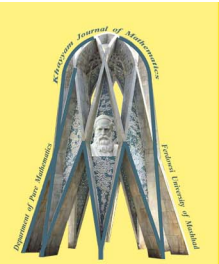
Khayyam J. Math. 10 (2024), no. 2, 309-326
DOI: 10.22034/kjm.2024.459109.3212



Khayyam Journal of Mathematics
emis.de/journals/KJM
kjm-math.org

ON THE BOUNDEDNESS OF SEMICLASSICAL PSEUDODIFFERENTIAL OPERATORS ON TRIEBEL–LIZORKIN SPACES

OMAR FAROUK AID[1], CHAHRAZED HARRAT[1,2] AND ABDERRAHMANE SENOUSSAOUI[1*]

Communicated by B. Kuzma

Abstract. In this paper, we study a specific class of h-pseudo-differential operators with Hörmander's symbol. This class of operators can be extended as a bounded linear mapping on Triebel–Lizorkin spaces.

1. Introduction

When solving linear partial differential equations, which frequently arise in this field of research, pseudo-differential operators are helpful tools. For a function $u \in \mathcal{S}(\mathbb{R}^d)$, a pseudo-differential operator can be written locally in the form

$$I_a u(x) = \iint_{\mathbb{R}^{2d}} e^{i\langle x-y,\xi\rangle} a(x,\xi) u(y) dy d\xi,$$

where a is a symbol. The most widely used class of symbols is those, created by Hormander in [10], named $S^m_{\rho,\delta}$ class (with $m \in \mathbb{R}$ and $0 \leq \rho, \delta \leq 1$) that consists of elements $a \in \mathcal{C}^\infty(\mathbb{R}^{2d})$ such that for any multi-indices $\alpha, \beta \in \mathbb{N}^d$, there exists $C_{\alpha,\beta} > 0$,

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C_{\alpha,\beta} \langle \xi \rangle^{m+\delta|\alpha|-\rho|\beta|},$$

where

$$\langle \xi \rangle := \left(1 + |\xi|^2\right)^{1/2}.$$

Date: Received: 26 May 2024; Revised: 4 October 2024; Accepted: 6 October 2024.
* Corresponding author.
2020 *Mathematics Subject Classification.* Primary 35S30; Secondary 35S05, 47G30.
Key words and phrases. Pseudodifferential operators, Hardy-Littlewood maximal operator and Triebel–Lizorkin spaces.

Research on the boundedness of these operators has been done in great detail; see [2, 4, 6, 7, 14, 21, 17].

The regularity of pseudo-differential operators defined by symbols $a(x,\xi)$ without any regularity assumptions in the x variable was initiated by Kenig and Staubach [11]. After then, this kind of operators was investigated by many authors; see [1, 3, 5, 17, 16, 18]. We would mention that the boundedness on Triebel–Lizorkin spaces was established by Park [15].

In the semiclassical case, an h-pseudo-differential operator has the following form:

$$Op_h(a)u(x) = (2\pi h)^{-n} \iint e^{ih^{-1}\langle x-y,\xi\rangle} a(x,y,\xi)u(y)dyd\xi, \quad u \in \mathcal{S}(\mathbb{R}^n), \tag{1.1}$$

where $h \in]0,h_0]$ is a semiclassical parameter.

A fundamental principle of h-pseudo-differential operator is that linear differential operators, known as semiclassical Schrodinger operators, are used to represent quantum systems including atoms, molecules, solids, and, to a certain extent, nuclei and even stars. For a description of this theory, we point the reader toward the widely used text of Landau and Lifshitz [12]. The Schrodinger operator for a quantum system is the linear partial differential operator

$$H = -\frac{h^2}{2m}\Delta + V.$$

The Laplace operator Δ is the second-order differential operator that, in Cartesian coordinates on $\mathbb{R}^n$, is given by

$$\Delta = \sum_{k=0}^{n} \frac{\partial^2}{\partial x_i^2}.$$

The constant m is the reduced mass of the system, and the semiclassical parameter h is called Planck's constant. The real-valued function V is called the potential.

The aim of this work is to extend results obtained in for a general class of semiclassical Fourier integral operators. Our investigation is to study the boundedness of h-FIOs on Lebesgue spaces with the same conditions on the phase functions and on the symbols are kept.

The aim of this work is to extend results obtained in [15] for a general class of operators $Op_h(a)$. Our investigation is to study the boundedness of $Op_h(a)$ on Triebel–Lizorkin spaces with the some conditions on the symbols are kept.

Let us now describe the plan of this article. In the second section we introduce the relevant notations and preliminaries that will be used throughout the paper.

Following that, we will go over some definitions and remarks concerning Triebel–Lizorkin spaces, which will serve as the starting point for our research. Finally, the last section is devoted to proving our main result.

2. Preliminaries

2.1. **Notations.** Throughout this paper, we denote the set of all natural numbers without 0 by $\mathbb{N}$, and $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$. Additionally, we consider $d \in \mathbb{N}$ in this paper

unless otherwise noted. For all $x, y, \xi \in \mathbb{R}^d$, we define

$$\langle x, \xi \rangle := \sum_{j=0}^{d} x_j \xi_j \quad \textit{and} \quad \widehat{d_h \xi} := (2\pi h)^{-n} d\xi.$$

First, we define the semiclassical Fourier transform $\mathcal{F}_h$ and its inverse $\mathcal{F}_h^{-1}$ by

$$\mathcal{F}_h u(\xi) := \int_{\mathbb{R}^d} e^{-ih^{-1}\langle x,\xi\rangle} u(x)\, dx, \quad \textit{and} \quad \mathcal{F}_h^{-1} u(x) := \int_{\mathbb{R}^d} e^{ih^{-1}\langle x,\xi\rangle} u(\xi)\, \widehat{d_h \xi},$$

where $u \in \mathcal{S}(\mathbb{R}^d)$. We scale partial derivatives with respect to a variable $x \in \mathbb{R}^d$ with the factor $-i$ are denoted by

$$D_x^\alpha := (-i)^{|\alpha|} \partial_x^\alpha := (-i)^{|\alpha|} \partial_{x_1}^{\alpha_1} \dots \partial_{x_d}^{\alpha_d},$$

where $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$ is a multi-index and $|\alpha| = \sum_{j=1}^d \alpha_j$ is the length of α.

Considering two Freshet spaces E and F, the set $\mathcal{L}(E, F)$ contains of all bounded linear operators $L : E \to F$. If $E = F$, then we also just write $\mathcal{L}(E)$.

Furthermore, we note by $B(x, r)$ the Euclidean ball centered at x with radius $r > 0$, and by $m\left(B(x, r)\right)$ the Lebesgue measure of $B(x, r)$.

For two nonnegative quantities A and B, the notation $A \lesssim B$ means that $A \leq CB$ for some unspecified constant $C > 0$.

Finally, let us recall the Hardy–Littlewood maximal operator $\mathcal{M}$ defined by

$$\mathcal{M}u(x) = \sup_{r>0} \frac{1}{m\left(B(x,r)\right)} \int_{B(x,r)} |u(y)| dy, \quad u \in L^1_{loc}(\mathbb{R}^d).$$

2.2. **Triebel–Lizorkin spaces.** Let Φ be a set of functions $\{\varphi_k\}$ on $\mathbb{R}^d$ satisfying

- $\varphi_k \in \mathcal{C}^\infty(\mathbb{R}^d)$ for all $k \in \mathbb{N}$,
- $Supp\left(\varphi_0\right) \subset \left\{x : |x| \leq 2\right\}$,
- $Supp\left(\varphi_k\right) \subset \left\{x : 2^{k-1} \leq |x| \leq 2^{k+1}\right\}$ for all $k \in \mathbb{N}$,
- $\sum_{k=0}^\infty \varphi_k(x) = 1$ for all $x \in \mathbb{R}^d$,
- $\sup_{x\in\mathbb{R}^d} \sup_{k\in\mathbb{N}_0} 2^{k|\alpha|} \left|\partial^\alpha \varphi_k(x)\right| < \infty$ for all $\alpha \in \mathbb{N}^d$.

