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BILINEAR NEURAL NETWORK METHOD FOR OBTAINING THE EXACT ANALYTICAL SOLUTIONS TO NONLINEAR EVOLUTION EQUATIONS AND ITS APPLICATION TO KdV EQUATION

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Abstract. Most conventional test functions for solving nonlinear partial differential equations can be built using a single and double hidden layer neural network model via the bilinear neural network method, as is well known. The neural network test function model for the KdV equation is extended to the "$2-3-1$" and "$2-2-2-1$" models in this research. To find analytical solutions to the KdV equation, some new test functions are developed by giving some unique activation functions. By choosing some parameters, lump wave, bright, kink soliton solutions and various soliton solutions are generated. The relationship between the frequency velocity of the wave and the wave intensity as well as the stability analysis of the obtained solutions are studied which provide us a fresh perspective on how to explore the model in detail. The dynamical features of these waves are shown using two-dimensional graphs, three-dimensional graphs and contour plots.

1. Introduction

Exact solutions of differential equation are necessary for comprehending complex physical phenomena. In recent years, water wave models or water wave-type models have gotten a huge amount of interest in the domains of both physics and mathematics [7, 31]. In natural science, there are many nonlinear partial differential equations that can be employed to express water wave phenomena. One of these is the well-known Korteweg-de Vries (KdV) equation. The KdV

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equation is used in a variety of scientific fields, including plasma physics, internal waves, shallow water and stratified flows and hydrodynamics. It also allows for Hamiltonian and Lagrangian structures, as well as integrability, resulting in cnoidal wave and exact analytical solitary wave solutions [40]. The KdV model is expressed as

$$u_{xxx} + 6uu_x + u_t = 0. \tag{1.1}$$

The model has two parts, each with its own set of characteristics: the shock part $u_t + 6uu_x = 0$ which describes the waves sharpening, while $u_t + u_{xxx} = 0$ is the linear part, describes dissipative and spreading of waves. However, they can be used to describe stable traveling waves when used together [48, 18]. The KdV equation is the first model used to introduce the idea of complexiton solutions, which is known by merging trigonometric and exponential waves. Many scholars have been fascinated by the complexity of various coupled systems. As a result, this field has evolved into a valuable tool for describing a variety of important events in engineering and science. Analytical solutions may adequately describe the physical properties of a natural system, and each system solution corresponds to a certain process. As a result, approaches for creating accurate solutions, such as analytical solution, plays an important role in understanding more complex problems [57, 14]. The solutions to the nonlinear partial differential equations are said to be solitons, which explain a single moving wave's solutions. Due to their being produced by a perfect balance between nonlinearity and dispersion in a nonlinear medium, optical solitons are extensively used in electromagnetics and telecommunications. Optical fibers are nonlinear Schrodinger equation solutions. Solitons exhibit extended and particle-like structures, such as cosmic strings and domain barriers, these structures have implications for the study of the early universe cosmology. The $sec(x+t)$, $tan(t+x)$ etc are periodic traveling wave solutions, which are known to be periodic solutions [33, 56]. Nonlinear evolution equations (NLEEs) are researched in a wide range of fields in theoretical physics and applied mathematics, including biology and other engineering fields. The study of exact NLEE solutions as scientific ways of understanding concepts will aid in the clarification of these phenomena. There are a number of successful approaches for obtaining accurate NLEE solutions, including , the new homoclinic test approach [23, 47, 46], the symbolic computation method [24], the Bernoulli sub-equation function method [10], the modified sub-equation technique[11], the generalized exponential rational function approach [13], Jacobi elliptic function expansion method [2], sine-Gordon expansion technique [3], the Lie group approach [29], the modified Kudryashov technique [37], the q-homotopy analysis transform approach[42], the modified extended tanh technique [6], the modified simple equation approach [5], the φ^6-model expansion method[44, 22, 27, 25, 28], the Laplace transform method [43], the extended auxiliary equation technique [1], the modified expansion technique [9], the bilinear neural network approach [26] and so on, have been introduced in the last few decades. The Hirota technique is a strong tool for discovering N-soliton solutions for NLEEs that was discovered by Hirota in 1971, which does not require complex mathematical computations. Additionally, the Hirota method has been successfully applied to the other kind of nonlinear PDEs [57, 8], creating a transformation into new variables with the intention of making multi-soliton solutions appear in a particularly simple manner was the main objective. Many alternative methods, such as the inverse scattering transformation and other many dressing techniques, can be used to obtain multi-soliton solutions. Hirota's approach is superior than the mentioned techniques and others since it is algebraic rather than analytic[19, 34]. Furthermore, Hirota's approach allows us to generate multi-soliton solution of the nonlineer partial differential models provided that they can be transformed into bilinear form, for that reason, soliton solutions are possible for any equation written in bilinear form. Shallow water waves on surfaces have been well mathematically represented by the KdV model. Several approaches, including the inverse scattering transformation [54], the Bilinear method [55], the Darboux transformation [21], the variational approximation approach[38], the novel $(G\prime/G)$-expansion approach[30, 12] and the leading order analysis technique [35] are just a few of the many approaches that researchers have tried to find accurate solutions to the KdV model. The bilinear neural network (**BNN**) approach, which was first suggested lately, is a new method through the use of bilinear approach [20, 41, 17, 39, 15, 16, 32] and neural network model for determining the exact analytical solution of NLEEs, by combining the Hirota bilinear form and the neural network model with single layer or many hidden layers, solutions like lump solution, period solution, interaction solution between lump and arbitrary function, two-wave and three-wave solutions, breather-type kink soliton solution, rational function solutions, traveling wave solutions, breather solutions, periodic wave solutions, lump-type solutions and so on can all be built using this method [49, 51, 36, 50, 52, 53]. So far, no study has been done on the KdV equation using the "$2-3-1$" and "$2-2-2-1$" neural network models. The KdV model is investigated in this study, we devise a test function approach that enriches the conventional use of single- and double-hidden layer neural network models. The structure of this paper is as follows. In section 2, a brief description of Hirota bilinearization is given. The KdV equation's bilinear transformation is achieved in section 3, which is accompanied by the model's Hirota bilinear form, the bilinear form is then justified by a theorem. In section 4, we introduce the neural network model and the tensor formula that goes with it. In section 5, the periodic solutions of the KdV model is studied using the "$2-3-1$" and "$2-2-2-1$" neural network models and the 2-dimensional, contour and 3-dimensional graphs of some of the derived results are plotted. In section 6, the physical dynamics of the exact solutions are assessed and conclusions are made in section 7.

2. Hirota Bilinear Form

In order to create various solutions to the integralable non-linear evolution models, Hirota [20] is the first person suggested the bilinear method in 1971. This method's fundamental idea is to use a dependent variable transformation to modify the non-linear evolution models into a form where the unknown function seems bilinear. The bilinear approach is among the most crucial and widely applied approaches for solving nonlinear partial differential equations. The foundation of this approach is the use of appropriate transformations to turn nonlinear models into bilinear forms. Then, by using the bilinear forms, it is possible to deduce various exact solutions such as periodic soliton solutions, kink solutions, rational solutions, lump solutions, breather solutions and N-soliton solutions [41, 17, 39, 15]. For a given set of function of a real variable say x:

$$D_x\left(q.p\right) = \left(\partial_x - \partial_{x'}\right) q(x)p(x^{'})|_{x'=x}, \tag{2.1}$$

$$D_x^a D_t^b\left(q.p\right) = \left(\partial_x - \partial_{x'}\right)^a \left(\partial_t - \partial_{t'}\right)^b q(x,t)p(x^{'},t^{'})\;|_{x'=x,t'=t}, \tag{2.2}$$

here $a, b \geq 0$ and $a + b \geq 1$ [38, 45]. When $q = p$

$$D_x^{2a-1} q.q = 0, \tag{2.3}$$

$$D_x^{2a} q.q = \sum_{k=0}^{2a} (-1)^{2a-k} \binom{2a}{k} \left(\partial_x^k q\right)\left(\partial_x^{2a-k} q\right), \tag{2.4}$$

contrarily, bilinear partial derivative structure is equivalent to

$$D_x^b D_t^a\left(q.q\right) = \sum_{i=0}^{b}\sum_{j=0}^{a} (-1)^{b-i+a-j} \binom{b}{i}\binom{a}{j} \left(\partial_x^i \partial_t^j q\right)\left(\partial_x^{b-i}\partial_t^{a-j} q\right), \tag{2.5}$$

for $a, b \geq 1$, we introduce a logarithmic transformation function $p(x,t) = p$ which is given as [57, 34, 41]

$$u\left(x,t\right) = 2\left(\ln p\right)_{xx}, \tag{2.6}$$

here 2 is chosen to cancel some terms for the purpose of getting a quadratic equation in p.

