Khayyam J. Math. 10 (2024), no. 2, 249-255
DOI: 10.22034/kjm.2024.395718.2852

Khayyam Journal of Mathematics
emis.de/journals/KJM
kjm-math.org

ON THE GENERALIZED NOTION OF AMENABLE LOCALLY COMPACT GROUPS

SEYEDEH SOMAYEH JAFARI

Communicated by H.R. Ebrahimi Vishki

ABSTRACT. In this note, we study some spaces associated with a locally compact group such that it has a positive action on the predual of a W^*-algebra. We investigate the existence of invariant means on these spaces and present several characterizations when the group acts amenably on the predual of a W^*-algebra.

1. Introduction

Throughout G is a locally compact group with the unit and fixed left Haar-measure. Suppose that $L^1(G)$ is the group algebra of G with the convolution product as defined in [5]. We recall that the convolution product of two complex-valued functions f and g on G is defined as follows:

$$f * g(s) = \int_G f(t)g(t^{-1}s)\, dt \quad (s \in G),$$

when the integral makes sense. It is known that the Lebesgue space $L^\infty(G)$, equipped with the locally essential supremum norm, can be identified by the first dual space of $L^1(G)$ under the pairing

$$\langle x,\, \phi\rangle = \int_G x(s)\phi(s)\, ds \qquad (x \in L^\infty(G),\, \phi \in L^1(G)).$$

A locally compact group G is called amenable if there is a mean on $L^\infty(G)$, which is left translation invariant.

Date: Received: 3 May 2023; Revised: 29 May 2024; Accepted: 11 June 2024.
2020 *Mathematics Subject Classification.* Primary 46H25; Secondary 43A07.
Key words and phrases. Amenable group, bounded linear operator, locally compact group, module, W^*-algebra.

One of the most important objects of study in the category of harmonic analysis is the Von-Neumann algebra. Many of the central objects of study in abstract harmonic analysis, such as amenable theory, are in this class. Sakai (1971) showed that Von-Neumann algebras can also be defined abstractly as W^*-algebras. It is known that the predual of a W^*-algebra is in fact unique up to isomorphism. On the other hand, amenability, which is a very distinctive property for locally compact groups, was defined firstly for discrete groups by Von-Neumann [10]. The definition of amenability for arbitrary locally compact groups was later given by Day [2]. Especially after Day's work, this subject has extended at a fast pace. Amenable groups have been studied from various points of view. A general notion of an amenable action on the predual of a W^*-algebra developed by Stokke [9] in 2004.

The main motivation of this paper is the existence of a vast body of results on equivalents of locally compact groups equipped with the amenable property. We show that G acts amenably on the predual of a W^*-algebra if and only if there exists a weakly compact positive operator with some special properties.

2. The results

Let $\mathcal{M}$ be a W^*-algebra with predual $\mathcal{M}_*$ and the identity element $e_{\mathcal{M}}$. Let also $\mathcal{M}_*$ be a left Banach G-module through $G\times\mathcal{M}_* \longrightarrow \mathcal{M}_*$ by $(s,\,\phi)\mapsto s\cdot\phi$, as defined in [7]. Then dual action makes $\mathcal{M}$ as the right G-module, not necessarily Banach. Moreover, $\mathcal{M}_*$ is a left neo-unital Banach $L^1(G)$-module by the following integral in the weak sense:

$$f\cdot\phi=\int_G s\cdot\phi\, f(s)\,ds \quad (f\in L^1(G),\, \phi\in\mathcal{M}_*).$$

Furthermore, $\mathcal{M}$ is a right Banach $L^1(G)$-module, which is given by making in turn the definition

$$\langle x\cdot f,\,\phi\rangle=\langle x,\,f\cdot\phi\rangle \quad (x\in\mathcal{M},\, f\in L^1(G)\,\text{and}\,\phi\in\mathcal{M}_*).$$

Note that $L^1(G)$ is a left Banach G-module with the action $s\cdot\phi=l_{s^{-1}}\phi$ for all $s\in G$ and $\phi\in L^1(G)$, where $l_{s^{-1}}\phi$ is the left translation of ϕ by s^{-1}. Using this fact, Stokke [9] introduced a unified approach under which the standard techniques used to develop the basic theory of amenable groups as follows. Before dealing, we need to recall that a bounded linear functional M on $\mathcal{M}$ is said to be a state if it satisfies $\|M\|=M(e_{\mathcal{M}})=1$.

