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THE MINKOWSKI INEQUALITY ASSOCIATED WITH CONSTANT PROPORTIONAL FRACTIONAL INTEGRALS

BARIŞ ÇELİK[1], ERHAN SET[2], AHMET OCAK AKDEMİR[3] * AND PRAVEEN AGARWAL[4,5]

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Abstract. The aim of this present investigation is establishing Minkowski fractional integral inequalities and certain other fractional integral inequalities by employing the constant proportional fractional integral operator. The inequalities presented in this paper are more general than the existing classical inequalities cited.

1. Introduction

The concept of classical calculus of derivatives and integrals for integer orders is recently extended to be fractional calculus, which is a generalized form of classical integrals and derivatives in case of non-integer order. In last few decades, the fractional calculus theory receives more attention due to significant applications in various fields, such as computer networking, biology, physics, fluid dynamics, signal processing, image processing, control theory, and other fields; see, for example, [7, 10, 13, 11, 36, 40]. Because of the importance of fractional calculus, many researchers have shown their intense interest in this see [5, 12, 17]. One of the widespread approaches among authors is the use of fractional integrals and derivatives operators. As a consequence, many different kinds of fractional integrals and derivatives have been realized, such as the Riemann–Liouville, conformable, Weyl types, Liouville, Saigo, Hadamard, Katugampola, and some other types can be found in Kilbas, Srivastava, and Trujillo [21].

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* Corresponding author.
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Integral inequalities have potential application in several areas of science: Technology, mathematics, chemistry, among others; especially, we point out initial value problems, the stability of linear transformation, integral differential equations, and impulse equations [14, 22, 23, 24, 25, 26, 28, 29]. Variants regarding fractional integral operators are of use in significant strategies amongst researchers and accumulate fertile functional applications in various areas of science; see [1, 2, 18, 19, 20, 31, 34]. On account of their potential results to be utilized for the presence of nontrivial and positive solutions of a distinct kind of fractional differential equations, our findings concerning fractional integrals are appreciable and essential.

In classical integral and differential equations, mathematical inequalities play a very dependable role and in the past several years, a number of useful and important mathematical inequalities invented by many authors. Inequalities, which involves fractional integrals and derivatives, are crucial in the study of a number of differential and integral equations, among which we mention [3, 4, 33, 39]. One of the most renowned and important integral inequality is given by Hermann Minkowski, which states that, for $l \geq 1$, if

$$0 < \int_c^d \eta_1^k(z)dz < \infty \quad and \quad 0 < \int_c^d \eta_2^k(z)dz\infty,$$

then

$$\left(\int_c^d (\eta_1(z)+\eta_2(z))^k dz\right)^{\frac{1}{l}} \leq \left(\int_c^d \eta_1^k(z)dz\right)^{\frac{1}{l}} + \left(\int_c^d \eta_2^k(z)dz\right)^{\frac{1}{l}}.$$

In the last few decades, this inequality has received considerable attention from many authors, and several articles have appeared in the literature. In 2006, Bougofa [8] presented an integral inequality concerning some reverse of Minkowski's inequality. In 2010, Dahmani [9] gave the reverse Minkowski and Hermite-Hadamard inequalities by mean of Riemann–Liouville fractional integral. In 2010, Set, Özdemir, and Dragomir [30] studied the inequalities of reverse Minkowski and Hermite–Hadamard involving two functions using the classical Riemann integral. In 2013, Chinchane and Pachpatte [15] and Taf and Brahim [37], established the reverse Minkowski's inequality via Saigo and the Hadamard fractional integral, respectively. In 2018 , da Vanterler, Sousa, and Capelas de Oliveira [16] presented the reverse Minkowski inequalities and some other related inequalities by mean of Katugampola fractional integral. In 2019, Rahman *et al.* [27] employed the generalized proportional fractional integral operators to establish the reverse Minkowski's inequality and other fractional inequalities. Very recently, in 2020, Set and Özdemir [32] presented the reverse Minkowski inequalities and some other related inequalities associated with mixed conformable fractional integral. More survey of some of the earlier and recent developments related to the inequality mentioned above can be found in [15, 27, 35, 38].

Motivated by the above mentioned background, our purpose in this paper is to use constant proportional fractional integral operator which is the classical Riemann–Liouville fractional integral of any function with respect to another function to establish and investigate some new fractional integral inequalities of

Minkowski's type. Moreover, we establish some new fractional inequalities related to reverse Minkowski's inequality via constant proportional fractional integral operator.

The article is organized as follows: In section 2, we recollect some notations, definitions, results and preliminary facts, which are used throughout this paper. In section 3, we give our main results of reverse Minkowski's inequality. In section 4, we present some other related results involving constant proportional fractional integral operator.

2. Preliminaries

Now, in this section, we give some basic definitions and properties of fractional integrals used to obtain and discuss our new results.

Definition 2.1. Let $\eta \in L[a,b]$. The Riemann–Liouville fractional integrals $J^{\alpha}_{a+}\eta$ and $J^{\alpha}_{b-}\eta$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$\mathbb{J}^{\alpha}_{a+}\eta(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} \eta(t)dt, \quad x > a,$$

and

$$\mathbb{J}^{\alpha}_{b-}\eta(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} \eta(t)dt, \quad x < b,$$

respectively, where $\Gamma(\alpha) = \int_0^{\infty} e^{-t}u^{\alpha-1}du$. Here is $\mathbb{J}^0_{a+}\eta(x) = \mathbb{J}^0_{b-}\eta(x) = \eta(x)$.

Dahmani [9] used the Riemann–Liouville fractional integrals to prove the following reverse Minkowski inequalities.