3. Bilinear Transformation of The KdV Equation

This section provides an overview on the KdV model's bilinear form. Equation (2.6) is substituted into (1.1) to produce the following

$$2\left(\ln p\right)_{xt} + 2\left(\ln p\right)_{xxxx} + 12\left(\ln p\right)_{xx}^2 = C, \tag{3.1}$$

where C is the constant of integration, when looking for a soliton equation, it is required to set C to be equal to zero, we get

$$2\left(p_{xt}p - p_x p_t\right) + 2\left(p_{xxxx}p - 4p_{xxx}p_x + 3p_{xx}^2\right) = 0. \tag{3.2}$$

The above equation can be written in terms of Hirota bilinear operator

$$D_x\left(D_x^3 + D_t\right) p.p = 0. \tag{3.3}$$

Theorem 3.1. *Obviously p meets (2.6) provided that $u\left(t,x\right) = 2\left(\ln p\right)_{xx}$ generates the solutions to the given (1.1)*

$$D_x\left(D_x^3 + D_t\right) p.p = 2\left(p_{xxxx}p - 4p_{xxx}p_x + 3p_{xx}^2\right) + 2\left(p_{xt}p - p_x p_t\right) = 0. \tag{3.4}$$

Following the presentation of Hirota's bilinear form of (1.1) and BNN method, numerous multiple-wave solutions can be easily obtained. We will also derive complexiton solutions in the sections below using Hirota's bilinear notation.

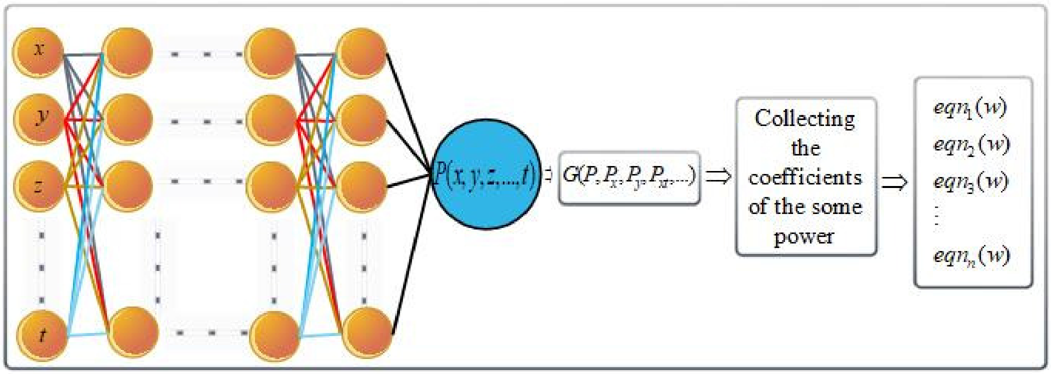

FIGURE 1. Algorithm flow chart of bilinear neural network technique.

4. Modeling a Neural Network and Corresponding Tensor Formula

In order to find precise analytical solution to the bilinear form of the KdV model (3.4), we build a neural network frame of solutions. The nonlinear neural network's tensor formula is shown below [26]:

$$p = W_{l_n,p} f_{l_n}(\xi_{l_n}), \tag{4.1}$$

for which $W_{l_n,p}$ is the weight coefficient of the neuron from l_n to p. The generalized activation parameter f can be chosen randomly as long as it satisfies the requirement that $0 \leq f_{l_n}(\xi)$ in the print layer. The equation $l_n = \{m_{n-1}, m_{n-1}+1, m_{n-1}+2, ..., n\}$ represents the nth layer space of the neural network model, below is a description of the term $f_{l_n}(\xi_{l_i})$:

$$\xi_{l_i} = W_{l_{i-1},l_i} f_{l_{i-1}}(\xi_{l_{i-1}}) + b_{l_i}, \qquad i = 1, 2, ..., n, \tag{4.2}$$

where $l_0 = \{z, y, x..., t\}$, $l_1 = \{1, 2, 3, 4..., m_1\}$,
$l_n = \{m_{n-1}, m_{n-1}+1, m_{n-1}+2, ..., m_i\}$, $i = 1, 2, ..., n-1$, b_{l_i} stands for a threshold, which can be considered as a constant. Figure 1 shows an intuitive representation of this neural network tensor model, while figure 2 shows the neural network model whose space layer is nth.

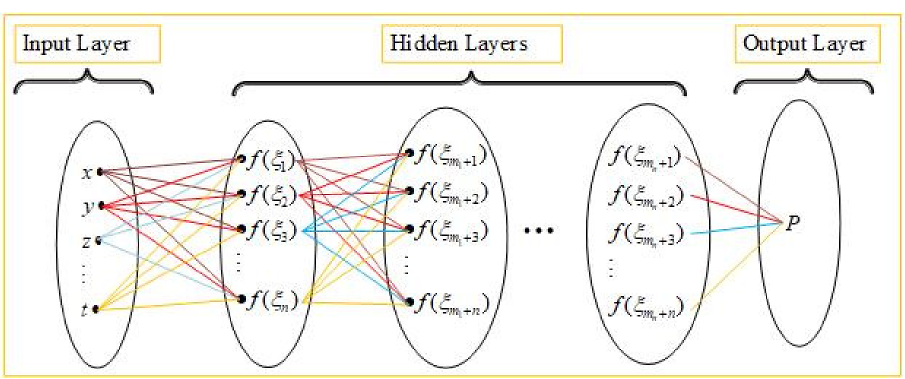

FIGURE 2. Algorithm flow chart of bilinear neural network of (1.1).

