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FACTORABLE STRONGLY (p, σ, q, ν)-NUCLEAR m-HOMOGENEOUS POLYNOMIALS

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ABSTRACT. In this paper, we study the ideal of strongly (p, σ)-continuous polynomials that extends the nowadays well-known ideal of Cohen strongly p-summing polynomials to the more general setting of the interpolated ideals of polynomials. We give an analogue of the Pietsch domination theorem among other results. Also, we introduce the polynomial ideal of factorable strongly (p, σ, q, ν)-nuclear polynomials in order to characterize those polynomials whose linearizations are (p, σ, q, ν)-nuclear.

1. Introduction

Matter [17] introduced the concept of (p, σ)-absolutely continuous operators for $1 \leq p < \infty$ and $0 \leq \sigma < 1$. This notion serves as a crucial analytical tool for examining properties such as super reflexivity within Banach spaces. Its development stems from an interpolation method pioneered by Jarchow and Matter [15]. The class of (p, σ)-absolutely continuous operators can be seen as an intermediate class situated between continuous operators and the well-known class of absolutely p-summing operators, offering a distinctive blend of characteristics from both. In the past decade, there has been a notable focus on exploring the extension of this concept into nonlinear frameworks (see, for instance, [3, 5, 12]).

In a recent paper [9], a new approach to (p, σ)-absolutely continuous operators was presented. The paper introduces and investigates the concept of (p, σ)-absolute continuity within the context of Bloch maps.

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The ideal of strongly (p, σ)-continuous linear operators, initially introduced and expanded into the multilinear context by Achour et al. [4], stands as an interpolated ideal situated between the ideal $\mathcal{D}_p$ of Cohen strongly p-summing linear operators and the ideal of all continuous operators. These operators characterize the adjoints of (p, σ)-absolutely continuous operators.

Pietsch [21] brought forth a multilinear approach to the theory of absolutely summing operators. Afterward, a multitude of papers have pursued this line of investigation, leading to the emergence of different summability concepts we mentioned [4, 10, 12, 14, 19]. Yet, extending summability properties into a non-linear context has presented challenges and sparked interest. The literature investigates various extensions of absolutely p-summing linear operators into the multilinear setting. Rueda and Sánchez Pérez [22] isolate the class of p-dominated polynomials that factor through a subspace of an L_p-space with the L_p norm, and based on this class, Pellegrino, Rueda, and Sánchez Pérez [19] introduced factorable strongly p-summing multilinear operators. These operators effectively maintain numerous fundamental properties from linear theory that previous extensions failed to capture. Building upon this achievement, Achour et al. [1] utilized similar ideas of factorable strongly summing polynomials/multilinear mappings. They defined the notion of factorable strongly p-nuclear m-homogeneous polynomials as a generalization of the ideal of Cohen p-nuclear linear operators introduced by Cohen [11].

The paper is intended to fulfill two objectives: Firstly, it introduces and investigates a polynomial version of the class of strongly (p, σ)-continuous linear operators. Secondly, it aims to define the class of factorable strongly (p, σ, q, ν)-nuclear homogeneous polynomials, establishing a significant connection to linear theory through polynomial linearization. It crucially establishes the relationship: a homogeneous polynomial is factorable strongly (p, σ, q, ν)-nuclear if and only if its linearization is a Cohen (p, σ, q, ν)-nuclear operator.

The paper is organized as follows: Section 2 is devoted to fixing the notation and recalling some definitions and basic facts. In Section 3, we expand the notion of strongly (p, σ)-continuous linear operators to the polynomial context. We show that this resulting class forms a polynomial ideal generated through the composition method. Additionally, we establish connections between these operators and their corresponding symmetric multilinear maps, while also comparing them to other well-known polynomial ideals. In Section 4, our focus is directed to the introduction and study of factorable strongly (p, σ, q, ν)-nuclear homogeneous polynomials. We describe these polynomials using polynomial linearizations and establish domination/factorization theorems for this class. Additionally, we characterize the adjoint operator related to these polynomials, alongside other results.

2. Notation and background

We use a standard notation on Banach spaces. Let n, n_1, n_2 and m be positive integers, and let $X, X_1, \ldots, X_m, Y$ be Banach spaces over the same scalar-field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. The closed unit ball of X is denoted B_X. We work with $1 \leq p \leq \infty$, and p^* denotes the conjugate of p; that is, $\frac{1}{p} + \frac{1}{p^*} = 1$. As usual, $l_p^n(X)$ is the

Banach space of all sequences $(x_i)_{1\leq i\leq n}$ in X with the norm

$$\|(x_i)_{i=1}^n\|_p = \left(\sum_{i=1}^n \|x_i\|^p\right)^{\frac{1}{p}},$$

and $l_p^{n,\omega}(X)$ denotes the Banach space of all sequences $(x_i)_{1\leq i\leq n}$ in X with the norm

$$\|(x_i)_{i=1}^n\|_{p,\omega} = \sup_{x^*\in B_{X^*}} \left(\sum_{i=1}^n |\langle x_i, x^*\rangle|^p\right)^{\frac{1}{p}},$$

where X^* is the topological dual of X.

We now turn our attention to multilinear mappings. Let $X_1,\dots,X_m,Y$ be Banach spaces. We denote by $\mathcal{L}(X_1,\dots,X_m;Y)$ the Banach space of all bounded m-linear operators T: $X_1\times\cdots\times X_m\to Y$ endowed with the norm

$$\|T\| = \sup_{\|x_1\|\leq 1,\dots,\|x_m\|\leq 1} \|T(x_1,\dots,x_m)\|.$$

If $Y=\mathbb{K}$, we write $\mathcal{L}(X_1,\dots,X_m)$. When $X_1=\cdots=X_m=X$ we simply write $\mathcal{L}(^mX;Y)$.

As usual, $X_1\otimes\cdots\otimes X_m$ denotes the m-fold tensor product of $X_1,\dots,X_m$, and if $X_1=\cdots=X_m=X$, we use the shorter notation $\otimes^m X := X\otimes\cdots\otimes X$. The projective norm π on $X_1\otimes\cdots\otimes X_m$ is defined by

$$\pi(u) := \inf\left\{\sum_{i=1}^n |\lambda_i| \left\|x_i^1\right\|\cdots\|x_i^m\| : u=\sum_{i=1}^n \lambda_i x_i^1\otimes\cdots\otimes x_i^m\right\},$$

where the infimum is taken over all representations of $u\in X_1\otimes\cdots\otimes X_m$. The completion of $X_1\otimes\cdots\otimes X_m$ when endowed with π is denoted by $X_1\widehat{\otimes}_\pi\cdots\widehat{\otimes}_\pi X_m$.

It is well know that, if $T: X_1\times\cdots\times X_m\to Y$ is a bounded m-linear operator, then there exists only one bounded linear operator

$$T_L : X_1\widehat{\otimes}_\pi\cdots\widehat{\otimes}_\pi X_m \to Y,$$

such that

$$T_L\left(x_i^1\otimes\cdots\otimes x_i^m\right) = T\left(x_i^1,\dots,x_i^m\right),$$

for all $x^j\in X_j, j=1,\dots,m$. Moreover, $\|T\|=\|T_L\|$. In other words, the mapping $T\leftrightarrow T_L$ from $\mathcal{L}(X_1,\dots,X_m;Y)$ to $\mathcal{L}\left(X_1\widehat{\otimes}_\pi\cdots\widehat{\otimes}_\pi X_m;Y\right)$ is an isometric isomorphism onto.

In particular, the spaces $\mathcal{L}(X_1,\dots,X_m)$ and $\left(X_1\widehat{\otimes}_\pi\cdots\widehat{\otimes}_\pi X_m\right)^*$ are isometrically isomorphic. For more details we refer to [23].

Consider now a Banach space X and the symmetric m-fold tensor product $\otimes^{m,s}X$, defined as the linear span of the elements $x\otimes\cdots\otimes x$ in $\otimes^m X$ with $x\in X$. We will require the following norm on this space:

The projective s-tensor norm π_s given by

$$\pi_s(z) = \inf\left\{\sum_{i=1}^k |\lambda_i| \|x_i\|^m : z=\sum_{i=1}^k \lambda_i x_i\otimes\cdots\otimes x_i\right\},$$

for $z\in\otimes^{m,s}X$. By $\widehat{\otimes}_{\pi_s}^{m,s}X$ we mean the completion of $\otimes_{\pi_s}^{m,s}X$.