Definition 2.1. A locally compact group G will be said to have positive action on $\mathcal{M}_*$ if $\mathcal{M}_*$ is a left Banach G-module such that

(a) $\|s\cdot\phi\|\le\|\phi\|$ for all $s\in G$ and $\phi\in\mathcal{M}_*$,
(b) $s\cdot\phi\in(\mathcal{M}_*)_1^+$ for all $s\in G$ and $\phi\in(\mathcal{M}_*)_1^+$,

where $(\mathcal{M}_*)_1^+$ is the collection of normal states on $\mathcal{M}$; that is, the collection of all W^*-continuous states on $\mathcal{M}_*$.

The reader can see more known examples of Definition 2.1 in [9, Example 1.2].

Hereafter, throughout, $\mathcal{M}$ is a W^*-algebra such that G has a positive action on the predual $\mathcal{M}_*$.

Definition 2.2. An element $x \in \mathcal{M}$ is said to be

(a) *uniformly continuous* if the mapping $G \longrightarrow (\mathcal{M}, \|.\|)$ that given by $s \mapsto x \cdot s$ is continuous.

(b) *weakly almost periodic* if the $\{x \cdot s \,|\, s \in G\}$ is relatively weakly compact.

The collection of all uniformly continuous elements denotes $UC(\mathcal{M})$, and the collection of all weakly almost periodic elements denotes $WAP(\mathcal{M})$.

Note that $UC(\mathcal{M})$ and $WAP(\mathcal{M})$ are norm closed and conjugate closed subspaces of $\mathcal{M}$ containing $e_{\mathcal{M}}$. According to [7], $x \in UC(\mathcal{M})$ if and only if the mapping $s \mapsto x \cdot s$ from G into $\mathcal{M}$ is weakly continuous. Moreover, we always have $UC(\mathcal{M}) = \mathcal{M} \cdot L^1(G)$; see [9]. This fact together with Mazur's theorem follows that $x \in UC(\mathcal{M})$ if and only if $x \cdot e_i \longrightarrow x$, if and only if $x \cdot e_i \longrightarrow x$ with respect to the weak topology of $\mathcal{M}$, where (e_i) is the bounded approximate identity of $L^1(G)$. Note that $UC(\mathcal{M})$ is weak*-dense in $\mathcal{M}$.

Proposition 2.3. *We always have $WAP(\mathcal{M}) \subseteq UC(\mathcal{M})$.*

Proof. Let $x \in WAP(\mathcal{M})$, and let X be the weak cluster of the set $\{x \cdot s \,|\, s \in G\}$ in $\mathcal{M}$. We claim that the mapping $s \mapsto x \cdot s$ from G into X is weakly continuous. Let $s_i \longrightarrow s$ in G. Since $\mathcal{M}_*$ is the left Banach G-module,

$$\langle x \cdot s_i,\, \phi\rangle = \langle x,\, s_i \cdot \phi\rangle \longrightarrow \langle x,\, s \cdot \phi\rangle = \langle x \cdot s,\, \phi\rangle$$

for all $\phi \in \mathcal{M}_*$. It means that $x \cdot s_i \longrightarrow x \cdot s$ with respect to the weak* topology of X (induced by $\mathcal{M}$). As known, weak topology is stronger than weak* topology. Also, X is compact with the stronger topology and Hausdorff with the weaker one. So, as a general result, these two topologies agree on X. Therefore, the proof is complete. $\square$

Remark 2.4. One might be made conditions that the reverse inclusion holds in Proposition 2.3. Here, we emphasize the following known examples as commutative and non-commutative W^*-algebras. These provide a comprehensive understanding of the subject matter.

