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MULTIFRAMELET SETS ON p-ADIC NUMBERS

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Abstract. This article explores various findings on multiframelets and multiframelet sets over one-dimensional p-adic numbers. A multiframelet is a sequence that resembles a frame, generated by a finite number of functions with a wavelet structure. Here multiframelet sets are studied extensively on $\mathbb{Q}_p$ with accompanying examples. It is known that a multiwavelet is a generator of a multiframelet, and as a result, a multiwavelet set is also a multiframelet set. However, the converse is not always true, as demonstrated in this article.

1. Introduction

Frames are a powerful tool in mathematics and engineering. They allow to express elements within the Hilbert space as a linear combination of frame elements. The coefficients of this representation are not unique, leading to frames being considered as an over-complete system. This property is significant as it distinguishes frames from traditional bases, such as Riesz bases or orthonormal bases, and opens up a wider range of applications in signal processing, image compression, and many other fields where over-completeness can be advantageous. The primary objective of frames is to offer a versatile substitute or extension to Riesz bases or orthonormal bases in Hilbert spaces. Duffin and Schaeffer [7] introduced frames to compute the coefficients in a linear combination of vectors from a linearly dependent spanning set. After several decades, Daubechies, Grossmann, and Meyer [5] connected frames with wavelets and Gabor systems. Heil [11] developed frame theory for general Hilbert space. Later, Grochenig [8] extended the frame theory to Banach space. In this paper, we delve into the concept of multiframelets, which offer a versatile approach to frames through

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wavelets. Multiframelets have gained considerable attention among mathematicians and engineers due to their extensive applications in signal processing, filter bank theory, and image processing. For a detailed discussion, readers are referred to [1, 2, 4, 5, 6, 7, 10, 17] and the references therein.

This article is arranged as follows: In Section 2, we discuss the basics of p-adic numbers and Fourier analysis on them. Multiframelet and its various properties on the p-adic setting are discussed in Section 3. Finally, we explore different properties of multiframelet sets along with pertinent examples in Section 4.

2. Preliminaries

We now provide a brief overview of p-adic analysis. For a thorough understanding of this topic, see references [14, 16]. For a prime number p, the p-adic norm on $\mathbb{Q}$ is defined by

$$|x|_p = \begin{cases} 0 & \text{if } x = 0, \\ p^{-\gamma} & \text{if } x = p^{\gamma}\frac{m}{n},\ p \nmid mn,\ \text{and } m, n, \gamma \in \mathbb{Z}. \end{cases}$$

This norm satisfies strong triangle property $|x+y|_p \leq \max\{|x|_p, |y|_p\}$, and the equality holds if $|x|_p \neq |y|_p$. Hence, this norm is non-Archimedean. The completion of $\mathbb{Q}$ with respect to the metric topology induced by $|\cdot|_p$ is said to be the set of p-adic numbers and denoted by $\mathbb{Q}_p$. Every p-adic number x $(\neq 0)$ can be written as $x = \sum\limits_{j=\gamma}^{\infty} x_j p^j$, where $x_j \in \{0, 1, \ldots, p-1\}$ with $x_\gamma \neq 0$ and $\gamma \in \mathbb{Z}$. Fractional part of x is $\{x\}_p := \sum\limits_{j=\gamma}^{-1} x_j p^j$. Thence, $\{x\}_p = 0$ if and only if $\gamma \geq 0$. Moreover, $\{0\}_p := 0$. The set of p-adic integers is $\mathbb{Z}_p := \{x \in \mathbb{Q}_p : \{x\}_p = 0\}$, and the set of fractional numbers is $I_p := \{x \in \mathbb{Q}_p : \{x\}_p = x\}$. Therefore, I_p can also be represented as $\left\{\frac{a_{-\gamma}}{p^{\gamma}} + \frac{a_{-\gamma+1}}{p^{\gamma-1}} + \cdots + \frac{a_{-1}}{p} \in \mathbb{Q}_p : 0 \leq a_l < p,\ l \in \mathbb{Z},\ \gamma \in \mathbb{N}\right\}$. Define $\mathbb{Z}_p^* := \{x \in \mathbb{Q}_p : |x|_p = 1\}$. Then $\mathbb{Z}_p^* = \mathbb{Z}_p \setminus p\mathbb{Z}_p$, and $\{p^j \mathbb{Z}_p^* : j \in \mathbb{Z}\}$ forms a partition of $\mathbb{Q}_p \setminus \{0\}$. The additive character χ_p is defined by $\chi_p(x) := e^{2\pi i \{x\}_p}$ on the field $\mathbb{Q}_p$.