A map $P : X \to Y$ is an m-homogeneous polynomial if there exists an m-linear operator $A : X \times \cdots \times X \to Y$ such that $P(x) = A(x, \overset{(m)}{\ldots}, x)$ for every $x \in X$. The space of all continuous m-homogeneous polynomials from X to Y is denoted by $\mathcal{P}\left({}^m X; Y\right)$ $(\mathcal{P}\left({}^m X\right)$ when $Y = \mathbb{K})$, and becomes a Banach space when endowed with the norm $\|P\| = \sup_{x \in B_X} \|P(x)\|$.

When $m = 1$ the space $\mathcal{P}\left({}^1 X; Y\right)$ is nothing but $\mathcal{L}(X; Y)$, the space of all bounded linear operators from X to Y. For the general theory of homogeneous polynomials we refer to [18].

There are several useful mappings associated to a polynomial $P \in \mathcal{P}\left({}^m X; Y\right)$:

- The m-linear symmetric mapping associated to P is the unique symmetric m-linear operator $\widehat{P} : X \times \cdots \times X \to Y$ such that $P(x) = \widehat{P}(x, \overset{(m)}{\ldots}, x)$ for every $x \in X$. The value of $\widehat{P}$ is given by the polarization formula:
$$\widehat{P}(x_1, \ldots, x_m) = \frac{1}{m! 2^m} \sum_{\epsilon_1, \ldots, \epsilon_m = \pm 1} \epsilon_1 \cdots \epsilon_m P(\epsilon_1 x_1 + \cdots + \epsilon_m x_m)$$
for all $x_1, \ldots, x_m \in X$.
- The adjoint of P is the linear operator $P^* : Y^* \to \mathcal{P}\left({}^m X\right)$ given by $P^*\left(y^*\right)(x) := y^*(P(x))$, for $y^* \in Y^*$ and $x \in X$. It is well known that $(u \circ P)^* = P^* \circ u^*$, for all $u \in \mathcal{L}(Y; Z)$, where Z is a Banach space.
- The linearization of P is the unique linear operator $P_L : \widehat{\otimes}_{\pi_s}^{m,s} X \to Y$ such that $P_L(x \otimes \cdots \otimes x) = P(x)$, for all $x \in X$. Ryan [23] proved that the correspondence $P \leftrightarrow P_L$ establishes an isometric isomorphism between the space $\mathcal{P}\left({}^m X\right)$ and the strong dual of $\widehat{\otimes}_{\pi_s}^{m,s} X$.

Consider the canonical m-homogeneous polynomial $\delta_m : X \to \widehat{\otimes}_{\pi_s}^{m,s} X$ defined by $\delta_m(x) = x \otimes \cdots \otimes x$, that is, $P = P_L \circ \delta_m$. The scalar case deserves special attention: when $Y = \mathbb{K}$ we call $\Delta_m : \mathcal{P}\left({}^m X\right) \to \left(\widehat{\otimes}_{\pi_s}^{m,s} X\right)^*$ the isometric isomorphism $\Delta_m(P) = P_L$ for all $P \in \mathcal{P}\left({}^m X\right)$. Note that $\Delta_m^{-1} = \delta_m^*$.

The space of (p, σ)-weakly summable sequences was introduced in [16] by López Molina and Sánchez Pérez. We recall some properties of this space. Let $1 \leq p < \infty$ and let $0 \leq \sigma < 1$. Let $(x_i)_{i=1}^n \subset X$. Define
$$\delta_{p,\sigma}\left((x_i)_{i=1}^n\right) = \sup_{x^* \in B_{X^*}} \left(\sum_{i=1}^n \left(|\langle x_i, x^* \rangle|^{1-\sigma} \|x_i\|^{\sigma} \right)^{\frac{p}{1-\sigma}} \right)^{\frac{1-\sigma}{p}}$$
and
$$H_{p,\sigma}(X) = \left\{ (x_i)_{i=1}^n \subset X : \delta_{p,\sigma}\left((x_i)_{i=1}^n\right) < \infty \right\}.$$

For the extreme cases $\sigma = 1$ and $p = \infty$, we define also for all $0 \leq t \leq 1$ and $1 \leq q \leq \infty$
$$\delta_{q,1}\left((x_i)_{i=1}^n\right) = \delta_{\infty,t}\left((x_i)_{i=1}^n\right) = \sup_{1 \leq i \leq n} \|x_i\| = \left\|(x_i)_{i=1}^n\right\|_{\infty}.$$

The sequence $(x_i)_{i=1}^n$ is said to be (p,σ)-weakly summable if it belongs to the vector space $\ell^{p,\sigma}(X) = \operatorname{span}(H_{p,\sigma}(X))$, equipped with the norm:

$$\|(x_i)_{i=1}^n\|_{p,\sigma} = \inf\left\{\sum_{l=1}^{k}\delta_{p,\sigma}\left(\left(x_i^l\right)_{i=1}^\infty\right) : (x_i)_{i=1}^n = \sum_{l=1}^{k}\left(x_i^l\right)_{i=1}^n, \left(x_i^l\right)_{i=1}^n \in H_{p,\sigma}(X)\right\}.$$

The completion will be denoted as usual by $\hat{\ell}^{p,\sigma}(X)$. We have the inclusions

$$\ell_{\frac{p}{1-\sigma}}(X) \subset \ell^{p,\sigma}(X) \subset \ell_{\frac{p}{1-\sigma},\omega}(X)$$

with

$$\|(x_i)_{i=1}^n\|_{\frac{p}{1-\sigma},\omega} \leq \|(x_i)_{i=1}^n\|_{p,\sigma} \leq \|(x_i)_{i=1}^n\|_{\frac{p}{1-\sigma}},$$

for all $(x_i)_{i=1}^n \in \ell_{\frac{p}{1-\sigma}}(X)$.

We recall some known concepts related to summability of linear and some non-linear operators.

- [17] Let X, Y be Banach spaces, let $1 \leq p < \infty$, and let $0 \leq \sigma < 1$. We say that $T \in \mathcal{L}(X,Y)$ is a (p,σ)-absolutely continuous, in the symbol $T \in \Pi_{p,\sigma}(X,Y)$, if there exists $C > 0$ such that for every $(x_i)_{i=1}^n \subset X$ we have

$$\|(T(x_i))_{i=1}^n\|_{\frac{p}{1-\sigma}} \leq C\delta_{p,\sigma}\left((x_i)_{i=1}^n\right). \tag{2.1}$$

  The norm of T is defined by

$$\pi_{p,\sigma}(T) = \inf\{C > 0\text{: satisfying the inequality (2.1) }\}.$$

- [4] Let $1 < p, r < \infty$ and let $0 \leq \sigma < 1$ be such that $\frac{1}{r} + \frac{1-\sigma}{p^*} = 1$. An operator $T \in \mathcal{L}(X,Y)$ between Banach spaces is strongly (p,σ)-continuous if there is a constant $C > 0$ such that for every $(x_i)_{i=1}^n \subset X$ and $(y_i^*)_{i=1}^n \subset Y^*$ the following inequality holds

$$\|(\langle T(x_i), y_i^*\rangle)_{i=1}^n\|_1 \leq C\,\|(x_i)_{i=1}^n\|_r\,\delta_{p^*,\sigma}\left((y_i^*)_{i=1}^n\right). \tag{2.2}$$

  The class of all strongly (p,σ)-continuous linear operators from X into Y is denoted by $\mathcal{D}_p^\sigma(X,Y)$ and by $d_p^\sigma(T)$ the strongly (p,σ)-continuous norm, which is defined by $d_p^\sigma(T) = \inf C$, where the infimum is taken over all constants C satisfying the inequality (2.2). This class is as an interpolated ideal positioned between the ideal $\mathcal{D}_p$ of Cohen strongly p-summing linear operators, introduced by Cohen [11], and the ideal of all continuous linear operators. Furthermore, it characterizes operators whose adjoints are (p^*,σ)-absolutely continuous; that is,

$$\mathcal{D}_p^\sigma = (\Pi_{p^*,\sigma})^{dual} = \{T \in \mathcal{L}(X,Y) : T^* \in \Pi_{p^*,\sigma}(Y^*, X^*)\}.$$

