *T is uniform norm preserving.*
(b) *If $f, g \in A$, then $|f| \leq |g|$ on X if and only if $|T(f) \leq |T(g)|$ on Y.*

Then the following assertions hold:

(i) *For each $y \in Y$, there exists $z \in X$ such that $\mathcal{F}_y(B) \subseteq T(\mathcal{F}_z(A))$.*
(ii) *For each $x \in X$, there exists a unique element y_η in Y_η such that $T(\mathcal{F}_x(A)) = \mathcal{F}_y(B)$.*

Proof. The surjectivity of $T : A \to B$ implies that there exists a map $S : B \to A$ such that $T \circ S = I_B$, the identity map on B. Since T is uniform norm preserving, for each $h \in B$ we have

$$\|h\|_Y = \|(T \circ S)(h)\|_Y = \|T(S(h))\|_Y = \|S(h)\|_X .$$

Therefore, S is a uniform norm preserving map. Let $h, k \in B$ with $|h| \leq |k|$ on Y. Then $|T(S(h))| \leq |T(S(k))|$ on Y. It follows that $|S(h)| \leq |S(k)|$ on X by (b). By Lemma 2.1, for each $y \in Y$, there exists $x \in X$ such that $S(\mathcal{F}_y(B)) \subseteq \mathcal{F}_x(A)$, which implies that $\mathcal{F}_y(B) \subseteq T(\mathcal{F}_x(A))$ since $T \circ S = I_B$. Hence, (i) holds.

To prove (ii), let $x \in X$. By Lemma 2.1, there exists $y \in Y$ such that

$$T(\mathcal{F}_x(A)) \subseteq \mathcal{F}_y(B). \tag{2.2}$$

By (i), there exists $z \in X$ such that

$$\mathcal{F}_y(B) \subseteq T(\mathcal{F}_z(A)). \tag{2.3}$$

Using (2.2) and (2.3), we get

$$T(\mathcal{F}_x(A)) \subseteq T(\mathcal{F}_z(A)). \tag{2.4}$$

By (2.4) and Lemma 2.2, we deduce that $z_\tau = x_\tau$, which implies that

$$T(\mathcal{F}_z(A)) = T(\mathcal{F}_x(A)). \tag{2.5}$$

Applying (2.5) and (2.3), we get

$$\mathcal{F}_y(B) \subseteq T(\mathcal{F}_x(A)). \tag{2.6}$$

According to (2.2) and (2.6), we deduce that there exists $y_\eta \in Y_\eta$ such that $T(\mathcal{F}_x(A)) = \mathcal{F}_y(B)$. The uniqueness of y_η follows from Lemma 2.3. Hence, (ii) holds. □

Lemma 2.5. *Let (X, d) be a compact metric space, let τ be a Lipschitz involution on X, and let A be a real subalgebra of $C(X, \tau)$ that contains* $\mathrm{Lip}(X, d, \tau)$*. Then*

$$|f(x)| = \inf\{\|fg\|_X : g \in \mathcal{P}_x(\mathrm{Lip}(X, d, \tau))\} = \inf\{\|fg\|_X : g \in \mathcal{F}_x(A)\}$$

for all $f \in A$ and $x \in X$.

Proof. Let $f \in A$ and $x \in X$ be given. Assume that $g \in \mathcal{F}_x(A)$. Then $g \in A$, $\|g\|_X = 1$ and $|g(x)| = 1$. Hence,

$$|f(x)| = |f(x)||g(x)| = |fg(x)| \leq \|fg\|_X .$$

Therefore, $|f(x)|$ is a lower bound for $\{\|fg\|_X : g \in \mathcal{F}_x(A)\}$, and so $|f(x)|$ is a lower bound for $\{\|fg\|_X : g \in \mathcal{P}_x(\mathrm{Lip}(X, d, \tau))\}$ since $\mathcal{P}_x(\mathrm{Lip}(X, d, \tau))$ is a subset of $\mathcal{F}_x(A)$.

Let $\varepsilon > 0$ be given. Take

$$U_\varepsilon = \{z \in X : |f(z)| < |f(x)| + \varepsilon\} .$$

Then U_ε is a τ-invariant subset of X, $x \in U_\varepsilon$, and U_ε is an open set in (X, d). We first assume that $U_\varepsilon = X$. In this case, take $g_\varepsilon = 1_X$. Then $g_\varepsilon \in \mathcal{P}_x(\mathrm{Lip}(X, d, \tau))$ and so $g_\varepsilon \in \mathcal{F}_x(A)$. Furthermore, $\|fg_\varepsilon\|_X < |f(x)| + \varepsilon$ since X is a compact set in (X, d) and for each $z \in X$, we have $|f(z)g_\varepsilon(z)| < |f(x)| + \varepsilon$. We now assume that U_ε is a proper subset of X. Then $X \setminus U_\varepsilon$ is a nonempty compact set in (X, d) and $x_\tau \cap (X \setminus U_\varepsilon) = \emptyset$. Therefore, $d(X \setminus U_\varepsilon, x_\tau) > 0$, where $d(X \setminus U_\varepsilon, x_\tau) = \inf\{d(s, t) : s \in X \setminus U_\varepsilon, t \in x_\tau\}$. Define the function $f_\varepsilon : X \to \mathbb{C}$ by

$$f_\varepsilon(z) = \max\left\{0, 1 - \frac{d(z, x_\tau)}{d(X \setminus U_\varepsilon, x_\tau)}\right\} \qquad (z \in X),$$

where $d(z, x_\tau) = \inf\{d(z, t) : t \in x_\tau\} = \min\{d(z, x), d(z, \tau(x))\}$. It is easy to see that $f_\varepsilon \in \mathrm{Lip}(X, d)$, $0 \leq f_\varepsilon(z) \leq 1$ for all $x \in X$ and $f_\varepsilon(x) = f_\varepsilon(\tau(x)) = 1$. Furthermore, if $z \in X \setminus U_\varepsilon$, then $d(z, x_\tau) \geq d(X \setminus U_\varepsilon, x_\tau) > 0$, which implies that $f_\varepsilon(z) = 0$. Take $g_\varepsilon = f_\varepsilon \cdot \tau^*(f_\varepsilon)$. Then $g_\varepsilon \in \mathrm{Lip}(X, d, \tau)$, $0 \leq g_\varepsilon(z) \leq 1$ for all $z \in X$, $g_\varepsilon(x) = g_\varepsilon(\tau(x)) = 1$ and $g_\varepsilon(z) = 0$ for all $z \in X \setminus U_\varepsilon$. Clearly, $g_\varepsilon \in \mathcal{P}_x(\mathrm{Lip}(X, d, \tau))$. If $z \in U_\varepsilon$, then

$$|f(z)g_\varepsilon(z)| = |f(z)||g_\varepsilon(z)| \leq |f(z)| < |f(x)| + \varepsilon.$$

If $z \in X \setminus U_\varepsilon$, then

$$|f(z)g_\varepsilon(z)| = 0 < |f(x)| + \varepsilon.$$

Therefore, $|f(z)g_\varepsilon(z)| < |f(x)| + \varepsilon$ for all $z \in X$. This implies that $\|fg_\varepsilon\|_X < |f(x)| + \varepsilon$ since $|fg_\varepsilon|$ is a real-valued continuous function on (X, d) and X is a compact set in (X, d). In fact, it is shown that for each $\varepsilon > 0$, there exists a function $g_\varepsilon \in \mathcal{P}_x(\mathrm{Lip}(X, d, \tau))$ such that $\|fg_\varepsilon\|_X < |f(x)| + \varepsilon$. Therefore,

$$|f(x)| = \inf\{\|fg\|_X : g \in \mathcal{P}_x(\mathrm{Lip}(X, d, \tau))\}.$$

This implies that

$$|f(x)| = \inf\{\|fg\|_X : g \in \mathcal{F}_x(A)\}$$

since $|f(x)|$ is a lower bound for $\{\|fg\|_X : g \in \mathcal{F}_x(A)\}$ and $\mathcal{P}_x(\mathrm{Lip}(X, d, \tau))$ is a subset of $\mathcal{F}_x(A)$. □

Lemma 2.6. *Let (X, d) be a compact metric space, let $\tau : X \to X$ be a Lipschitz involution on (X, d), let A be a real subalgebra of $C(X, \tau)$ containing $\mathrm{Lip}(X, d, \tau)$, and let $f, g \in A$. If $\|fh\|_X \leq \|gh\|_X$ for all $h \in \mathcal{P}(\mathrm{Lip}(X, d, \tau))$, then $|f(x)| \leq |g(x)|$ for all $x \in X$.*

Note that Lemma 2.6 is an extended version of [1, Lemma 2.7(i)] with exactly the same proof since $\mathrm{Lip}(X, d, \tau)$ is a subset of A.

Theorem 2.7. *Let $T : A \to B$ be a surjective multiplicatively uniform norm preserving map. Then there exists a unique bijective map $\Psi : X_\tau \to Y_\eta$ such that $|f(x)| = |T(f)(y)|$ for all $f \in A$, $x \in X$ and $y \in \Psi(x_\tau)$.*

Proof. We divide the proof into five parts.

Part 1. *T is a uniform norm preserving map.*

Let $f \in A$. Since T is multiplicatively uniform norm preserving, we get

$$0 \leq \|T(f)\|_Y^2 = \left\|T(f)^2\right\|_Y = \left\|f^2\right\|_X = \|f\|_X^2.$$

Therefore, $\|T(f)\|_Y = \|f\|_X$. Hence, Part 1 holds.

Note that Part 1 is valid without assuming T is a surjective map.

Part 2. *For any $f, g \in A$, $|f| \leq |g|$ on X if and only if $|T(f)| \leq |T(g)|$ on Y.*

We first assume that $f, g \in A$ with $|f| \leq |g|$ on X. Let $k \in \mathcal{P}(\mathrm{Lip}(Y, \rho, \eta))$ be arbitrary. The surjectivity of T implies that $k = T(h)$ for some $h \in A$. Since

$$|f(x)h(x)| = |f(x)||h(x)| \leq |g(x)||h(x)| = |g(x)h(x)| \leq \|gh\|_X$$

for all $x \in X$, we deduce that $\|fh\|_X \leq \|gh\|_X$. It follows that $\|T(f)T(h)\|_Y \leq \|T(g)T(h)\|_Y$ and so

$$\|T(f)k\|_Y \leq \|T(g)k\|_Y. \tag{2.7}$$

Since (2.7) holds for all $k \in \mathcal{P}(\mathrm{Lip}(Y, \rho, \eta))$, we deduce that $|T(f)| \leq |T(g)|$ on Y by Lemma 2.6.