5. Exact Soliton Solutions to KdV Equation

5.1. Solutions to the KdV equation using a single hidden layer neural network. The bilinear form of the KdV equation (3.4) has exact analytical solutions that can be found using a "$2-3-1$" neural network model, which indicates that there are two neurons in the input layer l_0, three neurons in the hidden layer l_1 and one neuron in the print layer p. Figure 3 shows how to understand the "$2-3-1$" paradigm intuitively. Using $l_0 = \{x, t\}$, $l_1 = \{1, 2, 3\}$ as the respective neuron in each layer and p as the generalized activation function, by choosing $f(\xi_1) = {\xi_1}^2$, $f(\xi_2) = {\xi_2}^2$ and $f(\xi_3) = c$, we have:

$$p = W_{3,p}f(\xi_3) + W_{2,p}f(\xi_2) + W_{1,p}f(\xi_1) + b, \tag{5.1.1}$$

where

$$\begin{aligned} \xi_1 &= W_{t,1}t + W_{x,1}x, \\ \xi_2 &= W_{t,2}t + W_{x,2}x, \\ \xi_3 &= W_{t,3}t + W_{x,3}x, \end{aligned} \tag{5.1.2}$$

here $W_{i,j}(i = 1, 2, 3, x, t, \quad j = 1, 2, 3, p \quad i \neq j)$ and b are constants that will be obtained later . A sophisticated equation can be generated by substituting

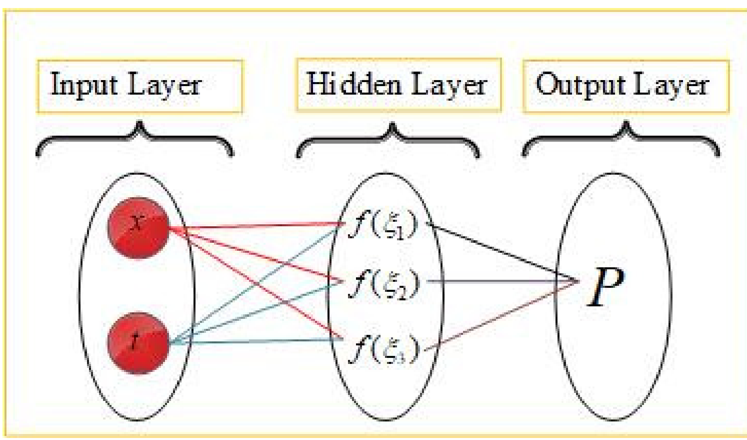

Figure 3. Framework for the "$2-3-1$" bilinear neural network of (1.1)

(5.1.1) into (3.4). The underdetermined set of nonlinear mathematical equations is derived by collecting the coefficients of the same terms. The coefficient solutions are achieved by using a computer program to solve this system of nonlinear mathematical equations are as follows.

Set 1. When

$$W_{x,2} = -\frac{\mathrm{i}W_{x,1}\sqrt{W_{1,p}}}{\sqrt{W_{2,p}}}, \quad b = -cW_{3,p}, \quad W_{t,1} = -\frac{\mathrm{i}W_{t,2}\sqrt{W_{2,p}}}{\sqrt{W_{1,p}}}. \tag{5.1.3}$$

Through the bilinear transformation, we can acquire the accurate lump wave solution to the KdV model by substituting set 1 into (2.6) using (5.1.1) and

(5.1.2),

$$u_1(x,t) = -\frac{2\begin{pmatrix} 2W_{x,1}W_{1,p}\left(W_{x,1}x - \frac{itW_{t,2}\sqrt{W_{2,p}}}{\sqrt{W_{1,p}}}\right) \\ -2iW_{x,1}\sqrt{W_{1,p}}\sqrt{W_{2,p}}\left(tW_{t,2} - \frac{iW_{x,1}x\sqrt{W_{1,p}}}{\sqrt{W_{2,p}}}\right) \end{pmatrix}^2}{\begin{pmatrix} W_{1,p}\left(W_{x,1}x - \frac{itW_{t,2}\sqrt{W_{2,p}}}{\sqrt{W_{1,p}}}\right)^2 \\ +W_{2,p}\left(tW_{t,2} - \frac{iW_{x,1}x\sqrt{W_{1,p}}}{\sqrt{W_{2,p}}}\right)^2 \end{pmatrix}^2}. \tag{5.1.4}$$

Set 2. When

$$W_{x,2} = -\frac{\mathrm{i}W_{x,1}\sqrt{W_{1,p}}}{\sqrt{W_{2,p}}}, \quad W_{t,1} = \frac{\mathrm{i}W_{t,2}\sqrt{W_{2,p}}}{\sqrt{W_{1,p}}}, \tag{5.1.5}$$

through the bilinear transformation, we can acquire the exact generalized lump wave solution to the KdV equation by substituting set 2 into (2.6) using (5.1.1) and (5.1.2),

$$u_2(x,t) = -\frac{2\begin{pmatrix} 2W_{x,1}W_{1,p}\left(W_{x,1}x + \frac{itW_{t,2}\sqrt{W_{2,p}}}{\sqrt{W_{1,p}}}\right) \\ -2iW_{x,1}\sqrt{W_{1,p}}\sqrt{W_{2,p}}\left(tW_{t,2} - \frac{iW_{x,1}x\sqrt{W_{1,p}}}{\sqrt{W_{2,p}}}\right) \end{pmatrix}^2}{\begin{pmatrix} cW_{3,p} + W_{1,p}\left(W_{x,1}x + \frac{itW_{t,2}\sqrt{W_{2,p}}}{\sqrt{W_{1,p}}}\right)^2 \\ +W_{2,p}\left(tW_{t,2} - \frac{iW_{x,1}x\sqrt{W_{1,p}}}{\sqrt{W_{2,p}}}\right)^2 + b \end{pmatrix}^2}. \tag{5.1.6}$$

We substitute the lump wave solution (5.1.4) and (5.1.6) into the KdV equation (1.1) to demonstrate the accuracy of the results. The simplified results reveal the accuracy of the result by obtaining left side of (1.1) to be equal to zero using a mathematical package. It proves the exact lump wave solution (5.1.4) and (5.1.6) are exact and error-free. It demonstrates the benefit of this approach as well when compared to the classical approach that can only provide approximate results.

5.2. **Double Hidden Layers Neural Network Nolutions to the KdV Equation.** The bilinear equation (1.1) has accurate analytical solutions that may be found using a "$2-2-2-1$" neural network model, which indicates that there are two neurons in input layer l_0, two neurons in each hidden layer l_2 and l_3 then one neuron in the last layer p. Figure 4 shows how to understand the "$2-2-2-1$" paradigm intuitively. Using $l_0 = \{x,t\}$, $l_1 = \{1,2\}$ and $l_2 = \{3,4\}$ as the respective neuron in each layer and p as the generalized activation function, by choosing $f(\xi_1) = \cos(\xi_1)$, $f(\xi_2) = e^{\xi_2}$, $f(\xi_3) = {\xi_3}^2$ and $f(\xi_4) = {\xi_4}^2$, we have:

$$p = W_{3,p}f(\xi_3) + W_{4,p}f(\xi_4), \tag{5.2.1}$$

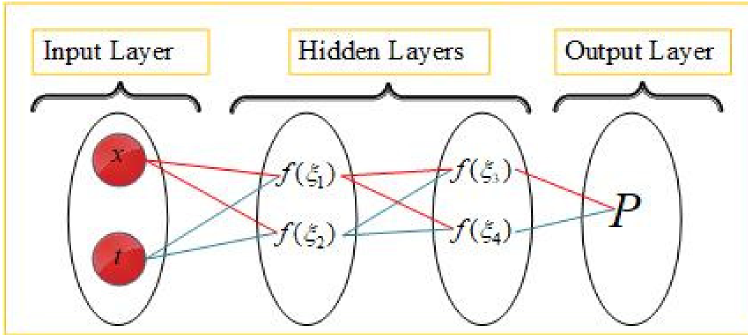

FIGURE 4. The framework for equation (1.1)'s "$2-2-2-1$" bilinear neural network.

where

$$\begin{aligned}\xi_1 &= W_{t,1}t + W_{x,1}x + a,\\ \xi_2 &= W_{t,2}t + W_{x,2}x + b,\\ \xi_3 &= W_{1,3}f(\xi_1) + W_{2,3}f(\xi_2) + c,\\ \xi_4 &= W_{1,4}f(\xi_1) + W_{2,4}f(\xi_2) + d,\end{aligned} \tag{5.2.2}$$

where $W_{i,j}(i = 1, 2, 4, x, t \quad j = 1, 2, 3, 4, p \quad j \neq i)$, a, b, c and d are constants that will be found later. A sophisticated equation can be generated by substituting (5.2.1) into (3.4). The underdetermined set of nonlinear mathematical equations can be derived by collecting the coefficients of the same terms. The coefficient solutions are obtained by using mathematical package to solve this system of nonlinear algebraic equations are as follows.