A ball of radius p^{γ}, where $\gamma \in \mathbb{Z}$ and center at $c \in \mathbb{Q}_p$, is given by $B_\gamma(c) := \{x \in \mathbb{Q}_p : |x - c|_p \leq p^{\gamma}\}$. Each ball $B_\gamma(c)$ is both open and compact, and here it is very interesting to note that every point in the ball is also a center of the ball. Moreover, any two balls are either nested or disjoint. The set $\mathbb{Q}_p$ is totally disconnected, locally compact, and lacks isolated points. Being a locally compact abelian group, it possesses a Haar measure dx on $\mathbb{Q}_p$, which is nonnegative and translation invariant, that is, $d(x+t) = dx$, for all $t \in \mathbb{Q}_p$ and $\int\limits_{\mathbb{Z}_p} dx = 1$. Furthermore, $d(ax) = |a|_p\, dx$, for all $a \in \mathbb{Q}_p \setminus \{0\}$. The Hilbert space containing all square integrable, $\mathbb{C}$-valued functions defined on $\mathbb{Q}_p$ is denoted as $L^2(\mathbb{Q}_p)$. Then the inner product on $L^2(\mathbb{Q}_p)$ is defined by $\langle g, h\rangle := \int\limits_{\mathbb{Q}_p} g(x)\overline{h(x)}dx$, for all $g, h \in L^2(\mathbb{Q}_p)$. The Fourier transform of

$f \in L^2(\mathbb{Q}_p)$ is defined by $\widehat{f}(\xi) := \int\limits_{\mathbb{Q}_p} \chi_p(\xi x) f(x) dx, \ \forall\ \xi \in \mathbb{Q}_p$. This Fourier transform satisfies the following useful equality, namely, Parseval theorem [16],

$$\langle g, h\rangle = \langle \widehat{g}, \widehat{h}\rangle, \quad \text{for all} \quad g, h \in L^2(\mathbb{Q}_p).$$

Throughout this article, we denote the finite set $\{1, 2, \cdots, L\}$ as $\mathfrak{L}$, the characteristic function of any set E as $\mathbb{1}_E$, μ as Haar measure, and $p^{\frac{j}{2}} g(p^{-j} \cdot -a)$ as $g_{j,a}$ for any function g.

3. Multiframelet on $\mathbb{Q}_p$

Before diving into multiframelet sets in next section, we will provide an overview on multiframelets in this section.

Definition 3.1 (Multiframelet)**.** For finitely many $f^{(1)}, f^{(2)}, \ldots, f^{(L)} \in L^2(\mathbb{Q}_p)$, $\{f^{(l)}_{j,a} := p^{\frac{j}{2}} f^{(l)}(p^{-j} \cdot -a) : j \in \mathbb{Z},\ a \in I_p,\ l \in \mathfrak{L}\}$ is said to be a multiframelet of order L if there are positive numbers A, B satisfying

$$A \left\lVert h\right\rVert^2 \leq \sum_{l\in\mathfrak{L}} \sum_{a\in I_p} \sum_{j\in\mathbb{Z}} |\langle h, f^{(l)}_{j,a}\rangle|^2 \leq B \left\lVert h\right\rVert^2, \quad \text{for all} \quad h \in L^2(\mathbb{Q}_p). \tag{3.1}$$

The set $\{f^{(1)}, f^{(2)}, \ldots, f^{(L)}\}$ is said to be the generator of this multiframelet. Note that lower bound A and upper bound B are not unique. Furthermore, if it is feasible to choose A and B such that $A = B$, then the multiframelet is said to be a tight multiframelet. In this case, the multiframelet is also termed as an A-tight multiframelet. Moreover, if $A = B = 1$, then it is said to be a Parseval multiframelet. It is noteworthy that every multiwavelet is a Parseval multiframelet, but not vice versa.

Example 3.2. Multiwavelet given by Kozyrev [13] as

$$\theta_k(x) = \chi_p\left(\frac{kx}{p}\right) \mathbb{1}_{\mathbb{Z}_p}(x), \text{where } 1 \leq k \leq p-1,$$

is a multiframelet generator of order $(p-1)$.

Remark 3.3. Example 3.2 is also a generator of the corresponding Parseval multiframelet. Moreover, every multiwavelet also generates a Parseval multiframelet.