- [4] Let $1 \leq p, r < \infty$ and $0 \leq \sigma < 1$ such that $\frac{1}{r} + \frac{1-\sigma}{p^*} = 1$. An m-linear mapping $T : X_1 \times \cdots \times X_m \to Y$ is strongly (p,σ)-continuous if there is a constant $C > 0$ such that for any $x_1^j, \ldots, x_n^j \in X_j, (1 \leq j \leq m)$ and any $y_1^*, \ldots, y_n^* \in Y^*$, we have

$$\left\|\left(\left\langle T\left(x_i^1, \ldots, x_i^m\right), y_i^*\right\rangle\right)_{i=1}^n\right\|_1 \leq C\left(\sum_{i=1}^{n}\prod_{j=1}^{m}\left\|x_i^j\right\|^r\right)^{\frac{1}{r}}\delta_{p^*,\sigma}\left((y_i^*)_{i=1}^n\right)$$

for all choices of $n \in \mathbb{N}$. The collection of all strongly (p,σ)-continuous m-linear maps $X_1 \times \cdots \times X_m \to Y$ will be denoted by $\mathcal{D}_p^{m,\sigma}(X_1,\ldots,X_m;Y)$

- According to [13], a linear operator $T \in \mathcal{L}(X,Y)$ is (p,σ,q,v)-nuclear $\left(1 < p,q < \infty \text{ and } 0 \le \sigma, v < 1 \text{ such that } \frac{1-\sigma}{p} + \frac{1-v}{q} = 1\right)$, if there exists a constant $C > 0$ such that for every $(x_i)_{i=1}^n \subset X$ and $(y_i^*)_{i=1}^n \subset Y^*$ the following inequality holds:

$$\left\|\left(\left\langle T(x_i), y_i^*\right\rangle\right)_{i=1}^n\right\|_1 \le C\delta_{p,\sigma}\left((x_i)_{i=1}^n\right)\delta_{q,v}\left((y_i^*)_{i=1}^n\right). \tag{2.3}$$

The class of all (p,σ,q,v)-nuclear operators between X and Y is denoted by $\mathcal{N}_{p,\sigma,q,v}(X,Y)$. The infimum of all the constant C in the inequality (2.3) defines a norm on $\mathcal{N}_{p,\sigma,q,v}(X,Y)$ denoted by $\mathcal{N}_{p,q}^{\sigma,v}$.

- [6] A polynomial $P \in \mathcal{P}(^mX;Y)$ is Cohen strongly p-summing if there is a constant $C > 0$ such that for n, all $x_1,\ldots,x_n \in X$ and all $y_1^*,\ldots,y_n^* \in Y^*$,

$$\sum_{i=1}^n \left|\left\langle P(x_i), y_i^*\right\rangle\right| \le C\left(\sum_{i=1}^n \|x_i\|^{mp}\right)^{\frac{1}{p}} \left\|(y_i^*)_i\right\|_{p^*,\omega}.$$

The class of such polynomials is denoted $\mathcal{P}_{Coh}^p(^mX,Y)$ and endowed with the norm d_p given by the smallest constant $C \ge 0$ satisfying the above inequality.

- [1] Let $P \in \mathcal{P}(^mX;Y)$ and let $1 \le p \le \infty$. We say that P is factorable strongly p-nuclear if there exists a constant $C \ge 0$ such that for all $n_1, n_2 \in \mathbb{N}$, all $(x_{i,k})_{1\le i\le n_2, 1\le k\le n_1} \subset X$, all $(y_k^*)_{1\le k\le n_1} \subset Y^*$, and all scalars $\lambda_{i,k}$, for $1 \le i \le n_2, 1 \le k \le n_1$, we have

$$\sum_{k=1}^{n_1}\left|\sum_{i=1}^{n_2}\lambda_k^i\left\langle P(x_{i,k}), y_k^*\right\rangle\right| \le C \sup_{\|q\|\le 1, q\in\mathcal{P}(^mX)}\left(\sum_{k=1}^{n_1}\left|\sum_{i=1}^{n_2}\lambda_k^i q(x_{i,k})\right|^p\right)^{\frac{1}{p}}\left\|(y_k^*)_k\right\|_{p^*,\omega}.$$

The class of all factorable strongly p-nuclear m-homogeneous polynomials from X to Y is denoted by $\mathcal{P}_{p,N}^{fs}(^mX;Y)$ and endowed with the norm $\pi_{p,N}^{fs}$ given by the infimum of all constants C as above.

- [2] Let $P \in \mathcal{P}(^mX;Y)$ and let $1 \le p \le \infty$. We say that P is Cohen p-nuclear if there is a constant $C \ge 0$ such that for all n, all $x_1,\ldots,x_n \in X$ and all $y_1^*,\ldots,y_n^* \in Y^*$,

$$\sum_{i=1}^n \left|\left\langle P(x_i), y_i^*\right\rangle\right| \le C \sup_{x^*\in B_{X^*}}\left(\sum_{i=1}^n \left|\left\langle x_i, x^*\right\rangle\right|^{mp}\right)^{\frac{1}{p}}\left\|(y_i^*)_i\right\|_{p^*,\omega}. \tag{2.4}$$

The class of such polynomials is denoted by $\mathcal{P}_{p,N}^c(^mX,Y)$. The infimum of the constants $C \ge 0$ such that inequality (2.4) holds, defines a norm $\pi_{p,N}$ on $\mathcal{P}_{p,N}^c(^mX,Y)$.

The following results show how the aforementioned concepts are related.

Proposition 2.1. [7, Theorem 2] *Let $1 < p, r < \infty$ and $0 \le \sigma < 1$ be such that $\frac{1}{r} + \frac{1-\sigma}{p^*} = 1$ and let $T \in \mathcal{L}(X;Y)$. Then*

1. $\mathcal{N}_{p,\sigma,q,\nu}(X,Y) \subset \Pi_{p,\sigma}(X,Y)$,

2. $\mathcal{N}_{p,\sigma,q,\nu}(X,Y) \subset \mathcal{D}^{\nu}_{q^*}(X,Y)$.

Proposition 2.2. *Let $1 \leq p < \infty$ and $P \in \mathcal{P}\left({}^m X;Y\right)$. Then*

1. $\mathcal{P}^{c}_{p,N}\left({}^m X;Y\right) \subset \mathcal{P}^{p}_{Coh}\left({}^m X;Y\right)$,
2. $\mathcal{P}^{fs}_{p,N}\left({}^m X;Y\right) \subset \mathcal{P}^{p}_{Coh}\left({}^m X;Y\right)$.

3. Strongly (p,σ)-continuous m-homogeneous polynomials

In this section, we broaden the notion of strongly (p,σ)-continuous linear operators to polynomial context, for which the resulting vector space $\mathcal{P}^{\sigma}_{p}\left({}^m X;Y\right)$ of strongly (p,σ)-continuous m-homogeneous polynomials is a normed (Banach) ideal of polynomials. Furthermore, we bring forth Pietsch's domination theorem within this class. We conclude that the generation of $\mathcal{P}^{\sigma}_{p}$ is achieved through the utilization of the composition method derived from the operator ideal $\mathcal{D}^{\sigma}_{p}$ of all strongly (p,σ)-continuous linear operators.

Definition 3.1. Let $1 < p,r < \infty$ and $0 \leq \sigma < 1$, such that $\frac{1}{r} + \frac{1-\sigma}{p^*} = 1$ and let $m \in \mathbb{N}$. An m-homogeneous polynomial $P : X \longrightarrow Y$ is called strongly (p,σ)-continuous if there is a constant $C > 0$ such that for any $x_1,\ldots,x_n \in X$ and any $y^*_1,\ldots,y^*_n \in Y^*$, we have

$$\sum_{i=1}^{n} \left|\langle P(x_i), y^*_i\rangle\right| \leq C \left(\sum_{i=1}^{n} \|x_i\|^{mr}\right)^{\frac{1}{r}} \delta_{p^*,\sigma}\left((y^*_i)_{i=1}^{n}\right). \tag{3.1}$$

Let $\mathcal{P}^{\sigma}_{p}\left({}^m X;Y\right)$ denote the space of all strongly (p,σ)-continuous m-homogeneous polynomial, which is a Banach space with the norm d^{σ}_{p}, the smallest constant C such that (3.1) holds. If $\sigma = 0$, we have $\mathcal{P}^{0}_{p}\left({}^m X;Y\right) = \mathcal{P}^{coh}_{p}\left({}^m X;Y\right)$.