(a). Let $\mathcal{M} = L^\infty(G)$ and $\mathcal{M}_* = L^1(G)$ with $s \cdot \phi = l_{s^{-1}}\phi$ for all $s \in G$ and $\phi \in L^1(G)$. Then $UC(\mathcal{M}) = LUC(G)$ and $WAP(\mathcal{M}) = WAP(G)$, where $LUC(G)$ and $WAP(G)$ have their usual meaning; see, for example, [1]. It is well known that $LUC(G) = WAP(G)$ if and only if G is compact.

(b). Let (π, H) be a unitary representation of G as defined in [6], let $\mathcal{M} = B(H)$, which consists of all bounded linear operators on H, and let $\mathcal{M}_* = Tr(H)$, which consists of all Trace class operators on H. Then it is also of interest to note $B(H)$ has a positive action on $Tr(H)$ that given by

$$s \cdot T = \pi(s)T\pi(s^{-1}) \quad (s \in G,\, T \in Tr(H)).$$

In this case, $UC(\mathcal{M}) = UCB(\pi)$ and $WAP(\mathcal{M}) = WAP(\pi)$. Moreover, $UCB(\pi) = WAP(\pi)$ if and only if the topological center induced by π is maximal; see [6] for definitions and more details.

Definition 2.5. A state M on $\mathcal{M}$ is called

(a) an invariant mean if $\langle M,\, x \cdot s\rangle = \langle M,\, x\rangle$ for all $x \in \mathcal{M}$ and $s \in G$.

(b) a topological invariant mean if $\langle M, x \cdot f \rangle = \langle M, x \rangle$ for all $x \in \mathcal{M}$ and $f \in L^1(G)_1^+$, where

$$L^1(G)_1^+ = \{f \in L^1(G) \,|\, f \geq 0, \int_G f = 1\}.$$

Also, we say that G acts amenably on $\mathcal{M}_*$ if there exists an invariant mean on $\mathcal{M}$.

As mentioned in [9, Lemma 1.9], we have

(a) every topological invariant mean on $\mathcal{M}$ (resp. $UC(\mathcal{M})$) is an invariant mean on $\mathcal{M}$ (resp. $UC(\mathcal{M})$),
(b) every invariant mean on $UC(\mathcal{M})$ is a topological invariant mean on $UC(\mathcal{M})$.

Considering Remark 2.4, we have respectively the following assertions:

(a) G acts amenably on $L^1(G)$ if and only if G is amenable.
(b) G acts amenably on $Tr(H)$ if and only if (π, H) is amenable.

Note that part (a) of Definition 2.1 can be defined on every norm closed, conjugate closed subspace X of $\mathcal{M}$ containing $e_{\mathcal{M}}$ such that $X \cdot s \subseteq X$ for all $s \in G$. Also, part (b) can be defined on every norm closed, conjugate closed subspace X of $\mathcal{M}$ containing $e_{\mathcal{M}}$ such that $X \cdot f \subseteq X$ for all $f \in L^1(G)$. When G acts amenably on $\mathcal{M}_*$, we can say that the positive action on $\mathcal{M}_*$ is amenable. In fact, the above notion, introduced and studied by Stokke [9], presents a unified approach that develops the basic theory of amenable groups. Stokke [9, Proposition 1.10] proved that the following statements are equivalent:

(a) There is a topological invariant mean on $\mathcal{M}$,
(b) there is an invariant mean on $\mathcal{M}$,
(c) there is an invariant mean on $UC(\mathcal{M})$,
(d) there is a topological invariant mean on $UC(\mathcal{M})$.