Theorem 2.2. [9] *Let $\alpha > 0$, let $p \geq 1$, and let η_1, η_2 be two positive functions on $[0,\infty)$, such that for all $t > 0$, $\mathbb{J}^{\alpha}\eta_1^p(t) < \infty$ and $\mathbb{J}^{\alpha}\eta_2^p(t) < \infty$. If $0 < \psi \leq \frac{\eta_1(\tau)}{\eta_2(\tau)} \leq \Psi$, $\tau \in [0,t]$, then*

$$\left[\mathbb{J}^{\alpha}\eta_1^p(t)\right]^{\frac{1}{p}} + \left[\mathbb{J}^{\alpha}\eta_2^p(t)\right]^{\frac{1}{p}} \leq \frac{1+\Psi(\psi+2)}{(\psi+1)(\Psi+1)} \left[\mathbb{J}^{\alpha}(\eta_1+\eta_2)^p(t)\right]^{\frac{1}{p}}.$$

Theorem 2.3. [9] *Let $\alpha > 0$, let $p \geq 1$, and let η_1, η_2 be two positive functions on $[0,\infty)$, such that for all $t > 0$, $\mathbb{J}^{\alpha}\eta_1^p(t) < \infty$, $\mathbb{J}^{\alpha}\eta_2^p(t) < \infty$. If $0 < \psi \leq \frac{\eta_1(\tau)}{\eta_2(\tau)} \leq \Psi$, $\tau \in [0,t]$, then*

$$\left[\mathbb{J}^{\alpha}\eta_1^p(t)\right]^{\frac{2}{p}} + \left[\mathbb{J}^{\alpha}\eta_2^p(t)\right]^{\frac{2}{p}} \geq \left(\frac{(\Psi+1)(\psi+1)}{\Psi} - 2\right) \left[\mathbb{J}^{\alpha}\eta_1^p(t)\right]^{\frac{1}{p}} \left[\mathbb{J}^{\alpha}\eta_2^p(t)\right]^{\frac{1}{p}}.$$

In [6], the following proportional derivative operator was defined:

$${}^{P}D_{\alpha}f(t) := K_1(\alpha,t)f(t) + K_0(\alpha,t)f'(t),$$

where K_1 and K_0 are functions of $\alpha \in [0,1]$ and $t \in \mathbb{R}$ satisfying certain conditions, and where f is a differentiable function of $t \in \mathbb{R}$. This operator arises naturally in control theory, and it relates to the large and expanding theory of conformable derivatives.

Furthermore, we recall from [6] the following general non-fractional differential operator, which has been called "proportional" or "conformable":

$$^{P}D_{\alpha}f(t) = K_1(\alpha, t)f(t) + K_0(\alpha, t)f'(t),$$

where K_0 and K_1 are functions of the variable t and the parameter $\alpha \in [0, 1]$, which satisfy the following conditions for all $t \in \mathbb{R}$:

$$\lim_{a\to 0^+} K_0(\alpha, t) = 0; \quad \lim_{\alpha\to 1^-} K_0(\alpha, t) = 1; \quad K_0(\alpha, t) \neq 0, \ \alpha \in (0, 1]; \tag{2.1}$$

$$\lim_{a\to 0^+} K_1(\alpha, t) = 1; \quad \lim_{\alpha\to 1^-} K_1(\alpha, t) = 0; \quad K_1(\alpha, t) \neq 0, \ \alpha \in [0, 1). \tag{2.2}$$

This can be seen as a generalization of the standard differentiation operator $Df(t) = f'(t)$, depending on an arbitrary parameter α, which is useful in control theory [6].

We shall also be interested in the particular case where the functions K_0 and K_1 are constant with respect to t, depending only on α. Let us denote this case by CP for "constant proportional":

$$^{CP}D_{\alpha}f(t) = K_1(\alpha)f(t) + K_0(\alpha)f'(t).$$

Lemma 2.4. [6] *The inverse of the proportional derivative operator* $^{P}D_{\alpha}$ *is given by*

$$^{P}_{a}\mathcal{I}_{\alpha}f(t) = \int_a^t \exp\left[-\int_u^t \frac{K_1(\alpha, s)}{K_0(\alpha, s)}ds\right] \frac{f(u)}{K_0(\alpha, u)}du,$$

and this satisfies the following inversion relations:

$$^{P}D_{\alpha}\, {}^{P}_{a}\mathcal{I}_{\alpha}f(t) = f(t), \quad {}^{P}_{a}\mathcal{I}_{\alpha}\, {}^{P}D_{\alpha}f(t) - \exp\left(-\int_a^t \frac{K_1(\alpha, s)}{K_0(\alpha_s)}ds\right) f(a).$$

Particularly, for the constant-coefficient operator $^{CP}D_{\alpha}$, the integral formula

$$^{CP}_{a}\mathcal{I}_{\alpha}f(t) = \frac{1}{K_0(\alpha)} \int_a^t \exp\left[-\frac{K_1(\alpha)}{K_0(\alpha)}(t-u)\right] f(u)du, \tag{2.3}$$

and the inversion relations are

$$^{CP}D_{\alpha}\, {}^{CP}_{a}\mathcal{I}_{\alpha}f(t) = f(t), \qquad {}^{CP}_{a}\mathcal{I}_{\alpha}\, {}^{CP}D_{\alpha}f(t) = f(t) - \exp\left(-\frac{K_1(\alpha)}{K_2(\alpha)}(t-a)\right) f(a).$$

Note that, if $f(a) = 0$, then the operators $^{P}D_{\alpha}$, ${}^{P}_{a}\mathcal{I}_{\alpha}$ and $^{CP}D_{\alpha}$, ${}^{CP}_{a}\mathcal{I}_{\alpha}$ form pairs of two-sided inverses to each other.

3. Reverse Minkowski inequality involving constant proportional fractional integral operators

In this section, we use the constant proportional fractional integral operator to develop reverse Minkowski integral inequalities.

Theorem 3.1. *Let $\alpha \in [0,1]$, let $p \geq 1$, and let η_1, η_2 be two positive functions on $[0,\infty)$, such that for all $t > 0$, ${}^{CP}\mathcal{I}_\alpha \eta_1^p(t) < \infty$ and ${}^{CP}\mathcal{I}_\alpha \eta_2^p(t) < \infty$. If $0 < \psi \leq \frac{\eta_1(\tau)}{\eta_2(\tau)} \leq \Psi$, for for all $\tau \in [0,t]$, then the following inequality holds:*

$$\left({}^{CP}\mathcal{I}_\alpha \eta_1^p(t)\right)^{\frac{1}{p}} + \left({}^{CP}\mathcal{I}_\alpha \eta_2^p(t)\right)^{\frac{1}{p}} \leq \kappa_1 \left({}^{CP}\mathcal{I}_\alpha (\eta_1+\eta_2)^p(t)\right)^{\frac{1}{p}} \tag{3.1}$$

with $\kappa_1 = \dfrac{\Psi(\psi+1)+\Psi+1}{(\psi+1)(\Psi+1)}$.