Let us first give an example of a strongly (p,σ)-continuous m-homogeneous polynomial. Let $1 < p,r < \infty$ and $0 \leq \sigma < 1$, such that $\frac{1}{r} + \frac{1-\sigma}{p^*} = 1$, let $m \in \mathbb{N}$, let $u : X \to Y$ be a strongly (p,σ)-continuous linear operator, and let $\varphi \in X^*$. The polynomial $P : X \to Y : P(x) = \varphi^{m-1}(x)u(x)$ is strongly (p,σ)-continuous. Indeed, for $x_1,\ldots,x_n \in X, y^*_1,\ldots,y^*_n \in Y^*$,

$$\begin{aligned}
\sum_{i=1}^{n} \left|\langle P\left(x_i\right), y^*_i\rangle\right| &= \sum_{i=1}^{n} \left|\left\langle \varphi^{m-1}\left(x_i\right) u\left(x_i\right), y^*_i\right\rangle\right| \\
&= \sum_{i=1}^{n} \left|\left\langle u\left(\varphi^{m-1}\left(x_i\right) x_i\right), y^*_i\right\rangle\right| \\
&\leq d^{\sigma}_{p}(u)\|\varphi\|^{m-1} \left(\sum_{i=1}^{n} \|x_i\|^{mr}\right)^{1/r} \delta_{p^*,\sigma}\left((y^*_i)_{i=1}^{n}\right).
\end{aligned}$$

So, P is a strongly (p,σ)-continuous m-homogeneous polynomial and $d^{\sigma}_{p}(P) \leq \|\varphi\|^{m-1} d^{\sigma}_{p}(u)$.

The polynomial version of the Pietsch domination theorem goes as follows.

Theorem 3.2. *Let $1 < p, r < \infty$ and $0 \leq \sigma < 1$ be such that $\frac{1}{r} + \frac{1-\sigma}{p^*} = 1$. Let $m \in \mathbb{N}$. An m-homogeneous polynomial $P \in \mathcal{P}({}^m X, Y)$ is strongly (p, σ)-continuous if and only if there is a Radon probability measure μ on $B_{Y^{**}}$ and $C > 0$ such that, for all $x \in X$ and $y^* \in Y^*$*

$$|\langle P(x), y^* \rangle| \leq C \|x\|^m \left(\int_{B_{Y^{**}}} \left(|\langle y^*, \varphi \rangle|^{1-\sigma} \|y^*\|^{\sigma} \right)^{\frac{p^*}{1-\sigma}} d\mu(y^{**}) \right)^{\frac{1-\sigma}{p^*}}. \tag{3.2}$$

Moreover, in this case $d_p^{\sigma}(P) = \inf\{C > 0 : C$ satisfies (3.2)$\}$.

Proof. Suppose that P is strongly (p, σ)-continuous. We will apply an unified abstract version of Piestch domination theorem (see [20, Theorem 4.6]). For it, consider the functions

$$S : \mathcal{P}({}^m X; Y) \times \mathbb{R} \times X \times Y^* \to [0, \infty), \quad S(P, b, x, y^*) = |\langle P(x), y^* \rangle|,$$

$$R_1 : B_{Y^{**}} \times \mathbb{R} \times X \to [0, \infty), \quad R_1(\varphi, b, x) = \|x\|^m,$$

and

$$R_2 : B_{Y^{**}} \times \mathbb{R} \times Y^* \to [0, \infty), \quad R_2(\varphi, b, y^*) = |\langle y^*, \varphi \rangle|^{1-\sigma} \|y^*\|^{\sigma}.$$

Then P is R_1-R_2-S-abstract $(r, \frac{p^*}{1-\sigma})$-summing (see [20, Definition 4.4]) since

$$\begin{aligned}
&\sum_{i=1}^{n} S(P, b_i, x_i, y_i^*) \\
&\quad = \sum_{i=1}^{n} |\langle P(x_i), y_i^* \rangle| \\
&\quad \leq d_p^{\sigma}(P) \left(\sum_{i=1}^{n} \|x_i\|^{mr} \right)^{\frac{1}{r}} \sup_{\varphi \in B_{Y^{**}}} \left(\sum_{i=1}^{n} \left(|\langle y_i^*, \varphi \rangle|^{1-\sigma} \|y_i^*\|^{\sigma} \right)^{\frac{p^*}{1-\sigma}} \right)^{\frac{1-\sigma}{p^*}} \\
&\quad = d_p^{\sigma}(P) \left(\sum_{i=1}^{n} R_1(\varphi, b_i, x_i)^r \right)^{\frac{1}{r}} \sup_{\varphi \in B_{Y^{**}}} \left(\sum_{i=1}^{n} R_2(\varphi, b_i, y_i^*)^{\frac{p^*}{1-\sigma}} \right)^{\frac{1-\sigma}{p^*}} \\
&\quad \leq d_p^{\sigma}(P) \sup_{\varphi \in B_{Y^{**}}} \left(\sum_{i=1}^{n} R_1(\varphi, b_i, x_i)^r \right)^{\frac{1}{r}} \sup_{\varphi \in B_{Y^{**}}} \left(\sum_{i=1}^{n} R_2(\varphi, b_i, y_i^*)^{\frac{p^*}{1-\sigma}} \right)^{\frac{1-\sigma}{p^*}}.
\end{aligned}$$

Then, by [20, Theorem 4.6], there exist $C > 0$ and Borel regular probability measures μ and λ on $B_{Y^{**}}$ such that

$$\begin{aligned}
S(P, b, x, y^*) \leq\, & C \left(\int_{B_{Y^{**}}} R_1(\varphi, b, x)^r d\lambda(y^{**}) \right)^{\frac{1}{r}} \\
& \times \left(\int_{B_{Y^{**}}} R_2(\varphi, b, y^*)^{\frac{p^*}{1-\sigma}} d\mu(y^{**}) \right)^{\frac{1-\sigma}{p^*}}.
\end{aligned}$$

For all b in $\mathbb{R}$, x in X and y^* in Y^*, and therefore

$$|\langle P(x), y^*\rangle| \leq C\|x\|^m \left(\int_{B_{Y^{**}}} \left(|\langle y^*, \varphi\rangle|^{1-\sigma} \|y^*\|^{\sigma}\right)^{\frac{p^*}{1-\sigma}} d\mu\left(y^{**}\right)\right)^{\frac{1-\sigma}{p^*}}.$$

For all x in X and y^* in Y^*. Furthermore, we have

$$|\langle P(x), y^*\rangle| \leq d_p^{\sigma}(P)\|x\|^m \left(\int_{B_{Y^{**}}} \left(|\langle y^*, \varphi\rangle|^{1-\sigma} \|y^*\|^{\sigma}\right)^{\frac{p^*}{1-\sigma}} d\mu\left(y^{**}\right)\right)^{\frac{1-\sigma}{p^*}},$$

for all x in X and y^* in Y^*.

Conversely, if (3.2) holds, then given $n \in \mathbb{N}$, $x_1, \dots, x_n \in X$; $y_1^*, \dots, y_n^* \in Y^*$. Then the Hölder's inequality gives

$$\begin{aligned}
&\sum_{i=1}^{n} |\langle P(x_i), y_i^*\rangle| \\
&\quad\leq C \sum_{i=1}^{n} \|x_i\|^m \left(\int_{B_{Y^{**}}} \left(|\langle y_i^*, \varphi\rangle|^{1-\sigma} \|y_i^*\|^{\sigma}\right)^{\frac{p^*}{1-\sigma}} d\mu\left(y^{**}\right)\right)^{\frac{1-\sigma}{p^*}} \\
&\quad\leq C \left(\sum_{i=1}^{n} \|x_i\|^{mr}\right)^{\frac{1}{r}} \left(\sum_{i=1}^{n} \int_{B_{Y^{**}}} \left(|\langle y_i^*, \varphi\rangle|^{1-\sigma} \|y_i^*\|^{\sigma}\right)^{\frac{p^*}{1-\sigma}} d\mu\left(y^{**}\right)\right)^{\frac{1-\sigma}{p^*}} \\
&\quad= C \left(\sum_{i=1}^{n} \|x_i\|^{mr}\right)^{\frac{1}{r}} \left(\int_{B_{Y^{**}}} \sum_{i=1}^{n} \left(|\langle y_i^*, \varphi\rangle|^{1-\sigma} \|y_i^*\|^{\sigma}\right)^{\frac{p^*}{1-\sigma}} d\mu\left(y^{**}\right)\right)^{\frac{1-\sigma}{p^*}} \\
&\quad\leq C \left(\sum_{i=1}^{n} \|x_i\|^{mr}\right)^{\frac{1}{r}} \delta_{p^*,\sigma}\left((y_i^*)_{i=1}^{n}\right).
\end{aligned}$$

Hence $P \in \mathcal{P}_p^{\sigma}\left({}^m X; Y\right)$ with $d_p^{\sigma}(P) \leq C$. □

A direct consequence of the previous theorem is the following corollary.