Proposition 3.4. *Elements of a multiframelet with upper bound B are bounded above by $\sqrt{B}$.*

Proof. For a fixed l', a', j', we have

$$\|f^{(l')}_{j',a'}\|^4 = |\langle f^{(l')}_{j',a'}\ ,\ f^{(l')}_{j',a'}\rangle|^2 \leq \sum_{l\in\mathfrak{L}} \sum_{a\in I_p} \sum_{j\in\mathbb{Z}} |\langle f^{(l')}_{j',a'}\ ,\ f^{(l)}_{j,a}\rangle|^2 \leq B\|f^{(l')}_{j',a'}\|^2,$$

that is, $\|f^{(l')}_{j',a'}\|^2 \leq B$, or $\|f^{(l')}_{j',a'}\| \leq \sqrt{B}$.

Therefore, elements of a multiframelet are bounded above by square root of its upper bound. ∎

The following result demonstrates the completeness of multiframelets, which is derived from the lower multiframelet condition.

Proposition 3.5. *Every multiframelet is complete in $L^2(\mathbb{Q}_p)$.*

Proof. We will prove this by contradiction. Let $V = \overline{\operatorname{span}}\{f_{j,a}^{(l)} : j \in \mathbb{Z},\ a \in I_p,\ l \in \mathfrak{L}\}$. Then $L^2(\mathbb{Q}_p) = V \oplus V^\perp$. If $V^\perp \neq \{0\}$, then we can choose $w \in V^\perp \setminus \{0\}$ satisfying

$$A\,\|w\|^2 \leq \sum_{l\in\mathfrak{L}} \sum_{a\in I_p} \sum_{j\in\mathbb{Z}} |\langle w, f_{j,a}^{(l)}\rangle|^2 = 0.$$

This implies $w = 0$, which is a contradiction. Hence, $L^2(\mathbb{Q}_p) = V$. Thus, a multiframelet is complete. □

Remark 3.6. In view of Proposition 3.5, it is clear that every multiframelet is extension of an orthonormal basis.

Definition 3.7 (Multiframelet sequence)**.** If a sequence $\{f_{j,a}^{(l)} : j \in \mathbb{Z},\ a \in I_p,\ l \in \mathfrak{L}\}$ forms multiframelet for $\overline{\operatorname{span}}\{f_{j,a}^{(l)} : j \in \mathbb{Z},\ a \in I_p,\ l \in \mathfrak{L}\}$, then it is said to be a multiframelet sequence.

Note that every multiframelet is also a multiframelet sequence. The application of a unitary operator preserves the multiframelet sequences, as demonstrated in the following theorem.

Theorem 3.8. *Let $\{f_{j,a}^{(l)} : j \in \mathbb{Z},\ a \in I_p,\ l \in \mathfrak{L}\}$ be a multiframelet sequence having bounds A, B, and let $\mathcal{U}$ be a unitary operator on $L^2(\mathbb{Q}_p)$. Then $\{\mathcal{U}f_{j,a}^{(l)} : j \in \mathbb{Z},\ a \in I_p,\ l \in \mathfrak{L}\}$ is a multiframelet sequence with the same bounds.*

Proof. Let $g \in \operatorname{span}\{\mathcal{U}f_{j,a}^{(l)} : j \in \mathbb{Z},\ a \in I_p,\ l \in \mathfrak{L}\}$. Then there exists $h \in \operatorname{span}\{f_{j,a}^{(l)} : j \in \mathbb{Z},\ a \in I_p,\ l \in \mathfrak{L}\}$ such that $g = \mathcal{U}h$. As $\mathcal{U}$ is unitary, we have

$$\sum_{l\in\mathfrak{L}} \sum_{a\in I_p} \sum_{j\in\mathbb{Z}} |\langle g, \mathcal{U}f_{j,a}^{(l)}\rangle|^2 = \sum_{l\in\mathfrak{L}} \sum_{a\in I_p} \sum_{j\in\mathbb{Z}} |\langle \mathcal{U}h, \mathcal{U}f_{j,a}^{(l)}\rangle|^2 = \sum_{l\in\mathfrak{L}} \sum_{a\in I_p} \sum_{j\in\mathbb{Z}} |\langle h, f_{j,a}^{(l)}\rangle|^2.$$