Definition 2.1. Let $s \in \mathbb{R}$ and let $0 < p, q \leq \infty$. Then the Triebel–Lizorkin space $F_p^{s,q}(\mathbb{R}^d)$ is defined by

$$F_p^{s,q} = \left\{ u \in \mathcal{S}'\left(\mathbb{R}^d\right) : \|u\|_{F_p^{s,q}} < \infty \right\}, \tag{2.1}$$

where

$$\|u\|_{F_p^{s,q}} = \left\| \left\{ 2^{ks} \mathcal{F}^{-1}\left(\varphi_k \widehat{u}\right) \right\} \right\|_{L^P(l^q)} \quad \text{if } p < \infty,$$

and

$$\|u\|_{F_\infty^{s,q}} = \left\| \mathcal{F}^{-1}\left(\varphi_0 \widehat{u}\right) \right\|_{L^\infty} + \sup_{Q : l(Q) < 1} \left(\frac{1}{|Q|} \int_Q \sum_{k=-\log_2 l(Q)} \left| 2^{ks} \mathcal{F}^{-1}\left(\varphi_k \widehat{u}\right)(x) \right|^q dx \right)^{\frac{1}{q}},$$

where the supremum is taken over all dyadic cubes Q of side-length $l(Q) < 1$.

Here the exponent s is the order of $F_p^{s,q}$, q is called summation exponent, and p is called integration exponent.

Remark 2.2. As shown in [20, 2.3.2] the Triebel–Lizorkin spaces do not depend on the choice of Φ.

In what follows, we fix such a system and use a notation $\phi_k(x) = \mathcal{F}^{-1}(\varphi_k)(x)$, where $\{\phi_k\}_{k\in\mathbb{N}_0}$ is a Littlewood–Paley partition of unity. That is, ϕ_0 and ϕ are smooth functions satisfying

- $Supp\left(\widehat{\phi_0}\right) \subset \{\xi : |\xi| \leq 2\}$,
- $Supp\left(\widehat{\phi}\right) \subset \{\xi : 2^{-1} \leq |\xi| \leq 2\}$,
- $\sum_{k=0}^{\infty} \widehat{\phi_k}(\xi) = 1$ for $\xi \in \mathbb{R}^d$, where

$$\phi_k(x) = 2^{kd}\phi(2^k x) \quad \text{for all } k \geq 1.$$

Then the (quasi-)norms on the spaces are

$$\|u\|_{F_p^{s,q}} = \left\| \left(\sum_{k=0}^{\infty} \left(2^{sk} |\phi_k * u|\right)^q \right)^{\frac{1}{q}} \right\|_{L^p}, \quad p < \infty$$

and

$$\|u\|_{F_\infty^{s,q}} = \|\Gamma_0 u\|_{L^\infty} + \sup_{l(P)<1} \left(\frac{1}{|P|} \int_P \sum_{k=-\log_2 l(P)} 2^{ks} |\phi_k * u(x)|^q \, dx \right)^{\frac{1}{q}}.$$

According to those norms, the spaces are quasi-Banach spaces (Banach spaces if $p \geq 1, q \geq 1$).

Lemma 2.3. [19, Lemma 2.1] *and* [22, Theorem 3.6]. *Let $A > 0$, let $s \in \mathbb{R}$, let $0 < p < \infty$, $0 < q \leq \infty$, and let $\{u_j\}_{j=0}^{\infty}$ be a sequence in $S'(\mathbb{R}^d)$ satisfying $Supp\,(\widehat{u_0}) \subset \{\xi : |\xi| \leq A\}$ and $Supp\,(\widehat{u_j}) \subset \{\xi : 2^j A^{-1} \leq |\xi| \leq 2^j A\}$ for $j \geq 1$.*

If $\left\|\{2^{s_j} u_j\}_{j=0}^{\infty}\right\|_{l^q(L_p)} < \infty$, then $\sum_{j=0}^{\infty} u_j$ converges in $\mathcal{S}'(\mathbb{R}^d)$ to some $u \in F_p^{s,q}$ and

$$\|u\|_{F_p^{s,q}} \precsim_A \left\|\{2^{s_j} u_j\}_{j=0}^{\infty}\right\|_{l^q(L_p)}.$$

Lemma 2.4. [19, 13] *Let $A > 0$ and let $v \in \mathcal{S}'(\mathbb{R}^d)$ and $b(x,\xi)$ be a function on $\mathbb{R}^d \times \mathbb{R}^d$ such that $Supp\,(\widehat{v}) \subset \left\{\xi : |\xi| \leq 2^k A\right\}$ and $b(x,\xi) = 0$ for $|\xi| > 2^k A$. Then for all $0 < r \leq 1$,*

$$|T_b v(x)| \precsim_A \left\|b\left(x, 2^k \cdot\right)\right\|_{W^{c^{(r,d)},1}} \mathcal{M}_r v(x),$$

where $c^{(r,d)}$ is the smallest integer greater than $\frac{d}{r}$ and $W^{c^{(r,d)},1}$ is the Sobolev space with

$$\|u\|_{W^{c^{(r,d)},1}} = \sum_{|\alpha| \leq c^{(r,d)}} \|\partial^\alpha u\|_{L^1}.$$

Definition 2.5. Let $0 < p \leq 1$, $0 < q \leq \infty$, and let $s \in \mathbb{R}$. A sequence of complex numbers $r = \{r_Q\}_{\substack{Q\in\mathcal{D}\\ l(Q)\leq 1}}$ is called an $\infty - atom$ for $F_p^{s,q}$ if there exists a dyadic cube Q_0 such that

$$r_Q = 0 \text{ if } \ Q \nsubseteq Q_0$$

and

$$\|g^{s,q}(r)\|_{L^\infty} \leq |Q_0|^{\frac{-1}{p}}. \tag{2.2}$$

2.3. **A general class of symbols.**

Definition 2.6. Let m, ρ, and δ be real numbers; $\rho, \delta \in [0,1]$. Then the space of symbols $S^m_{\rho,\delta}\left(\mathbb{R}^{2d}\right)$ is the set of all smooth functions $a : \mathbb{R}^{2d} \to \mathbb{C}$ such that

$$\left|\partial_\xi^\alpha \partial_x^\beta a\left(x,\xi\right)\right| \leq C_{\alpha,\beta}\langle\xi\rangle^{\mu-\rho|\alpha|+\delta|\beta|} \tag{2.3}$$

uniformly in $x, \xi \in \mathbb{R}^d$ for all $\alpha, \beta \in \mathbb{N}_0^d$. Moreover, let

$$|a|_k^{\mu,\rho} := \max_{|\alpha|+|\beta|\leq k} \sup_{x,\xi\in\mathbb{R}^d} \langle\xi\rangle^{-\mu+\rho|\alpha|-\delta|\beta|}\left|\partial_\xi^\alpha \partial_x^\beta a\left(x,\xi\right)\right|, \quad k \in \mathbb{N},$$

be the associated sequence of semi-norms.

Remark 2.7. For short we write $S^m_{\rho,\delta}$ instead of $S^m_{\rho,\delta}\left(\mathbb{R}^{2d}\right)$.

Proposition 2.8. *Let $m_1, m_2 \in \mathbb{R}$ and let $\rho, \delta \in [0,1]$*

(1) *if $a \in S^{m_1}_{\rho,\delta}$, then $\partial_\xi^\alpha \partial_x^\beta a \in S^{m_1-\rho|\alpha|+\delta|\beta|}_{\rho,\delta}$;*
(2) *if $a \in S^{m_1}_{\rho,\delta}$ and $b \in S^{m_2}_{\rho,\delta}$, then $ab \in S^{m_1+m_2}_{\rho,\delta}$.*

Proof. The proof is based on Leibniz's formula. □

Now, we take an interest in giving a sense of the integral of type

$$\left[Op_h\left(a\right)u\right](x) = \iint_{\mathbb{R}^{2d}} e^{ih^{-1}\langle x-y,\xi\rangle} a\left(x,\xi\right)u\left(y\right)dy\widehat{d_h\xi}, \quad u \in \mathcal{S}\left(\mathbb{R}^d\right), \tag{2.4}$$

where $h \in]0, h_0]$ and $a \in S^m_{\rho,\delta}$.