Set 1. When

$$\begin{aligned}&W_{1,3} = 0, W_{2,3} = \frac{\mathrm{i}W_{2,4}\sqrt{W_{4,p}}}{\sqrt{W_{3,p}}}, W_{t,1} = 4{W_{x,1}}^3, W_{x,2} = -\frac{\mathrm{i}W_{x,1}}{\sqrt{2}},\\ &W_{t,2} = -\frac{\mathrm{i}W_{x,1}^3}{2\sqrt{2}}, \quad W_{1,4} = 0,\end{aligned} \tag{5.2.3}$$

through the bilinear transformation, the new analytical solution of the KdV equation is gained by substituting set 1 into (2.6) using (5.2.1) and (5.2.2),

$$u_3(x,t) = -\frac{2W_{x,1}^2 W_{2,4}\sqrt{W_{4,p}}e^{b+\frac{iW_{x,1}\left(tW_{21}^2+2x\right)}{2\sqrt{2}}}\left(d\sqrt{W_{4,p}} - ic\sqrt{W_{3,p}}\right)}{\left(2e^b W_{2,4}\sqrt{W_{4,p}} + e^{\frac{iW_{x,1}\left(tW_{x,1}^2+2x\right)}{2\sqrt{2}}}\left(d\sqrt{W_{4,p}} - ic\sqrt{W_{3,p}}\right)\right)^2}. \tag{5.2.4}$$

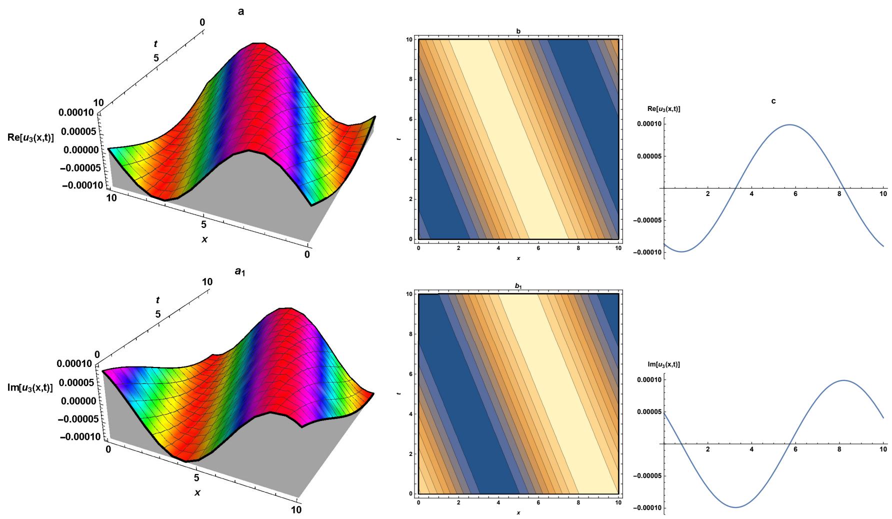

FIGURE 5. The 3-dimensional (a), (a_1), contour (b),(b_1) and 2-dimensional (c),(c_1) graphs of (5.2.4) for $W_{3,p} = 0.6, W_{4,p} = 0.4, W_{x,1} = 0.9, W_{2,4} = 2.9, b = 9, c = 4, d = 3$ which represents the solution at a given time.

Set 2. When

$$\begin{aligned} &W_{t,1} = 0, W_{x,1} = 0, W_{t,2} = -{W_{x,2}}^3, \\ &W_{1,4} = \frac{-3cW_{1,3}W_{3,p} - \mathrm{i}dW_{1,3}\sqrt{W_{3,p}}\sqrt{W_{4,p}}}{-\mathrm{i}c\sqrt{W_{3,p}}\sqrt{W_{4,p}} + 3dW_{4,p}}, \\ &W_{2,4} = \frac{\mathrm{i}W_{2,3}\sqrt{W_{3,p}}}{\sqrt{W_{4,p}}}, \end{aligned} \tag{5.2.5}$$

through the bilinear transformation, we can acquire the new analytical solution of the KdV equation by substituting set 2 into (2.6) using (5.2.1) and (5.2.2),

$$u_4(x,t) = \frac{\left[\begin{array}{c} 4W_{x,2}^2 W_{2,3}\sqrt{W_{3,p}}e^{b+tW_{x,2}^3+W_{x,2}x} \\ \left(c^2W_{3,p} + 2icd\sqrt{W_{3,p}}\sqrt{W_{4,p}} + 3d^2W_{4,p}\right) \\ \left(\sqrt{W_{3,p}}\left(c - 2W_{1,3}\cos(a)\right) + 3id\sqrt{W_{4,p}}\right) \end{array}\right]}{\left[\left(\begin{array}{c} e^{tW_{x,2}^3}\left(c\sqrt{W_{3,p}} - id\sqrt{W_{4,p}}\right) \\ \left(\sqrt{W_{3,p}}\left(c - 2W_{1,3}\cos(a)\right) + 3id\sqrt{W_{4,p}}\right) \\ +2W_{2,3}\sqrt{W_{3,p}}e^{b+W_{x,2}x}\left(c\sqrt{W_{3,p}} + 3id\sqrt{W_{4,p}}\right) \end{array}\right)^2\right]}. \tag{5.2.6}$$

Set 3. When

$$\begin{aligned} &W_{2,3} = \frac{\mathrm{i}W_{2,4}\sqrt{W_{4,p}}}{\sqrt{W_{3,p}}}, W_{x,1} = 0, W_{t,2} = -{W_{x,2}}^3, \\ &W_{1,4} = \frac{(-i)c^2w_{1,3}\sqrt{w_{4,p}}w_{3,p}^{3/2} - 2cdw_{1,3}w_{4,p}w_{3,p} + id^2w_{1,3}\sqrt{w_{3,p}}w_{4,p}^{3/2}}{-c^2w_{3,p}w_{4,p} + 2icd\sqrt{w_{3,p}}w_{4,p}^{3/2} + d^2w_{4,p}^2}, \end{aligned} \tag{5.2.7}$$

through the bilinear transformation, we can acquire the new accurate analytical solution of the KdV equation by substituting set 3 into (2.6) using (5.2.1) and

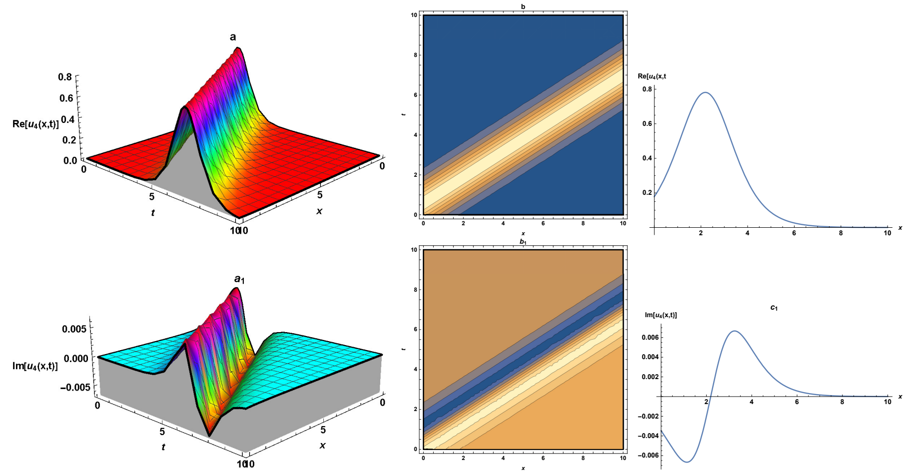

FIGURE 6. The 3-dimensional (a), (a_1), contour (b),(b_1) and 2-dimensional (c),(c_1) graphs of (5.2.6) for $W_{3,p} = 0.6, W_{4,p} = 0.4, W_{t,2} = 1.2, W_{x,2} = 1.25, W_{1,3} = 8, W_{2,3} = 0.9, a = 5, b = 2, c = 4, d = 3$ which (a) represents the bright soliton and (a_1) represents thr kink soliton solution.