Since $\{f_{j,a}^{(l)} : j \in \mathbb{Z},\ a \in I_p,\ l \in \mathfrak{L}\}$ is a multiframelet sequence having bounds A, B, we have

$$\begin{aligned}
&A\|h\|^2 \leq \sum_{l\in\mathfrak{L}} \sum_{a\in I_p} \sum_{j\in\mathbb{Z}} |\langle h, f_{j,a}^{(l)}\rangle|^2 \leq B\|h\|^2,\\
\text{or}\quad &A\|\mathcal{U}h\|^2 \leq \sum_{l\in\mathfrak{L}} \sum_{a\in I_p} \sum_{j\in\mathbb{Z}} |\langle \mathcal{U}^{-1}g, f_{j,a}^{(l)}\rangle|^2 \leq B\|\mathcal{U}h\|^2,\\
\text{or}\quad &A\|g\|^2 \leq \sum_{l\in\mathfrak{L}} \sum_{a\in I_p} \sum_{j\in\mathbb{Z}} |\langle \mathcal{U}^*g, f_{j,a}^{(l)}\rangle|^2 \leq B\|g\|^2,\\
\text{or}\quad &A\|g\|^2 \leq \sum_{l\in\mathfrak{L}} \sum_{a\in I_p} \sum_{j\in\mathbb{Z}} |\langle g, \mathcal{U}f_{j,a}^{(l)}\rangle|^2 \leq B\|g\|^2.
\end{aligned}$$

In view of [9, Proposition 3], the above inequality holds for $g \in \overline{\operatorname{span}}\{\mathcal{U}f_{j,a}^{(l)} : j \in \mathbb{Z},\ a \in I_p,\ l \in \mathfrak{L}\}$. Therefore, $\{\mathcal{U}f_{j,a}^{(l)} : j \in \mathbb{Z},\ a \in I_p,\ l \in \mathfrak{L}\}$ is a multiframelet sequence with bounds A, B. □

4. Multiframelet set

Similar to multiwavelet sets, multiframelet sets determine multiframelets. In this section, we introduce multiframelet sets on $\mathbb{Q}_p$ and explore the properties of Parseval multiframelet sets through relevant examples.

Definition 4.1 (Multiframelet set)**.** A measurable set $\mathfrak{F} \subset \mathbb{Q}_p$ is said to be a multiframelet set of order L if $\mathfrak{F} = \bigcup\limits_{l \in \mathfrak{L}} F_l$, where $\widehat{f^{(l)}} = \mathbb{1}_{F_l}$, for $l \in \mathfrak{L}$ and $\{f_{j,a}^{(l)} : j \in \mathbb{Z},\ a \in I_p,\ l \in \mathfrak{L}\}$ is a multiframelet for $L^2(\mathbb{Q}_p)$.

Example 4.2. The set $\mathfrak{F}_p := B_0\left(-\frac{1}{p}\right) \cup \cdots \cup B_0\left(-\frac{p-1}{p}\right)$ is a multiframelet set of order $(p-1)$.

Remark 4.3. It is readily apparent that $\mathfrak{F}_p$ is open and compact but not connected. Furthermore, every multiwavelet set is a Parseval multiframelet set. However, the converse is not true, as evident from the following two consecutive examples.

Example 4.4 (Parseval multiframelet set)**.** The set $(1+p\mathbb{Z}_p) \cup (2+p\mathbb{Z}_p) \cup \cdots \cup (p-1+p\mathbb{Z}_p)$ is a Parseval multiframelet set for $L^2(\mathbb{Q}_p)$.

Proof. Consider $\widehat{f^{(l)}} = \mathbb{1}_{l+p\mathbb{Z}_p}$, for $l = 1, 2, \ldots, p-1$. We know that the set containing characteristic functions of balls is complete in $L^2(\mathbb{Q}_p)$, as $\mathbb{Q}_p$ is locally compact abelian group and $\{f_{j,a}^{(l)} : j \in \mathbb{Z},\ a \in I_p,\ 1 \le l \le p-1\}$ is invariant under translations and dilations. Therefore, in order to prove that $\{f_{j,a}^{(l)} : j \in \mathbb{Z},\ a \in I_p,\ 1 \le l \le p-1\}$ is a Parseval multiframelet, it is sufficient to check the Parseval multiframelet condition for $\mathbb{1}_{\mathbb{Z}_p}$. Now,