3. The boundedness of $Op_h\left(a\right)$ on Triebel–Lizorkin spaces

Theorem 3.1. *Let $0 < \rho < 1$, $0 < p < \infty, 0 < q, t \leq \infty$, and $s_1, s_2 \in \mathbb{R}$. Suppose that $m \in \mathbb{R}$ satisfies*

$$m - s_1 + s_2 \leq -d(1-\rho)\left|\frac{1}{2} - \frac{1}{p}\right| \tag{3.1}$$

and $a \in S^m_{\rho,\rho}$. Then $Op_h\left(a\right)$ maps $F_p^{s_1,q}$ into $F_p^{s_2,t}$ in the following cases:

(1) *if $m - s_1 + s_2 < -d(1-\rho)\left|\frac{1}{2} - \frac{1}{p}\right|$,*
(2) *if $p = 2$, $q \leq 2 \leq t$, and $m = 0$,*
(3) *if $0 < p < 2$, $p \leq t \leq \infty, 0 < q \leq \infty$, and $m - s_1 + s_2 = -d(1-\rho)\left(\frac{1}{p} - \frac{1}{2}\right)$,*
(4) *if $2 < p < \infty, 0 < t \leq \infty$, $0 < q \leq p$, and $m - s_1 + s_2 = -d(1-\rho)\left(\frac{1}{2} - \frac{1}{p}\right)$.*

Proof. The first statement of theorem is simply from Holder's inequality and the embedding theorem $L^{p_1} \subset L^{p_2}$ when $p_1 \leq p_2$. The second one is an immediate consequence of (2.3). Thus we shall prove just the third and the last statement. Assume

$$m - s_1 + s_2 \leq -d(1-\rho)\left|\frac{1}{2} - \frac{1}{p}\right| \tag{3.2}$$

and $p \neq 2$.

It suffices to show that if $a(x,\xi)$ has compact support in x variable. Then

$$\|Op_h(a) f\|_{F_p^{s_2,t}} \leq C \|f\|_{F_p^{s_1,q}},$$

where a constant C is independent of the compact support. To be specific pick a smooth function γ, which is identically 1 near a neighborhood of the origin and compactly supported. Then

$$a^\tau(x,\xi) = a(x,\xi)\gamma(\frac{x}{2^\tau}),\ \tau > 0 \tag{3.3}$$

belongs to $S^m_{\rho,\rho}$ uniformly in τ. For fixed $x \in \mathbb{R}^d$,

$$|\phi_k * Op_h(a) f(x) - \phi_k * Op_h(a^\tau) f(x)| = \left|Op_h(a) f(\phi_k(x-\cdot)(1-\gamma(\frac{\cdot}{2^\tau})))\right|.$$

Since $Op_h(a) f \in \mathcal{S}'(\mathbb{R}^d)$, this is bounded by

$$C \sum_{|\alpha|\leq N} \left\|(1+|\cdot|)^L \partial^\alpha \left[\phi_k(x-\cdot)(1-\gamma(\frac{\cdot}{2^\tau}))\right]\right\|_{L^\infty} \tag{3.4}$$

for some constants C, L, $N > 0$, independent of k, τ, and x. For τ sufficiently large so that $|x| < 100 \cdot 2^\tau$, (3.4) is less than $C_{N,M} 2^{-k(M-N-d)} 2^{-\tau(M-L)}$ for $M > N$ because $(1-\gamma(\frac{y}{2^\tau}))$ and any derivatives of it vanish on $\{y; |y| \lesssim 2^\tau\}$. This gives

$$\left\|\left\{2^{sk}\left(\phi_k * Op_h(a) f(x) - \phi_k * Op_h(a^\tau) f(x)\right)\right\}\right\|_{l^q(\mathbb{Z})} \lesssim C_{N,M} 2^{-\tau(M-L)}$$

for sufficiently large $M > 0$, and thus we obtain

$$\lim_{\tau\to\infty} \left\|\left\{2^{sk}\phi_k * Op_h(a^\tau) f(x)\right\}\right\|_{l^q(\mathbb{Z})} = \left\|\left\{2^{sk}\phi_k * Op_h(a) f(x)\right\}\right\|_{l^q(\mathbb{Z})}$$

for fixed $x \in \mathbb{R}^d$. Once $Op_h(a^\tau)$ is a bounded operator, then

$$\begin{aligned} \|Op_h(a) f\|_{F_p^{s,q}} &\leq \liminf_{\tau\to\infty} \|Op_h(a^\tau) f\|_{F_p^{s,q}} \\ &\lesssim \|f\|_{F_p^{s,q}} \end{aligned}$$

by Fatou's lemma.

Therefore we may assume that $a(x,\xi)$ has a compact support in x variable. Since each estimate after this will be done independently of the compact support indeed, we require this compact support assumption just to do integration by part in order to acquire (3.11) the uniformity would be ensured.

We may also assume $s_1 = s_2 = 0$ due to the composition property of pseudo-differential operators.

The idea of our proof is based on the paradifferential technique. One splits $a \in S^m_{\rho,\rho}$ into three symbols as follows. Let

$$a_{j,k}(x,\xi) = \begin{cases} \phi_j * a(\cdot,\xi)(x)\widehat{\phi_k}(\xi), & j,k \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{3.5}$$

Then we decompose the symbol $a(x,\xi)$ as

$$\begin{aligned} a(x,\xi) &= \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} a_{j,k}(x,\xi) \\ &= \sum_{j=3}^{\infty}\sum_{k=0}^{j-3} a_{j,k}(x,\xi) + \sum_{k=0}^{\infty}\sum_{j=k-2}^{k+2} a_{j,k}(x,\xi) + \sum_{k=3}^{\infty}\sum_{j=0}^{k-3} a_{j,k}(x,\xi) \\ &= \gamma_1(x,\xi) + \gamma_2(x,\xi) + \gamma_3(x,\xi) \end{aligned}$$

and proceed by estimating each term separately.

Then our claim is that for any $s, m \in \mathbb{R}$ and $0 < t \leq \infty$, $Op_h(\gamma_1)$ and $Op_h(\gamma_2)$ satisfy the estimates

$$\|Op_h(\gamma_1) f\|_{F_p^{0,t}} \lesssim \|f\|_{F_p^{s,t}} \tag{3.6}$$

and

$$\|Op_h(\gamma_2) f\|_{F_p^{0,t}} \lesssim \|f\|_{F_p^{s,t}} \tag{3.7}$$

which, of course, imply both operators map $F_p^{0,q}$ into $F_p^{0,t}$ for $0 < q \leq \infty$.

Proof of (3.6). It follows in the same way as in [19], which is based on the following two lemmas.

Observe that

$$Op_h(a_{j,k})f(x) = Op_h(a_{j,k})(f * \widetilde{\phi_k})(x),$$

where $\widetilde{\phi_k} = \phi_{k-1} + \phi_k + \phi_{k+1}$ and the Fourier transform of $u_j = \sum_{k=0}^{j-3} Op_h(a_{j,k})f$ is supported in the annulus $\{\eta : 2^{j-2} \leq |\eta| \leq 2^{j+2}\}$. By using Lemma 2.3, we obtain

$$\|Op_h(\gamma_1)f\|_{F_p^{0,t}} \lesssim \left\| \left(\sum_{j=3}^{\infty} \left| \sum_{k=0}^{j-3} Op_h(a_{j,k})(f * \widetilde{\phi_k}) \right|^t \right)^{\frac{1}{t}} \right\|_{L^p} \tag{3.8}$$

and then apply Lemma 2.4 to get

$$\left| Op_h(a_{j,k})(f * \widetilde{\phi_k})(x) \right| \lesssim \left\| a_{j,k}(x, 2^k \cdot) \right\|_{W^{c(r,d),1}} \mathcal{M}_r(f * \widetilde{\phi_k})(x)$$

for $0 < r < \min\{1, p, t\}$. For each multi-index α_1 and $|\xi| \sim 1$, using the cancellation condition of ϕ_j, we see that

$$\begin{aligned} \left| \phi_j * \partial_\xi^{\alpha_1} a(\cdot, 2^k\xi)(x) \right| &\lesssim \sum_{|\beta|=N} \int_{\mathbb{R}^d} |\phi_j(y)| \, |y|^N \, \kappa(x, y, 2^k\xi) dy \\ &\lesssim C_N 2^{k(m-\rho|\alpha_1|+\rho N)-jN}, \end{aligned}$$

with

$$\kappa(x, y, 2^k\xi) = \int_0^1 \cdots \int_0^1 t_1 \dots t_N \left| \partial_x^\beta \partial_\xi^{\alpha_1} a\left(x - t_1 \dots t_N y, 2^k\xi\right) \right| dt_1 \dots dt_N,$$

for any positive integer N.