(5.2.2),

$$u_5(x,t) = \frac{4W_{x,2}^2 W_{2,4}\sqrt{W_{4,p}}e^{b+tW_{x,2}^3+W_{x,2}x}\left(d\sqrt{W_{4,p}} - ic\sqrt{W_{3,p}}\right)}{\left(2W_{2,4}\sqrt{W_{4,p}}e^{b+W_{x,2}x} + e^{tW_{x,2}^3}\left(d\sqrt{W_{4,p}} - ic\sqrt{W_{3,p}}\right)\right)^2}. \tag{5.2.8}$$

Set 4. When

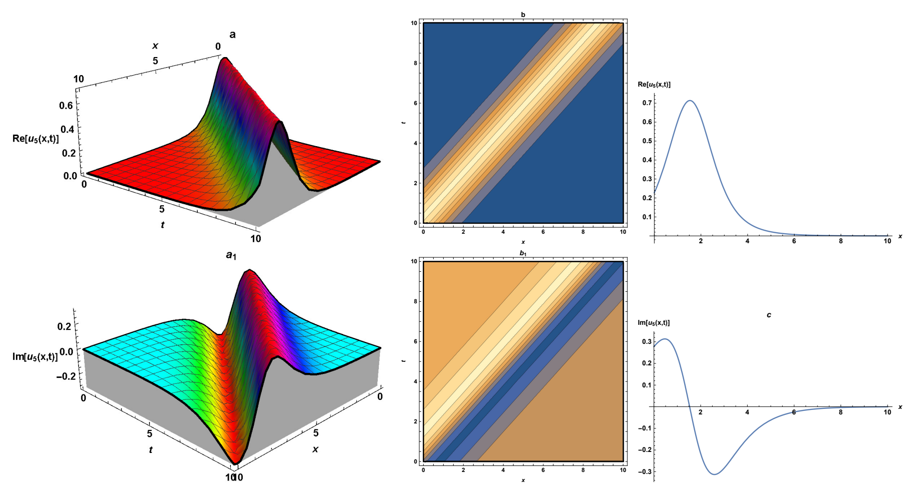

FIGURE 7. The 3-dimensional (a), (a_1), contour (b),(b_1) and 2-dimensional (c),(c_1) graphs of (5.2.8) for $W_{3,p} = 6, W_{4,p} = 0.2, W_{x,1} = 0.3, W_{x,2} = 0.95, W_{2,4} = 1, b = 2, c = 2, d = 3$ which (a) represents the bright soliton and (a_1) represents the kink soliton solution.

$$W_{1,3} = 0, W_{t,1} = 4{W_{x,1}}^3, W_{x,2} = -iW_{x,1},$$
$$W_{t,2} = -4i{W_{x,1}}^3, W_{1,4} = 0, W_{2,4} = -\tfrac{cW_{2,3}W_{3,p}}{dW_{4,p}}, \tag{5.2.9}$$

through the bilinear transformation, we acquire the accurate analytical solution of the KdV equation by substituting set 4 into (2.6) using (5.2.1) and (5.2.2),

$$u_6(x,t) = -\frac{8d^2W_{x,1}^2W_{2,3}^2W_{3,p}W_{4,p}e^{2b+2iW_{x,1}\left(4tW_{x,1}^2+x\right)}}{\left(e^{2b}W_{2,3}^2W_{3,p} + d^2W_{4,p}e^{2iW_{x,1}\left(4tW_{x,1}^2+x\right)}\right)^2}. \tag{5.2.30}$$

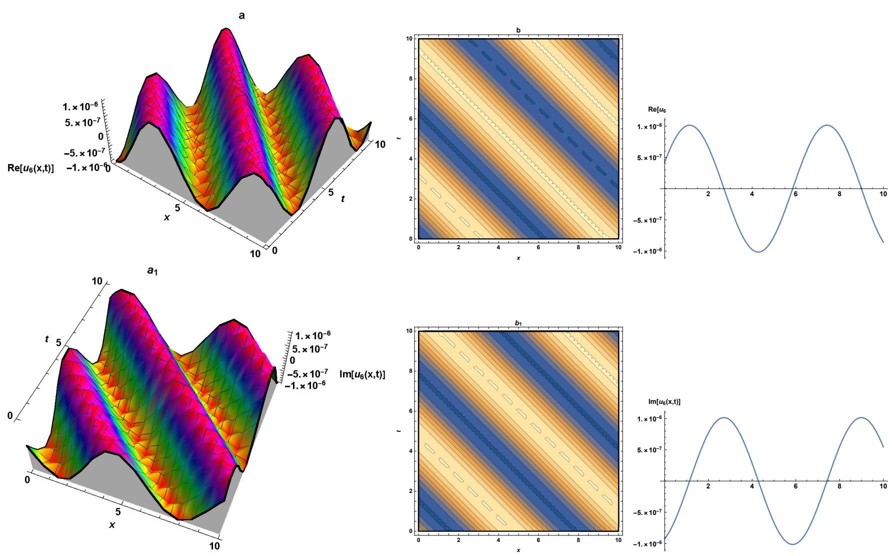

FIGURE 8. The 3-dimensional (a), (a_1), contour (b),(b_1) and 2-dimensional (c),(c_1) graphs of (5.2.30) for $W_{3,p} = 3, W_{4,p} = 1, W_{x,1} = 0.5, W_{2,3} = 0.3, b = 9, d = 3$ which represents the travelling wave solution.

Set 5. When

$$W_{1,3} = 0, W_{2,3} = -\tfrac{dW_{2,4}W_{4,p}}{cW_{3,p}}, W_{t,2} = -4{W_{x,2}}^3,$$
$$W_{t,1} = \tfrac{8W_{x,1}^4-12W_{x,1}^2W_{x,2}^2-3W_{x,2}^4}{2W_{x,1}}, W_{1,4} = 0, \tag{5.2.31}$$

through the bilinear transformation, we obtain the accurate analytical solution of the KdV equation by substituting set 5 into (2.6) using (5.2.1) and (5.2.2),

$$u_7(x,t) = \frac{8c^2W_{x,2}^2W_{2,4}^2W_{3,p}W_{4,p}e^{2\left(b+4tW_{x,2}^3+W_{x,2}x\right)}}{\left(W_{2,4}^2W_{4,p}e^{2(b+W_{x,2}x)} + c^2e^{8tW_{x,2}^3}W_{3,p}\right)^2}. \tag{5.2.32}$$

Set 6. When

$$W_{1,3} = 0, W_{2,3} = -\tfrac{dW_{2,4}W_{4,p}}{cW_{3,p}}, W_{t,1} = 4{W_{x,1}}^3,$$
$$W_{t,2} = -4i{W_{x,1}}^3, W_{1,4} = 0, W_{x,2} = -i{W_{x,1}}^3 \tag{5.2.33}$$