Corollary 3.3. *If $1 < p \leq q < \infty$, then $\mathcal{P}_q^{\sigma}\left({}^m X; Y\right) \subset \mathcal{P}_p^{\sigma}\left({}^m X; Y\right)$.*

The next proposition follows from Corollary 3.6 combined with [4, Proposition 4.5] and [8, Proposition 3.2].

Proposition 3.4. *The polynomial $P \in \mathcal{P}\left({}^m X, Y\right)$ is strongly (p, σ)-continuous if and only if its associated symmetric m-linear operator $\widehat{P} \in \mathcal{L}\left({}^m X, Y\right)$ is also strongly (p, σ)-continuous.*

We show in what follows that the polynomial ideal generated by the composition method from the operator ideal $\mathcal{D}_p^{\sigma}$ coincides with the space of strongly (p, σ)-continuous m-homogeneous polynomials.

Proposition 3.5. *Let $1 < p, r < \infty$ and $0 \leq \sigma < 1$ be such that $\frac{1}{r} + \frac{1-\sigma}{p^*} = 1$. Let $P \in \mathcal{P}({}^m X; Y)$ and let P_L be its linearization. The following properties are equivalent:*

(i) *The polynomial P belongs to $\mathcal{P}_p^{\sigma}\left({}^m X; Y\right)$.*

(ii) *The operator* P_L *belongs to* $\mathcal{D}_p^\sigma\left(\widehat{\otimes}_{\pi_s}^{m,s}X,Y\right)$.

In this case, $d_p^\sigma(P)=d_p^\sigma(P_L)$.

Proof. Suppose that $P_L\in\mathcal{D}_p^\sigma\left(\widehat{\otimes}_{\pi_s}^{m,s}X,Y\right)$. For $x_1,\dots,x_n\in X$ and $y_1^*,\dots,y_n^*\in Y^*$, we have

$$\begin{aligned}\sum_{i=1}^{n}\left|\left\langle P\left(x_i\right),y_i^*\right\rangle\right| &= \sum_{i=1}^{n}\left|\left\langle P_L\left(x_i\otimes\cdots\otimes x_i\right),y_i^*\right\rangle\right| \\ &\leq d_p^\sigma(P_L)\left(\sum_{i=1}^{n}\pi_s\left(x_i\otimes\cdots\otimes x_i\right)^r\right)^{\frac{1}{r}}\delta_{p^*,\sigma}\left(\left(y_i^*\right)_{i=1}^n\right) \\ &\leq d_p^\sigma(P_L)\left(\sum_{i=1}^{n}\|x_i\|^{mr}\right)^{\frac{1}{r}}\delta_{p^*,\sigma}\left(\left(y_i^*\right)_{i=1}^n\right).\end{aligned}$$

So $P\in\mathcal{P}_p^\sigma\left({}^mX;Y\right)$ and $d_p^\sigma(P)\leq d_p^\sigma(P_L)$.

Conversely, suppose that P is strongly (p,σ)-continuous. Let $v\in\otimes_{\pi_s}^{m,s}X$ and let $y^*\in Y^*$. Suppose that $v=\sum\limits_{i=1}^{n}x_i\otimes\ldots\otimes x_i$. Then

$$\begin{aligned}\left|\left\langle P_L(v),y^*\right\rangle\right| &\leq \sum_{i=1}^{n}\left|\left\langle P\left(x_i\right),y^*\right\rangle\right| \\ &\leq \sum_{i=1}^{n}d_p^\sigma(P)\|x_i\|^m\left(\int\limits_{B_{Y^{**}}}\left(\left|\left\langle y^*,\varphi\right\rangle\right|^{1-\sigma}\|y^*\|^\sigma\right)^{\frac{p^*}{1-\sigma}}d\mu\left(y^{**}\right)\right)^{\frac{1-\sigma}{p^*}} \\ &\leq d_p^\sigma(P)\left(\sum_{i=1}^{n}\|x_i\|^m\right)\left(\int\limits_{B_{Y^{**}}}\left(\left|\left\langle y^*,\varphi\right\rangle\right|^{1-\sigma}\|y^*\|^\sigma\right)^{\frac{p^*}{1-\sigma}}d\mu\left(y^{**}\right)\right)^{\frac{1-\sigma}{p^*}}.\end{aligned}$$

Taking the infimum over all representations of v we obtain

$$\left|\left\langle P_L(v),y^*\right\rangle\right|\leq d_p^\sigma(P)\pi_s\left(v\right)\left(\int_{B_{Y^{**}}}\left(\left|\left\langle y^*,\varphi\right\rangle\right|^{1-\sigma}\|y^*\|^\sigma\right)^{\frac{p^*}{1-\sigma}}d\mu(y^{**})\right)^{\frac{1-\sigma}{p^*}}.$$

Therefore, P_L is strongly (p,σ)-continuous and $d_p^\sigma(P_L)\leq d_p^\sigma(P)$. □

As an immediate consequence of the preceding proposition, we get the following corollary.

Corollary 3.6. *The polynomial* P *is strongly* (p,σ)*-continuous if and only if there exist a strongly* (p,σ)*-continuous operator* u *and a polynomial* Q *such that* $P=u\circ Q$*, that is,* $\mathcal{P}_p^\sigma\left({}^mX;Y\right)=\mathcal{D}_p^\sigma\circ\mathcal{P}\left({}^mX;Y\right)$.

Corollary 3.7. *The class of strongly* (p,σ)*-continuous* m*-homogeneous polynomials constitutes a Banach ideal of* m*-homogeneous polynomials.*

The application of Proposition 3.5 enables us to establish links between the class of strongly (p,σ)-continuous m-homogeneous polynomials and other well-known classes.

Theorem 3.8. *Let $1 < p, r < \infty$ and $0 \leq \sigma < 1$ be such that $\frac{1}{r} + \frac{1-\sigma}{p^*} = 1$ and let $P \in \mathcal{P}\left({}^m X; Y\right)$.*

(1) *If $P \in \mathcal{P}^p_{Coh}\left({}^m X, Y\right)$, then $P \in \mathcal{P}^\sigma_p\left({}^m X, Y\right)$ and $d^\sigma_p(P) \leq d_p(P)$.*
(2) *If $P \in \mathcal{P}^c_{p,N}\left({}^m X, Y\right)$, then $P \in \mathcal{P}^\sigma_p\left({}^m X, Y\right)$ and $d^\sigma_p(P) \leq \pi_{p,N}(P)$.*
(3) *If $P \in \mathcal{P}^{fs}_{p,N}\left({}^m X; Y\right)$, then $P \in \mathcal{P}^\sigma_p\left({}^m X, Y\right)$ and $d^\sigma_p(P) \leq \pi^{fs}_{p,N}(P)$.*

Proof. (1) Suppose that P is a Cohen strongly p-summing m-homogeneous polynomial. According to [6, Proposition 3.2], P_L is Cohen strongly p-summing. Additionally, [4, Proposition 3.6] implies that P_L is strongly (p, σ)-continuous. Applying Proposition 3.5, we deduce that P is strongly (p, σ)-continuous.

(2) Since $\mathcal{P}^c_{p,N}\left({}^m X, Y\right) \subset \mathcal{P}^p_{Coh}\left({}^m X, Y\right)$, the result directly follows from (1).

(3) Consider P as a factorable strongly p-nuclear m-homogeneous polynomial. According to [1, Theorem 3.6], P_L is Cohen p-nuclear. By combining [11, Theorem 2.2.1] with [4, Proposition 3.6], we deduce that P_L is strongly (p, σ)-continuous. By Proposition 3.5, we conclude that P is a strongly (p, σ)-continuous m-homogeneous polynomial. □

Achour et al. [4, Theorem 4.8] established that a linear operator is strongly (p, σ)-continuous if and only if its adjoint is absolutely (p^*, σ)-continuous. The following result is a polynomial counterpart to this theorem.