$$
\begin{aligned}
\sum_{l=1}^{p-1} \sum_{a \in I_p} \sum_{j \in \mathbb{Z}} |\langle f_{j,a}^{(l)}, \mathbb{1}_{\mathbb{Z}_p} \rangle|^2 &= \sum_{l=1}^{p-1} \sum_{a \in I_p} \sum_{j \in \mathbb{Z}} |\langle \widehat{f_{j,a}^{(l)}}, \widehat{\mathbb{1}}_{\mathbb{Z}_p} \rangle|^2 \\
&= \sum_{l=1}^{p-1} \sum_{a \in I_p} \sum_{j \in \mathbb{Z}} p^{-j} \left| \int\limits_{\mathbb{Z}_p} \chi_p(p^j a\xi) \mathbb{1}_{p^{-j}(l+p\mathbb{Z}_p)}(\xi) d\xi \right|^2 \\
&= \sum_{l=1}^{p-1} \sum_{a \in I_p} \sum_{\substack{j \in \mathbb{Z}, \\ j \le 0}} p^{-j} \left| \int\limits_{p^{-j}(l+p\mathbb{Z}_p)} \chi_p(p^j a\xi) d\xi \right|^2 \\
&= \sum_{l=1}^{p-1} \sum_{\substack{a \in I_p, \\ |a|_p \le p}} \sum_{\substack{j \in \mathbb{Z}, \\ j \le 0}} p^{-j} \left| p^{j-1} \chi_p(al) \right|^2 = 1 = \|\mathbb{1}_{\mathbb{Z}_p}\|^2.
\end{aligned}
$$

Therefore, $(1+p\mathbb{Z}_p) \cup (2+p\mathbb{Z}_p) \cup \cdots \cup (p-1+p\mathbb{Z}_p)$ is a Parseval multiframelet set for $L^2(\mathbb{Q}_p)$. □

Each component of the multiframelet set in Example 4.4 has measure $\frac{1}{p}$ $(\neq 1)$, which disqualifies it as a multiwavelet set for $L^2(\mathbb{Q}_p)$. We will now provide a more general example of Parseval multiframelet sets.

Example 4.5. For $m \in \mathbb{N} \cup \{0\}$, $p^m \bigcup\limits_{l=1}^{p-1} B_0\left(-\frac{l}{p}\right)$ is a Parseval multiframelet set for $L^2(\mathbb{Q}_p)$.

Proof. Note that, $p^m \bigcup\limits_{l=1}^{p-1} B_0\left(-\frac{l}{p}\right) = \bigcup\limits_{l=1}^{p-1}(lp^{m-1} + p^m\mathbb{Z}_p)$. We prove this by using same argument as given in Example 4.4. Therefore, it is enough to check that, $\sum\limits_{l=1}^{p-1}\sum\limits_{a\in I_p}\sum\limits_{j\in\mathbb{Z}} |\langle f^{(l)}_{j,a}, \mathbb{1}_{\mathbb{Z}_p}\rangle|^2 = \|\mathbb{1}_{\mathbb{Z}_p}\|^2$, where $\widehat{f^{(l)}} = \mathbb{1}_{lp^{m-1}+p^m\mathbb{Z}_p}$, for $1 \leq l \leq p-1$. Now,

$$\begin{aligned}
\sum_{l=1}^{p-1}\sum_{a\in I_p}\sum_{j\in\mathbb{Z}} |\langle f^{(l)}_{j,a}, \mathbb{1}_{\mathbb{Z}_p}\rangle|^2 &= \sum_{l=1}^{p-1}\sum_{a\in I_p}\sum_{j\in\mathbb{Z}} |\langle \widehat{f^{(l)}_{j,a}}, \widehat{\mathbb{1}}_{\mathbb{Z}_p}\rangle|^2 \\
&= \sum_{l=1}^{p-1}\sum_{a\in I_p}\sum_{j\in\mathbb{Z}} p^{-j}\left|\int\limits_{\mathbb{Z}_p\cap p^{-j}(lp^{m-1}+p^m\mathbb{Z}_p)} \chi_p(p^j a\xi)d\xi\right|^2 \\
&= \sum_{l=1}^{p-1}\sum_{a\in I_p}\sum_{\substack{j\in\mathbb{Z},\\ j\leq m-1}} p^{-j}\left|\int\limits_{p^{-j}(lp^{m-1}+p^m\mathbb{Z}_p)} \chi_p(p^j a\xi)d\xi\right|^2 \\
&= \sum_{l=1}^{p-1}\sum_{a\in I_p}\sum_{\substack{j\in\mathbb{Z},\\ j\leq m-1}} p^{j-2m}\left|\chi_p(alp^{m-1})\mathbb{1}_{\mathbb{Z}_p}(ap^m)\right|^2 \\
&= \sum_{l=1}^{p-1}\sum_{\substack{a\in I_p,\\ |a|_p\leq p^m}}\sum_{\substack{j\in\mathbb{Z},\\ j\leq m-1}} p^{j-2m} = 1 = \|\mathbb{1}_{\mathbb{Z}_p}\|^2.
\end{aligned}$$

Therefore, $(p^{m-1} + p^m\mathbb{Z}_p) \cup (2p^{m-1} + p^m\mathbb{Z}_p) \cup \cdots \cup ((p-1)p^{m-1} + p^m\mathbb{Z}_p)$ is a Parseval multiframelet set for $L^2(\mathbb{Q}_p)$. □

It is pertinent to mention that, the set of all multiwavelet sets $\subset$ the set of all Parseval multiframelet sets $\subset$ the set of all tight multiframelet sets $\subset$ the set of all multiframelet sets. However, the next result demonstrates a correlation between multiframelet bounds and the measure of the corresponding components in multiframelet sets.