We now assume that $f, g \in A$ with $|T(f)| \leq |T(g)|$ on Y. Let $h \in \mathcal{P}(\mathrm{Lip}(X, d, \tau))$ be arbitrary. Then $T(h) \in B$, and for each $y \in Y$ we have

$$\begin{aligned}|T(f)(y)T(h)(y)| &= |T(f)(y)||T(h)(y)| \leq |T(g)(y)||T(h)(y)| \\ &= |T(g)(y)T(h)(y)| \leq \|T(g)T(h)\|_Y.\end{aligned}$$

It follows that $\|T(f)T(h)\|_Y \leq \|T(g)T(h)\|_Y$, which implies that

$$\|fh\|_X \leq \|gh\|_X. \tag{2.8}$$

Since (2.8) holds for all $h \in \mathcal{P}(\mathrm{Lip}(X, d, \tau))$, we deduce that $|f| \leq |g|$ on X by Lemma 2.6. Hence, Part 2 holds.

Part 3. *Define the map $\Psi : X_\tau \to Y_\eta$ by $\Psi(x_\tau) = y_\eta$, where $x \in X$ and y_η is the unique element of Y_η such that $T(\mathcal{F}_x(A)) = \mathcal{F}_y(B)$. Then Ψ is well-defined and it is a bijective map.*

According to Parts 1 and 2, the surjective map T satisfies the conditions of Lemma 2.4, and hence, Ψ is well-defined.

To prove the injectivity of Ψ, let $x, z \in X$ with $\Psi(x_\tau) = \Psi(z_\tau)$. Then $T(\mathcal{F}_x(A)) = T(\mathcal{F}_z(A))$. By Lemma 2.2, we deduce that $x_\tau = z_\tau$. Hence, Ψ is injective.

To prove the surjectivity of Ψ, let $y \in Y$. By (i) of Lemma 2.4, there exists $x \in X$ such that

$$\mathcal{F}_y(B) \subseteq T(\mathcal{F}_x(A)). \tag{2.9}$$

By (ii) of Lemma 2.4, there exists a unique element $w_\eta \in Y_\eta$ such that

$$T(\mathcal{F}_x(A)) = \mathcal{F}_w(B). \tag{2.10}$$

It follows that $\Psi(x_\tau) = w_\eta$. Therefore, by (2.9) and (2.10), we get $\mathcal{F}_y(B) \subseteq \mathcal{F}_w(B)$, which implies that $y_\eta = w_\eta$ by Lemma 2.3. Thus, $y_\eta = \Psi(x_\tau)$ and so Ψ is surjective. Hence, Part 3 holds.

Part 4. *$|f(x)| = |T(f)(y)|$ for all $f \in A$, $x \in X$ and $y \in \Psi(x_\tau)$.*

Let $f \in A$, $x \in X$ and $y \in \Psi(x_\tau)$. According to the definition of Ψ, we get

$$T(\mathcal{F}_x(A)) = \mathcal{F}_y(B).$$

It follows that

$$\begin{aligned} |f(x)| &= \inf\{\|fg\|_X : g \in \mathcal{F}_x(A)\} \\ &= \inf\{\|T(f)T(g)\|_Y : T(g) \in \mathcal{F}_y(B)\} \\ &= \inf\{\|T(f)k\|_Y : k \in \mathcal{F}_y(B)\} \\ &= |T(f)(y)|, \end{aligned}$$

by using Lemma 2.5 and the surjectivity of T. Hence, Part 4 holds.

Part 5. *Ψ is unique, that is, if $\Phi : X_\tau \to Y_\eta$ satisfies $|f(x)| = |T(f)(y)|$ for all $f \in A$, $x \in X$ and $y \in \Phi(x_\tau)$, then $\Phi = \Psi$ on X_τ.*

Let $\Phi : X_\tau \to Y_\eta$ be a map satisfying $|f(x)| = |T(f)(y)|$ for all $f \in A$, $x \in X$ and $y \in \Phi(x_\tau)$. We show that $\Phi = \Psi$ on X_τ. Let $x \in X$ be given. Assume that $y \in Y$ with $\Psi(x_\tau) = y_\eta$ and $w \in Y$ with $\Phi(x_\tau) = w_\eta$. Then $y \in \Psi(x_\tau)$ and $w \in \Phi(x_\tau)$. We claim that

$$T(\mathcal{F}_x(A)) = \mathcal{F}_w(B). \tag{2.11}$$

Let $f \in \mathcal{F}_x(A)$. Then

$$|T(f)(y)| = |f(x)| = |T(f)(w)|. \tag{2.12}$$

The definition of Ψ implies that $T(f) \in \mathcal{F}_y(B)$. Therefore, from (2.12) we get

$$|T(f)(w)| = \|T(f)\|_Y = 1.$$

This implies that $T(f) \in \mathcal{F}_w(B)$. Hence,

$$T(\mathcal{F}_x(A)) \subseteq \mathcal{F}_w(B). \tag{2.13}$$

Let $k \in \mathcal{F}_w(B)$. Then

$$|k(w)| = \|k\|_Y = 1. \tag{2.14}$$

The surjectivity of T implies that $k = T(g)$ for some $g \in A$. By (i) of Lemma 2.4, we get

$$\|g\|_X = \|k\|_Y = 1. \tag{2.15}$$

On the other hand,

$$|g(x)| = |T(g)(w)| = |k(w)|. \tag{2.16}$$

According to (2.16), (2.14), and (2.15), we get $|g(x)| = 1 = \|g\|_X$. Thus $g \in \mathcal{F}_x(A)$ and so $k \in T(\mathcal{F}_x(A))$. Therefore,

$$\mathcal{F}_w(B) \subseteq T(\mathcal{F}_x(A)). \tag{2.17}$$

Hence, our claim is justified by (2.13) and (2.17). According to (2.11) and the definition of Ψ, we get $w_\eta = \Psi(x_\tau)$. Therefore, $\Phi(x_\tau) = \Psi(x_\tau)$. Since this equality holds for all $x \in X$, we deduce that $\Phi = \Psi$ on X_τ. Hence, Part 5 holds. □

3. Nonsymmetric uniform norm preserving maps

Throughout this section, we assume that A is a real subalgebra of $C(X,\tau)$ that contains $\mathrm{Lip}(X,d,\tau)$ and that B is a real subalgebra of $C(Y,\eta)$ that contains $\mathrm{Lip}(Y,\rho,\eta)$ whenever (X,d) and (Y,ρ) are compact metric spaces and τ and η are Lipschitz involutions on X and Y, respectively.

We first give a description of surjective mappings $T : A \to B$ satisfying the nonsymmetric uniform norm condition

$$\|T(f)T(g) - 1_Y\|_Y = \|fg - 1_X\|_X$$

for all $f, g \in A$. For such a map T, we first prove that $(T(1_X))^2 = 1_Y$ and that T is a multiplicatively uniform norm preserving map. Next, in the case that $A = \mathrm{Lip}(X,d,\tau)$ and $B = \mathrm{Lip}(Y,\rho,\eta)$, we show that T is an essentially weighted composition operator on $A_{\mathbb{R}}$, the set of all $f \in A$ for which f is real-valued.

Lemma 3.1. *Let $T : A \to B$ be a map satisfying $\|T(f)T(g) - 1_Y\|_Y = \|fg - 1_X\|_X$ for all $f, g \in A$. Then*

- (i) *$(T(\alpha 1_X))^2 = 1_Y$, where $\alpha \in \{-1, 1\}$. In particular, $T(1_X) = 1_Y$ or $T(1_X) = -1_Y$ whenever Y is connected.*
- (ii) *$T(\alpha^{-1}1_X)T(\alpha 1_X) = 1_Y$ for all $\alpha \in \mathbb{R} \setminus \{0\}$.*

Proof. To prove (i), let $\alpha \in \{-1, 1\}$. Then

$$\left\|(T(\alpha 1_X))^2 - 1_Y\right\|_Y = \left\|(\alpha 1_X)^2 - 1_X\right\|_X = \|0_X\|_X = 0,$$

which implies that $(T(\alpha 1_X))^2 = 1_Y$.

If Y is connected, then $T(1_X)(Y)$ is a connected set in complex plane $\mathbb{C}$. Therefore, $T(1_X)(y) = 1$ for all $y \in Y$ or $T(1_X)(y) = -1$ for all $y \in Y$. Hence, (i) holds.

To prove (ii), let $\alpha \in \mathbb{R} \setminus \{0\}$. Since

$$\left\|T(\alpha^{-1}1_X)T(\alpha 1_X) - 1_Y\right\|_Y = \left\|\alpha^{-1}1_X\alpha 1_X - 1_X\right\|_X = \|0_X\|_X = 0,$$

we deduce that $T(\alpha^{-1}1_X)T(\alpha 1_X) = 1_Y$. Hence, (ii) holds. □

Lemma 3.2. *Let $T : A \to B$ be a surjective map satisfying $\|T(f)T(g) - 1_Y\|_Y = \|fg - 1_X\|_X$ for all $f, g \in A$. If $f \in A$ and $y \in Y$, then $|T(\alpha f)(y)| = |\alpha||T(f)(y)|$ for all $\alpha \in \mathbb{R}$. In particular, $|T(\alpha 1_X)| = |\alpha| 1_Y$ for all $\alpha \in \mathbb{R}$.*

Proof. To prove the assertion, we first show that the following statement holds:

If $g \in A$ and $y \in Y$, then $|T(\beta g)(y)| \leq |\beta||T(g)(y)|$ for all $\beta \in \mathbb{R}$.