Theorem 3.9. *Let $1 < p, r < \infty$ and $0 \leq \sigma < 1$ be such that $\frac{1}{r} + \frac{1-\sigma}{p^*} = 1$. A polynomial $P \in \mathcal{P}\left({}^m X; Y\right)$ is strongly (p, σ)-continuous if and only if the adjoint operator $P^* : Y^* \to \mathcal{P}\left({}^m X\right)$ is (p^*, σ)-absolutely continuous. In this case, $d^\sigma_p(P) = \pi_{p^*,\sigma}\left(P^*\right)$.*

As a consequence of the above and [4, Corollary 3.8], we get the following result.

Corollary 3.10. *Let $1 < p, r < \infty$ and $0 \leq \sigma < 1$ be such that $\frac{1}{r} + \frac{1-\sigma}{p^*} = 1$. A polynomial $P \in \mathcal{P}\left({}^m X; Y\right)$ is strongly (p, σ)-continuous if and only if the operator $P^{**} : \mathcal{P}\left({}^m X\right)^* \to Y^{**}$ is strongly (p, σ)-continuous. In this case, $d^\sigma_p(P) = d^\sigma_p\left(P^{**}\right)$.*

4. Factorable strongly (p, σ, q, ν)-nuclear m-homogeneous polynomials

In this section, we introduce the class of factorable strongly (p, σ, q, ν)-nuclear homogeneous polynomials. The goal is to describe, in terms of summability, homogeneous polynomials whose linearizations are (p, σ, q, ν)-nuclear. This characterization establishes a robust connection between the theory of (p, σ, q, ν)-nuclear linear operators and non-linear factorable strongly (p, σ, q, ν)-nuclear homogeneous polynomials, marking a notable improvement over previous approaches.

Definition 4.1. Let $P \in \mathcal{P}\left({}^m X; Y\right)$, let $1 \leq p, q < \infty$ and $0 \leq \sigma, v < 1$ be such that $\frac{1-\sigma}{p} + \frac{1-v}{q} = 1$, and let $m \geq 1$. We say that P is factorable strongly (p, σ, q, ν)-nuclear if there exists a constant $C > 0$ such that for all

$n_1, n_2 \in \mathbb{N}$, all $(x_{i,k})_{1\le i\le n_2, 1\le k\le n_1} \subset X$, all $(y_k^*)_{1\le k\le n_1} \subset Y^*$, and all scalars $\lambda_k^i, 1 \le i \le n_2, 1 \le k \le n_1$, we have

$$\begin{aligned}\sum_{k=1}^{n_1} & \left|\sum_{i=1}^{n_2} \lambda_k^i \left\langle P\left(x_{i,k}\right), y_k^*\right\rangle\right| \\ & \le C \sup_{\phi\in B_{\mathcal{P}(^m X)}} \left(\sum_{k=1}^{n_1}\left(\left|\sum_{i=1}^{n_2} \lambda_k^i \phi\left(x_{i,k}\right)\right|^{1-\sigma}\left(\sum_{i=1}^{n_2}\left|\lambda_k^i\right|\left\|x_{i,k}\right\|^m\right)^{\sigma}\right)^{\frac{p}{1-\sigma}}\right)^{\frac{1-\sigma}{p}} \\ & \quad \times \delta_{q,\nu}\left(\left(y_k^*\right)_{k=1}^{n_1}\right).\end{aligned}$$

The class of all factorable strongly (p,σ,q,ν)-nuclear m-homogeneous polynomials from X to Y is denoted by $\mathcal{P}_{p,\sigma,q,\nu,N}^{Fst}\left(^m X;Y\right)$ and endowed with the norm $N_{p,\sigma,q,\nu}^{Fst}$ given by the infimum of all constants $C>0$ as above.

From the definition, the ensuing statements become readily apparent:

(1) If we set $m = n_2 = 1$, then we regain the class of (p,σ,q,ν)-nuclear linear operators $\mathcal{N}_{p,\sigma,q,\nu}\left(X_1, Y\right)$.
(2) When $\sigma = \nu = 0$, we obtain $\mathcal{P}_{p,0,p^*,0,N}^{Fst}\left(^m X;Y\right) = \mathcal{P}_{p,N}^{fs}\left(^m X;Y\right)$.

The following result justifies the introduction of factorable strongly (p,σ,q,ν)-nuclear polynomials by establishing a direct connection with their linearizations.

Theorem 4.2. *Let $1 \le p,q < \infty$ and $0 \le \sigma, v < 1$ be such that $\frac{1-\sigma}{p} + \frac{1-v}{q} = 1$, let $m \ge 1$, and let $P \in \mathcal{P}\left(^m X;Y\right)$. The following assertions are equivalent:*

(i) *P is factorable strongly (p,σ,q,ν)-nuclear.*
(ii) *$P_L : \widehat{\otimes}_{\pi_s}^{m,s} X \to Y$ is (p,σ,q,ν)-nuclear.*

In this case, $N_{p,q}^{\sigma,\nu}\left(P_L\right) = N_{p,\sigma,q,\nu}^{Fst}(P)$.

Proof. $(i) \Rightarrow (ii)$ Suppose that P is a factorable strongly (p,σ,q,ν)-nuclear polynomial, and take $(u_k)_{1\le k\le n_1} \subset \otimes^{m,s} X$ and $(y_k^*)_{1\le k\le n_1} \subset Y^*$. Then, completing the sum with zeros if necessary, there is a natural number n_2 so that we can write each $u_k = \sum_{i=1}^{n_2} \lambda_k^i x_{i,k} \otimes \cdots \otimes x_{i,k}$ for some $x_{i,k} \in X$ and scalars λ_k^i. Let $\varepsilon > 0$. Choose representations of u_k such that

$$\sum_{i=1}^{n_2}\left|\lambda_k^i\right|\left\|x_{i,k}\right\|^m \le \pi_s\left(u_k\right) + \varepsilon.$$

Since P is factorable strongly (p,σ,q,ν)-nuclear it follows that

$$\begin{aligned}&\sum_{k=1}^{n_1}\left|\left\langle P_L\left(u_k\right), y_k^*\right\rangle\right| \\ &\quad = \sum_{k=1}^{n_1}\left|\sum_{i=1}^{n_2} \lambda_k^i \left\langle P\left(x_{i,k}\right), y_k^*\right\rangle\right| \\ &\quad \le N_{p,\sigma,q,\nu}^{Fst}(P)\end{aligned}$$

$$
\begin{aligned}
&\times \sup_{\phi \in B_{\mathcal{P}(^m X)}} \left(\sum_{k=1}^{n_1} \left(\left| \sum_{i=1}^{n_2} \lambda_k^i \phi\left(x_{i,k}\right) \right|^{1-\sigma} \left(\sum_{i=1}^{n_2} \left|\lambda_k^i\right| \left\|x_{i,k}\right\|^m \right)^{\sigma} \right)^{\frac{p}{1-\sigma}} \right)^{\frac{1-\sigma}{p}} \\
&\times \delta_{q,\nu}\left((y_k^*)_{k=1}^{n_1}\right) \\
\leq\ & N_{p,\sigma,q,\nu}^{Fst}(P) \\
&\times \sup_{\phi \in B_{\mathcal{P}(^m X)}} \left(\sum_{k=1}^{n_1} \left(\left| \sum_{i=1}^{n_2} \lambda_k^i \phi\left(x_{i,k}\right) \right|^{1-\sigma} \left(\pi_s\left(u_k\right) + \varepsilon\right)^{\sigma} \right)^{\frac{p}{1-\sigma}} \right)^{\frac{1-\sigma}{p}} \\
&\times \delta_{q,\nu}\left((y_k^*)_{k=1}^{n_1}\right) \\
=\ & N_{p,\sigma,q,\nu}^{Fst}(P) \\
&\times \sup_{\phi_L \in B_{\left(\widehat{\otimes}_{\pi_s}^{m,s} X\right)^*}} \left(\sum_{k=1}^{n_1} \left(\left| \phi_L \left(\sum_{i=1}^{n_2} \lambda_k^i x_{i,k} \otimes \cdots \otimes x_{i,k} \right) \right|^{1-\sigma} \left(\pi_s\left(u_k\right) + \varepsilon\right)^{\sigma} \right)^{\frac{p}{1-\sigma}} \right)^{\frac{1-\sigma}{p}} \\
&\times \delta_{q,\nu}\left((y_k^*)_{k=1}^{n_1}\right).
\end{aligned}
$$

Letting ε tend to 0, we get

$$
\begin{aligned}
\sum_{k=1}^{n_1} & \left|\left\langle P_L\left(u_k\right), y_k^*\right\rangle\right| \\
& \leq N_{p,\sigma,q,\nu}^{Fst}(P) \sup_{\phi_L \in B_{\left(\widehat{\otimes}_{\pi_s}^{m,s} X\right)^*}} \left(\sum_{k=1}^{n_1} \left(\left|\phi_L\left(u_k\right)\right|^{1-\sigma} \pi_s\left(u_k\right)^{\sigma} \right)^{\frac{p}{1-\sigma}} \right)^{\frac{1-\sigma}{p}} \\
& \quad \times \delta_{q,\nu}\left((y_k^*)_{k=1}^{n_1}\right).
\end{aligned}
$$

By the density of $\otimes^{m,s} X$ in $\widehat{\otimes}_{\pi_s}^{m,s} X$, we conclude that the operator P_L is (p,σ,q,ν)-nuclear and $N_{p,q}^{\sigma,\nu}\left(P_L\right) \leq N_{p,\sigma,q,\nu}^{Fst}(P)$.