Proof. Using the condition $\frac{\eta_1(\tau)}{\eta_2(\tau)} \leq \Psi$, $\tau \in [0,t]$, $t > 0$, we can write

$$(\Psi+1)^p \eta_1^p(\tau) \leq \Psi^p (\eta_1(\tau)+\eta_2(\tau))^p. \tag{3.2}$$

Multiplying by $\dfrac{1}{K_0(\alpha)} \exp\left[-\dfrac{K_1(\alpha)}{K_0(\alpha)}(t-\tau)\right]$ both sides of (3.2) and integrating with respect to the variable τ, we have

$$\begin{aligned}
&\frac{(\Psi+1)^p}{K_0(\alpha)} \int_0^t \exp\left[-\frac{K_1(\alpha)}{K_0(\alpha)}(t-\tau)\right] \eta_1^p(\tau) d\tau \\
&\leq \frac{\Psi^p}{K_0(\alpha)} \int_0^t \exp\left[-\frac{K_1(\alpha)}{K_0(\alpha)}(t-\tau)\right] (\eta_1+\eta_2)^p(\tau) d\tau,
\end{aligned} \tag{3.3}$$

which can be written as

$${}^{CP}\mathcal{I}_\alpha \eta_1^p(t) \leq \frac{\Psi^p}{(\Psi+1)^p} \, {}^{CP}\mathcal{I}_\alpha (\eta_1+\eta_2)^p(t). \tag{3.4}$$

Consequently, we can write

$$\left({}^{CP}\mathcal{I}_\alpha \eta_1^p(t)\right)^{\frac{1}{p}} \leq \frac{\Psi}{\Psi+1} \left({}^{CP}\mathcal{I}_\alpha (\eta_1+\eta_2)^p(t)\right)^{\frac{1}{p}}. \tag{3.5}$$

On the other hand, as $\psi\eta_1(\tau) \leq \eta_2(\tau)$, we have

$$\left(1+\frac{1}{\psi}\right)^p \eta_2^p(\tau) \leq \left(\frac{1}{\psi}\right)^p (\eta_1(\tau)+\eta_2(\tau))^p. \tag{3.6}$$

Furthermore, multiplying by $\dfrac{1}{K_0(\alpha)} \exp\left[-\dfrac{K_1(\alpha)}{K_0(\alpha)}(t-\tau)\right]$ both sides of (3.6) and integrating with respect to the variable τ, we have

$$\left({}^{CP}\mathcal{I}_\alpha \eta_2^p(t)\right)^{\frac{1}{p}} \leq \frac{1}{\psi+1} \left({}^{CP}\mathcal{I}_\alpha (\eta_1+\eta_2)^p(t)\right)^{\frac{1}{p}}. \tag{3.7}$$

Adding the inequalities (3.5) and (3.7), we obtain the inequality (3.1). □

Eq. (3.1) is the so-called reverse Minkowski's inequality associated with the constant proportional fractional integral.

Theorem 3.2. *Let $\alpha \in [0,1]$, let $p \geq 1$, and let η_1, η_2 be two positive functions on $[0,\infty)$, such that for all $t > 0$, ${}^{CP}\mathcal{I}_\alpha \eta_1^p(t) < \infty$ and ${}^{CP}\mathcal{I}_\alpha \eta_2^p(t) < \infty$. If $0 < \psi \leq \frac{\eta_1(\tau)}{\eta_2(\tau)} \leq \Psi$, for for all $\tau \in [0,t]$, then*

$$\left({}^{CP}\mathcal{I}_\alpha \eta_1^p(t)\right)^{\frac{2}{p}} + \left({}^{CP}\mathcal{I}_\alpha \eta_2^p(t)\right)^{\frac{2}{p}} \geq \kappa_2 \left({}^{CP}\mathcal{I}_\alpha \eta_1^p(t)\right)^{\frac{1}{p}} \left({}^{CP}\mathcal{I}_\alpha \eta_2^p(t)\right)^{\frac{1}{p}} \tag{3.8}$$

with $\kappa_2 = \dfrac{(\Psi+1)(\psi+1)}{\Psi} - 2.$

Proof. Carrying out the product between (3.5) and (3.7), we have

$$\frac{(\Psi+1)(\psi+1)}{\Psi}\left({}^{CP}\mathcal{I}_\alpha\eta_1^p(t)\right)^{\frac{1}{p}}\left({}^{CP}\mathcal{I}_\alpha\eta_2^p(t)\right)^{\frac{1}{p}} \le \left({}^{CP}\mathcal{I}_\alpha(\eta_1+\eta_2)^p(t)\right)^{\frac{2}{p}}. \tag{3.9}$$

Using the Minkowski's inequality, on the right side of (3.9), we have

$$\frac{(\Psi+1)(\psi+1)}{\Psi}\left({}^{CP}\mathcal{I}_\alpha\eta_1^p(t)\right)^{\frac{1}{p}}\left({}^{CP}\mathcal{I}_\alpha\eta_2^p(t)\right)^{\frac{1}{p}} \le \left(\left({}^{CP}\mathcal{I}_\alpha\eta_1^p(t)\right)^{\frac{1}{p}} + \left({}^{CP}\mathcal{I}_\alpha\eta_2^p(t)\right)^{\frac{1}{p}}\right)^2. \tag{3.10}$$

So, from (3.10), we conclude that

$$\left(\frac{(\Psi+1)(\psi+1)}{\Psi} - 2\right)\left({}^{CP}\mathcal{I}_\alpha\eta_1^p(t)\right)^{\frac{1}{p}}\left({}^{CP}\mathcal{I}_\alpha\eta_2^p(t)\right)^{\frac{1}{p}} \le \left({}^{CP}\mathcal{I}_\alpha\eta_1^p(t)\right)^{\frac{2}{p}} + \left({}^{CP}\mathcal{I}_\alpha\eta_2^p(t)\right)^{\frac{2}{p}}.$$

□

4. Other Related inequalities via constant proportional fractional integral operator

This section is devoted to deriving certain related inequalities involving constant proportional fractional integral operator.