Let $g \in A$ and $y \in Y$. Assume that $\beta \in \mathbb{R}$. Let $\varepsilon > 0$ be given. Since $T(g) \in C(Y, \eta)$ and $C(Y, \eta) = \overline{\mathrm{Lip}(Y, \rho, \eta)}^{\|\cdot\|_Y}$, we deduce that there exists a function $h_\varepsilon \in \mathrm{Lip}(Y, \rho, \eta)$ such that

$$\|T(g) - h_\varepsilon\|_Y < \frac{\varepsilon}{4(1 + |\beta|)}. \tag{3.1}$$

By (3.1), we get

$$|h_\varepsilon(y)| < \frac{\varepsilon}{4(1 + |\beta|)} + |T(g)(y)|. \tag{3.2}$$

Define the function $h : Y \to \mathbb{C}$ by $h = h_\varepsilon - h_\varepsilon(y)1_Y$. Then $h \in \mathrm{Lip}(Y, \rho, \eta)$ and $h(y) = 0$. By Lemma 2.5, there exists a function $k \in \mathcal{P}_y(\mathrm{Lip}(Y, \rho, \eta))$ such that

$$\|hk\|_Y < \frac{\varepsilon}{4(1 + |\beta|)}. \tag{3.3}$$

According to the fact of the definition of h, (3.1), and (3.2), we get

$$\|T(g) - h\|_Y < \frac{\varepsilon}{4(1 + |\beta|)} + |T(g)(y)|. \tag{3.4}$$

Let $n \in \mathbb{N}$, and Take $k_n = nk$. Then $k_n \in B$ and $k_n(y) = n$. The surjectivity of T implies that there exists a function $g_n \in A$ such that $T(g_n) = k_n$. According to (3.4) and (3.3), we get

$$\begin{aligned}
n|T(\beta g)(y)| - 1 &= |nT(\beta g)(y)| - 1 = |k_n(y)T(\beta g)(y)| - 1 \\
&= |T(g_n)(y)T(\beta g)(y)| - 1 \leq |T(g_n)(y)T(\beta g)(y) - 1| \\
&\leq \|T(g_n)T(\beta g) - 1_Y\|_Y = \|\beta g_n g - 1_X\|_X \\
&= \|\beta(g_n g - 1_X) + \beta 1_X - 1_X\|_X \leq \|\beta(g_n g - 1_X)\|_X + |\beta| + 1 \\
&= |\beta| \, \|g_n g - 1_X\|_X + |\beta| + 1 = |\beta| \, \|T(g_n)T(g) - 1_Y\|_Y + |\beta| + 1 \\
&\leq |\beta| \, \|T(g_n)T(g)\|_Y + 2|\beta| + 1 = |\beta| \, \|k_n T(g)\|_Y + 2|\beta| + 1 \\
&= n|\beta|\|kT(g)\|_Y + 2|\beta| + 1 = n|\beta|\|k(T(g) - h) + kh\|_Y + 2|\beta| + 1 \\
&\leq n|\beta|(\|k\|_Y\|T(g) - h\|_Y + \|kh\|_Y) + 2|\beta| + 1 \\
&= n|\beta|(\|T(g) - h\|_Y + \|kh\|_Y) + 2|\beta| + 1 \\
&\leq n|\beta|(\frac{\varepsilon}{4(1 + |\beta|)} + \frac{\varepsilon}{4(1 + |\beta|)} + |T(g)(y)|) + 2|\beta| + 1 \\
&= n|\beta||T(g)(y)| + \frac{n|\beta|\varepsilon}{2(1 + |\beta|)} + 2|\beta| + 1 \\
&< n|\beta||T(g)(y)| + \frac{n\varepsilon}{2} + 2|\beta| + 1.
\end{aligned}$$

Thus,

$$n|T(\beta g)(y)| < n|\beta||T(g)(y)| + \frac{n\varepsilon}{2} + 2(|\beta| + 1),$$

and so

$$|T(\beta g)(y)| < |\beta||T(g)(y)| + \frac{\varepsilon}{2} + \frac{2(|\beta| + 1)}{n}. \tag{3.5}$$

Since the inequality (3.5) holds for all $n \in \mathbb{N}$, we deduce that

$$|T(\beta g)(y)| \leq |\beta||T(g)(y)| + \frac{\varepsilon}{2}.$$

Therefore,

$$|T(\beta g)(y)| < |\beta||T(g)(y)| + \varepsilon. \tag{3.6}$$

Since the inequality (3.6) holds for all $\varepsilon > 0$, we conclude that

$$|T(\beta g)(y)| \leq |\beta||T(g)(y)|.$$

Therefore, the mentioned statement holds.

Let $f \in A$ and $y \in Y$ be given. By the mentioned statement, we get

$$|T(\alpha f)(y)| \leq |\alpha||T(f)(y)|. \tag{3.7}$$

We now show that

$$|\alpha||T(f)(y)| \leq |T(\alpha f)(y)|. \tag{3.8}$$

It is clear that (3.8) holds for $\alpha = 0$. Assume that $\alpha \in \mathbb{R} \setminus \{0\}$. According to the mentioned statement, we have

$$|T(f)(y)| = |T(\alpha^{-1}\alpha f)(y)| \leq |\alpha^{-1}||T(\alpha f)(y)| = |\alpha|^{-1}|T(\alpha f)(y)|$$

and so (3.8) holds. By (3.7) and (3.8), we get

$$|T(\alpha f)(y)| = |\alpha||T(f)(y)|.$$

□

Proposition 3.3. *Let $T : A \to B$ be a surjective map satisfying $\|T(f)T(g) - 1_Y\|_Y = \|fg - 1_X\|_X$ for all $f, g \in A$. Then*

(i) *T is a multiplicatively uniform norm preserving map.*

(ii) *There exists a unique bijective map $\Psi : X_\tau \to Y_\eta$ such that $|f(x)| = |T(f)(y)|$ for all $f \in A$, $x \in X$ and $y \in \Psi(x_\tau)$.*

Proof. To prove (i), let $f, g \in A$. It is immediate from Lemma 3.2 that for each $n \in \mathbb{N}$, we have

$$\|T(nf)T(g)\|_Y = n\,\|T(f)T(g)\|_Y\,. \tag{3.9}$$

Thus, we get

$$\begin{aligned} n\,\|fg\|_X - 1 &\leq \|nfg - 1_X\|_X = \|T(nf)T(g) - 1_Y\|_Y \\ &\leq \|T(nf)T(g)\|_Y + 1 = n\,\|T(f)T(g)\|_Y + 1 \end{aligned}$$

for all $n \in \mathbb{N}$. Therefore, $\|fg\|_X \leq \|T(f)T(g)\|_Y + \frac{2}{n}$ for all $n \in \mathbb{N}$. This implies that

$$\|fg\|_X \leq \|T(f)T(g)\|_Y\,. \tag{3.10}$$

Since (3.9) holds for all $n \in \mathbb{N}$, we get

$$\begin{aligned} n\,\|T(f)T(g)\|_Y - 1 &= \|T(nf)T(g)\|_Y - 1 \leq \|T(nf)T(g) - 1_Y\|_Y \\ &= \|nfg - 1_X\|_X \leq n\,\|fg\|_X - 1 \end{aligned}$$

for all $n \in \mathbb{N}$. Therefore, $\|T(f)T(g)\|_Y \leq \|fg\|_X + \frac{2}{n}$ for all $n \in \mathbb{N}$. This implies that

$$\|T(f)T(g)\|_Y \leq \|fg\|_X\,. \tag{3.11}$$

According to (3.10) and (3.11), we get

$$\|T(f)T(g)\|_Y = \|fg\|_X . \tag{3.12}$$

Since (3.12) holds for all $f, g \in A$, we deduce that T is a multiplicatively uniform norm preserving map. Hence, (i) holds.

By (i), T is a multiplicatively uniform norm preserving map. Therefore, (ii) holds by Theorem 2.7. □

Definition 3.4. Let $T : A \to B$ be a surjective map satisfying $\|T(f)T(g) - 1_Y\|_Y = \|fg - 1_X\|_X$ for all $f, g \in A$. By (ii) of Proposition 3.3, there exists a unique bijective map $\Psi : X_\tau \to Y_\eta$ such that $|f(x)| = |T(f)(y)|$ for all $f \in A$, $x \in X$ and $y \in \Psi(x_\tau)$. Such a map is called the *associated map* to T.

Now, we give a straightforward lemma that will make easier the reading of the proofs.

Lemma 3.5. *Let $\alpha, \beta \in \mathbb{C}$.*

(i) *If $|\alpha - 1| = |\beta| + 1$ and $|\alpha| = |\beta|$, then $\alpha = -|\beta|$.*
(ii) *If $|\beta| = |\alpha|$, $|\beta - 1| \leq |\alpha - 1|$ and $|\beta + 1| \leq |\alpha + 1|$, then $\beta = \alpha$ or $\beta = \overline{\alpha}$.*

Proposition 3.6. *Let $T : A \to B$ be a surjective map satisfying $\|T(f)T(g) - 1_Y\|_Y = \|fg - 1_X\|_X$ for all $f, g \in A$. If $f, g \in A$, then $-1 \in \mathrm{Ran}_{\pi,X}(fg)$ if and only if $-1 \in \mathrm{Ran}_{\pi,Y}(T(f)T(g))$.*

Proof. We first assume that $f, g \in A$ with $-1 \in \mathrm{Ran}_{\pi,X}(fg)$. Then there exists $x \in X$ such that $-1 = f(x)g(x)$ and $\|fg\|_X = |f(x)g(x)| = 1$. Therefore,

$$2 = \|fg - 1_X\|_X = \|T(f)T(g) - 1_Y\|_Y .$$

It follows that there exists $y \in Y$ such that $|T(f)(y)T(g)(y) - 1| = 2$. Therefore,

$$2 \leq |T(f)(y)T(g)(y)| + 1 \leq \|T(f)T(g)\|_Y + 1 = \|fg\|_X + 1 = 2$$

since T is a multiplicatively uniform norm preserving map by (i) of Proposition 3.3. Thus $|T(f)(y)T(g(y))| = 1$. Since $|T(f)(y)T(g)(y) - 1| = |f(x)g(x)| + 1$ and $|T(f)(y)T(g)(y)| = |f(x)g(x)|$, by (i) of Lemma 3.5, we deduce that

$$T(f)(y)T(g)(y) = -|f(x)g(x)| = -1.$$

Hence, $-1 \in \mathrm{Ran}(T(f)T(g))$. It follows that $-1 \in \mathrm{Ran}_{\pi,Y}(T(f)T(g))$ since $\|T(f)T(g)\|_Y = \|fg\|_X = 1$.