Here, we used Taylor's theorem

$$g(x-y)=\sum_{|\beta|=1}\int_0^1\partial^\beta g(x-ty)dt(-y)^\beta$$

N times for the first inequality. Then this proves

$$\begin{aligned}\left\|a_{j,k}(x,2^k\cdot)\right\|_{W^{c^{(r,d)},1}} &\lesssim \sum_{|\alpha|\leq c^{(r,d)}}\int_{|\xi|\sim 1}\left|\partial_\xi^\alpha\left(\phi_j*\partial_\xi^{\alpha_1}a(\cdot,2^k\xi)(x)\,\widetilde{\phi}(\xi)\right)\right|d\xi\\ &\lesssim 2^{k\left(m+c^{(r,d)}(1-\rho)+\rho N\right)-jN}.\end{aligned}$$

Therefore

$$\left|Op_h(a_{j,k})(f*\widetilde{\phi_k})(x)\right|\lesssim 2^{k\left(m+c^{(r,d)}(1-\rho)+\rho N\right)-jN}\mathcal{M}_r(f*\widetilde{\phi_k})(x), \tag{3.9}$$

and finally, we have

$$\begin{aligned}\|Op_h(\gamma_1)f\|_{F_p^{0,t}} &\lesssim \left\|\left(\sum_{j=3}^{\infty}2^{-jNt}\left(\sum_{k=0}^{j-3}2^{k\left(m+c^{(r,d)}(1-\rho)+\rho N\right)}\mathcal{M}_r(f*\widetilde{\phi_k})\right)^t\right)^{\frac{1}{t}}\right\|_{L^p}\\ &\lesssim \left\|\left(\left(\sum_{k=0}^{\infty}2^{kt\left(m+c^{(r,d)}(1-\rho)+\rho N\right)}\mathcal{M}_r(f*\widetilde{\phi_k})\right)^t\sum_{j=k+3}^{\infty}2^{-jtN}\right)^{\frac{1}{t}}\right\|_{L^p}\\ &\lesssim \left\|\left(\sum_{k=0}^{\infty}2^{kt\left(m+c^{(r,d)}(1-\rho)+N(1-\rho)\right)}\left|(f*\widetilde{\phi_k})\right|^t\right)^{\frac{1}{t}}\right\|_{L^p}\\ &\lesssim \|f\|_{F_p^{m+c^{(r,d)}(1-\rho)+N(1-\rho),t}}\end{aligned}$$

for N sufficiently large and $0<\epsilon<N$, where the third inequality follows from the $L^p(l^t)$ boundedness of maximal operator $\mathcal{M}_r$ with $0<r<p$.

For all $s\in\mathbb{R}$, choose N sufficiently large so that

$$m+c^{(r,d)}(1-\rho)+N((1-\rho)<s.$$

This ends the proof of (3.6).

Proof of (3.7) Now, we consider the operator $Op_h(\gamma_2)$. By setting

$$a_k(z,\eta):=\sum_{j=k-2}^{k+2}a_{j,k}(z,\eta)=\Phi_k*a_k(.,\eta)(z)\widehat{\phi}_k(\eta),$$

where $\Phi_k=\sum_{j=k-2}^{k+2}\phi_j$, we express $Op_h(\gamma_2)$ as

$$Op_h(\gamma_2)=\sum_{j=0}^{9}Op_h(a_k)+\sum_{j=10}^{\infty}Op_h(a_k). \tag{3.10}$$

The finite sum of operators clearly satisfies (3.7). For each $k \geq 10$ the kernel of $Op_h(a_k)$ is

$$\begin{aligned} K_k(x,y) &= \int_{\mathbb{R}^d} a_k(x,\xi) e^{2\pi(x-y,\xi)} d\xi \\ &= \int_{\mathbb{R}^{3d}} a(z,\xi) \widehat{\Phi}_k(\eta) \widehat{\phi}_k(\xi) e^{-2\pi(z,\eta)} e^{2\pi(x-y,\xi)} dx d\eta d\xi. \end{aligned}$$

Observe that in the integral the variables ξ and η live in $|\xi| \sim 2^k$ and $|\eta| \sim 2^k$. Then it has the size estimate

$$|K_k(x,y)| \lesssim C_{J,N} 2^{-Jk} \frac{1}{(1+|x-y|)^N} \tag{3.11}$$

for any $J > 0$ and $N > 0$.

The idea to get (3.11) is to apply a technique of oscillatory integrals by integrating by parts with respect to each variable. First, we perform this with respect to the z-variable and then carry out a similar process on the η-variable. Now, (3.10) is dominated by

$$C_M 2^{-k(2M-d)} \int_{\mathbb{R}^d} \left| \int_{|\xi|} (I-\Delta_Z)^M a(z,\xi) e^{2\pi(x-y,\xi)} \widehat{\phi}_k(\xi) d\xi \right| \frac{1}{(1+|x-z|)^{2d}} dz,$$

for any $M > 0$.

If $|x-y| \leq 1$, then

$$\left| \int_{|\xi| \sim 2^k} (I-\Delta_Z)^M a(z,\xi) e^{2\pi(x-y,\xi)} \widehat{\phi}_k(\xi) d\xi \right| \lesssim \int_{|\xi| \sim 2^k} (1+|\xi|)^{m+2\rho M} \, d\xi \lesssim 2^{k(m+2\rho M+d)}.$$

If $|x-y| > 1$, then we do integration by parts in ξ-variable to get

$$\begin{aligned} \left| \int_{|\xi| \sim 2^k} (I-\Delta_Z)^M a(z,\xi) e^{2\pi(x-y,\xi)} \widehat{\phi}_k(\xi) d\xi \right| &\lesssim \frac{C_{N,M}}{|x-y|^N} \int_{|\xi| \sim 2^k} (1+|\xi|)^{m-\rho N+2\rho M} \, d\xi \\ &\lesssim C_{N,M} \frac{2^{k(m-\rho N+2\rho M+d)}}{|x-y|^N}. \end{aligned}$$

These yield (3.11) by choosing M and N sufficiently large.

Since $\widehat{Op_h(a_k) f}$ is supported in $\left\{ |\eta| \leq 2^{k+4} \right\}$, we have

$$\begin{aligned} \left\| \sum_{k=10}^{\infty} Op_h(a_k) f \right\|_{F_p^{0,t}} &\leq \left\| \left(\sum_{j=0}^{\infty} \left(\sum_{k=10}^{\infty} \left| \phi_j * Op_h(a_k) \left(f * \widehat{\phi}_j \right) \right| \right)^t \right)^{\frac{1}{t}} \right\|_{L^p} \\ &\lesssim \left\| \left(\sum_{j=0}^{13} \left(\sum_{k=10}^{\infty} \left| \phi_j * Op_h(a_k) \left(f * \widehat{\phi}_j \right) \right| \right)^t \right)^{\frac{1}{t}} \right\|_{L^p} \qquad (3.12) \\ &\quad + \left\| \left(\sum_{j=14}^{\infty} \left(\sum_{k=j-4}^{\infty} \left| \phi_j * Op_h(a_k) \left(f * \widehat{\phi}_j \right) \right| \right)^t \right)^{\frac{1}{t}} \right\|_{L^p} \qquad (3.13) \end{aligned}$$