Theorem 4.1. *Let $\alpha \in [0,1]$, let $p > 1$, let $\frac{1}{p} + \frac{1}{q} = 1$, and let η_1, η_2 be two positive functions on $[0,\infty)$, such that ${}^{CP}\mathcal{I}_\alpha\eta_1^p(t) < \infty$ and ${}^{CP}\mathcal{I}_\alpha\eta_1^p(t) < \infty$. If $0 < \psi \le \frac{\eta_1(\tau)}{\eta_2(\tau)} \le \Psi$, for for all $\tau \in [0,t]$, then the following inequality holds:*

$$\left({}^{CP}\mathcal{I}_\alpha\eta_1(t)\right)^{\frac{1}{p}}\left({}^{CP}\mathcal{I}_\alpha\eta_2(t)\right)^{\frac{1}{q}} \le \left(\frac{\Psi}{\psi}\right)^{\frac{1}{pq}}\left({}^{CP}\mathcal{I}_\alpha(\eta_1^{\frac{1}{p}}(t)\eta_2^{\frac{1}{p}}(t)\right). \tag{4.1}$$

Proof. Using the condition $\frac{\eta_1(\tau)}{\eta_2(\tau)} \le \Psi$, $\tau \in [0,t]$ with $t > 0$, we have

$$\eta_1(\tau) \le \Psi\eta_2(\tau) \Rightarrow \eta_2^{\frac{1}{q}}(\tau) \ge \Psi^{-\frac{1}{q}}\eta_1^{\frac{1}{q}}(\tau). \tag{4.2}$$

Multiplying by $\eta_1^{\frac{1}{p}}(\tau)$ both sides of (4.2), we can rewrite it as follows:

$$\eta_1^{\frac{1}{p}}(\tau)\eta_2^{\frac{1}{q}}(\tau) \ge \Psi^{-\frac{1}{q}}\eta_1(\tau). \tag{4.3}$$

Now, multiplying by $\dfrac{1}{K_0(\alpha)}\exp\left[-\dfrac{K_1(\alpha)}{K_0(\alpha)}(t-\tau)\right]$ both sides of (4.3) and integrating with respect to the variable τ, we have

$$\begin{aligned}&\frac{\Psi^{-\frac{1}{q}}}{K_0(\alpha)}\int_0^t \exp\left[-\frac{K_1(\alpha)}{K_0(\alpha)}(t-\tau)\right]\eta_1(\tau)d\tau \\ &\le \frac{1}{K_0(\alpha)}\int_0^t \exp\left[-\frac{K_1(\alpha)}{K_0(\alpha)}(t-\tau)\right]\eta_1^{\frac{1}{p}}(\tau)\eta_2^{\frac{1}{q}}(\tau)d\tau.\end{aligned} \tag{4.4}$$

So, the inequality follows

$$\Psi^{-\frac{1}{pq}}\left({}^{CP}\mathcal{I}_\alpha\eta_1(t)\right)^{\frac{1}{p}} \le \left({}^{CP}\mathcal{I}_\alpha(\eta_1^{\frac{1}{p}}(t)\eta_2^{\frac{1}{q}}(t))\right)^{\frac{1}{p}}. \tag{4.5}$$

On the other hand, we have

$$\psi^{\frac{1}{p}}\eta_2^{\frac{1}{p}}(\tau) \le \eta_1^{\frac{1}{p}}(\tau), \quad t > 0. \tag{4.6}$$

Multiplying by $\eta_2^{\frac{1}{q}}(\tau)$ both sides of (4.6) and using the relation $\frac{1}{p}+\frac{1}{q}=1$, we have

$$\psi^{\frac{1}{p}}\eta_2(\tau) \le \eta_1^{\frac{1}{p}}(\tau)\eta_2^{\frac{1}{q}}(\tau). \tag{4.7}$$

Multiplying by $\dfrac{1}{K_0(\alpha)}\exp\left[-\dfrac{K_1(\alpha)}{K_0(\alpha)}(t-\tau)\right]$ both sides of (4.7) and integrating with respect to the variable τ, we have

$$\psi^{\frac{1}{pq}}\left({}^{CP}\mathcal{I}_\alpha\eta_2(t)\right)^{\frac{1}{q}} \le \left({}^{CP}\mathcal{I}_\alpha(\eta_1^{\frac{1}{p}}(t)\eta_2^{\frac{1}{q}}(t))\right)^{\frac{1}{p}}. \tag{4.8}$$

Evaluating the product between (4.5) and (4.8) and using the relation $\frac{1}{p}+\frac{1}{q}=1$, we conclude that

$$\left({}^{CP}\mathcal{I}_\alpha\eta_1(t)\right)^{\frac{1}{p}}\left({}^{CP}\mathcal{I}_\alpha\eta_2(t)\right)^{\frac{1}{q}} \le \left(\frac{\Psi}{\psi}\right)^{\frac{1}{pq}}\left({}^{CP}\mathcal{I}_\alpha(\eta_1^{\frac{1}{p}}(t)\eta_2^{\frac{1}{q}}(t))\right)^{\frac{1}{p}}.$$

□

Theorem 4.2. *Let $\alpha\in[0,1]$, let $p>1$, let $\frac{1}{p}+\frac{1}{q}=1$, and let η_1,η_2 be two positive functions on $[0,\infty)$, such that ${}^{CP}\mathcal{I}_\alpha\eta_1^p(t)<\infty$ and ${}^{CP}\mathcal{I}_\alpha\eta_2^p(t)<\infty$. If $0<\psi\le\frac{\eta_1(\tau)}{\eta_2(\tau)}\le\Psi$, for for all $\tau\in[0,t]$, then the following inequality holds:*

$${}^{CP}\mathcal{I}_\alpha\eta_1(t)\eta_2(t) \le \kappa_3\left({}^{CP}\mathcal{I}_\alpha(\eta_1^p+\eta_2^p)(t)\right)+\kappa_4\left({}^{CP}\mathcal{I}_\alpha(\eta_1^q+\eta_2^q)(t)\right), \tag{4.9}$$

with $\kappa_3=\dfrac{2^{p-1}\Psi^p}{p(\Psi+1)^p}$ *and* $\kappa_4=\dfrac{2^{q-1}}{q(\psi+1)^q}$.