We now assume that $-1 \in \mathrm{Ran}_{\pi,Y}(T(f)T(g))$. Then there exists $y \in Y$ such that $-1 = T(f)(y)T(g)(y)$ and $\|T(f)T(g)\|_X = |T(f)(y)T(g)(y)| = 1$. Therefore,

$$2 = \|T(f)T(g) - 1_Y\|_Y = \|fg - 1_X\|_X .$$

It follows that there exists $x \in X$ such that $|f(x)g(x) - 1| = 2$. Therefore,

$$2 \leq |f(x)g(x)| + 1 \leq= \|fg\|_X + 1 = \|T(f)T(g)\|_Y + 1 = 2$$

since T is a multiplicatively uniform norm preserving map by (i) of Proposition 3.3. Thus $|f(x)g(x))| = 1$. Since $|f(x)g(x) - 1| = |T(f)(y)T(g)(y)| + 1$ and $|T(f)(y)T(g)(y)| = |f(x)g(x)|$, by (i) of Lemma 3.5, we deduce that

$$f(x)g(x) = -|T(f)(y)T(g)(y)| = -1.$$

Therefore, $-1 \in \mathrm{Ran}_{\pi,X}(fg)$. It follows that $-1 \in \mathrm{Ran}_{\pi,Y}(fg)$ since $\|fg\|_X = \|T(f)T(g)\|_Y = 1$. $\square$

Proposition 3.7. *Let T be a surjective map from A to B satisfying $\|T(f)T(g) - 1_Y\|_Y = \|fg - 1_X\|_X$ for all $f, g \in A$ and let $\Psi : X_\tau \to Y_\eta$ be the associated map to T. If $x \in X$, then $T(-f)(y) = -\overline{T(f)(y)}$ for all $f \in \mathcal{P}_x(A)$ and $y \in \Psi(x_\tau)$.*

Proof. We divide the proof into two claims.

Claim 1. *If $x \in X$ and $g \in A$ with $g(x) = g(\tau(x)) = 1$ and $|g(z)| < 1$ for all $z \in X \setminus \{x, \tau(x)\}$, then $T(\alpha f)(y) = -\overline{T(-\alpha g)(y)}$ for all $\alpha \in \{-1, 1\}$, $f \in \mathcal{P}_x(A)$ and $y \in \Psi(x_\tau)$.*

Let $x \in X$ and $g \in A$ with $g(x) = g(\tau(x)) = 1$ and $|g(z)| < 1$ for all $z \in X \setminus \{x, \tau(x)\}$. Assume that $\alpha \in \{-1, 1\}$ and $f \in \mathcal{P}_x(A)$. It is easy to see that $-1 \in \mathrm{Ran}_{\pi,X}((\alpha f)(-\alpha g))$. Therefore, by Proposition 3.6, we have $-1 \in \mathrm{Ran}_{\pi,Y}(T(\alpha f)T(-\alpha g))$. This implies that there exists $w \in Y$ such that

$$T(\alpha f)(w)T(-\alpha g)(w) = -1, \tag{3.13}$$

and so

$$|T(\alpha f)(w)T(-\alpha g)(w)| = 1. \tag{3.14}$$

The surjectivity of $\Psi : X_\tau \to Y_\eta$ yields that there exists $z \in X$ such that $w_\eta = \Psi(z_\tau)$, and so $w \in \Psi(z_\tau)$. By (ii) of Proposition 3.3 and the fact that $|\alpha| = 1$, we have

$$|T(\alpha f)(w)| = |f(z)|, \qquad |T(-\alpha g)(w)| = |g(z)|. \tag{3.15}$$

According to (3.14) and (3.15), we get

$$|f(z)g(z)| = 1. \tag{3.16}$$

If $z \notin x_\tau$, then $|g(z)| < 1$, which implies that

$$|f(z)g(z)| = |f(z)||g(z)| \leq \|f\|_X |g(z)| \leq |g(z)| < 1$$

that contradicts (3.16). Therefore, $z \in x_\tau$ and so $w \in \Psi(x_\tau)$. According to $T(\alpha f), T(-\alpha g) \in B$, $-1 \in \mathbb{R}$, and (3.13), we get

$$T(\alpha f)(\eta(w))T(-\alpha g)(\eta(w)) = -1. \tag{3.17}$$

Let $y \in \Psi(x_\tau)$ be given. Then $y \in w_\eta = \{w, \eta(w)\}$. Therefore, by (3.13) and (3.17), we deduce that

$$T(\alpha f)(y)T(-\alpha g)(y) = -1. \tag{3.18}$$

By (ii) of Proposition 3.3, we have

$$|T(-\alpha g)(y)| = |-\alpha g(x)| = 1, \tag{3.19}$$

since $x \in X$, $y \in \Psi(x_\tau)$, $-\alpha g \in A$, $|\alpha| = 1$ and $g(x) = 1$. According to (3.18) and (3.19), we get

$$T(\alpha f)(y) = -\overline{T(-\alpha g)(y)}.$$

Hence, Claim 1 holds.

Claim 2. *If $x \in X$, then $T(-f)(y) = -\overline{T(f)(y)}$ for all $f \in \mathcal{P}_x(A)$ and $y \in \Psi(x_\tau)$.*

Let $x \in X$. By [1, Lemma 2.1], there exists a function $g_x \in \mathrm{Lip}(X, d, \tau)$ such that $g_x(x) = g_x(\tau(x)) = 1$ and $0 \leq g_x(z) < 1$ for all $z \in X \setminus \{x, \tau(x)\}$. Clearly, $g_x \in \mathcal{P}_x(A)$. Let $f \in \mathcal{P}_x(A)$ and $y \in \Psi(x_\tau)$ be given. By Claim 1, we have

$$T(\alpha f)(y) = -\overline{T(-\alpha g_x)(y)}, \qquad T(\alpha g_x)(y) = -\overline{T(-\alpha g_x)(y)}$$

for all $\alpha \in \{-1, 1\}$. Therefore, we get

$$T(f)(y) = -\overline{T(-g_x)(y)} = T(g_x)(y) = \overline{\overline{T(g_x)(y)}} = -\overline{T(-f)(y)}.$$

Hence, Claim 2 holds. $\square$

Proposition 3.8. *Let $T : A \to B$ be a surjective map satisfying $\|T(f)T(g) - 1_Y\|_Y = \|fg - 1_X\|_X$ for all $f, g \in A$. Then $T(\alpha 1_X) = \alpha T(1_X)$ for all $\alpha \in \mathbb{R}$.*

Proof. We first prove the assertion for $\alpha = -1$. Let $\Psi : X_\tau \to Y_\eta$ be the associated map to T. Assume that $y \in Y$. The surjectivity of Ψ implies that there exists $x \in X$ such that $y_\eta = \Psi(x_\tau)$. According to $1_X \in \mathcal{P}_x(A)$, $y \in \Psi(x_\tau)$ and $T(1_X)(y) \in \mathbb{R}$, by Proposition 3.7, we deduce that $T(-1_X)(y) = -T(1_X)(y)$. Since this equality holds for all $y \in Y$, we conclude that $T(-1_X) = -T(1_X)$.

We now prove the assertion for all $\alpha \in \mathbb{R}$. Let $\alpha \in \mathbb{R}$ be given. Assume that $y \in Y$. By Lemma 3.2, we have

$$|T(\alpha 1_X)(y)| = |\alpha|. \tag{3.20}$$

On the other hand,

$$\begin{aligned} |T(\alpha 1_X)(y)T(1_X)(y) - 1| &\leq \|T(\alpha 1_X)T(1_X) - 1_Y\|_Y \\ &= \|(\alpha 1_X)1_X - 1_X\|_X = \|(\alpha - 1)1_X\|_X \\ &= |\alpha - 1| \, \|1_X\|_X = |\alpha - 1|. \end{aligned} \tag{3.21}$$

Since $T(-1_X)(y) = -T(1_X)(y)$, we deduce that

$$\begin{aligned} |T(\alpha 1_X)(y)T(1_X)(y) + 1| &= |(-T(\alpha 1_X)T(1_X))(y) - 1| \\ &= |(T(\alpha 1_X)T(-1_X))(y) - 1| \leq \|T(\alpha 1_X)T(-1_X) - 1_Y\|_Y \\ &= \|(\alpha 1_X)(-1_X) - 1_X\|_X = \| - (\alpha + 1)1_X\|_X \\ &= |\alpha + 1|\|1_X\|_X = |\alpha + 1|. \end{aligned} \tag{3.22}$$

According to (3.20), (3.21), (3.22), and (ii) of Lemma 3.5, we conclude that

$$T(\alpha 1_X)(y)T(1_X)(y) = \alpha, \tag{3.23}$$

since $\alpha \in \mathbb{R}$. By (3.23) and $(T(1_X))^2 = 1_Y$, we get

$$T(\alpha 1_X)(y) = \alpha T(1_X)(y). \tag{3.24}$$

Since (3.24) holds for each $y \in Y$, we deduce that $T(\alpha 1_X) = \alpha T(1_X)$. $\square$

Proposition 3.9. *Let $T : A \to B$ be a surjective map satisfying $\|T(f)T(g) - 1_Y\|_Y = \|fg - 1_X\|_X$ for all $f, g \in A$. Then*

$$\mathbb{R} \cap \mathrm{Ran}_{\pi,X}(f) = \mathbb{R} \cap \mathrm{Ran}_{\pi,Y}(T(1_X)T(f))$$

for all $f \in A$.