Let $\sigma > \max\left\{\frac{d}{p}, \frac{d}{t}\right\}$, and choose $N > \sigma$. Then

$$\begin{aligned}\left|\phi_j * Op_h(a_k)\left(f * \widehat{\phi}_j\right)(x)\right| &\leq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left|\phi_j(x-z)\left\|K_k(z,y)\right\| f * \widehat{\phi}_j(y)\right| dydz \\ &\lesssim \mathcal{M}_{\sigma,2^k}\left(f * \widehat{\phi}_j(x)\right)\iint_{\mathbb{R}^{2d}} V_{\sigma,j,k}(x,y,z)dydz \\ &\lesssim 2^{-k(J-\sigma)}\mathcal{M}_{\sigma,2^k}\left(f * \widehat{\phi}_j(x)\right),\end{aligned}$$

with

$$V_{\sigma,j,k}(x,y,z) = \left(1+2^k|x-y|\right)^{\sigma}|\phi_j(x-z)|\left\|K_k(z,y)\right\|,$$

by the size estimate (3.11). Thus for $\epsilon > 0$

$$\begin{aligned}(3.12) \quad &\lesssim \left\|\sum_{k=10}^{\infty} 2^{-k(J-\sigma)}\mathcal{M}_{\sigma,2^k}\left(f * \widehat{\phi}_j\right)\right\|_{L^p} \\ &\lesssim \left\|\left(\sum_{k=10}^{\infty} 2^{-kt(J-\sigma-\epsilon)}\left(\mathcal{M}_{\sigma,2^k}\left(f * \widehat{\phi}_j\right)\right)^t\right)^{\frac{1}{t}}\right\|_{L^p} \\ &\lesssim \left\|\left(\sum_{k=10}^{\infty} 2^{-kt(J-\sigma-\epsilon)}\left|f * \widehat{\phi}_j\right|^t\right)^{\frac{1}{t}}\right\|_{L^p} \lesssim \|f\|_{F_p^{-(J-\sigma-\epsilon),t}}.\end{aligned}$$

Similarly, we also get

$$(3.13) \lesssim \|f\|_{F_p^{-(J-\sigma-\epsilon),t}}.$$

Then (3.7) follows by choosing J satisfying $-(J-\sigma-\epsilon) < s$.

Boundedness of $Op_h(\gamma_3)$. We write a symbol $\gamma_3 \in S^m_{\rho,\rho}$ as

$$\gamma_3(x,\xi) = \sum_{k=3}^{\infty} b_k(x,\xi).$$

where $b_k(x,\xi) = \sum_{j=0}^{k-3} a_{j,k}(x,\xi)$.

We observe that since $\widehat{Op_h(b_k)u}$ is supported in $\left\{\xi:\ 2^{k-2} \leq |\xi| \leq 2^{k+2}\right\}$, for $0 < r < \infty$,

$$\|Op_h(b_k)u\|_{L^r} \lesssim \|Op_h(b_k)u\|_{h^r}.$$

In addition,

$$b_k(x,\xi) = \left(\sum_{j=0}^{k-3}\phi_j\right) * a(.,\xi)(x)$$

is an $S^m_{\rho,\rho}$ symbol with a constant, which is independent of k, and thus for $u \in S'\left(\mathbb{R}^d\right)$,

$$\|Op_h(b_k)u\|_{h^r} \leq C_r 2^{k\left(d(1-\rho)\left(\left|\frac{1}{r}-\frac{1}{2}\right|-\left|\frac{1}{p}-\frac{1}{2}\right|\right)\right)}\|\phi_k * u\|_{h^r},$$

which implies

$$\|Op_h(b_k)u\|_{L^r} \leq C_r 2^{k\left(d(1-\rho)\left(\left|\frac{1}{r}-\frac{1}{2}\right|-\left|\frac{1}{p}-\frac{1}{2}\right|\right)\right)}\|\phi_k * u\|_{h^r}. \tag{3.14}$$

By the same reasoning as (3.8), we have

$$\|Op_h(\gamma_3)f\|_{F_p^{0,t}} \lesssim \left\| \left(\sum_{k=3}^{\infty} |Op_h(b_k)f|^t \right)^{\frac{1}{t}} \right\|_{L^p}.$$

Therefore in order to conclude the proof of Theorem 3.1, it is enough to show that

$$\left\| \left(\sum_{k=3}^{\infty} |Op_h(b_k)f|^t \right)^{\frac{1}{t}} \right\|_{L^p} \lesssim \|f\|_{F_p^{0,q}}. \tag{3.15}$$

We shall study just the cases $0 < p \leq 1$ and $2 < p < \infty$.

Indeed, for $1 < p < 2$,

$$\begin{aligned}
\|Op_h(a)\, f\|_{F_p^{0,t}} &= \|\{\phi_k * Op_h(a)\, f\}\|_{L^p(l^p)} \\
&= \sup_{\|\{g_k\}\|_{L^{p'}(l^{p'})}\leq 1} \left| \int_{\mathbb{R}^d} \sum_{k=0}^{\infty} \phi_k * Op_h(a)\, f(x) g_k(x) dx \right| \\
&= \sup_{\|\{g_k\}\|_{L^{p'}(l^{p'})}\leq 1} \left| \int_{\mathbb{R}^d} \sum_{j=0}^{\infty} \phi_j * f(x) \widetilde{\phi_j} * \left(Op_h(a)^* \left(\sum_{k=0}^{\infty} \phi_k * g_k \right) \right) dx \right| \\
&\lesssim \|f\|_{F_p^{0,\infty}} \sup_{\|\{g_k\}\|_{L^{p'}(l^{p'})}\leq 1} \left\| Op_h(a)^* \sum_{j=0}^{\infty} \phi_j * g_j \right\|_{F_p^{0,1}} \\
&\lesssim \|f\|_{F_p^{0,\infty}} \sup_{\|\{g_k\}\|_{L^{p'}(l^{p'})}\leq 1} \left\| \sum_{k=0}^{\infty} \phi_k * g_k \right\|_{F_{p'}^{0,p'}} \leq \|f\|_{F_p^{0,\infty}}.
\end{aligned}$$

Here everything makes sense because the infinite sum of $\phi_k * g_k \in \mathcal{S}'$ due to Lemma 2.3 with the estimate

$$\left\| \left(\sum_{k=0}^{\infty} |\phi_k * g_k|^{p'} \right)^{\frac{1}{p'}} \right\|_{L^{p'}} \leq \|g_k\|_{L^{p'}(l^{p'})} \leq 1.$$

Proof of (3.15); the case $0 < p \leq 1$. We need to prove (3.15) with $t = p$ and $q = \infty$. It is natural to use an adaption of the atomic decomposition of Triebel–Lizorkin spaces. Alternatively, one can characterize Triebel–Lizorkin spaces $F_p^{s,q}$ with the associated sequence spaces $F_p^{s,q}$ via the Frazier and Jawerth φ-transform, and then use atomic decomposition for the sequence spaces. We follow the latter approach and recall definitions.

The following atomic decomposition of $F_p^{s,q}$ substitutes the atomic decomposition of h^p.

Lemma 3.2. [8] [9, 6.6.3]. *Suppose* $0 < p \leq 1$, $0 < q \leq \infty$, *and*

$$b = \{b_Q\}_{\substack{Q\in\mathcal{D} \\ l(Q)\leq 1}} \in F_p^{s,q}.$$

Then there exist $C_{d,p,q} > 0$, *a sequence of scalars* $\{\lambda_j\}$, *and a sequence of* ∞*-atoms* $r_j = \{r_{j,Q}\}_{\substack{Q\in\mathcal{D}\\ l(Q)\leq 1}}$ *for* $F_p^{s,q}$ *such that*

$$b = \{b_Q\} = \sum_{j=1}^{\infty} \lambda_j \{r_{j,Q}\} = \sum_{j=1}^{\infty} \lambda_j r_j$$

and such that

$$\left(\sum_{j=1}^{\infty} |\lambda_j|^p\right)^{\frac{1}{p}} \leq C_{d,p,q} \|b\|_{F_p^{s,q}}.$$

Moreover,

$$\|b\|_{F_p^{s,q}} \approx \inf\left\{\left(\sum_{j=1}^{\infty} |\lambda_j|^p\right)^{\frac{1}{p}} : s = \sum_{j=1}^{\infty} \lambda_j r_j,\ r_j \text{ is an } \infty - \text{atom for } F_p^{s,q}\right\}.$$

By Lemma 3.2, $f \in F_p^{0,\infty}$ can be decomposed with $\{b_Q\}_{\substack{Q\in\mathcal{D}\\ l(Q)\leq 1}} \in F_p^{0,\infty}$, and there exist a sequence of scalars $\{\lambda_j\}$ and a sequence of ∞-atoms $r_j = \{r_{j,Q}\}$ for $F_p^{0,\infty}$ such that

$$f(x) = \sum_{\substack{Q\in\mathcal{D}\\ l(Q)\leq 1}} b_Q \vartheta^Q(x) = \sum_{j=1}^{\infty} \lambda_j \sum_{\substack{Q\in\mathcal{D}\\ l(Q)\leq 1}} r_{j,Q}\vartheta^Q(x).$$