Proof. Using the hypothesis, we have the following identity:

$$(\Psi+1)^p\eta_1^p(\tau) \le \Psi^p(\eta_1+\eta_2)^p(\tau). \tag{4.10}$$

Multiplying by $\dfrac{1}{K_0(\alpha)}\exp\left[-\dfrac{K_1(\alpha)}{K_0(\alpha)}(t-\tau)\right]$ both sides of (4.10) and integrating with respect to the variable τ, we get

$$\begin{aligned}&\frac{(\Psi+1)^p}{K_0(\alpha)}\int_0^t\exp\left[-\frac{K_1(\alpha)}{K_0(\alpha)}(t-\tau)\right]\eta_1^p(\tau)d(\tau)\\ &\le\frac{\Psi^p}{K_0(\alpha)}\int_0^t\exp\left[-\frac{K_1(\alpha)}{K_0(\alpha)}(t-\tau)\right](\eta_1+\eta_2)^p(\tau)d\tau.\end{aligned}$$

In this way, we have

$${}^{CP}\mathcal{I}_\alpha\eta_1^p(t) \le \frac{\Psi^p}{(\Psi+1)^p}\,{}^{CP}\mathcal{I}_\alpha(\eta_1+\eta_2)^p(t). \tag{4.11}$$

On the other hand, as $0<\psi<\frac{\eta_1(\tau)}{\eta_2(\tau)}$, $\tau\in(0,t)$, we have

$$(\psi+1)^q\eta_2^q(\tau) \le (\eta_1+\eta_2)^q(\tau). \tag{4.12}$$

Again, multiplying by $\frac{1}{K_0(\alpha)}\exp\left[-\frac{K_1(\alpha)}{K_0(\alpha)}(t-\tau)\right]$ both sides of (4.12) and integrating with respect to the variable τ, we get

$$ {}^{CP}\mathcal{I}_\alpha \eta_2^q(t) \leq \frac{1}{(\psi+1)^q}\, {}^{CP}\mathcal{I}_\alpha (\eta_1+\eta_2)^q(t). \tag{4.13}$$

Considering Young's inequality, [23]

$$\eta_1(\tau)\eta_2(\tau) \leq \frac{\eta_1^p(\tau)}{p} + \frac{\eta_2^q(\tau)}{q}, \tag{4.14}$$

multiplying by $\frac{1}{K_0(\alpha)}\exp\left[-\frac{K_1(\alpha)}{K_0(\alpha)}(t-\tau)\right]$ both sides of (4.14) and integrating with respect to the variable τ, we have

$$ {}^{CP}\mathcal{I}_\alpha(\eta_1\eta_2)(t) \leq \frac{1}{p}\left({}^{CP}\mathcal{I}_\alpha \eta_1^p(t)\right) + \frac{1}{q}\left({}^{CP}\mathcal{I}_\alpha \eta_2^q(t)\right). \tag{4.15}$$

Thus, using (4.11), (4.13) and (4.15), we get

$$\begin{aligned} {}^{CP}\mathcal{I}_\alpha(\eta_1\eta_2)(t) &\leq \frac{1}{p}\left({}^{CP}\mathcal{I}_\alpha \eta_1^p(t)\right) + \frac{1}{q}\left({}^{CP}\mathcal{I}_\alpha \eta_2^q(t)\right) \\ &\leq \frac{\Psi^p}{p(\Psi+1)^p}\left({}^{CP}\mathcal{I}_\alpha(\eta_1+\eta_2)^p(t)\right) \\ &\quad + \frac{1}{q(\psi+1)^q}\left({}^{CP}\mathcal{I}_\alpha(\eta_1+\eta_2)^q(t)\right). \end{aligned} \tag{4.16}$$

Using the following inequality, $(a+b)^r \leq 2^{p-1}(a^r+b^r)$, $r>1$, $a,b\geq 0$, we get

$$ {}^{CP}\mathcal{I}_\alpha(\eta_1+\eta_2)^p(t) \leq 2^{p-1}\, {}^{CP}\mathcal{I}_\alpha(\eta_1^p+\eta_2^p)(t) \tag{4.17}$$

and

$$ {}^{CP}\mathcal{I}_\alpha(\eta_1+\eta_2)^q(t) \leq 2^{q-1}\, {}^{CP}\mathcal{I}_\alpha(\eta_1^q+\eta_2^q)(t). \tag{4.18}$$

Thus, replacing (4.17) and (4.18) at (4.16), we conclude that

$$ {}^{CP}\mathcal{I}_\alpha(\eta_1\eta_2)(t) \leq \frac{2^{p-1}\Psi^p}{p(\Psi+1)^p}\left({}^{CP}\mathcal{I}_\alpha(\eta_1^p+\eta_2^p)(t)\right) + \frac{2^{q-1}}{q(\psi+1)^q}\left({}^{CP}\mathcal{I}_\alpha(\eta_1^q+\eta_2^q)(t)\right).$$

□

Theorem 4.3. *Let $\alpha\in[0,1]$, let $p\geq 1$, and let η_1,η_2 be two positive functions on $[0,\infty)$, such that ${}^{CP}\mathcal{I}_\alpha\eta_1^p(t)<\infty$ and ${}^{CP}\mathcal{I}_\alpha\eta_2^p(t)<\infty$. If $0<c<\psi\leq\frac{\eta_1(\tau)}{\eta_2(\tau)}\leq\Psi$, for for all $\tau\in[0,t]$, then the following inequality holds:*

$$\begin{aligned} \frac{\Psi+1}{\Psi-c}\left({}^{CP}\mathcal{I}_\alpha(\eta_1(t)-c\eta_2(t))\right)^{\frac{1}{p}} &\leq \left({}^{CP}\mathcal{I}_\alpha\eta_1^p(t)\right)^{\frac{1}{p}} + \left({}^{CP}\mathcal{I}_\alpha\eta_2^p(t)\right)^{\frac{1}{p}} \\ &\leq \frac{\psi+1}{\psi-c}\left({}^{CP}\mathcal{I}_\alpha(\eta_1(t)-c\eta_2(t))\right)^{\frac{1}{p}}. \end{aligned} \tag{4.19}$$

Proof. By the hypothesis $0<c<\psi\leq\Psi$, so

$$\psi c \leq \Psi c \Rightarrow \psi c+\psi \leq \psi c+\Psi \leq \Psi c+\Psi \Rightarrow (\Psi+1)(\psi-c)\leq(\psi+1)(\Psi-c).$$

Thus, we conclude that

$$\frac{\Psi+1}{\Psi-c} \leq \frac{\psi+1}{\psi-c}.$$

Also, we have

$$\psi - c \leq \frac{\eta_1(\tau) - c\eta_2(\tau)}{\eta_2(\tau)} \leq \Psi - c,$$

which implies

$$\frac{(\eta_1(\tau) - c\eta_2(\tau))^p}{(\Psi-c)^p} \leq \eta_2^p(\tau) \leq \frac{(\eta_1(\tau) - c\eta_2(\tau))^p}{(\psi-c)^p}. \tag{4.20}$$