Proof. Let $f \in A$. If $\mathbb{R} \cap \mathrm{Ran}_{\pi,X}(f) = \emptyset$, then

$$\mathbb{R} \cap \mathrm{Ran}_{\pi,X}(f) \subseteq \mathbb{R} \cap \mathrm{Ran}_{\pi,Y}(T(1_X)T(f)).$$

Assume that $\mathbb{R} \cap \mathrm{Ran}_{\pi,X}(f) \neq \emptyset$ and $\alpha \in \mathbb{R} \cap \mathrm{Ran}_{\pi,X}(f)$. Then $\alpha \in \mathbb{R}$ and $\alpha \in \mathrm{Ran}_{\pi,X}(f)$. If $\alpha = 0$, then $\|f\|_X = |\alpha| = 0$ and so $\|T(f)\|_Y = 0$, which implies that $T(1_X)T(f) = 0_Y$ and so $\alpha \in \mathrm{Ran}_{\pi,Y}(T(1_X)T(f))$. Assume that $\alpha \neq 0$. Then $-\alpha^{-1}f \in A$ and $-1 \in \mathrm{Ran}_{\pi,X}(-\alpha^{-1}f) = \mathrm{Ran}_{\pi,X}((-\alpha^{-1}1_X)f)$ since $\alpha \in \mathbb{R} \setminus \{0\}$. By Proposition 3.6, we deduce that $-1 \in \mathrm{Ran}_{\pi,Y}(T(-\alpha^{-1}1_X)T(f))$. Hence, $-1 \in \mathrm{Ran}_{\pi,Y}(-\alpha^{-1}T(1_X)T(f))$ since $T(-\alpha^{-1}1_X) = -\alpha^{-1}T(1_X)$ by Proposition 3.9. It follows that $\alpha \in \mathrm{Ran}_{\pi,Y}(T(1_X)T(f))$. Therefore,

$$\mathbb{R} \cap \mathrm{Ran}_{\pi,X}(f) \subseteq \mathbb{R} \cap \mathrm{Ran}_{\pi,Y}(T(1_X)T(f)). \tag{3.25}$$

If $\mathbb{R} \cap \mathrm{Ran}_{\pi,Y}(T(1_X)T(f)) = \emptyset$, then

$$\mathbb{R} \cap \mathrm{Ran}_{\pi,Y}(T(1_X)T(f)) \subseteq \mathbb{R} \cap \mathrm{Ran}_{\pi,X}(f).$$

Assume that $\mathbb{R} \cap \mathrm{Ran}_{\pi,Y}(T(1_X)T(f)) \neq \emptyset$ and $\alpha \in \mathrm{Ran}_{\pi,Y}(T(1_X)T(f))$. If $\alpha = 0$, then $\|T(f)\|_Y = \|T(1_X)T(f)\|_Y = 0$ (because $T(1_X)(y) \in \{-1, 1\}$ for all $y \in Y$) and so $\|f\|_X = 0$, which implies that $\alpha \in \mathbb{R} \cap \mathrm{Ran}_{\pi.X}(f)$. Assume that $\alpha \neq 0$. Then $-\alpha^{-1}f \in A$ and $-1 \in \mathrm{Ran}_{\pi,Y}(-\alpha^{-1}T(1_X)T(f))$. Thus, $-1 \in \mathrm{Ran}_{\pi,Y}(T(-\alpha^{-1}1_X)T(f))$ since $T(-\alpha^{-1}1_X) = -\alpha^{-1}T(1_X)$ by Proposition 3.8. Therefore, we get $-1 \in \mathrm{Ran}_{\pi,X}((-\alpha^{-1}1_X)f) = \mathrm{Ran}_{\pi,X}(-\alpha^{-1}f)$ according to Proposition 3.6. It follows that $\alpha \in \mathbb{R} \cap \mathrm{Ran}_{\pi,X}(f)$. Therefore,

$$\mathbb{R} \cap \mathrm{Ran}_{\pi,Y}(T(1_X)T(f)) \subseteq \mathbb{R} \cap \mathrm{Ran}_{\pi,X}(f). \tag{3.26}$$

Thus the assertion holds by (3.25) and (3.26). □

Theorem 3.10. *Suppose that $A = \mathrm{Lip}(X, d, \tau)$ and $B = \mathrm{Lip}(Y, \rho, \eta)$. Let T be a surjective map from A to B satisfying $\|T(f)T(g) - 1_Y\|_Y = \|fg - 1_X\|_X$ for all $f, g \in A$ and let $\Psi : X_\tau \to Y_\eta$ be the associated map to T. If $x \in X$ and $y \in \Psi(x_\tau)$, then $T(f)(y) = T(1_X)(y)f(x)$ for all $f \in A$ with $f(x) \in \mathbb{R}$.*

Proof. Assume that $x \in X$ and $y \in \Psi(x_\tau)$. Let $f \in A$ with $f(x) \in \mathbb{R}$. We first assume that $f(x) = 0$. By (ii) of Proposition 3.3, we have $|T(f)(y)| = |f(x)| = 0$. Therefore,

$$T(f)(y) = 0 = T(1_X)(y)f(x).$$

We now assume that $f(x) \neq 0$. Take $\alpha = f(x)$. Then $\alpha^{-1}f \in A$ and $-\alpha^{-1}f(x) = -1 \neq 0$. By [1, Theorem 2.5(i)], there exists $h \in \mathcal{P}_x(\mathrm{Lip}(X, d, \tau))$ with

$$M(-\alpha^{-1}fh) = M(h) = x_\tau, \tag{3.27}$$

and

$$\mathrm{Ran}_{\pi.X}(-\alpha^{-1}fh) = \{-\alpha^{-1}f(x), -\alpha^{-1}f(\tau(x))\} = \{-1\}. \tag{3.28}$$

By (3.28) and Proposition 3.6, we have $-1 \in \mathrm{Ran}_{\pi,X}(T(f)T(-\alpha^{-1}h))$. Therefore, there exists $w \in Y$ such that

$$T(f)(w)T(-\alpha^{-1}h)(w) = -1. \tag{3.29}$$

This implies that

$$|T(f)(w)T(-\alpha^{-1}h)(w)| = 1. \tag{3.30}$$

The surjectivity of $\Psi : X_\tau \to Y_\eta$ implies that there exists $z \in X$ such that $w_\eta = \Psi(z_\tau)$ and so $w \in \Psi(z_\tau)$. By (ii) of Proposition 3.3, we have

$$|T(f)(w)| = |f(z)|, \qquad |T(-\alpha^{-1}h)(w)| = |(-\alpha^{-1}h)(z)|. \tag{3.31}$$

According to (3.31) and (3.30), we get $|-\alpha^{-1}f(z)h(z)| = 1$. This implies that $|-\alpha^{-1}f(z)h(z)| = |-1| = |-\alpha^{-1}f(x)| = \|-\alpha^{-1}fh\|_X$. Thus $z \in M(-\alpha^{-1}fh)$ and so $z \in x_\tau$ by (3.27). Therefore, $z_\tau = x_\tau$, which implies that $w_\eta = y_\eta$ Thus, $y \in w_\eta$ because $y \in \Psi(x_\tau)$. Since $T(f), T(\alpha^{-1}h) \in B$ and $-1 \in \mathbb{R}$, by (3.29), we get

$$T(f)(\eta(w))T(\alpha^{-1}h)(\eta(w)) = -1. \tag{3.32}$$

By (3.30), (3.32) and $y \in w_\eta$, we get

$$T(f)(y)T(-\alpha^{-1}h)(y) = -1. \tag{3.33}$$

Since $\mathrm{Ran}_{\pi,X}(-\alpha^{-1}h) = \{-\alpha^{-1}\}$ and $\alpha \in \mathbb{R}$, by Proposition 3.9, we conclude that

$$\mathbb{R} \cap \mathrm{Ran}_{\pi,Y}(T(1_X)T(-\alpha^{-1}h)) = \left\{-\alpha^{-1}\right\}.$$

It follows that there exists $y' \in Y$ such that

$$T(1_X)(y')T(-\alpha^{-1}h)(y') = -\alpha^{-1}. \tag{3.34}$$

According to $T(1_X) \in B$, $T(\alpha^{-1}h) \in B$, $-\alpha^{-1} \in \mathbb{R}$ and (3.34), we get

$$T(1_X)(\eta(y'))T(-\alpha^{-1}h)(\eta(y')) = -\alpha^{-1}. \tag{3.35}$$

The surjectivity of $\Psi : X_\tau \to Y_\eta$ implies that there exists $x' \in Y$ such that $y'_\eta = \Psi(x'_\tau)$. By (ii) of Proposition 3.3, we get

$$|T(1_X)(y')T(-\alpha^{-1}h)(y')| = |1_X(x')\alpha^{-1}h(x')| = |\alpha^{-1}||h(x')|. \tag{3.36}$$

According to (3.36) and (3.34), we conclude that $|h(x')| = 1 = \|h\|_X$. This implies that $x' \in M(h) = x_\tau$. Therefore, $x'_\tau = x_\tau$ and so that $y'_\eta = \Psi(x'_\tau) = \Psi(x_\tau) = y_\eta$, which implies that $y \in y'_\eta = \{y', \eta(y')\}$. Hence, by (3.34) and (3.35), we get $T(1_X)(y)T(-\alpha^{-1}h)(y) = -\alpha^{-1}$. This implies that

$$T(-\alpha^{-1}h)(y) = -\alpha^{-1}T(1_X)(y) \tag{3.37}$$

since $(T(1_X)(y))^2 = 1$. According to (3.33) and (3.37), we get

$$-\alpha^{-1}T(1_X)(y)T(f)(y) = -1.$$

This implies that $T(1_X)(y)T(f)(y) = \alpha$. Therefore, $T(f)(y) = T(1_X)(y)f(x)$ since $(T(1_X)(y))^2 = 1$ and $\alpha = f(x)$. □

The following result is an immediate consequence of Theorem 3.10.

Corollary 3.11. *Let $A = \mathrm{Lip}(X, d, \tau)$, let $B = \mathrm{Lip}(Y, \rho, \eta)$, and let $T : A \to B$ be a surjective map satisfying $\|T(f)T(g) - 1_Y\|_Y = \|fg - 1_X\|_X$ for all $f, g \in A$. Then the following assertions hold:*

(i) $T(A_\mathbb{R}) \subseteq B_\mathbb{R}$.
(ii) $T(\alpha f) = \alpha T(f)$ *for all* $\alpha \in \mathbb{R}$ *and* $f \in A_\mathbb{R}$.