Proposition 4.6. *Suppose that $\{f^{(l)}_{j,a} : j \in \mathbb{Z},\ a \in I_p,\ l \in \mathfrak{L}\}$ is a multiframelet for $L^2(\mathbb{Q}_p)$ having upper bound B, where $\widehat{f^{(l)}} = \mathbb{1}_{F_l}$ and F_l is a measurable subset of $\mathbb{Q}_p$, for all $l \in \mathfrak{L}$. Then measure of each F_l is less than or equal to B.*

Proof. In view of Proposition 3.4 and the Parseval theorem, we have

$$\|\widehat{f^{(l)}_{j,a}}\|^2 \leq B, \quad \text{for all } j \in \mathbb{Z}, a \in I_p, l \in \mathfrak{L}.$$

Now, $\|\widehat{f^{(l)}_{j,a}}\|^2 = \int\limits_{\mathbb{Q}_p} |p^{-\frac{j}{2}}\chi_p(p^j a\xi)\widehat{f^{(l)}}(p^j\xi)|^2 d\xi = \int\limits_{\mathbb{Q}_p} p^{-j}\mathbb{1}_{F_l}(p^j\xi)d\xi = \mu(F_l)$. □

Theorem 4.7. *If $\{f^{(l)}_{j,a} : j \in \mathbb{Z},\ a \in I_p,\ l \in \mathfrak{L}\}$ is a Parseval multiframelet for $L^2(\mathbb{Q}_p)$, where $\widehat{f^{(l)}} = \mathbb{1}_{F_l}$ and F_l is a measurable subset of $\mathbb{Q}_p$, for all $l \in \mathfrak{L}$, then*

$$\|g\|^2 = \sum_{l\in\mathfrak{L}} \sum_{a\in I_p} \sum_{j\in\mathbb{Z}} p^{-j} \left| \int\limits_{p^{-j}F_l} \chi_p(p^j a\xi)\ \overline{\widehat{g}(\xi)}\ d\xi \right|^2, \quad \textit{for all} \ \ g \in L^2(\mathbb{Q}_p).$$

Proof. As $\{f^{(l)}_{j,a} : j \in \mathbb{Z},\ a \in I_p,\ l \in \mathfrak{L}\}$ is a Parseval multiframelet,

$$\|g\|^2 = \sum_{l\in\mathfrak{L}} \sum_{a\in I_p} \sum_{j\in\mathbb{Z}} |\langle f^{(l)}_{j,a}\ , g\rangle|^2, \quad \text{for all} \ \ g \in L^2(\mathbb{Q}_p).$$

Now, using Parseval theorem, we get

$$\begin{aligned}
\|g\|^2 &= \sum_{l\in\mathfrak{L}} \sum_{a\in I_p} \sum_{j\in\mathbb{Z}} |\langle \widehat{f^{(l)}_{j,a}}\ , \widehat{g}\rangle|^2 \\
&= \sum_{l\in\mathfrak{L}} \sum_{a\in I_p} \sum_{j\in\mathbb{Z}} \left| \int\limits_{\mathbb{Q}_p} p^{-\frac{j}{2}}\ \chi_p(p^j a\xi)\ \widehat{f^{(l)}}(p^j\xi)\ \overline{\widehat{g}(\xi)}\ d\xi \right|^2 \\
&= \sum_{l\in\mathfrak{L}} \sum_{a\in I_p} \sum_{j\in\mathbb{Z}} \left| \int\limits_{p^{-j}F_l} p^{-\frac{j}{2}}\ \chi_p(p^j a\xi)\ \overline{\widehat{g}(\xi)}\ d\xi \right|^2 \\
&= \sum_{l\in\mathfrak{L}} \sum_{a\in I_p} \sum_{j\in\mathbb{Z}} p^{-j} \left| \int\limits_{p^{-j}F_l} \chi_p(p^j a\xi)\ \overline{\widehat{g}(\xi)}\ d\xi \right|^2 .
\end{aligned}$$