Again, we have

$$\frac{1}{\Psi} \leq \frac{\eta_2(\tau)}{\eta_1(\tau)} \leq \frac{1}{\psi} \Rightarrow \frac{\psi-c}{c\psi} \leq \frac{\eta_1(\tau) - c\eta_2(\tau)}{c\eta_1(\tau)} \leq \frac{\Psi-c}{c\Psi},$$

which implies

$$\left(\frac{\Psi}{\Psi-c}\right)^p (\eta_1(\tau) - c\eta_2(\tau))^p \leq \eta_1^p(\tau) \leq \left(\frac{\psi}{\psi-c}\right)^p (\eta_1(\tau) - c\eta_2(\tau))^p. \tag{4.21}$$

Multiplying by $\frac{1}{K_0(\alpha)} \exp\left[-\frac{K_1(\alpha)}{K_0(\alpha)}(t-\tau)\right]$ both sides of (4.20) and integrating with respect to the variable τ, we have

$$\begin{aligned}
&\frac{1}{(\Psi-c)^p K_0(\alpha)} \int_0^t \exp\left[-\frac{K_1(\alpha)}{K_0(\alpha)}(t-\tau)\right] (\eta_1(\tau) - c\eta_2(\tau))^p d\tau \\
&\leq \frac{1}{K_0(\alpha)} \int_0^t \exp\left[-\frac{K_1(\alpha)}{K_0(\alpha)}(t-\tau)\right] \eta_2^p(\tau) d\tau \\
&\leq \frac{1}{(\psi-c)^p K_0(\alpha)} \int_0^t \exp\left[-\frac{K_1(\alpha)}{K_0(\alpha)}(t-\tau)\right] (\eta_1(\tau) - c\eta_2(\tau))^p d\tau.
\end{aligned}$$

In this way, we obtain

$$\begin{aligned}
\frac{1}{\Psi-c}\left({}^{CP}\mathcal{I}_\alpha(\eta_1(t) - c\eta_2(t))^p\right)^{\frac{1}{p}} &\leq \left({}^{CP}\mathcal{I}_\alpha \eta_2^p(t)\right)^{\frac{1}{p}} \\
&\leq \frac{1}{\psi-c}\left({}^{CP}\mathcal{I}_\alpha(\eta_1(t) - c\eta_2(t))^p\right)^{\frac{1}{p}}. \quad (4.22)
\end{aligned}$$

Realizing the same procedure as in (4.21), we have

$$\begin{aligned}
\frac{\Psi}{\Psi-c}\left({}^{CP}\mathcal{I}_\alpha(\eta_1(t) - c\eta_2(t))^p\right)^{\frac{1}{p}} &\leq \left({}^{CP}\mathcal{I}_\alpha \eta_1^p(t)\right)^{\frac{1}{p}} \\
&\leq \frac{\psi}{\psi-c}\left({}^{CP}\mathcal{I}_\alpha(\eta_1(t) - c\eta_2(t))^p\right)^{\frac{1}{p}}. \quad (4.23)
\end{aligned}$$

Adding (4.22) and (4.23), we conclude that

$$\begin{aligned}
\frac{\Psi+1}{\Psi-c}\left({}^{CP}\mathcal{I}_\alpha(\eta_1(t) - c\eta_2(t))^p\right)^{\frac{1}{p}} &\leq \left({}^{CP}\mathcal{I}_\alpha \eta_1^p(t)\right)^{\frac{1}{p}} + \left({}^{CP}\mathcal{I}_\alpha \eta_2^p(t)\right)^{\frac{1}{p}} \\
&\leq \frac{\psi+1}{\psi-c}\left({}^{CP}\mathcal{I}_\alpha(\eta_1(t) - c\eta_2(t))^p\right)^{\frac{1}{p}}.
\end{aligned}$$

□

Theorem 4.4. *Let $\alpha \in [0,1]$, let $p \geq 1$, and let η_1, η_2 be two positive functions on $[0,\infty)$, such that ${}^{CP}\mathcal{I}_\alpha \eta_1^p(t) < \infty$ and ${}^{CP}\mathcal{I}_\alpha \eta_2^p(t) < \infty$. If $0 \leq a \leq \eta_1(\tau) \leq \mathcal{A}$ and $0 \leq b \leq \eta_2(\tau) \leq \mathcal{B}$ for for all $\tau \in [0,t]$, then the following inequality holds:*

$$\left({}^{CP}\mathcal{I}_\alpha \eta_1^p(t)\right)^{\frac{1}{p}} + \left({}^{CP}\mathcal{I}_\alpha \eta_2^p(t)\right)^{\frac{1}{p}} \leq \kappa_5 \left({}^{CP}\mathcal{I}_\alpha (\eta_1+\eta_2)^p(t)\right)^{\frac{1}{p}}, \tag{4.24}$$

with $\kappa_5 = \dfrac{\mathcal{A}(a+\mathcal{B}) + \mathcal{B}(\mathcal{A}+b)}{(\mathcal{A}+b)(a+\mathcal{B})}$.

Proof. By hypothesis, it follows that

$$\frac{1}{\mathcal{B}} \leq \frac{1}{\eta_2(\tau)} \leq \frac{1}{b}. \tag{4.25}$$

Realizing the product between (4.25) and $0 \leq a \leq \eta_1(\tau) \leq \mathcal{A}$, we have

$$\frac{a}{\mathcal{B}} \leq \frac{\eta_1(\tau)}{\eta_2(\tau)} \leq \frac{\mathcal{A}}{b}. \tag{4.26}$$

From (4.26), we get

$$\eta_2^p(\tau) \leq \left(\frac{\mathcal{B}}{a+\mathcal{B}}\right)^p (\eta_1(\tau)+\eta_2(\tau))^p \tag{4.27}$$

and

$$\eta_1^p(\tau) \leq \left(\frac{\mathcal{A}}{b+\mathcal{A}}\right)^p (\eta_1(\tau)+\eta_2(\tau))^p. \tag{4.28}$$