Theorem 3.12. *Let $A = \mathrm{Lip}(X, d, \tau)$, let $B = \mathrm{Lip}(Y, \rho, \eta)$, and let $T : A \to B$ be a surjective map satisfying $\|T(f)T(g) - 1_Y\|_Y = \|fg - 1_X\|_X$ for all $f, g \in A$. Then $(T(1_X))^2 = 1_Y$, and there exists a bijective map $\varphi : Y \to X$ with $\tau \circ \varphi = \varphi \circ \eta$ on Y such that φ is continuous at each $y \in Y$ with $\eta(y) = y$, φ^{-1} is continuous at each $x \in X$ with $\tau(x) = x$, and*

$$T(f)(y) = T(1_X)(y) f(\varphi(y)) \tag{3.38}$$

for all $y \in Y$ and $f \in A$ with $f(\varphi(y)) \in \mathbb{R}$. In particular, (3.38) holds for all $f \in A_{\mathbb{R}}$ and $y \in Y$.

Proof. Let $\Psi : X_\tau \to Y_\eta$ be the associated map to T. We divide the proof into nine steps.

Step 1. *If $x \in X$ with $\tau(x) \neq x$ and $g \in A$ with $g(x) = i$ and $|g(z)| < 1$ for all $z \in X \setminus \{x, \tau(x)\}$, then $T(g)(y) \in \{-i, i\}$ for all $y \in \Psi(x_\tau)$.*

Let $x \in X$ with $\tau(x) \neq x$ and let $g \in A$ with $g(x) = i$ and $|g(z)| < 1$ for all $z \in X \setminus \{x, \tau(x)\}$. Then $\|g\|_X = 1$ and $(g(x))^2 = -1$. Therefore, $\|g^2 - 1_X\|_X = 2$ and so $\|(T(g))^2 - 1_Y\|_Y = 2$. This implies that there exists $w \in Y$ such that

$$|(T(g)(w))^2 - 1| = 2. \tag{3.39}$$

The surjectivity of Ψ implies that there exists $z \in X$ with $\Psi(z_\tau) = w_\eta$ and so $w \in \Psi(z_\tau)$. By (ii) of Proposition 3.3, we get

$$|T(g)(w)| = |g(z)|. \tag{3.40}$$

We claim that $z_\tau = x_\tau$. Otherwise, $z \in X \setminus \{x, \tau(x)\}$ and so $|g(z)| < 1$. Therefore, according to (3.39) and (3.40) we get

$$2 \leq |T(g)(w)|^2 + 1 = |g(z)|^2 + 1 < 2$$

which is impossible. Hence, our claim is justified. Thus $w \in \Psi(x_\tau)$, and so by (ii) of Proposition 3.3, we have $|T(g)(w)| = |g(x)| = 1$. Hence, by (i) of Lemma 3.5, we deduce that $(T(g)(w))^2 = -|g(x)| = -1$. Thus, $T(g)(w) \in \{-i, i\}$ and so $T(g)(\eta(w)) \in \{-i, i\}$. Therefore, $T(g)(y) \in \{-i, i\}$ for all $y \in \Psi(x_\tau)$ since $\Psi(x_\tau) = w_\eta = \{w, \eta(w)\}$. Hence, Step 1 holds.

Step 2. *If $x \in X$ with $\tau(x) \neq x$ and $g \in A$ with $g(x) = i$ and $|g(z)| < 1$ for all $z \in X \setminus \{x, \tau(x)\}$, then $T(f)(y) = T(g)(y)$ for all $f \in i\mathcal{P}_x(A)$ and $y \in \Psi(x_\tau)$.*

Let $x \in X$ with $\tau(x) \neq x$ and $g \in A$ with $g(x) = i$ and $|g(z)| < 1$ for all $z \in X \setminus \{x, \tau(x)\}$. Assume that $f \in i\mathcal{P}_x(A)$. Then $f \in A$, $\|f\|_X = 1$ and $f(x) = i$. Therefore, $\|fg - 1_X\|_X = 2$ and so $\|T(f)T(g) - 1_Y\|_Y = 2$. This implies that there exists $w \in Y$ such that

$$|T(f)(w)T(g)(w) - 1| = 2. \tag{3.41}$$

According to the surjectivity of Ψ, there exists $z \in X$ such that $w_\eta = \Psi(z_\tau)$. By (ii) of Proposition 3.3, we get

$$|f(z)| = |T(f)(w)|, \qquad |g(z)| = |T(g)(w)|. \tag{3.42}$$

We claim that $|g(z)| = 1$. Otherwise, $|g(z)| < 1$ since $|g(z)| \leq \|g\|_X = 1$. According to (3.41) and (3.42), we get

$$2 \leq |T(f)(w)||T(g)(w)| + 1 = |f(z)||g(z)| + 1 \leq |g(z)| + 1 < 2,$$

which is impossible. Hence, our claim is justified. Thus $z_\tau = x_\tau$ and so $w \in \Psi(x_\tau)$. Therefore,

$$|T(f)(w)T(g)(w)| = |f(z)||g(z)| = |f(x)||g(x)| = |f(x)g(x)|.$$

According to (3.41) and $|f(x)g(x)| = |i^2| = 1$, we get

$$|T(f)(w)T(g)(w) - 1| = 1 + |f(x)g(x)|.$$

Hence, by (i) of Lemma 3.5, we deduce that

$$T(f)(w)T(g)(w) = -|f(x)g(x)| = -1. \tag{3.43}$$

According to $T(f) \in B$, $T(g) \in B$, $-1 \in \mathbb{R}$ and (3.43), we get

$$T(f)(\eta(w))T(g)(\eta(w)) = -1. \tag{3.44}$$

Let $y \in \Psi(x_\tau)$ be given. Then $y_\eta = \Psi(x_\tau) = w_\eta$ and so $y \in w_\eta$. Therefore, by (3.43) and (3.44), we get

$$T(f)(y)T(g)(y) = -1, \tag{3.45}$$

and according to (3.42), we have

$$|T(g)(y)| = |g(z)| = |g(x)| = 1. \tag{3.46}$$

By (3.45) and (3.46), we get

$$T(f)(y) = -\overline{T(g)(y)}. \tag{3.47}$$

Since $y \in \Psi(x_\tau)$, by Step 1, we have $T(g)(y) \in \{-i, i\}$ and so

$$T(g)(y) = -\overline{T(g)(y)}. \tag{3.48}$$

According to (3.47) and (3.48), we get $T(f)(y) = T(g)(y)$. Hence, Step 2 holds.

Step 3. *If $x \in X$ with $\tau(x) \neq x$ and $f \in i\mathcal{P}_x(A)$, then $T(f)(y) \in \{-i, i\}$ for all $y \in \Psi(x_\tau)$.*

Let $x \in X$ with $\tau(x) \neq x$ and $f \in i\mathcal{P}_x(A)$. By [1, Lemma 2.3], there exists a function $g_x \in \operatorname{Lip}(X, d, \tau)$ with $g_x(x) = i$ and $|g_x(z)| < 1$ for all $z \in X \setminus \{x, \tau(x)\}$. Therefore, by Step 1, we get $T(g_x)(y) \in \{-i, i\}$ for all $y \in \Psi(x_\tau)$. Furthermore, by Step 2, we get $T(f)(y) = T(g_x)(y)$ for all $y \in \Psi(x_\tau)$. Thus, $T(f)(y) \in \{-i, i\}$ for all $y \in \Psi(x_\tau)$, and so Step 3 holds.

Step 4. *If $x \in X$ with $\tau(x) \neq x$, then there exists a function $g_x \in i\mathcal{P}_x(A)$ such that $T(f)(y) = T(g_x)(y)$ for all $f \in i\mathcal{P}_x(A)$ and $y \in \Psi(x_\tau)$.*

Let $x \in X$ with $\tau(x) \neq x$. By [1, Lemma 2.3], there exists a function $g_x \in \operatorname{Lip}(X, d, \tau)$ such that $g_x(x) = i$ and $|g_x(z)| < 1$ for all $z \in X \setminus \{x, \tau(x)\}$. Clearly, $g_x \in i\mathcal{P}_x(A)$. Therefore, by Step 2, we have $T(f)(y) = T(g_x)(y)$ for all $f \in i\mathcal{P}_x(A)$ and $y \in \Psi(x_\tau)$. Hence, Step 4 holds.

Step 5. *If $x \in X$ with $\tau(x) \neq x$, then $T(-f)(y) = -T(f)(y)$ for all $f \in i\mathcal{P}_x(A)$ and $y \in \Psi(x_\tau)$.*

Let $x \in X$ with $\tau(x) \neq x$. Assume that $f \in i\mathcal{P}_x(A)$ and $y \in \Psi(x_\tau)$. By Step 3, we have $T(f)(y) \in \{-i.i\}$. Clearly, $f \in i\mathcal{P}_x(A)$ implies that $-f \in i\mathcal{P}_{\tau(x)}(A)$. Since $\eta(y) \in \Psi((\tau(x))_\tau)$, by Step 2, we get $T(-f)(\eta(y)) \in \{-i, i\}$

and so $T(-f)(y) \in \{-i, i\}$. Therefore, $T(-f)(y) = T(f)(y)$ or $T(-f)(y) = -T(f)(y)$. We claim that $T(-f)(y) = -T(f)(y)$. Otherwise, we have

$$T(f)(y)T(-f)(y) = -1.$$

Therefore,

$$\begin{aligned} 2 &= |T(f)(y)T(-f)(y) - 1| \leq \|T(f)T(-f) - 1_Y\|_Y \\ &\leq \|T(f)\|_Y \|T(-f)\|_Y + 1 = \|f\|_X \| - f\|_X + 1 \\ &= \|f\|_X^2 + 1 = 2, \end{aligned}$$

and so $\|T(f)T(-f) - 1_Y\|_Y = 2$, which implies that $\|-f^2 - 1_X\|_X = 2$. Hence, there exists $z \in X$ such that

$$|(f(z))^2 + 1| = | - (f(z))^2 - 1| = 2. \tag{3.49}$$

Since $|(f(z))^2| = |f(z)|^2 \leq (\|f\|_X)^2 = 1$, according to (3.49) we deduce that $(f(z))^2 = 1$, which implies that $f(z) \in \{-1, 1\}$. Therefore, $f(z) \in \mathrm{Ran}_{\pi,X}(f)$ since $|f(z)| = 1 = \|f\|_X$. Thus, $f(z) \in \{-i, i\}$ since $\mathrm{Ran}_{\pi,X}(f) = \{-i, i\}$. Therefore, we arrive at a contradiction. Hence, our claim is justified and so Step 5 holds.