Hence our assertion is tenable. ☐

Corollary 4.8. *Let $\{f^{(l)}_{j,a} : j \in \mathbb{Z},\ a \in I_p,\ l \in \mathfrak{L}\}$ be a Parseval multiframelet for $L^2(\mathbb{Q}_p)$. If $\widehat{f^{(l)}} = \mathbb{1}_{F_l}$, where F_l is a measurable subset of $\mathbb{Q}_p$, for all $l \in \mathfrak{L}$, then*

$$\sum_{l\in\mathfrak{L}} \sum_{a\in I_p} \sum_{j\in\mathbb{Z}} p^j \left| \int\limits_{p^j\mathbb{Z}_p\cap F_l} \chi_p(a\xi) d\xi \right|^2 = 1.$$

Shukla, Maury and Mittal [15] studied semi-orthogonal Parseval wavelets on local fields of positive characteristic. We now define semi-orthogonal multiframelet on $\mathbb{Q}_p$ as follows.

Definition 4.9 (Semi-orthogonal multiframelet). A multiframelet $\{f^{(l)}_{j,a} : j \in \mathbb{Z},\ a \in I_p,\ l \in \mathfrak{L}\}$ is said to be a semi-orthogonal multiframelet if

$$\overline{\text{span}}\{f^{(l)}_{j,a} : a \in I_p,\ l \in \mathfrak{L}\} \perp \overline{\text{span}}\{f^{(l)}_{j',a} : a \in I_p,\ l \in \mathfrak{L}\}, \ \text{for all}\ j \neq j'.$$

Theorem 4.10. *Let $\{f_{j,a}^{(l)} : j \in \mathbb{Z},\ a \in I_p,\ l \in \mathfrak{L}\}$ be a collection of functions in $L^2(\mathbb{Q}_p)$ such that $\widehat{f^{(l)}} = \mathbb{1}_{F_l}$ and let $\mathfrak{F} := \bigcup\limits_{l \in \mathfrak{L}} F_l$ be the union of mutually disjoint measurable subsets of $\mathbb{Q}_p$. If $\{f_{j,a}^{(l)} : j \in \mathbb{Z},\ a \in I_p,\ l \in \mathfrak{L}\}$ is a semi-orthogonal multiframelet for $L^2(\mathbb{Q}_p)$, then*

(i) *$\{p^j \mathfrak{F} : j \in \mathbb{Z}\}$ is a measurable partition of $\mathbb{Q}_p$, and*

(ii) $\sum\limits_{l \in \mathfrak{L}} \int\limits_{\mathbb{Q}_p} \widehat{f^{(l)}}(\xi) \frac{d\xi}{|\xi|_p} = 1 - \frac{1}{p}$.

Proof. We know that any multiframelet is complete in $L^2(\mathbb{Q}_p)$. As $\mathfrak{F}$ is a disjoint union, we have

$$\overline{\text{span}}\{f_{0,a}^{(l)} : a \in I_p,\ l \in \mathfrak{L}\} = \{g \in L^2(\mathbb{Q}_p) : \text{supp } (\widehat{g}) \subset \mathfrak{F}\}$$

and furthermore,

$$\overline{\text{span}}\{f_{j,a}^{(l)} : j \in \mathbb{Z},\ a \in I_p,\ l \in \mathfrak{L}\} = \{g \in L^2(\mathbb{Q}_p) : \text{supp } (\widehat{g}) \subset p^{-j}\mathfrak{F}\}.$$

Since $\{f_{j,a}^{(l)} : j \in \mathbb{Z},\ a \in I_p,\ l \in \mathfrak{L}\}$ is semi-orthogonal and complete,

$$\bigoplus_{j \in \mathbb{Z}} \overline{\text{span}}\{f_{j,a}^{(l)} : a \in I_p,\ l \in \mathfrak{L}\} = L^2(\mathbb{Q}_p).$$

Thus $\{p^j \mathfrak{F} : j \in \mathbb{Z}\}$ is a measurable partition of $\mathbb{Q}_p$.