Theorem 2.6. *A locally compact group G is amenable if and only if every positive action of G on the predual of a W^*-algebra $\mathcal{M}$ is amenable.*

Proof. Suppose that G is amenable and that $\phi \in (\mathcal{M}_*)_1^+$. For each $x \in \mathcal{M}$, we define the complex-valued function ψ_x on G by $\psi_x(s) = \langle x, s \cdot \phi \rangle$ for all $s \in G$. Clearly, ψ_x is bounded, and since $\mathcal{M}_*$ is a left Banach G-module, ψ_x is continuous. Therefore, $\psi_x \in L^\infty(G)$. Define now $M : \mathcal{M} \longrightarrow \mathbb{C}$ by $M(x) = m(\psi_x)$ for all $x \in \mathcal{M}$, where m is a left invariant mean on $L^\infty(G)$. One can readily see that M is an invariant mean on $\mathcal{M}$. For the converse, we regard the W^*-algebra $\mathcal{M} = L^\infty(G)$ with $\mathcal{M}_* = L^1(G)$. □

The above theorem has also been proved in [9, Corollary 1.11] by Day's fixed theorem [3, Theorem 3.3.5].

For each M in the dual of $\mathcal{M}$, we define the linear mapping ω_M from $\mathcal{M}$ into $L^\infty(G)$ as follows:

$$\langle \omega_M(x), f \rangle = \langle M, x \cdot f \rangle \quad (x \in \mathcal{M}, f \in L^1(G)).$$

One can easily see that ω_M is bounded and $\|\omega_M\| \leq \|M\|$.

Remark 2.7. Let $x \in \mathcal{M}$ and $M \in \mathcal{M}^*$. As some useful features, we have the following statements:

(a) $\omega_M(x) \in LUC(G)$ if $x \in UC(\mathcal{M})$.
(b) $\omega_M(x) \in WAP(G)$ if $x \in WAP(\mathcal{M})$.

To prove part (a), let $s_i \longrightarrow s$ in G. Then

$$\begin{aligned}\|l_{s_i}\omega_M(x) - l_s\omega_M(x)\|_\infty &= \|\omega_M(x \cdot s_i) - \omega_M(x \cdot s)\|_\infty \\ &\leq \|M\| \, \|x \cdot s_i - x \cdot s\|.\end{aligned}$$

It follows that part (a) holds. For the second one, note that

$$\{l_s\omega_M(x) \,|\, s \in G\} = \omega_M(\{x \cdot s \,|\, s \in G\}). \tag{2.1}$$

Now, let $x \in WAP(\mathcal{M})$ and $M \in \mathcal{M}^*$. Then $x \in UC(\mathcal{M})$ by Proposition 2.3, and so the set (2.1) lies in $LUC(G)$ by part (a). On the other hand, $\omega_M : UC(\mathcal{M}) \longrightarrow LUC(G)$ is also continuous when both $UC(\mathcal{M})$ and $LUC(G)$ carry their weak topologies. So, a standard argument shows that every weak-closure point of the set (2.1) can be seen by $\omega_M(y)$ for some y in the weak-closure of $\{x \cdot s \,|\, s \in G\}$. It readily yields that part (b) holds.

Proposition 2.8. *There always exists an invariant mean on $WAP(\mathcal{M})$.*

Proof. Suppose that $M \in \mathcal{M}^*$ and that $x \in \mathcal{M}$. Using the fact that every state in $\mathcal{M}^*$ can be approximated by normal states in $\mathcal{M}_*$ in weak*-topology, [9, Lemma 1.3, part (1)], the mapping ω_M possesses the following properties:

(a) $\omega_M(x) \geq 0$ if $x \geq 0$,
(b) $\omega_M(x \cdot s) = l_s\omega_M(x)$.