Step 6. *If $x \in X$ and $y \in \Psi(x_\tau)$, then $\tau(x) \neq x$ if and only if $\eta(y) \neq y$.*

Let $x \in X$ and $y \in \Psi(x_\tau)$. We first assume that $\tau(x) \neq x$. Then $i\mathcal{P}_x(A) \neq \emptyset$ by [1, Lemma 2.3]. Let $f \in i\mathcal{P}_x(A)$. By Step 2, we have $T(f)(y) \in \{-i, i\}$. This implies that $y \neq \eta(y)$.

Now we assume that $y \neq \eta(y)$. Then $i\mathcal{P}_y(B) \neq \emptyset$ by [1, Lemma 2.3]. Let $k \in i\mathcal{P}_y(B)$. The surjectivity of T implies that there exists a function $f \in A$ such that $T(f) = k$. We claim that $\tau(x) \neq x$. Otherwise, $f(x) \in \mathbb{R}$. By Theorem 3.10, we have $T(f)(y) = T(1_X)(y)f(x)$, which implies that $k(y) = T(1_X)(y)f(x)$. Hence, $f(x) \in \{-i, i\}$ since $T(1_X)(y) \in \{-1, 1\}$ and $k(y) = i$. Therefore, $f(x) \notin \mathbb{R}$, which contradicts $f(x) \in \mathbb{R}$. Hence, our claim is justified and so Step 6 holds.

Step 7. *If $x \in X$ with $\tau(x) \neq x$ and $f \in i\mathcal{P}_x(A)$, then there exists $y \in \Psi(x_\tau)$ such that $T(1_X)(y)T(f)(y) = i$.*

Let $f \in i\mathcal{P}_x(A)$ be given. Assume that $w \in \Psi(x_\tau)$. Then $\Psi(x_\tau) = w_\eta$. By Step 3, we have $T(f)(w) \in \{-i, i\}$, which implies that $T(1_X)(w)T(f)(w) \in \{-i, i\}$ since $T(1_X)(y) \in \{-1, 1\}$. Choose $y = w$ if $T(1_X)(w)T(f)(w) = i$ and $y = \eta(w)$ if $T(1_X)(w)T(f)(w) = -i$. Then $y \in \Psi(x_\tau)$ and $T(1_X)(y)T(f)(y) = i$. Hence, Step 7 holds.

Step 8. *If $x \in X$ with $\tau(x) \neq x$, then there exists a unique $y \in \Psi(x_\tau)$ such that $T(1_X)(y)T(f)(y) = i$ for all $f \in i\mathcal{P}_x(A)$.*

Let $x \in X$ with $\tau(x) \neq x$. By Step 4, there exists a function $g_x \in i\mathcal{P}_x(A)$ such that $T(f)(w) = T(g_x)(w)$ for all $f \in i\mathcal{P}_x(A)$ and $w \in \Psi(x_\tau)$. By Step 7, there exists $y \in \Psi(x_\tau)$ such that $T(1_X)(y)T(g_x)(y) = i$. Therefore, there exists $y \in \Psi(x_\tau)$ such that $T(1_X)(y)T(f)(y) = i$ for all $f \in i\mathcal{P}_x(A)$.

To prove the uniqueness of y, let $y' \in \Psi(x_\tau)$ such that $T(1_X)(y')T(f)(y') = i$ for all $f \in i\mathcal{P}_x(A)$. Note that $\eta(y) \neq y$ by Step 6. We claim that $y' = y$. Otherwise, $y' = \eta(y)$. Let $f \in i\mathcal{P}_x(A)$ be given. According to $T(1_X)(y), T(1_X)(y') \in \mathbb{R}$, we get

$$\begin{aligned} i &= \overline{T(1_X)(y)T(f)(y)} = T(1_X)(\eta(y))\overline{T(f)(\eta(y))} = T(1_X)(y')\overline{T(f)(y')} \\ &= \overline{T(1_X)(y')T(f)(y')} = \bar{i} = -i, \end{aligned}$$

which is impossible. Hence, our claim is justified and so Step 8 holds.

Step 9. $(T(1_X))^2 = 1_Y$ *and there exists a bijective map* $\varphi : Y \to X$ *with* $\tau \circ \varphi = \varphi \circ \eta$ *on* Y *such that* φ *is continuous at each* $y \in Y$ *with* $\eta(y) = y$ *and* φ^{-1} *is continuous at each* $x \in X$ *with* $\tau(x) = x$ *and* $T(f)(y) = T(1_X)(y)f(\varphi(y))$ *for all* $y \in Y$ *and* $f \in A$ *with* $f(\varphi(y)) \in \mathbb{R}$.

By Lemma 3.1, we have $(T(1_X))^2 = 1_Y$. Note that if $x \in X$ with $\tau(x) = x$, then $\Psi(x_\tau)$ is singleton by Step 6. Define the map $\psi : X \to Y$ as follows. In the case that $x \in X$ satisfies $\tau(x) = x$, let $\psi(x)$ be the unique element of $\Psi(x_\tau)$ and in the case that $\tau(x) \neq x$, let $\psi(x)$ be the unique element y of $\Psi(x_\tau)$ satisfying $T(1_X)(y)T(f)(y) = i$ for all $f \in i\mathcal{P}_x(A)$. Then ψ is well-defined by Steps 6 and 8. We now show that $\psi \circ \tau = \eta \circ \psi$ on X. First assume that $x \in X$ with $\tau(x) = x$. By Step 6, we have $\psi(x) = \eta(\psi(x))$ and so

$$(\psi \circ \tau)(x) = \psi(\tau(x)) = \psi(x) = \eta(\psi(x)) = (\eta \circ \psi)(x).$$

Now assume that $x \in X$ with $\tau(x) \neq x$. By Step 6, we have $\psi(x) \neq \eta(\psi(x))$. By the definition of ψ, we have $\Psi(x_\tau) = \{\psi(x), \eta(\psi(x))\}$. Since $\tau(x) \in x_\tau$, we deduce that $\psi(\tau(x)) \in \Psi(x_\tau)$. Therefore, $\psi(\tau(x)) \in \{\psi(x), \eta(\psi(x))\}$. We claim that $\psi(\tau(x)) \neq \psi(x)$. Otherwise, we have $\psi(\tau(x)) = \psi(x)$. Let $g \in i\mathcal{P}_x(A)$. Then $T(1_X)(\psi(x))T(g)(\psi(x)) = i$ by the definition of ψ. Furthermore, $-g \in i\mathcal{P}_{\tau(x)}(A)$. Since $\tau(\tau(x)) = x \neq \tau(x)$, we deduce that $T(1_X)(\psi(\tau(x))T(-g)(\psi(\tau(x))) = i$ by the definition of ψ . Therefore, according to $\psi(\tau(x)) = \psi(x)$ and Step 5, we get

$$i = T(1_X)(\psi(x)))T(-g)(\psi(x)) = -T(1_X)(\psi(x))T(g)(\psi(x)) = -i,$$

which is impossible. Hence, our claim is justified. It follows that $(\psi \circ \tau)(x) = (\eta \circ \psi)(x)$. Since this equality holds for all $x \in X$, we deduce that $\psi \circ \tau = \eta \circ \psi$ on X.

We show that ψ is injective. To get a contradiction, let $x, z \in X$ with $x \neq z$ and $\psi(x) = \psi(z)$. We claim that $x_\tau \cap z_\tau = \emptyset$. Otherwise, we have $z = \tau(x)$. This implies that

$$\psi(x) = \psi(z) = \psi(\tau(x)) = \eta(\psi(x))$$

because $\psi \circ \tau = \eta \circ \psi$ on X. Since $\psi(x) \in \Psi(x_\tau)$, by Step 6, we deduce that $x = \tau(x)$, which contradicts $\tau(x) \neq x$. Hence, our claim is justified. By [1, Lemma 2.1], there exists $g \in \mathrm{Lip}(X, d, \tau)$ with $g(x) = g(\tau(x)) = 1$ and $0 \leq g(t) < 1$ for all $t \in X \setminus \{x, \tau(x)\}$. Since $z \in X \setminus \{x, \tau(x)\}$, we have $0 \leq g(z) < 1$. According to $g(x) \in \mathbb{R}$, $\psi(x) \in \Psi(x_\tau)$, $g(z) \in \mathbb{R}$, $\psi(z) \in \Psi(z_\tau)$ and Theorem 3.10, we get

$$\begin{aligned} T(1_X)(\psi(x)) &= T(1_X)(\psi(x))g(x) = T(g)(\psi(x)) \\ &= T(g)(\psi(z)) = T(1_X)(\psi(z))g(z). \end{aligned}$$

It follows that $|T(1_X)(\psi(x))| < 1$ since $0 \leq g(z) < 1$. This contradicts $(T(1_X))^2 = 1_Y$. Therefore, ψ is injective.

To prove the surjectivity of ψ, let $y \in Y$. The surjectivity of $\Psi : X_\tau \to Y_\eta$ implies that there exists $x \in X$ such that $\Psi(x_\tau) = y_\eta$. The definition of ψ implies that $\psi(x) \in y_\eta = \{y, \eta(y)\}$. Therefore, either $y = \psi(x)$ or $y = \eta(\psi(x)) = \psi(\tau(x))$ since $\eta \circ \psi = \psi \circ \tau$ on X. Hence, ψ is surjective.