Moreover, it is known that $\{p^j \mathbb{Z}_p^* : j \in \mathbb{Z}\}$ is a partition of $\mathbb{Q}_p \setminus \{0\}$. Since the collections $\{F_l : l \in \mathfrak{L}\}$ and $\{p^j \mathfrak{F} : j \in \mathbb{Z}\}$ consist of mutually disjoint sets, therefore

$$\begin{aligned}
\sum_{l \in \mathfrak{L}} \int\limits_{\mathbb{Q}_p} \widehat{f^{(l)}}(\xi) \frac{d\xi}{|\xi|_p} &= \sum_{l \in \mathfrak{L}} \int\limits_{\mathbb{Q}_p} \mathbb{1}_{F_l}(\xi) \frac{d\xi}{|\xi|_p} \\
&= \sum_{l \in \mathfrak{L}} \sum_{j \in \mathbb{Z}} \int\limits_{p^j \mathbb{Z}_p^*} \mathbb{1}_{F_l}(\xi) \frac{d\xi}{|\xi|_p} \\
&= \sum_{j \in \mathbb{Z}} \int\limits_{\mathbb{Z}_p^*} \mathbb{1}_{p^j \mathfrak{F}}(\xi) \frac{d\xi}{|\xi|_p} \\
&= \int\limits_{\mathbb{Z}_p^*} \mathbb{1}_{\mathbb{Q}_p}(\xi) \frac{d\xi}{|\xi|_p} = \mu(\mathbb{Z}_p^*) = 1 - \frac{1}{p}.
\end{aligned}$$

□

The following theorem provides a constructive representation of the multiframelet condition through corresponding multiframelet set.

Theorem 4.11. *Let $\{f_{j,a}^{(l)} : j \in \mathbb{Z},\ a \in I_p,\ l \in \mathfrak{L}\}$ be such that $\widehat{f^{(l)}} = \mathbb{1}_{F_l}$, for each $l \in \mathfrak{L}$, where each F_l is a measurable subset of $\mathbb{Q}_p$. Then $\bigcup\limits_{l \in \mathfrak{L}} F_l$ is a multiframelet* *set having bounds A, B if and only if*

$$A\|h\|^2 \le \sum_{l\in\mathfrak{L}}\sum_{a\in I_p}\sum_{j\in\mathbb{Z}} p^{-j}|\langle h, \chi_p(p^j a\cdot)\mathbb{1}_{p^{-j}F_l}\rangle|^2 \le B\|h\|^2, \quad \text{for all} \;\; h\in L^2(\mathbb{Q}_p).$$

Proof. As the Fourier transform is a unitary operator, multiframelet condition (3.1) is equivalent to

$$A\|h\|^2 \le \sum_{l\in\mathfrak{L}}\sum_{a\in I_p}\sum_{j\in\mathbb{Z}} |\langle h, \widehat{f_{j,a}^{(l)}}\rangle|^2 \le B\|h\|^2$$

$$\text{or,}\quad A\|h\|^2 \le \sum_{l\in\mathfrak{L}}\sum_{a\in I_p}\sum_{j\in\mathbb{Z}} |\langle h, p^{-\frac{j}{2}}\chi_p(p^j a\cdot)\widehat{f^{(l)}}(p^j\cdot)\rangle|^2 \le B\|h\|^2$$

$$\text{or,}\quad A\|h\|^2 \le \sum_{l\in\mathfrak{L}}\sum_{a\in I_p}\sum_{j\in\mathbb{Z}} p^{-j}|\langle h, \chi_p(p^j a\cdot)\mathbb{1}_{p^{-j}F_l}\rangle|^2 \le B\|h\|^2, \tag{4.1}$$

for all $h\in L^2(\mathbb{Q}_p)$. $\square$

Corollary 4.12. *Suppose that $\bigcup_{l\in\mathfrak{L}} F_l \subset \mathbb{Q}_p$ is a multiframelet set corresponding to a multiframelet having bounds A, B. Then*

$$A \le \sum_{l\in\mathfrak{L}}\sum_{a\in I_p}\sum_{j\in\mathbb{Z}} p^{-j}|\langle \mathbb{1}_{\mathbb{Z}_p}, \chi_p(p^j a\cdot)\mathbb{1}_{p^{-j}F_l}\rangle|^2 \le B.$$

Proof. Considering $h = \mathbb{1}_{\mathbb{Z}_p}$, (4.1) reduces to

$$A \le \sum_{l\in\mathfrak{L}}\sum_{a\in I_p}\sum_{j\in\mathbb{Z}} p^{-j}|\langle \mathbb{1}_{\mathbb{Z}_p}, \chi_p(p^j a\cdot)\mathbb{1}_{p^{-j}F_l}\rangle|^2 \le B.$$

$\square$

5. Conclusion

The results presented in this article represented a theoretical advancement in multiframelet analysis, specifically focusing on multiframelet sets over $\mathbb{Q}_p$. These developments expanded the potential applications of this field in mathematical physics, signal processing, string theory, quantum field theory and other fields.

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