Also, $e_{\mathcal{M}} \cdot f = e_{\mathcal{M}}$ for all $f \in L^1(G)^+_1$ by [9, Lemma 1.3, part (2)], and so

$$e_{\mathcal{M}} \cdot f = e_{\mathcal{M}} \int_G f(s)\, ds \quad (f \in L^1(G)).$$

It follows that $\omega_M(e_{\mathcal{M}}) = 1$. On the other hand, Remark 2.7 ensures that $\omega_M(WAP(\mathcal{M})) \subseteq WAP(G)$. Now, let m be the unique invariant mean on $WAP(G)$. Then one can readily check that $m \circ \omega_M$ is an invariant mean on $WAP(\mathcal{M})$. $\square$

Unlike $WAP(G)$, as observed in the proof of Proposition 2.8, $WAP(\mathcal{M})$ may admits more than one invariant mean. In [1], it is seen that there exists a unique invariant mean on $WAP(\pi)$ if and only if (π, H) is irreducible. However, we do not know yet for which $\mathcal{M}$ is the invariant mean on $WAP(\mathcal{M})$ unique.

Proposition 2.9. *Let for each $x \in UC(\mathcal{M})$, the map $f \mapsto x \cdot f$ from $L^1(G)$ into $UC(\mathcal{M})$ be weakly compact. Then G acts amenably on $\mathcal{M}_*$.*

Proof. According to [4, Corollary 3.4, (i)], the norm closure of the convex hull of the set $\{x \cdot s \,|\, s \in G\}$ is equal to the norm cluster of the set $\{x \cdot f \,|f \in L^1(G)^+_1\}$. This fact follows that

$$\{x \cdot s \,|\, s \in G\}^{-w} \subseteq \{x \cdot f \,|f \in L^1(G),\, \|f\|_1 \leq 1\}^{-w}.$$

Therefore, $x \in WAP(\mathcal{M})$. So, $UC(\mathcal{M}) = WAP(\mathcal{M})$. Now, Proposition 2.8 completes the proof. $\square$

The converse of the above proposition is not valid, in general; for example, when $\mathcal{M} = L^\infty(G)$ for each non-compact amenable group G.

Note that a bounded linear functional M on $\mathcal{M}$ (or $UC(\mathcal{M})$) is a state if and only if M satisfies in any pair of the following conditions:

(a) M is positive; that is, $M(x) \geq 0$ for all positive elements $x \in \mathcal{M}$,
(b) $M(e_{\mathcal{M}}) = 1$,
(c) $\|M\| = 1$.

Theorem 2.10. *The following statements are equivalent:*

(a) *G acts amenably on $\mathcal{M}_*$,*
(b) *There exists a weakly compact positive operator $\omega : \mathcal{M} \longrightarrow L^\infty(G)$ of norm 1 such that $\omega(x \cdot s) = l_s\omega(x)$ for all $x \in \mathcal{M}$ and $s \in G$.*

Proof. (a) $\Longrightarrow$ (b). Suppose that G acts amenably on $\mathcal{M}_*$. Then there exists an invariant mean M on $\mathcal{M}$. Define $\omega(x) = M(x)1$ for all $x \in \mathcal{M}$. So, $m \circ \omega = m(1)M$ for all $m \in L^1(G)^{**}$. It follows effortlessly that

$$\omega^{**}(x^{**}) = x^{**}(M)1 \in L^\infty(G) \quad (x^{**} \in \mathcal{M}^{**}).$$

So, ω is weakly compact by [8, Theorem 3.5.8]. One can easily see that $\|\omega\| = \|M\| = 1$. Note that since M is a state, ω is positive by the above result. Also, one can readily see that $\omega(x \cdot s) = l_s\omega(x)$ for all $x \in \mathcal{M}$ and $s \in G$.

(b) $\Longrightarrow$ (a). By assumption, for each $x \in \mathcal{M}$

$$\{l_s\omega(x) |\, s \in G\} = \{\omega(x \cdot s) |\, s \in G\}$$

is relatively weakly compact in the weak topology of $L^\infty(G)$. It follows that $\omega(x) \in WAP(G)$. It is known that there exists a unique left invariant mean on $WAP(G)$; see, for example, [3]. We define $\langle M, x\rangle = \langle m, \omega(x)\rangle$ for all $x \in \mathcal{M}$. Then M is the invariant mean on $\mathcal{M}$. ☐

We end the work with the following result.