We show that ψ is continuous at each $x \in X$ with $\tau(x) = x$. Otherwise, there exist $x \in X$ with $\tau(x) = x$ and a sequence $\{x_n\}_{n=1}^{\infty}$ converging to x in (X, d), but $\{\psi(x_n)\}_{n=1}^{\infty}$ does not converge to $\psi(x)$ in (Y, ρ). Then there exist an $\varepsilon > 0$ and a strictly increasing sequence $\{\sigma(n)\}_{n=1}^{\infty}$ in $\mathbb{N}$ such that $\rho(\psi(x_{\sigma(n)}), \psi(x)) \geq \varepsilon$ for all $n \in \mathbb{N}$. Since $\tau(x) = x$, we have $\eta(\psi(x)) = \psi(x)$ by Step 6. Define the function $k_{\psi(x),\varepsilon} : Y \to \mathbb{C}$ by

$$k_{\psi(x),\varepsilon}(y) = \max\left\{0, 1 - \frac{\rho(\psi(x), y)}{\varepsilon}\right\} \qquad (y \in Y).$$

Then $k_{\psi(x),\varepsilon} \in \mathrm{Lip}(Y, \rho)$, $k_{\psi(x),\varepsilon}(\psi(x)) = 1$ and $k_{\psi(x),\varepsilon}(\psi(x_{\sigma(n)})) = 0$ for all $n \in \mathbb{N}$. Take $k = k_{\psi(x),\varepsilon}\eta^*(k_{\psi(x),\varepsilon})$. Then $k \in \mathrm{Lip}(Y, \rho, \eta) = B$. In addition, $k(\psi(x)) = 1$ and $k(\psi(x_{\sigma(n)})) = 0$ for all $n \in \mathbb{N}$. The surjectivity of T implies that there exists a function $g \in A$ such that $T(g) = k$. By (ii) of Proposition 3.3, we get

$$|g(x_{\sigma(n)})| = |T(g)(\psi(x_{\sigma(n)}))|$$

for all $n \in \mathbb{N}$. Therefore $g(x_{\sigma(n)}) = 0$ for all $n \in \mathbb{N}$. This implies that $g(x) = 0$. On the other hand, by (ii) of Proposition 3.3, we have

$$|g(x)| = |T(g)(\psi(x))| = |k(\psi(x))| = 1.$$

We arrive at a contradiction and so ψ is continuous at x.

We show that ψ^{-1} is continuous at each $y \in Y$ with $\eta(y) = y$. Otherwise, there exists $y \in Y$ with $\eta(y) = y$ and a sequence $\{y_n\}$ converging to y in (Y, ρ), but $\{\psi^{-1}(y_n)\}_{n=1}^{\infty}$ does not converge to $\psi(y)$ in (Y, ρ). Then there exist an $\varepsilon > 0$ and a strictly increasing sequence $\{\sigma(n)\}_{n=1}^{\infty}$ in $\mathbb{N}$ such that $d(\psi^{-1}(y_{\sigma(n)}), \psi^{-1}(y)) \geq \varepsilon$ for all $n \in \mathbb{N}$. Since $\eta(y) = y$, we have $\tau(\psi^{-1}(y)) = \psi^{-1}(y)$ by step 6. Define the function $g_{\psi^{-1}(y),\varepsilon} : X \to \mathbb{R}$ by

$$g_{\psi^{-1}(y),\varepsilon}(x) = \max\left\{0, 1 - \frac{d(\psi^{-1}(y), x)}{\varepsilon}\right\} \qquad (x \in X).$$

Then $g_{\psi^{-1}(y),\varepsilon} \in \mathrm{Lip}(X, d)$, $g_{\psi^{-1}(y),\varepsilon}(\psi^{-1}(y)) = 1$ and $g_{\psi^{-1}(y),\varepsilon}(\psi^{-1}(y_{\sigma(n)})) = 0$ for all $n \in \mathbb{N}$. Take $g = g_{\psi^{-1}(y),\varepsilon}\tau^*(g_{\psi^{-1}(y),\varepsilon})$. Then $g \in \mathrm{Lip}(X, d, \tau) = A$. In addition, $g(\psi^{-1}(y)) = 1$ and $g(\psi^{-1}(y_{\sigma(n)})) = 0$ for all $n \in \mathbb{N}$. By (ii) of Proposition 3.3, we get

$$|T(g)(y_{\sigma(n)})| = |g(\psi^{-1}(y_{\sigma(n)}))|$$

for all $n \in \mathbb{N}$. Therefore $T(g)(\psi^{-1}(y_{\sigma(n)})) = 0$ for all $n \in \mathbb{N}$. This implies that $T(g)(y) = 0$. On the other hand, by (ii) of Proposition 3.3, we have

$$|T(g)(y))| = |g(\psi^{-1}(y))| = 1.$$

We arrive at a contradiction and so ψ^{-1} is continuous at x.

Take $\varphi = \psi^{-1}$. Then φ is a bijective map from Y onto X, $\varphi \circ \eta = \tau \circ \varphi$ on Y, φ is continuous at each $y \in Y$ with $\eta(y) = y$ and φ^{-1} is continuous at each $x \in X$ with $\tau(x) = x$. Let $y \in Y$ and $f \in A$ with $f(\varphi(y)) \in \mathbb{R}$. Since $y = \psi(\varphi(y))$ and $\psi(\varphi(y)) \in \Psi(\{\varphi(y), \tau(\varphi(y))\})$, by Theorem 3.10, we have

$$T(f)(y) = T(1_X)(y) \cdot f(\varphi(y)).$$

Hence, Step 9 holds. $\square$

Corollary 3.13. *Let T be a surjective map from $\mathrm{Lip}_{\mathbb{R}}(X,d)$ to $\mathrm{Lip}_{\mathbb{R}}(Y,\rho)$ satisfying $\|T(f)T(g) - 1_Y\|_Y = \|fg - 1_X\|_X$ for all $f, g \in \mathrm{Lip}_{\mathbb{R}}(X,d)$. Then $(T(1_X))^2 = 1_Y$ and there exists a Lipschitz homeomorphism φ from (Y,ρ) to (X,d) such that $T(f) = T(1_X)\cdot(f\circ\varphi)$ for all $f \in \mathrm{Lip}_{\mathbb{R}}(X,d)$.*

Proof. Let τ and η be the identity maps on X and Y, respectively. Then $\mathrm{Lip}_{\mathbb{R}}(X,d) = \mathrm{Lip}(X,d,\tau)$ and $\mathrm{Lip}_{\mathbb{R}}(Y,\rho) = \mathrm{Lip}(Y,\rho,\eta)$. Thus, T is a surjective map from $\mathrm{Lip}(X,d,\tau)$ to $\mathrm{Lip}(Y,\rho,\eta)$ satisfying $\|T(f)T(g)-1_Y\|_Y = \|fg-1_X\|_X$ for all $f, g \in \mathrm{Lip}(X,d,\tau)$. By Theorem 3.12, $(T(1_X))^2 = 1_Y$ and there exists a bijective map $\varphi : Y \to X$ such that $T(f) = T(1_X)\cdot(f\circ\varphi)$ for all $f \in \mathrm{Lip}_{\mathbb{R}}(X,d)$, which implies that $T(f) = T(1_X)\cdot(f\circ\varphi)$ for all $f \in \mathrm{Lip}_{\mathbb{R}}(X,d)$. Define the map $S : \mathrm{Lip}_{\mathbb{R}}(X,d) \to \mathrm{Lip}_{\mathbb{R}}(Y,\rho)$ by

$$S(f) = T(1_X)\cdot T(f) \qquad (f \in \mathrm{Lip}_{\mathbb{R}}(X,d)).$$

Then $S(f) = f\circ\varphi$ for all $f \in \mathrm{Lip}_{\mathbb{R}}(X,d)$ since $(T(1_X))^2 = 1_Y$.

According to the surjectivity of T and $(T(1_X))^2 = 1_Y$, we conclude that S is surjective. The surjectivity of $\varphi : Y \to X$ implies that S is an injective map.

Define the map $\tilde{S} : \mathrm{Lip}(X,d) \to \mathrm{Lip}(Y,\rho)$ by

$$\tilde{S}(h) = S(f) + iS(g),$$

where $h \in \mathrm{Lip}(X,d)$, $f, g \in \mathrm{Lip}(X,d)$ and $h = f + ig$. Then $\tilde{S}$ is well-defined since $\mathrm{Lip}(X,d) = \mathrm{Lip}_{\mathbb{R}}(X,d) \oplus i\mathrm{Lip}_{\mathbb{R}}(X,d)$ and $\mathrm{Lip}(Y,\rho) = \mathrm{Lip}_{\mathbb{R}}(Y,\rho) \oplus i\mathrm{Lip}_{\mathbb{R}}(Y,\rho)$. It is easy to see that $\tilde{S}(h) = h\circ\varphi$ for all $h \in \mathrm{Lip}(X,d)$. Therefore, $\tilde{S}$ is an algebra homomorphism from $\mathrm{Lip}(X,d)$ to $\mathrm{Lip}(Y,\rho)$ and $\tilde{S}(1_X) = 1_Y$. It is easy to see that $\tilde{S}$ is a bijective map. By [17, Theorem 5.1], there exists a Lipschitz mapping θ from (Y,ρ) to (X,d) such that $\tilde{S}(h) = h\circ\theta$ for all $h \in \mathrm{Lip}(X,d)$. In addition, the injectivity of $\tilde{S}$ implies that θ is surjective and the surjectivity of $\tilde{S}$ implies that there exists a positive constant M such that $M\rho(y_1,y_2) \leq d(\theta(y_1),\theta(y_2))$ for all $y_1, y_2 \in Y$. Therefore, θ is bijective and θ^{-1} is a Lipschitz mapping from (X,d) to (Y,ρ). Since $h\circ\varphi = \tilde{S}(h) = h\circ\theta$ for all $h \in \mathrm{Lip}(X,d)$ and $\mathrm{Lip}(X,d)$ separates the points of X, we deduce that $\varphi = \theta$. Therefore, φ is a Lipschitz homeomorphism from (Y,ρ) to (X,d). $\square$

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