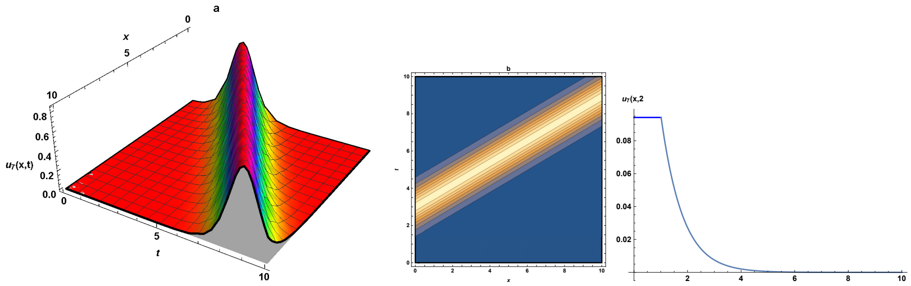

FIGURE 9. The 3-dimensional (a), contour (b) and 2-dimensional (c) graphs of (5.2.32) for $W_{3,p} = 6, W_{4,p} = 0.2, W_{x,1} = 3, W_{x,2} = 0.65, W_{2,4} = 1, b = 5, c = 1$ which represents a bright solution.

through the bilinear transformation, we acquire the accurate analytical solution of the KdV equation by substituting set 6 into (2.6) using (5.2.1) and (5.2.2),

$$u_8(x,t) = -\frac{8c^2W_{x,1}^2W_{2,4}^2W_{3,p}W_{4,p}e^{2b+2iW_{x,1}\left(4tW_{x,1}^2+x\right)}}{\left(e^{2b}W_{2,4}^2W_{4,p}+c^2W_{3,p}e^{2iW_{x,1}\left(4tW_{x,1}^2+x\right)}\right)^2}. \tag{5.2.34}$$

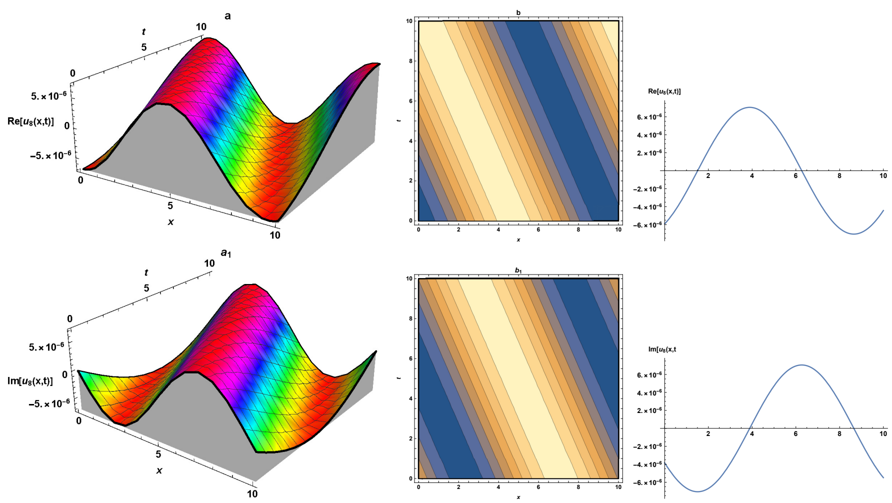

FIGURE 10. The 3-dimensional (a), (a_1), contours (b), (b_1) and 2-dimensional (c), (c_1) graphs of (5.2.34) for $W_{3,p} = 0.6, W_{4,p} = 0.4, W_{x,1} = 0.33, W_{2,4} = 2.9, b = 5, c = 1$ which represent the solitary wave solutions.

Set 7. When

$$\begin{aligned} &W_{1,3} = 0, W_{2,3} = \frac{\mathrm{i}W_{2,4}\sqrt{W_{4,p}}}{\sqrt{W_{3,p}}}, W_{t,2} = -{W_{x,2}}^3, \\ &W_{1,4} = 0, W_{t,1} = -2\left(-2W_{x,1}^3+3W_{x,1}W_{x,2}^2\right), \end{aligned} \tag{5.2.35}$$

through the bilinear transformation, we acquire the exact analytical solution of the KdV equation by substituting set 7 into (2.6) using (5.2.1) and (5.2.2),

$$u_9(x,t) = \frac{4W_{x,2}^2 W_{2,4}\sqrt{W_{4,p}}e^{b+tW_{x,2}^3+W_{x,2}x}\left(d\sqrt{W_{4,p}} - ic\sqrt{W_{3,p}}\right)}{\left(2W_{2,4}\sqrt{W_{4,p}}e^{b+W_{x,2}x} + e^{tW_{x,2}^3}\left(d\sqrt{W_{4,p}} - ic\sqrt{W_{3,p}}\right)\right)^2}. \tag{5.2.36}$$

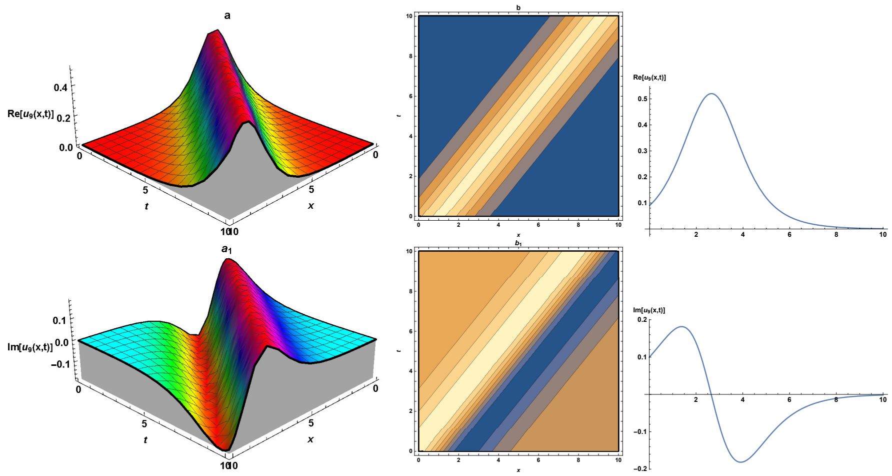

FIGURE 11. The 3-dimensional (a), (a_1), contour (b), (b_1) and 2-dimensional (c), (c_1) graphs of (5.2.36) for $W_{3,p} = 2, W_{4,p} = 1.6, W_{x,2} = 0.9, W_{2,4} = 2.9, b = -1, c = 4, d = 3$ which (a) represents the bright soliton and (a_1) represents the kink soliton solution.

Set 8. When

$$\begin{aligned} &W_{1,3} = 0, W_{2,3} = -\tfrac{dW_{2,4}W_{4,p}}{cW_{3,p}}, W_{t,2} = -4{W_{x,2}}^3, \\ &W_{1,4} = 0, W_{t,1} = 4{W_{x,1}}^3, \end{aligned} \tag{5.2.37}$$

through the bilinear transformation, we acquire the accurate analytical solution of the KdV equation by substituting set 8 into (2.6) using (5.2.1) and (5.2.2),

$$u_{10}(x,t) = \frac{8c^2 W_{x,2}^2 W_{2,4}^2 W_{3,p} W_{4,p} e^{2\left(b+4tW_{x,2}^3+W_{x,2}x\right)}}{\left(W_{2,4}^2 W_{4,p} e^{2(b+W_{x,2}x)} + c^2 e^{8tW_{x,2}^3} W_{3,p}\right)^2}, \tag{5.2.38}$$