$(ii) \Rightarrow (i)$ Let us suppose that the linearization $P_L : \widehat{\otimes}_{\pi_s}^{m,s} X \to Y$ is (p,σ,q,ν)-nuclear. Take $(x_{i,k})_{1\leq i\leq n_2, 1\leq k\leq n_1} \subset X$, $(y_k^*)_{1\leq k\leq n_1} \subset Y^*$ and scalars $(\lambda_k^i)_{1\leq i\leq n_2, 1\leq k\leq n_1}$. Then, for each $1 \leq k \leq n_1$ the element $u_k = \sum_{i=1}^{n_2} \lambda_k^i x_{i,k} \otimes \cdots \otimes x_{i,k}$ belongs to $\widehat{\otimes}_{\pi_s}^{m,s} X$.

Since $P_L : \widehat{\otimes}_{\pi_s}^{m,s} X \to Y$ is (p,σ,q,ν)-nuclear, we have

$$
\begin{aligned}
\sum_{k=1}^{n_1} & \left| \sum_{i=1}^{n_2} \lambda_k^i \left\langle P\left(x_{i,k}\right), y_k^*\right\rangle \right| \\
& = \sum_{k=1}^{n_1} \left|\left\langle P_L\left(u_k\right), y_k^*\right\rangle\right| \\
& \leq N_{p,q}^{\sigma,\nu}(P_L) \sup_{\phi_L \in B_{\left(\widehat{\otimes}_{\pi_s}^{m,s} X\right)^*}} \left(\sum_{k=1}^{n_1} \left(\left|\phi_L\left(u_k\right)\right|^{1-\sigma} \pi_s\left(u_k\right)^{\sigma} \right)^{\frac{p}{1-\sigma}} \right)^{\frac{1-\sigma}{p}} \delta_{q,\nu}\left((y_k^*)_{k=1}^{n_1}\right) \\
& = N_{p,q}^{\sigma,\nu}(P_L)
\end{aligned}
$$

$$
\begin{aligned}
&\times \sup_{\phi_L \in B_{\left(\widehat{\otimes}^{m,s}_{\pi_s} X\right)^*}} \left(\sum_{k=1}^{n_1} \left(\left| \phi_L \left(\sum_{i=1}^{n_2} \lambda_k^i x_{i,k} \otimes \cdots \otimes x_{i,k} \right) \right|^{1-\sigma} \pi_s\left(u_k\right)^{\sigma} \right)^{\frac{p}{1-\sigma}} \right)^{\frac{1-\sigma}{p}} \\
&\times \delta_{q,\nu}\left(\left(y_k^*\right)_{k=1}^{n_1} \right) \\
\leq\ & N_{p,q}^{\sigma,\nu}(P_L) \\
&\times \sup_{\phi_L \in B_{\left(\widehat{\otimes}^{m,s}_{\pi_s} X\right)^*}} \left(\sum_{k=1}^{n_1} \left(\left| \phi_L \left(\sum_{i=1}^{n_2} \lambda_k^i x_{i,k} \otimes \cdots \otimes x_{i,k} \right) \right|^{1-\sigma} \left(\sum_{i=1}^{n_2} \left|\lambda_k^i\right| \left\|x_{i,k}\right\|^m \right)^{\sigma} \right)^{\frac{p}{1-\sigma}} \right)^{\frac{1-\sigma}{p}} \\
&\times \delta_{q,\nu}\left(\left(y_k^*\right)_{k=1}^{n_1} \right) \\
=\ & N_{p,q}^{\sigma,\nu}(P_L) \sup_{\phi \in B_{\mathcal{P}(^m X)}} \left(\sum_{k=1}^{n_1} \left(\left| \sum_{i=1}^{n_2} \lambda_k^i \phi\left(x_{i,k}\right) \right|^{1-\sigma} \left(\sum_{i=1}^{n_2} \left|\lambda_k^i\right| \left\|x_{i,k}\right\|^m \right)^{\sigma} \right)^{\frac{p}{1-\sigma}} \right)^{\frac{1-\sigma}{p}} \\
&\times \delta_{q,\nu}\left(\left(y_k^*\right)_{k=1}^{n_1} \right).
\end{aligned}
$$

Hence P is factorable strongly (p,σ,q,ν)-nuclear and $N^{Fst}_{p,\sigma,q,\nu}(P) \leq N^{\sigma,\nu}_{p,q}(P_L)$. $\square$

The following corollary directly follows from the former theorem.

Corollary 4.3. *Let $P \in \mathcal{P}\left(^m X, Y\right)$. Then P is factorable strongly (p,σ,q,ν)-nuclear if and only if there exist a (p,σ,q,ν)-nuclear operator u and a polynomial Q such that $P = u \circ Q$, that is, $\mathcal{P}^{Fst}_{p,\sigma,q,\nu,N}\left(^m X; Y\right) = \mathcal{N}_{p,\sigma,q,\nu} \circ \mathcal{P}\left(^m X; Y\right)$.*

The following result follows from combining Theorem 4.2 with [8, Proposition 3.7].

Proposition 4.4. *The class of factorable strongly (p,σ,q,ν)-nuclear homogeneous polynomials constitutes a Banach ideal of m-homogeneous polynomials.*

Another consequence derived from Theorem 4.2 is the following outcome (in the linear case, the same result is proven, as outlined in [13, Theorem 4]).

Corollary 4.5 (Factorization Theorem)**.** *Let X, Y be Banach spaces. Then*

$$\mathcal{P}^{Fst}_{p,\sigma,q,\nu,N}\left(^m X; Y\right) = \mathcal{D}^{\nu}_{q^*} \circ \Pi_{p,\sigma} \circ \mathcal{P}\left(^m X; Y\right).$$

Now, we aim to describe factorable strongly (p,σ,q,ν)-nuclearity through a domination inequality.

Proposition 4.6. *Given $P \in \mathcal{P}\left(^m X; Y\right)$, let $1 < p, q < \infty$ and $0 \leq \sigma, v < 1$ be such that $\frac{1-\sigma}{p} + \frac{1-v}{q} = 1$, and let $m \geq 1$. Then P is factorable strongly (p,σ,q,ν)-nuclear if and only if there are a constant $C \geq 0$ and regular probability measures μ on $B_{\mathcal{P}(^m X)}$ and λ on $B_{Y^{**}}$ (both endowed with the weak star topology) such that for all $(x_i)_{1\leq i\leq n} \subset X$ and $y^* \in Y^*$, the following relation holds:*

$$\left| \sum_{i=1}^{n} \left\langle P\left(x_i\right), y^* \right\rangle \right| \ \leq\ C \left(\int_{B_{\mathcal{P}(^m X)}} \left(\left| \sum_{i=1}^{n} \phi\left(x_i\right) \right|^{1-\sigma} \left(\sum_{i=1}^{n} \left\|x_i\right\|^m \right)^{\sigma} \right)^{\frac{p}{1-\sigma}} d\mu(\phi) \right)^{\frac{1-\sigma}{p}}$$

$$\times\left(\int\limits_{B_{Y^{**}}}\left(\left|\varphi\left(y^{*}\right)\right|^{1-\nu}\left\|y^{*}\right\|^{\nu}\right)^{\frac{q}{1-\nu}}d\lambda(y^{**})\right)^{\frac{1-\sigma}{p}}. \quad (4.1)$$

Moreover, $N^{Fst}_{p,\sigma,q,\nu}(P)=\inf\{C>0 : C$ *satisfies the inequality* (4.1) $\}$.