Multiplying by $\dfrac{1}{K_0(\alpha)} \exp\left[-\dfrac{K_1(\alpha)}{K_0(\alpha)}(t-\tau)\right]$ both sides of (4.27) and integrating with respect to the variable τ, we have

$$\begin{aligned}
&\frac{1}{K_0(\alpha)} \int_0^t \exp\left[-\frac{K_1(\alpha)}{K_0(\alpha)}(t-\tau)\right] \eta_2^p(\tau) d\tau \\
&\leq \left(\frac{\mathcal{B}}{a+\mathcal{B}}\right)^p \frac{1}{K_0(\alpha)} \int_0^t \exp\left[-\frac{K_1(\alpha)}{K_0(\alpha)}(t-\tau)\right] (\eta_1(\tau)+\eta_2(\tau))^p d\tau.
\end{aligned}$$

Thus, it follows that

$$\left({}^{CP}\mathcal{I}_\alpha \eta_2^p(t)\right)^{\frac{1}{p}} \leq \frac{\mathcal{B}}{a+\mathcal{B}} \left({}^{CP}\mathcal{I}_\alpha (\eta_1+\eta_2)^p(t)\right)^{\frac{1}{p}}. \tag{4.29}$$

Similarly, we perform the calculations for (4.28), we get

$$\left({}^{CP}\mathcal{I}_\alpha \eta_1^p(t)\right)^{\frac{1}{p}} \leq \frac{\mathcal{A}}{b+\mathcal{A}} \left({}^{CP}\mathcal{I}_\alpha (\eta_1+\eta_2)^p(t)\right)^{\frac{1}{p}}. \tag{4.30}$$

Adding (4.29) and (4.30), we conclude that

$$\left({}^{CP}\mathcal{I}_\alpha \eta_1^p(t)\right)^{\frac{1}{p}} + \left({}^{CP}\mathcal{I}_\alpha \eta_2^p(t)\right)^{\frac{1}{p}} \leq \frac{\mathcal{A}(a+\mathcal{B}) + \mathcal{B}(b+\mathcal{A})}{(a+\mathcal{B})(b+\mathcal{A})} \left({}^{CP}\mathcal{I}_\alpha (\eta_1+\eta_2)^p(t)\right)^{\frac{1}{p}}.$$

□

Theorem 4.5. *Let $\alpha \in [0,1]$, let $p \geq 1$, and let η_1, η_2 be two positive functions on $[0,\infty)$, such that ${}^{CP}\mathcal{I}_\alpha \eta_1^p(t) < \infty$ and ${}^{CP}\mathcal{I}_\alpha \eta_2^p(t) < \infty$. If $0 < \psi \leq \frac{\eta_1(\tau)}{\eta_2(\tau)} \leq \Psi$, for for all $\tau \in [0,t]$, then*

$$\frac{1}{\Psi}\left({}^{CP}\mathcal{I}_\alpha \eta_1(t)\eta_2(t)\right) \leq \frac{1}{(\psi+1)(\Psi+1)}\left({}^{CP}\mathcal{I}_\alpha (\eta_1+\eta_2)^2(t)\right) \leq \frac{1}{\psi}\left({}^{CP}\mathcal{I}_\alpha \eta_1(t)\eta_2(t)\right). \tag{4.31}$$

Proof. Being $0 < \psi \leq \frac{\eta_1(\tau)}{\eta_2(\tau)} \leq \Psi$, we have

$$\eta_2(\tau)(\psi+1) \leq \eta_2(\tau) + \eta_1(\tau) \leq \eta_2(\tau)(\Psi+1). \tag{4.32}$$

Also, it follows that $\frac{1}{\Psi} \leq \frac{\eta_2(\tau)}{\eta_1(\tau)} \leq \frac{1}{\psi}$, which implies

$$\eta_1(\tau)\left(\frac{\Psi+1}{\Psi}\right) \leq \eta_2(\tau) + \eta_1(\tau) \leq \eta_1(\tau)\left(\frac{\psi+1}{\psi}\right). \tag{4.33}$$

Evaluating the product between (4.32) and (4.33), we have

$$\frac{\eta_1(\tau)\eta_2(\tau)}{\Psi} \leq \frac{(\eta_2(\tau)+\eta_1(\tau))^2}{(\psi+1)(\Psi+1)} \leq \frac{\eta_1(\tau)\eta_2(\tau)}{\psi}. \tag{4.34}$$

Multiplying by $\dfrac{1}{K_0(\alpha)} \exp\left[-\dfrac{K_1(\alpha)}{K_0(\alpha)}(t-\tau)\right]$ both sides of (4.34) and integrating with respect to the variable τ, we have

$$\begin{aligned}
&\frac{1}{\Psi K_0(\alpha)}\int_0^t \exp\left[-\frac{K_1(\alpha)}{K_0(\alpha)}(t-\tau)\right]\eta_1(\tau)\eta_2(\tau)d\tau \\
&\leq \frac{1}{(\psi+1)(\Psi+1)K_0(\alpha)}\int_0^t \exp\left[-\frac{K_1(\alpha)}{K_0(\alpha)}(t-\tau)\right](\eta_1(\tau)+\eta_2(\tau))^2 d\tau \\
&\leq \frac{1}{\psi K_0(\alpha)}\int_0^t \exp\left[-\frac{K_1(\alpha)}{K_0(\alpha)}(t-\tau)\right]\eta_1(\tau)\eta_2(\tau)d\tau.
\end{aligned}$$

Thus, we conclude that

$$\frac{1}{\Psi}\left({}^{CP}\mathcal{I}_\alpha \eta_1(t)\eta_2(t)\right) \leq \frac{1}{(\psi+1)(\Psi+1)}\left({}^{CP}\mathcal{I}_\alpha (\eta_1+\eta_2)^2(t)\right) \leq \frac{1}{\psi}\left({}^{CP}\mathcal{I}_\alpha \eta_1(t)\eta_2(t)\right).$$

□

Theorem 4.6. *Let $\alpha \in [0,1]$, let $p \geq 1$, and let η_1, η_2 be two positive functions on $[0,\infty)$, such that ${}^{CP}\mathcal{I}_\alpha \eta_1^p(t) < \infty$ and ${}^{CP}\mathcal{I}_\alpha \eta_2^p(t) < \infty$. If $0 < \psi \leq \frac{\eta_1(\tau)}{\eta_2(\tau)} \leq \Psi$, for for all $\tau \in [0,t]$, then the following inequality holds:*

$$\left({}^{CP}\mathcal{I}_\alpha \eta_1^p(t)\right)^{\frac{1}{p}} + \left({}^{CP}\mathcal{I}_\alpha \eta_2^p(t)\right)^{\frac{1}{p}} \leq 2\left({}^{CP}\mathcal{I}_\alpha \eta_3^p(\eta_1(t),\eta_2(t))\right)^{\frac{1}{p}},$$

where $\eta_3(\eta_1(\tau),\eta_2(\tau)) = \max\left\{\Psi\left[\left(\frac{\Psi}{\psi}+1\right)\eta_1(\tau) - \Psi\eta_2(\tau)\right], \frac{(\Psi+\psi)\eta_2(\tau)-\eta_1(\tau)}{\psi}\right\}$.