Set 9. When

$$\begin{aligned} &W_{1,3} = 0, W_{2,3} = -\tfrac{dW_{2,4}W_{4,p}}{cW_{3,p}}, W_{t,2} = -4{W_{x,2}}^3, \\ &W_{1,4} = 0, W_{t,1} = -2\left(-2W_{x,1}^3 + 3W_{x,1}W_{x,2}^2\right), \end{aligned} \tag{5.2.39}$$

through the bilinear transformation, we can acquire the accurate analytical solution to the KdV equation by substituting set 9 into (2.6) using (5.2.1) and (5.2.2),

$$u_{11}(x,t) = \frac{8c^2 W_{x,2}^2 W_{2,4}^2 W_{3,p} W_{4,p} e^{2\left(b+4tW_{x,2}^3+W_{x,2}x\right)}}{\left(W_{2,4}^2 W_{4,p} e^{2(b+W_{x,2}x)} + c^2 e^{8tW_{x,2}^3} W_{3,p}\right)^2}, \tag{5.2.40}$$

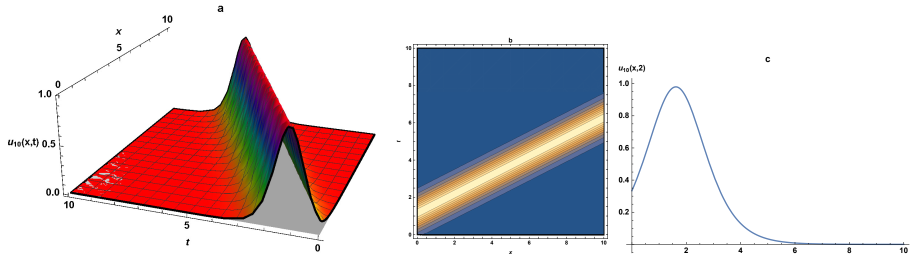

FIGURE 12. The 3-dimensional (a), contour (b) and 2-dimensional (c) graphs of (5.2.38) for $W_{3,p} = 0.2, W_{4,p} = 0.4, W_{t,2} = 1.2, W_{x,1} = 0.8, W_{x,2} = 0.7, W_{1,3} = 8, W_{2,3} = 0.8, W_{2,4} = 2.9, a = 0.1, b = 0.2, c = 1, d = 1$ which represents the bright soliton solution at a given time.

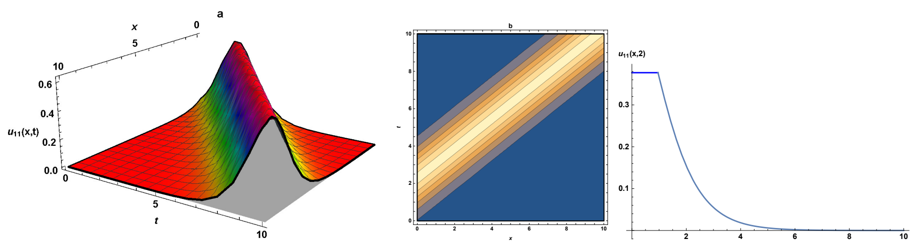

FIGURE 13. The 3-dimensional (a), contour (b) and 2-dimensional (c) graphs of (5.2.40) for $W_{3,p} = 0.2, W_{4,p} = 0.4, W_{t,2} = 1.2, W_{x,1} = 0.8, W_{x,2} = 0.56, W_{1,3} = 8, W_{2,3} = 0.8, W_{2,4} = 2.9, a = 0.1, b = 0.2, c = 1, d = 1$ which represents a bright soliton solution.

Set 10. When

$$\begin{aligned} &W_{1,3} = 0, W_{2,3} = -\tfrac{dW_{2,4}W_{4,p}}{cW_{3,p}}, W_{t,2} = -4{W_{x,2}}^3, \\ &W_{1,4} = 0, W_{t,1} = \tfrac{W_{x,1}^4 - 6W_{x,1}^2W_{x,2}^2 - 3W_{x,2}^4}{W_{x,1}}, \end{aligned} \tag{5.2.41}$$

through the bilinear transformation, we get the exact analytical solution of the KdV equation by substituting set 10 into (2.6) using (5.2.1) and (5.2.2),

$$u_{12}(x,t) = \frac{8c^2W_{x,2}^2W_{2,4}^2W_{3,p}W_{4,p}e^{2\left(b+4tW_{x,2}^3+W_{x,2}x\right)}}{\left(W_{2,4}^2W_{4,p}e^{2(b+W_{x,2}x)} + c^2e^{8tW_{x,2}^3}W_{3,p}\right)^2}, \tag{5.2.42}$$

6. STABILITY ANALYSIS

The stability analysis [45, 4] for the mathematical model (1.1) will be investigated in this part. Observing the perturbation solution of the type;

$$u\left(x,t\right) = r + \lambda v\left(x,t\right), \tag{6.1}$$

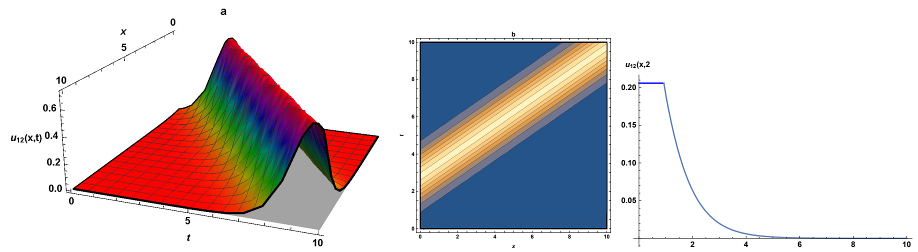

FIGURE 14. The 3-dimensional (a), contour (b) and 2-dimensional (c) graphs of (5.2.42) for $W_{3,p} = 0.2, W_{4,p} = 0.4, W_{t,2} = 0.2, W_{x,1} = 0.8, W_{x,2} = 0.6, W_{1,3} = 1, W_{2,3} = 0.8, W_{2,4} = 2.9, b = 1, c = 1$ which represents a bright soliton solution.

it is simple to observe that (1.1) has a steady state value for any real variable r, by using (6.1) and (1.1), the outcome of the relation is

$$\lambda v_t + 6r\lambda v_x + 6\lambda^2 v v_x + \lambda v_{xxx} = 0, \tag{6.2}$$

by linearizing the aforementioned equation in λ

$$\lambda v_t + 6r\lambda v_x + \lambda v_{xxx} = 0. \tag{6.3}$$

Suppose the preceding equation has solution that takes the following form

$$v(x,t) = e^{i(-\mu t + kx)}, \tag{6.4}$$

here k is said to be a normalized wave number, by using (6.4) and (6.3), the following is obtained

$$6rk - k^3 - \mu = 0,$$
$$\mu = -k\left(k^2 + 6r\right). \tag{6.5}$$

The solutions will appear to decay for the negative values of μ, as shown in (6.5). As a result, the dispersion remains stable.

7. Result and discussion

The function is discretized for nonlinear models using the classic neural network technique and then the original function is fitted with these distinct points to provide exact solutions, with the help of the BNN method, the analytical solutions for (1.1) are successfully derived. The "$2-3-1$" and "$2-2-2-1$" models are applied to the neural network model for the KdV equation. By specifying certain activation functions, such as $[f(\xi_1) = {\xi_1}^2, f(\xi_2) = {\xi_2}^2, f(\xi_3) = c]$ or $[f(\xi_1) = \cos(\xi_1), f(\xi_2) = e^{\xi_2}, f(\xi_3) = {\xi_3}^2, f(\xi_4) = {\xi_4}^2]$. New solitory wave solutions such as lump wave, bright, kink and other soliton solutions are obtained by choosing some special parameters. The results were illustrated by $2D$, $3D$ and contour graphs that displayed various periodic wave-forms at a given time t. **In a traveling wave solution, the coexistence of virtual and real components describes both the general features and finer details of wave behavior. Figures 5, 6, 7, 8, 10 and 11, presented in this paper represent snapshots of solutions with virtual and real components. The real part usually refers to more obvious features such as wave heights,**

amplitudes and the overall appearance of the waveform, while the virtual part represents finer details and phase changes. Analyzing both the real and virtual parts together can provide a more comprehensive understanding of wave propagation and behavior. In addition, traveling wave solutions without virtual parts are also produced by the method. Snapshots of these solutions are presented in figures 9, 12, 13 and 14. These graphs represent the general shape of the wave pattern and profile and the energy distribution, which are the physical effects of wave propagation. The method used prove to be efficient and effective for the extraction of soliton solutions of different types for the KdV equation. In the field of shallow water waves, this will be applied to analyze nonlinear processes.