Theorem 2.11. *The following statements are equivalent:*

(a) *G acts amenably on $\mathcal{M}_*$,*
(b) *There exists a weakly compact positive operator $\omega : UC(\mathcal{M}) \longrightarrow LUC(G)$ of norm 1 such that $\omega(x \cdot f) = \omega(x) \cdot f$ for all $x \in UC(\mathcal{M})$ and $f \in L^1(G)$.*

Proof. Let $M \in UC(\mathcal{M})^*$. Regard $\omega_M : UC(\mathcal{M}) \longrightarrow LUC(G)$. As noted earlier, $\|\omega_M\| \leq \|M\|$. To prove the reverse inequality, let (e_i) be an approximate identity of $L^1(G)$ bounded to 1. Then for each $x \in UC(\mathcal{M})$, we have

$$\|x \cdot e_i - x\| \longrightarrow 0,$$

and so

$$|\omega_M(x)(e_i)| \longrightarrow |\langle M, x\rangle|.$$

Furthermore, for each i, we have

$$\|\omega_M(x)\|_\infty \geq |\omega_M(x)(e_i)|.$$

Consequently, $\|\omega_M\| \geq \|M\|$. So, $\|\omega_M\| = \|M\|$.

Now, let G act amenably on $\mathcal{M}_*$. Choose a topological invariant mean M on $UC(\mathcal{M})$ by [9, Proposition 1.10]. Then

$$\langle \omega_M(x),\, f\rangle = \langle M,\, x\rangle \quad (x \in \mathcal{M},\, f \in L^1(G)_1^+).$$

So, $\omega_M(x) \geq 0$ if $x \geq 0$; that is, ω_M is positive. Similar to the proof of Theorem 2.10, ω_M is weakly compact, and also one can easily see that $\omega_M(x\cdot f) = \omega_M(x)\cdot f$ for all $x \in \mathcal{M}$ and $f \in L^1(G)$. So, part (a) implies (b). The converse is similar to the proof of Theorem 2.10, and the fact that the unique invariant mean on $WAP(G)$ is also a topological left invariant on $WAP(G)$. □

Acknowledgement. The author is grateful to the anonymous referees for their careful reading of the paper and valuable suggestions.

References

1. C. Chou and A.T. Lau, *Vector-valued invariant means on spaces of bounded operators associated to a locally compact group*, Illinois J. Math. **45** (2001) 581–602.
2. M.M. Day, *Means on semigroups and groups*, Bull. Amer. Math. Soc. **55** (1949) 1054–1055.
3. F.P. Greenleaf, *Invariant Means on Topological Groups and Their Applications*, Van Nostrand Mathematical Studies, 1969.
4. H.A. Hamouda, *Module maps and invariant subsets of Banach modules of locally compact groups.* Bull. Belg. Math. Soc. Simon Stevin. **21** (2014) 253–261.
5. E. Hewitt and K. Ross, *Abstract Harmonic Analysis I*, Springer, New York, 1970.
6. S.S. Jafari, *On topological centers induced by unitary representations*, Arch. Math. **117** (2021) 323–333.
7. B.E. Johnson, *Cohomology in Banach algebras*, 127, Memoirs of the American Mathematical Society, 1972.
8. R.E. Megginson, *An Introduction to Banach Space Theory*, Springer-Verlag, New York, 1998.
9. R. Stokke, *Quasi-central bounded approximate identities in group algebras*, Illinois J. Math. **48** (2004) 151–170.
10. J. Neumann, *Zur allgemeinen theorie des Masses*, Fund. Math. **13** (1929), no. 1, 73–116.

Department of Mathematics, Payame Noor University, Tehran, Iran.
Email address: ss.jafari@pnu.ac.ir