Proof. From the hypothesis $0 < \psi \leq \frac{\eta_1(\tau)}{\eta_2(\tau)} \leq \Psi$, for all $\tau \in [0,t]$, we have

$$0 < \psi \leq \Psi + \psi - \frac{\eta_1(\tau)}{\eta_2(\tau)} \tag{4.35}$$

and

$$\Psi + \psi - \frac{\eta_1(\tau)}{\eta_2(\tau)} \leq \Psi. \tag{4.36}$$

Thus, using (4.35) and (4.36), we get

$$\eta_2(\tau) < \frac{(\Psi + \psi)\eta_2(\tau) - \eta_1(\tau)}{\psi} \leq \eta_3(\eta_1(\tau), \eta_2(\tau)), \tag{4.37}$$

where $\eta_3(\eta_1(\tau), \eta_2(\tau)) = \max\left\{\Psi\left[\left(\frac{\Psi}{\psi} + 1\right)\eta_1(\tau) - \Psi\eta_2(\tau)\right], \frac{(\Psi+\psi)\eta_2(\tau)-\eta_1(\tau)}{\psi}\right\}$. Using the hypothesis, it follows that $0 < \frac{1}{\Psi} \leq \frac{\eta_2(\tau)}{\eta_1(\tau)} \leq \frac{1}{\psi}$. In this way, we obtain

$$\frac{1}{\Psi} \leq \frac{1}{\Psi} + \frac{1}{\psi} - \frac{\eta_2(\tau)}{\eta_1(\tau)} \tag{4.38}$$

and

$$\frac{1}{\Psi} + \frac{1}{\psi} - \frac{\eta_2(\tau)}{\eta_1(\tau)} \leq \frac{1}{\psi}. \tag{4.39}$$

Then, from (4.38) and (4.39), we have

$$\frac{1}{\Psi} \leq \frac{\left(\frac{1}{\psi} + \frac{1}{\Psi}\right)\eta_1(\tau) - \eta_2(\tau)}{\eta_1(\tau)} \leq \frac{1}{\psi},$$

which can be rewritten as

$$\begin{aligned}
\eta_1(\tau) &\leq \Psi\left(\frac{1}{\psi} + \frac{1}{\Psi}\right)\eta_1(\tau) - \Psi\eta_2(\tau) \\
&= \frac{\Psi(\Psi+\psi)\eta_1(\tau) - \Psi^2\psi\eta_2(\tau)}{\Psi\psi} \\
&= \left(\frac{\Psi}{\psi} + 1\right)\eta_1(\tau) - \Psi\eta_2(\tau) \\
&= \Psi\left[\left(\frac{\Psi}{\psi} + 1\right)\eta_1(\tau) - \Psi\eta_2(\tau)\right] \\
&\leq \eta_3(\eta_1(\tau), \eta_2(\tau)).
\end{aligned} \tag{4.40}$$

Thus, using (4.37) and (4.40), we can write

$$\eta_1^p(\tau) \leq \eta_3^p(\eta_1(\tau), \eta_2(\tau)) \tag{4.41}$$

and

$$\eta_2^p(\tau) \leq \eta_3^p(\eta_1(\tau), \eta_2(\tau)). \tag{4.42}$$

Multiplying by $\dfrac{1}{K_0(\alpha)} \exp\left[-\dfrac{K_1(\alpha)}{K_0(\alpha)}(t-\tau)\right]$ both sides of (4.41) and integrating with respect to the variable τ, we have

$$\begin{aligned}
&\frac{1}{K_0(\alpha)} \int_0^t \exp\left[-\frac{K_1(\alpha)}{K_0(\alpha)}(t-\tau)\right] \eta_1^p(\tau) d\tau \\
&\leq \frac{1}{K_0(\alpha)} \int_0^t \exp\left[-\frac{K_1(\alpha)}{K_0(\alpha)}(t-\tau)\right] \eta_3^p(\eta_1(\tau), \eta_2(\tau)) d\tau.
\end{aligned}$$

In this way, we obtain

$$\left({}^{CP}\mathcal{I}_\alpha \eta_1^p(t)\right)^{\frac{1}{p}} \leq \left({}^{CP}\mathcal{I}_\alpha \eta_3^p(\eta_1(t), \eta_2(t))\right)^{\frac{1}{p}}. \tag{4.43}$$

Using the same procedure as above, for (4.42), we have

$$\left({}^{CP}\mathcal{I}_\alpha \eta_2^p(t)\right)^{\frac{1}{p}} \leq \left({}^{CP}\mathcal{I}_\alpha \eta_3^p(\eta_1(t), \eta_2(t))\right)^{\frac{1}{p}}. \tag{4.44}$$

Thus, using (4.43) and (4.44), we conclude that

$$\left({}^{CP}\mathcal{I}_\alpha \eta_1^p(t)\right)^{\frac{1}{p}} + \left({}^{CP}\mathcal{I}_\alpha \eta_2^p(t)\right)^{\frac{1}{p}} \leq 2\left({}^{CP}\mathcal{I}_\alpha \eta_3^p(\eta_1(t), \eta_2(t))\right)^{\frac{1}{p}}.$$

□

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[1] Department of Mathematics, Faculty of Science and Arts, Ordu University, Ordu, Turkey

Email address: `bariscelik15@hotmail.com`

[2] Department of Mathematics, Faculty of Science and Arts, Ordu University, Ordu, Turkey
Email address: erhanset@yahoo.com

[3] Department of Mathematics, Faculty of Arts and Sciences, Ağri İbrahim Çeçen University, Ağri, Turkey
Email address: aocakakdemir@gmail.com

[4] International Center for Basic and Applied Sciences, Jaipur, 302029, India

[5] Department of Mathematics, Anand International College of Engineering, Jaipur, India
Email address: praveen.agarwal@anandice.ac.in