Then

$$\begin{aligned}
\|Op_h(\gamma_3)f\|_{F_p^{0,p}} &\lesssim \left\|\left(\sum_{k=3}^{\infty} |Op_h(b_k)f|^p\right)^{\frac{1}{p}}\right\|_{L^p} \qquad (3.16)\\
&= \left\|\left(\sum_{k=3}^{\infty}\left|\sum_{j=1}^{\infty}\lambda_j Op_h(b_k)\left(\sum_{\substack{Q\in\mathcal{D}\\ l(Q)\leq 1}} r_{j,Q}\vartheta^Q\right)\right|^p\right)^{\frac{1}{p}}\right\|_{L^p}\\
&\leq \left(\sum_{j=1}^{\infty}|\lambda_j|^p \int_{\mathbb{R}^d}\sum_{k=3}^{\infty}\left|Op_h(b_k)\left(\sum_{\substack{Q\in\mathcal{D}\\ l(Q)\leq 1}} r_{j,Q}\vartheta^Q\right)\right|^p dx\right)^{\frac{1}{p}}\\
&\lesssim \left(\sum_{j=1}^{\infty}|\lambda_j|^p\right)^{\frac{1}{p}} \sup_j\left\{\left(\sum_{k=3}^{\infty}\left\|Op_h(b_k)\left(\sum_{\substack{Q\in\mathcal{D}\\ l(Q)\leq 1}} r_{j,Q}\vartheta^Q\right)\right\|_{L^p}^p\right)^{\frac{1}{p}}\right\}.
\end{aligned}$$

by using the triangle inequality for $p \leq 1$ and $l^p \subset l^1$. Since

$$\left(\sum_{j=1}^{\infty}|\lambda_j|^p\right)^{\frac{1}{p}} \lesssim \|f\|_{F_p^{0,\infty}},$$

it suffices to show the supremum in (3.16) is bounded by a constant.

Let Q_0 be any dyadic cubes with side length $2^{-\mu}$ and let r_Q be an ∞-atom for $F_p^{0,\infty}$ with Q_0, and define

$$R_{Q_0,k}(x) = \sum_{\substack{Q\in\mathcal{D}_k, Q\subset Q_0 \\ l(Q)\le 1}} r_Q \vartheta^Q(x).$$

Then we obtain the desired result by showing

$$\left(\sum_{k=3}^{\infty}\left\| Op_h(b_k)\left(\sum_{\substack{Q\in\mathcal{D} \\ l(Q)\le 1}} r_{j,Q}\vartheta^Q\right)\right\|_{L^p}^p\right)^{\frac{1}{p}} \lesssim 1, \tag{3.17}$$

uniformly in Q_0. Note that, by (2.2) we have

$$|r_Q| \le |Q|^{\frac{1}{2}} |Q_0|^{\frac{-1}{p}}. \tag{3.18}$$

Let Q_0^* be a dilate of Q_0 by a factor of 10 and Q_0^{**} by a factor of $100\sqrt{d}$. Furthermore, denote by Q_0^p a dilate of Q_0 with side $10l(Q_0)^p$, by $\widetilde{Q_0^p}$ with side $100\sqrt{d}l(Q_0)^p$.

We first consider the case $l(Q_0) \le 2^{-3}$(i.e., $\mu \ge 3$). The condition $Q \subset Q_0$ in the definition of $R_{Q_0,k}$ ensures that $R_{Q_0,k}$ vanishes unless $\mu \le k$. Hence the summations is taken over $k \ge \mu$ in (3.18). For each k, we split the range of the integral into two parts. $\widetilde{Q_0^p}$ and $\left(\widetilde{Q_0^p}\right)^c$. That is,

$$\left(\sum_{k=3}^{\infty}\|Op_h(b_k)R_{Q_0,k}\|_{L^p}^p\right)^{\frac{1}{p}} \lesssim \Lambda_1^{\frac{1}{p}} + \Lambda_2^{\frac{1}{p}},$$

where

$$\Lambda_1 = \sum_{k=\mu}^{\infty}\|Op_h(b_k)R_{Q_0,k}\|_{L^p\left(\widetilde{Q_0^p}\right)}^p,$$

and

$$\Lambda_2 = \sum_{k=\mu}^{\infty}\|Op_h(b_k)R_{Q_0,k}\|_{L^p\left(\widetilde{Q_0^p}\right)^c}^p.$$

For the term corresponding to $\widetilde{Q_0^p}$, we use Holder's inequality, (3.14), and (3.18), and then we obtain that for $p < r$

$$\begin{aligned}
\|Op_h(b_k)R_{Q_0,k}\|_{L^p\left(\widetilde{Q_0^p}\right)}^p &\le \left|\widetilde{Q_0^p}\right|^{\frac{1}{p}-\frac{1}{r}} \|Op_h(b_k)R_{Q_0,k}\|_{L^r} \\
&\lesssim 2^{-\mu\rho d\left(\frac{1}{p}-\frac{1}{r}\right)} 2^{-kd(1-\rho)\left(\frac{1}{p}-\frac{1}{r}\right)} \|R_{Q_0,k}\|_{h^r} \\
&\lesssim 2^{-\mu\rho d\left(\frac{1}{p}-\frac{1}{r}\right)} 2^{-kd(1-\rho)\left(\frac{1}{p}-\frac{1}{r}\right)} \|\Gamma(Q)\|_{L^r} \\
&\le 2^{\mu d(1-\rho)\left(\frac{1}{p}-\frac{1}{r}\right)} 2^{-kd(1-\rho)\left(\frac{1}{p}-\frac{1}{r}\right)},
\end{aligned}$$

with

$$\Gamma(Q)=\left(\sum_{Q\in\mathcal{D}_k,Q\subset Q_0}\left(|r_Q|\,|Q|^{\frac{1}{2}}\,\chi_Q\right)^2\right)^{\frac{1}{2}}.$$

This proves

$$\left(\sum_{k=\mu}^{\infty}\|Op_h(b_k)R_{Q_0,k}\|^p_{L^p\left(\widetilde{Q_0^p}\right)}\right)^{\frac{1}{p}}\lesssim 1$$

uniformly in μ.

For the latter one, we split it into

$$\left(\sum_{k=\mu}^{\infty}\left\|Op_h(b_k)\left(\chi_{\left(Q_0^*\right)^C}R_{Q_0,k}\right)\right\|^p_{L^p\left(\widetilde{Q_0^p}\right)^c}\right)^{\frac{1}{p}} \tag{3.19}$$

and

$$\left(\sum_{k=\mu}^{\infty}\left\|Op_h(b_k)\left(\chi_{Q_0^*}R_{Q_0,k}\right)\right\|^p_{L^p\left(\widetilde{Q_0^p}\right)^c}\right)^{\frac{1}{p}}, \tag{3.20}$$

and our claim is that each part can be controlled by a constant independent of μ.

By applying (3.14) we write

$$(3.19)\lesssim\left(\sum_{k=\mu}^{\infty}\left\|\phi_k*\left(\chi_{\left(Q_0^*\right)^C}R_{Q_0,k}\right)\right\|^p_{h^p}\right)^{\frac{1}{p}}$$

and the Fourier support of ϕ_k gives the estimate

$$\begin{aligned}&\left\|\phi_k*\left(\chi_{\left(Q_0^*\right)^C}R_{Q_0,k}\right)\right\|_{h^p}\\ &\lesssim\left\|\phi_k^{(-1)}*\left(\chi_{\left(Q_0^*\right)^C}R_{Q_0,k}\right)\right\|_{L^p}+\left\|\phi_k^{(0)}*\left(\chi_{\left(Q_0^*\right)^C}R_{Q_0,k}\right)\right\|_{L^p}+\left\|\phi_k^{(1)}*\left(\chi_{\left(Q_0^*\right)^C}R_{Q_0,k}\right)\right\|_{L^p}.\end{aligned}$$

For each $j\in\{-1,0,1\}$ we have

$$\phi_k^{(-1)}=\phi_{k-1}*\phi_k,\ \ \phi_k^{(0)}=\phi_k*\phi_k,\ \text{ and }\ \phi_k^{(1)}=\phi_{k+1}*\phi_k.$$