Proof. By Theorem 4.2, P is factorable strongly (p,σ,q,ν)-nuclear if and only if its linearization P_L: $\widehat{\otimes}^{m,s}_{\pi_s}X\to Y$ is (p,σ,q,ν)-nuclear. In this case $N^{Fst}_{p,\sigma,q,\nu}(P)=N^{\sigma,\nu}_{p,q}(P_L)$. By the Pietsch domination theorem [13, Theorem 4 (iii)], there are $C>0$ and regular probability measures μ on $B_{\left(\widehat{\otimes}^{m,s}_{\pi_s}X\right)^*}$ and λ on $B_{Y^{**}}$ such that for all $u\in\widehat{\otimes}^{m,s}_{\pi_s}X$ and $y^*\in Y^*$, we have

$$\begin{aligned}\left|\left\langle P_L\left(u\right),y^{*}\right\rangle\right| &\leq C\left(\int_{B_{\left(\widehat{\otimes}^{m,s}_{\pi_s}X\right)^*}}\left(\left|\phi_L\left(u\right)\right|^{1-\sigma}\pi_s\left(u\right)^{\sigma}\right)^{\frac{p}{1-\sigma}}d\mu(\phi)\right)^{\frac{1-\sigma}{p}}\\ &\quad\times\left(\int_{B_{Y^{**}}}\left(\left|\varphi\left(y^{*}\right)\right|^{1-\nu}\left\|y^{*}\right\|^{\nu}\right)^{\frac{q}{1-\nu}}d\lambda(y^{**})\right)^{\frac{1-\sigma}{p}}. \quad (4.2)\end{aligned}$$

Moreover, $N^{\sigma\nu}_{p,q}\left(P_L\right)$ is the infimum of the constants C. Let $(x_i)_{1\leq i\leq n}$ in X. Applying (4.2) to the tensor $u=\sum_{i=1}^{n}x_i\otimes\cdots\otimes x_i$, we get

$$\begin{aligned}\left|\sum_{i=1}^{n}\left\langle P\left(x_i\right),y^{*}\right\rangle\right| &\leq C\times\left(\int\limits_{B_{\mathcal{P}(^mX)}}\left(\left|\sum_{i=1}^{n}\phi\left(x_i\right)\right|^{1-\sigma}\left(\sum_{i=1}^{n}\left\|x_i\right\|^{m}\right)^{\sigma}\right)^{\frac{p}{1-\sigma}}d\mu(\phi)\right)^{\frac{1-\sigma}{p}}\\ &\quad\times\left(\int\limits_{B_{Y^{**}}}\left(\left|\varphi\left(y^{*}\right)\right|^{1-\nu}\left\|y^{*}\right\|^{\nu}\right)^{\frac{q}{1-\nu}}d\lambda(y^{**})\right)^{\frac{1-\sigma}{p}}.\end{aligned}$$

□

Now, in the upcoming proposition, we establish a connection between the class of factorable strongly (p,σ,q,ν)-nuclear polynomials and that of strongly (q^*,ν)-continuous polynomials.

Proposition 4.7. *Consider* $1\leq p,q<\infty$ *and* $0\leq\sigma,v<1$ *be such that* $\frac{1-\sigma}{p}+\frac{1-v}{q}=1$, *and let* $m\geq 1$. *We then have* $\mathcal{P}^{Fst}_{p,\sigma,q,\nu,N}\left(^mX;Y\right)\subset\mathcal{P}^{\nu}_{q^*}\left(^mX;Y\right)$.

Moreover, $d^{\nu}_{q^*}(P)\leq N^{Fst}_{p,\sigma,q,\nu}(P)$ *such that* $P\in\mathcal{P}^{Fst}_{p,\sigma,q,\nu,N}\left(^mX;Y\right)$.

Proof. Suppose that P is a factorable strongly (p,σ,q,ν)-nuclear polynomial. According to Theorem 4.2, it follows that this is equivalent to P_L being (p,σ,q,ν)-nuclear. Additionally, applying [7, Theorem 2 (i)] in the linear case, we deduce that P_L is strongly (q^*,ν)-continuous. Finally, Proposition 3.5 gives the result, additionally, $d^{\nu}_{q^*}(p)\leq N^{Fst}_{p,\sigma,q,\nu}(p)$. □

In the next theorem, we characterize the adjoint operator for the class of factorable strongly (p,σ,q,ν)-nuclear m-homogeneous polynomials.

Theorem 4.8. *Let $1 < p, q < \infty$ and $0 \leq \sigma, v < 1$ be such that $\frac{1-\sigma}{p} + \frac{1-v}{q} = 1$, and let $m \geq 1$. A polynomial $P \in \mathcal{P}(^m X; Y)$ is factorable strongly (p, σ, q, ν)-nuclear if and only if the adjoint operator $P^* : Y^* \to \mathcal{P}(^m X)$ is (q, ν, p, σ)-nuclear.*

Moreover, $N^{Fst}_{p,\sigma,q,\nu}(P) = N^{\nu,\sigma}_{q,p}(P^*)$.

Proof. Assume first that $P : X \to Y$ is factorable strongly (p, σ, q, ν)-nuclear. By Theorem 4.2 its linearization $P_L : \widehat{\otimes}^{m,s}_{\pi_s} X \to Y$ is (p, σ, q, ν)-nuclear. By [13, Theorem 5] its adjoint $P^*_L : Y^* \to \left(\widehat{\otimes}^{m,s}_{\pi_s} X\right)^*$ is a (q, ν, p, σ)-nuclear operator. Consider the isometric isomorphism $\Delta_m : \mathcal{P}(^m X) \to \left(\widehat{\otimes}^{m,s}_{\pi_s} X\right)^*$ given by $\Delta_m(P) = P_L$. Since $P = P_L \circ \delta_m$, by duality we get $P^* = \delta^*_m \circ P^*_L = \Delta^{-1}_m \circ P^*_L$. The ideal property ensures that P^* is (q, ν, p, σ)-nuclear.

Conversely, assume that P^* is (q, ν, p, σ)-nuclear. Then the equality $P^*_L = \Delta_m \circ P^*$ and the ideal property gives that P^*_L is (q, ν, p, σ)-nuclear. Again, [13, Theorem 5] gives that P_L is (p, σ, q, ν)-nuclear, and by Theorem 4.2 we conclude that P is factorable strongly (p, σ, q, ν)-nuclear. Furthermore, $N^{Fst}_{p,\sigma,q,\nu}(P) = N^{\nu,\sigma}_{q,p}(P^*)$. ☐

Corollary 4.9. *Let $1 \leq p, q < \infty$ and $0 \leq \sigma, v < 1$ be such that $\frac{1-\sigma}{p} + \frac{1-v}{q} = 1$, and let $m \geq 1$. A polynomial $P \in \mathcal{P}(^m X; Y)$ is factorable strongly (p, σ, q, ν)-nuclear if and only if the linear operator $P^{**} : \mathcal{P}(^m X)^* \to Y^{**}$ is (p, σ, q, ν)-nuclear. In this case, $N^{Fst}_{p,\sigma,q,\nu}(P) = N^{\sigma,\nu}_{p,q}(P^{**})$.*

In [7, Corollary 4] the authors proved that the (p, σ, q, ν)-nuclear operators can be described in terms of a suitable tensor product. That is,

$$\mathcal{N}_{p,\sigma,q,v}(X, Y) = \left(X \otimes_{g^{(\sigma,\nu)}_{(p,q)}} Y^*\right)^*, \tag{4.3}$$

where $g^{(\sigma,\nu)}_{(p,q)}(.)$ is a reasonable cross norm on $X \otimes Y$ defined by

$$g^{(\sigma,\nu)}_{(p,q)}(u) = \inf \left\{\delta_{p,\sigma}\left((x_i)^n_{i=1}\right) \delta_{q,\nu}\left((y_i)^n_{i=1}\right)\right\},$$

the infimum being taken over all representations of $u = \sum^n_{i=1} x_i \otimes y_i \in X \otimes Y$.

From Theorem 4.2 and the isometric isomorphism (4.3) we extend this to m-homogeneous factorable strongly (p, σ, q, ν)-nuclear polynomials.

Corollary 4.10. *The spaces $\mathcal{P}^{Fst}_{p,\sigma,q,\nu,N}(^m X; Y)$ and $\left(\widehat{\otimes}^{m,s}_{\pi_s} X \otimes_{g^{(\sigma,\nu)}_{(p,q)}} Y^*\right)^*$ are isometrically isomorphic.*

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