In the process of gathering data for complex physical, biological or technical systems, time and cost relevance are both sought after. The usage of mathematical models that are based on fundamental definitions and theorems and employed in decision-making processes is crucial. However, it may encounter difficulties in generating solutions for the mathematical model. It has the potential to be faster and more energy-efficient than current conventional computer technology at solving computationally challenging problems. In parallel with the development of computer technology, mathematical solution techniques are also developing. In this work, the traveling wave solutions to the KdV model are derived for the first time with the recently proposed BNN method. The effects of solutions involving both x-spatial and t-time will provide insight into the nonlinear wave propagation. An illustrative investigation of the impacts of modest changes in shallow water will be provided by solutions whose effects lead to non-linear permanent form of waves. Waves, which are part of almost everything in the universe, are mechanisms through which energy is transferred from one place to another. The square of the amplitude of a mechanical wave determines its energy at any point. In the light of this basic definition, equation (5.2.6) which is the subject of our discussion will be considered as $|u|^2$ related to the amplitude of the traveling wave solution. It is highly important in mathematics that (5.2.6) has several physical parameters, allowing experimental researchers to generate a variety of scenarios. The impact of the traveling wave's frequency speed on its intensity has been picked as the discussion's central idea. By substituting five different values for the frequency velocity in (5.2.6) which represents the traveling wave, the graph below can be obtained in order to study the relationship between the wave intensity and the frequency velocity of the wave. The frequency velocity of the high-intensity soliton is $W_{x,2} = 1.7$, as shown in Figure 15, whereas the frequency velocity of the low-intensity soliton is $W_{x,2} = 1.08$. When Figure 15 is closely examined, it is clear that as the wave's frequency velocity falls, so does its intensity. As a result, it can be concluded that a soliton with low density has low frequency velocity and low energy.

The frequency velocity of the high-intensity soliton is $W_{x,2} = 1.7$, as shown in figure 15, whereas the frequency velocity of the low-intensity soliton is $W_{x,2} = 1.08$. When Figure 15 is closely examined, it is clear that as the wave's frequency velocity falls, so does its intensity. As a result, it can be concluded that a soliton

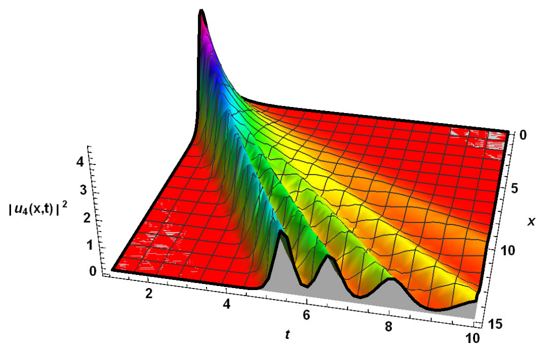

FIGURE 15. The response given of (5.2.6) for $1.08, 1.25, 1.4, 1.55, 1.7$ with the order of $W_{x,2}$ representing wave frequency velocity.

with low density has low frequency velocity as well as low energy. Additionally, the study's proposed hypothesis is that the results represent the traveling wave. The graph below can be evaluated using the time parameter t to verify this theory. Figure 16(a) depicts a soliton that propagates at a constant speed, maintains its

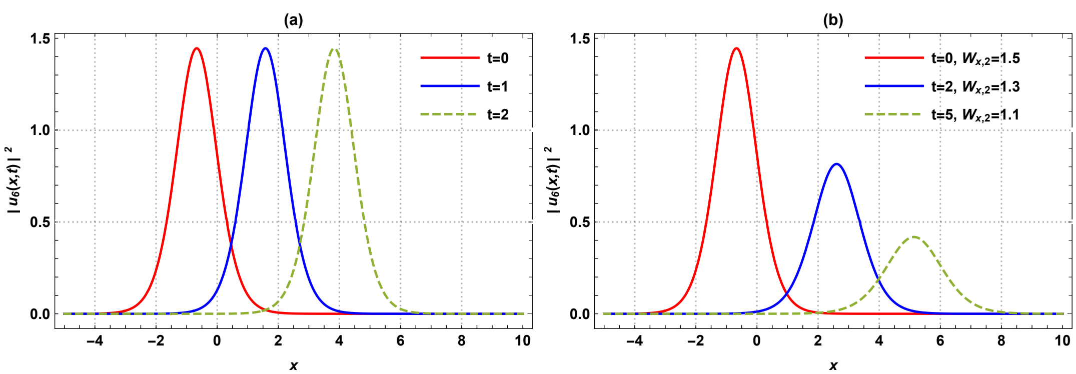

FIGURE 16. (a) Soliton property for different values of the time parameter t, (b) Diffusion property representing nonlinear distribution for different values of time t and frequency velocity $W_{x,2}$ parameters in (5.2.6).

shape and characteristics, and demonstrates single wave features. Additionally, for different values of time t and frequency velocity $W_{x,2}$, the diffusion phenomenon associated with nonlinear distribution can be analyzed in figure 16(b). As a result, analyses performed on one of the obtained solutions can be applied to the others. Solutions representing traveling wave under stable conditions can be actively used in both soliton and nonlinear scattering phenomena.

8. CONCLUSION

In this work, Hirota Bilinearization, the Hirota bilinearization of the provided model and BNN method have been briefly discussed. This method has an advantage over the traditional exact analytical method in the sense that it can create deep-layer composite functions rather than just basic linear superposition. The superiority of this technique over the traditional methodology is that it can produce exact analytical solutions without error rather than numerical solutions including errors. To determine the KdV equation's analytical solution via BNN method, new test functions have been developed. Other nonlinear partial differential equations can also have their exact analytical solutions obtained using these novel test functions. The "$2-3-1$" and "$2-2-2-1$" models are applied to the neural network model for obtaining the soliton solutions to the KdV equation. By specifying certain activation functions, such as $[f(\xi_1) = {\xi_1}^2, f(\xi_2) = {\xi_2}^2, f(\xi_3) = c]$ or $[f(\xi_1) = \cos(\xi_1), f(\xi_2) = e^{\xi_2}, f(\xi_3) = {\xi_3}^2, f(\xi_4) = {\xi_4}^2]$. New solitary wave solutions are obtained by choosing some special parameters and the stabilty analysis of the derived results has been studied. The three-dimensional, two-dimensional and contour plots offer good descriptions of the novel KdV equation solutions found in this research. Examining the correlation between wave frequency velocity and wave density as well as the stability study of the found solutions allowed for a thorough investigation of the equation. To get more thorough analytical solutions to the NPDE, attempt will be made to build some deeper test functions via BNN approach in the future. The development of deeper layer test functions will make computations more difficult and time consuming, which will impede our research. As a result, studying parallel algorithms is an excellent concept for future research projects.

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9. Data Availability Statement

The authors declare that data supporting the findings of this study are available within the article, the figures are concrete expression.

10. Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper. The authors declare that there are no conflicts of interests with publication of this work.

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