Also,

$$\begin{aligned}
&\left\|\phi_k^{(j)} * \left(\chi_{\left(Q_0^*\right)^C} R_{Q_0,k}\right)\right\|_{L^p}^p \\
&\leq \int_{\mathbb{R}^d} \left(\int_{y\in\left(Q_0^*\right)^C} \left|\phi_k^{(j)}(x-y)\right| \sum_{Q\in\mathcal{D}_k, Q\subset Q_0} |r_Q| \left|\vartheta^Q(y)\right| dy\right)^p dx \\
&\lesssim 2^{-kp\left(N-\frac{d}{2}\right)} \int_{\mathbb{R}^d} \left(\int_{y\in\left(Q_0^*\right)^C} \left|\phi_k^{(j)}(x-y)\right| \sum_{Q\in\mathcal{D}_k, Q\subset Q_0} \frac{|r_Q|}{|y-x_Q|^N} dy\right)^p dx \\
&\leq 2^{-kp\left(N-\frac{d}{2}\right)} \left(\sum_{Q\in\mathcal{D}_k, Q\subset Q_0} |r_Q|\right)^p \int_{\mathbb{R}^d} \left(\int_{y\in\left(Q_0^*\right)^C} \frac{\left|\phi_k^{(j)}(x-y)\right|}{|y-c_{Q_0}|^N} dy\right)^p dx \\
&\leq 2^{-kp(N-d)} 2^{\mu dp\left(\frac{1}{p}-1\right)} \int_{\mathbb{R}^d} \left(\int_{y\in\left(Q_0^*\right)^C} \frac{\left|\phi_k^{(j)}(x-y)\right|}{|y-c_{Q_0}|^N} dy\right)^p dx \\
&\lesssim 2^{-kp\left(N-2d+\frac{d}{p}-\frac{L}{p}\right)} 2^{\mu dp\left(\frac{1}{p}-1\right)} \left(\int_{y\in\left(Q_0^*\right)^C} \frac{1}{|y-c_{Q_0}|^{N-\frac{L}{p}}} dy\right)^p \\
&\lesssim 2^{-kp\left(N-2d+\frac{d}{p}-\frac{L}{p}\right)} 2^{\mu dp\left(\frac{1}{p}-1\right)} 2^{\mu p\left(N-\frac{L}{p}-d\right)},
\end{aligned}$$

for $L > d(1-p)$ and $N - \frac{L}{p} > d$. Here the third inequality follows from the fact that

$$|y - x_Q| \gtrsim |y - c_Q|$$

for $y \in (Q_0^*)^C$ and $Q \subset Q_0$, and the fourth one holds because of (3.18). Finally, we obtain (3.19) $\lesssim 1$.

For (3.20) let $K_k(x,y)$ be the kernel of $Op_h(b_k)$, and write

$$\left\|Op_h(b_k)\left(\chi_{Q_0^*} R_{Q_0,k}\right)\right\|_{L^p\left(\widetilde{Q_0^p}\right)^c} \leq \left[\int_{\widetilde{Q_0^p}^c} \left(\int_{Q_0^*} |K(x,y)| \left|R_{Q_0,k}(y)\right| dy\right)^p dx\right]^{\frac{1}{p}}.$$

By using Holder's inequality and (3.18), it is less than

$$\begin{aligned}
&2^{\mu\rho\left(\frac{L}{p}-\frac{d}{P}+d\right)} \|R_{Q_0,k}\|_{L^1} \sup_{y\in Q_0^*} \int_{\widetilde{Q_0^p}^c} |x-y|^{\frac{L}{p}} |K_k(x,y)| dx \\
&\leq 2^{\mu\rho\frac{L}{p}} 2^{\mu d(1-\rho)\left(\frac{1}{P}-1\right)} \sup_{y\in Q_0^*} \int_{\widetilde{Q_0^p}^c} |x-y|^{\frac{L}{p}} |K_k(x,y)| dx
\end{aligned}$$

for $L > d(1-p)$. Here we select $L > d(1-p)$ such that $\frac{L}{p}$ becomes an integer. Then it suffices to show

$$\sup_{y\in Q_0^*} \int_{\widetilde{Q_0^p}^c} |x-y|^{\frac{L}{p}} |K_k(x,y)| dx \lesssim 2^{-k\rho\frac{L}{p}} 2^{-kd(1-\rho)\left(\frac{1}{P}-1\right)} 2^{\epsilon(\mu-k)} \tag{3.21}$$

for some $\epsilon > 0$. Let $y \in Q_0^*$, by the Cauchy–Schwarz inequality and the fact that $|x - c_Q| \lesssim |x - y|$ we obtain

$$\int_{\widetilde{Q_0^p}^c} |x-y|^{\frac{L}{p}} |K_k(x,y)|\, dx \lesssim 2^{\mu\rho\left(|\alpha|-\frac{d}{2}\right)} \left(\int_{\mathbb{R}^d} \left| (x-y)^{\alpha+\beta} K_k(x,y) \right|^2 dx \right)^{\frac{1}{2}}$$

for a multi-index β with $|\beta| = \frac{L}{p}$. Define

$$c_k(y,\eta) = \int_{\mathbb{R}^d} K_k(x+y,y)\, e^{-2\pi i(x,\eta)} dx.$$

Then by Plancherel's theorem

$$\left(\int_{\mathbb{R}^d} \left| (x-y)^{\alpha+\beta} K_k(x,y) \right|^2 dx \right)^{\frac{1}{2}} = \left(\left| \int_{\mathbb{R}^d} \partial_\eta^{\alpha+\beta} c_k(y,\eta) \right|^2 d\eta \right)^{\frac{1}{2}}.$$

We observe that $\overline{c_k(y,\eta)}$ can be interpreted a symbol corresponding to the adjoint operator of T_{b_k} and therefore $c_k \in S^m_{\rho,\rho}$. Furthermore, η lives in the annulus $\left\{\eta :\ 2^{k-2} \leq |\eta| \leq 2^{k+2}\right\}$. Therefore we have

$$\int_{\widetilde{Q_0^p}^c} |x-y|^{\frac{L}{p}} |K_k(x,y)|\, dx \lesssim 2^{-k\rho\frac{L}{p}} 2^{-kd(1-\rho)\left(\frac{1}{P}-1\right)} 2^{\rho\left(|\alpha|-\frac{d}{2}\right)(\mu-k)},$$

which concludes (3.21).

Now, assume $l(Q_0) > 2^{-3}$ (i.e., $\mu < 3$). In this case, it is better to use the range $k \geq 3$ in our summations since $\mu \leq 2$. Then by repeating the above process, we see that

$$\left(\sum_{k=3}^{\infty} \|Op_h(b_k) R_{Q_0,k}\|^p_{L^p(Q_0^{**})} \right)^{\frac{1}{p}} \lesssim 1.$$

$$\left(\sum_{k=3}^{\infty} \left\| Op_h(b_k) \chi_{(Q_0^*)^c} R_{Q_0,k} \right\|^p_{L^p((Q_0^{**})^c)} \right)^{\frac{1}{p}} \lesssim 1,$$

and

$$\left(\sum_{k=3}^{\infty} \left\| Op_h(b_k) \chi_{Q_0^*} R_{Q_0,k} \right\|^p_{L^p((Q_0^{**})^c)} \right)^{\frac{1}{p}} \lesssim 1$$

uniformly in $\mu \leq 2$. This completes the proof of the case where $0 < p \leq 1$ in Theorem 3.1.

Proof of (3.15); the case $2 < p < \infty$. Suppose $2 < p < \infty$, and we prove (3.15) with $0 < t \leq \infty$ and $q = p$. Let $0 < a < d$, $\epsilon > 0$ and $1 < p_0 < p < \infty$.

Consider the integral operators T_k defined on $\mathcal{S}(\mathbb{R}^d)$ by

$$T_k f(x) = \int_{\mathbb{R}^d} K_k(x,y) f(y) dy,$$

where each K_k is a continuous and bounded kernel. Assume that T_k satisfy

$$\sup_{k>0} 2^{-k\frac{a}{P}} \|T_k\|_{L^p \to L^p} \leq A \tag{3.22}$$

and

$$\sup_{k>0} 2^{-k\frac{a}{P_0}} \|T_k\|_{L^{P_0}\to L^{P_0}} \le B_0. \tag{3.23}$$

Furthermore, let $\Gamma \ge 1$ and assume that for each cube Q, there is a measurable set ε_Q such that

$$|\varepsilon_Q| \le \Gamma \max\left\{|Q|^{1-\frac{a}{d}}, |Q|\right\},$$

and for every $k \in \mathbb{N}$ and every cube Q with $2^k l(Q) \ge 1$, we have

$$\sup_{x\in Q} \int_{\mathbb{R}^d\setminus\varepsilon_Q} |K_k(x,y)|\,dy \le B_1 \max\left\{\left(2^k l(Q)\right)^{-\epsilon}, 2^{-k\epsilon}\right\}. \tag{3.24}$$

Let $\mathcal{B} = B_0^{\frac{p_0}{p}}\left(A\Gamma^{\frac{1}{p}} + B_1\right)^{1-\frac{p_0}{p}}$. Then there is $C>0$ such that

$$\left\|\left(\sum_k 2^{\frac{kar}{p}} |\phi_k * T_k f_k|^q\right)^{\frac{1}{q}}\right\|_{L^p} \le CA\left[\log\left(3+\frac{\mathcal{B}}{A}\right)\right]^{\frac{1}{q}-\frac{1}{p}}\left(\sum_k \|f_k\|_{L^p}^p\right)^{\frac{1}{p}}.$$

We set $a = (1-\rho)\,d$ and for the cube Q with $l(Q) < 1$, choose ε_Q to be the cube with the same center, but with diameter $Cl(Q)^p$ for large C. If $l(Q) \ge 1$, then ε_Q is just a dilate of Q by a factor of large constant C. Let

$$\widetilde{b_k}(x,\xi) := (1+|\xi|^2)^{\frac{-d(1-\rho)}{2p}} b_k(x,\xi), \tag{3.25}$$

and define $T_k := Op_h(\widetilde{b_k})$. Then operators T_k obviously satisfy (3.22) and (3.15) with $p_0 = 2$.

Thus it suffices to show that (3.24) still holds with our kernel

$$K_k(x,y) = \int_{\mathbb{R}^d} \widetilde{b_k}(x,\xi) e^{2\pi i(x-y,\xi)} d\xi. \tag{3.26}$$

Fix $x \in Q$ and then for $y \in \mathbb{R}^d\setminus\varepsilon_Q$, we see $|x-y| \gtrsim l(\varepsilon_Q)$. Therefore

$$\begin{aligned}
\int_{\mathbb{R}^d\setminus\varepsilon_Q} |K_k(x,y)|\,dy &\le \int_{|x-y|\gtrsim l(\varepsilon_Q)} |K_k(x,y)|\,dy \\
&\lesssim l(\varepsilon_Q)^{-|\alpha|+\frac{d}{2}}\left(\int_{\mathbb{R}^d} \left|(x-y)^\alpha K_k(x-y)\right|^2 dy\right)^{\frac{1}{2}} \\
&= l(\varepsilon_Q)^{-|\alpha|+\frac{d}{2}}\left(\int_{\mathbb{R}^d} \left|\partial_\xi^\alpha \widetilde{b_k}(x-\xi)\right|^2 dy\right)^{\frac{1}{2}}
\end{aligned}$$

for any multi-indices α with $|\alpha| > \frac{d}{2}$ by the Cauchy–Schwarz inequality and Plancherel's theorem. Since $\widetilde{b_k} \in S^{-\frac{d(1-\rho)}{2}}_{\rho,\rho}$, this is bounded by

$$\left(2^{k\rho} l(\varepsilon_Q)\right)^{-|\alpha|+\frac{d}{2}}.$$

By choosing α satisfying $\rho\left(|\alpha|+\frac{d}{2}\right) > \epsilon$, we prove (3.24), and it concludes the proof of (3.15). □

Acknowledgement. The authors thank the anonymous reviewer and the editor for their suggestions for improvement of the presentation of this article.

References

1. O.F. Aid and A. Senoussaoui, *The boundedness of h-admissible Fourier integral operators on Bessel potential spaces*, Turk. J. Math., **43** (2019), no. 5, 2125–2141.
2. O.F. Aid and A. Senoussaoui, *H^s boundedness of a class of a Fourier integral operators*, Math. Slovaca, **71** (2021), no. 4, 889–902.
3. O.F. Aid and A. Senoussaoui, *The boundedness of a class of semiclassical Fourier integral operators on Sobolev space H^s*, Matematychni Studii, **56** (2021), no. 1, 61–66.
4. O.F. Aid and A. Senoussaoui, *The boundedness of Bilinear Fourier integral Operators on $L^2 \times L^2$*, Int. J. Nonlinear Anal. Appl. **13** (2022), no. 2, 1565–1575.
5. O.F. Aid and A. Senoussaoui, *L^2 -Hilbert-Schemidtness of Fourier integral operators with weighted symbols*, Mem. Differ. Equ. Math. Phys. **88** (2023), 1–11.
6. J. Alvarez and J. Hounie, *Estimates for the kernel and continuity properties of pseudo-differential operators*, Arkiv der Matematik, **28** (1990), 1–122.
7. S. Chanillo and A. Torchinsky, *Sharp function and weighted L p estimates for a class of pseudo-differential operators*, Arkiv der Matematik, **24** (1985), no. 1-2, 1–25.
8. M. Frazier and B. Jawerth, *Applications of the and wavelet transforms to the theory of function spaces*, Wavelets and Their Applications, 377–417, Jones and Bartlett, Boston, MA, 1992
9. L. Grafakos, *Modern Fourier Analysis*, Springer, 2008.
10. L. Hörmander, *The Weyl calculus of pseudodifferential operators*, Comm. Pure Applied. Math., **32** (1979), 359–443.
11. C.E. Kenig and W. Staubach, *$\hat{I}$-pseudodifferential operators and estimates for maximal oscillatory integrals*, Stud. Math., **183** (2007), 249–258.
12. L. D. Landau and E. M. Lifshitz, *Quantum Mechanics*, Pergamon Press, Oxford, 1977.
13. J. Marschall, *Nonregular pseudo-differential operators*, J. Anal. Anwendungen, **15** (1996), no. 1, 109–148.
14. S. Nishigaki, *Weighted norm inequalities for certain pseudo-differential operators*, Tokyo J. Math., **7** (1984), no. 1, 129–140.
15. B. Park, *On the boudedness of pseudo-diferential operators on Triebel–Lizorkin and Besov spaces*, J. Math. Anal. Appl., **461** (2018), no. 1, 544–576.
16. M.A. Ragusa, *Dirichlet problem in Morrey spaces for elliptic equations in non divergence form with VMO coefficients*, Proceedings of the Eighth International Colloquium on Differential Equations, (1998) 385–390.
17. M.A. Ragusa, *Elliptic boundary value problem in Vanishing Mean Oscillation hypothesis*, Commentat. Math. Univ. Carol., **40** (1999), no. 4, 651–663.
18. M.A. Ragusa, *On some trends on regularity results in Morrey spaces*, In AIP conference proceedings, vol. 1493, no. 1, pp. 770–777. American Institute of Physics, 2012.
    USA: Birkäuser, 1987.
19. J. Johnsen, *Domains of pseudo-differential operators: a case for the Triebel–Lizorkin spaces*, J. Funct. Spaces Appl., **3** (2005), 263–286.
20. H. Triebel, *Theory of Function Spaces*, Birkhauser, Basel-Boston-Stuttgart, 1983.
21. K. Yabuta, *Weighted norm inequalities for pseudodifferential operators*, Osaka J. Math., **23** (1986), no. 3, 703–723.
22. M. Yamazaki, *A quasi-homogeneous version of paradifferential operators, I. Boundedness on spaces of Besov type*, J. Fac. Sci. Univ. Tokyo Sect. IA, Math., **33** (1986), 131–174.

[1] Laboratory of Fundamental and Applied Mathematics of Oran (LMFAO), University of Oran 1, Ahmed Ben Bella. B.P. 1524 El M'naouar, Oran, Algeria.
Email address: aidomarfarouk@gmail.com; senoussaoui_abdou@yahoo.fr

[2] Department of Mathematics, Faculty of Mathematics and Computer Sciences, University of Sciences and Technology "M.B" of Oran, Algeria
Email address: harratchahrazed@gmail.com