Khayyam J. Math. 10 (2024), no. 2, 337-348

DOI: 10.22034/kjm.2024.475795.3304

Khayyam Journal of Mathematics

emis.de/journals/KJM
kjm-math.org

ON THE FRACTIONAL P-LAPLACIAN PROBLEM VIA YOUNG MEASURES

IHYA TALIBI[1*], FARAH BALAADICH[1], BRAHIM EL BOUKARI[1] AND JALILA EL GHORDAF[1]

Communicated by L. Barreira

ABSTRACT. In this paper, we study the existence results for a parabolic problem of the following form:

$$\begin{cases} \frac{\partial u}{\partial t} + (-\Delta)_p^s u = f & \text{in } \Omega \times (0,T), \\ u = 0 & \text{in } (\mathbb{R}^n \backslash \Omega) \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$, and $(-\Delta)_p^s$ is the fractional p-Laplacian. Under appropriate assumptions concerning the main functions, the existence of weak solutions is obtained by applying the Galerkin method combined with the theory of Young measures.

1. Introduction

The study of fractional Laplacian and nonlocal operators has received a lot of interest in recent years. This kind of operator arises in various applications such as population dynamics, continuum mechanics, image process, phase transition phenomena, Lévy processes, and game theory [4, 9, 10, 11, 22]. For this reason, it is particularly important to study equations where such nonlocal operators are involved. When the fractional p-Laplacian is concerned, the elliptic and parabolic cases have been actively studied in [15, 24, 25, 19, 27, 29] and many others.

Date: Received: 28 August 2024; Revised: 13 November 2024; Accepted: 14 November 2024.

* Corresponding author.

2020 *Mathematics Subject Classification.* Primary: 35A01; Secondary: 35K60, 35K55.

Key words and phrases. Weak solution, fractional p-Laplacian, Young measures, Galerkin method, approximation solution.

In this paper, we take into account the following parabolic problem:

$$\begin{cases} \frac{\partial u}{\partial t} + (-\Delta)_p^s u = f & \text{in } Q_T, \\ u = 0 & \text{in } (\mathbb{R}^n \backslash \Omega) \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded open domain of $\mathbb{R}^n$, $T > 0$ is constant, $Q_T = \Omega \times (0,T)$, f belongs to $L^{p'}(0,T;W_0^*)$, $p' = \frac{p}{p-1}$ and $L^{p'}(0,T;W_0^*)$ is a dual space of $L^p(0,T;W_0)$, in which W_0 will be introduced in Section 2. The fractional p-Laplacian operator $(-\Delta)_p^s u$ is defined as follows:

$$(-\Delta)_p^s u(x,t) = P.V \int_{\mathbb{R}^n} \frac{|u(x,t) - u(y,t)|^{p-2}(u(x,t) - u(y,t))}{|x-y|^{n+ps}} dy, \quad x \in \mathbb{R}^n,$$

where $P.V$, which stands for "in the principal value sense", is a frequently used abbreviation; for more information on these operators, see [9].

There are a large number of references in the literature studying problem (1.1) where the second member belongs to different spaces. Among all of them, the authors in [2] proved the existence of weak solutions if $(f, u_0) \in L^1(Q_T) \times L^1(\Omega)$, and has a nonnegative entropy solution if f and u_0 are nonnegative. Abdellaoui [1] proved the asymptotic behavior result of entropy solutions when the right-hand side does not depend on time.

When $p = 2$, several authors have studied the elliptic and parabolic cases: For the elliptic case, see [3, 16]; for the parabolic case, the authors in [18] studied the existence of solutions to problem same to (1.1), using duality and approximation arguments.

In [21], for $p \neq 2$ and $f \equiv 0$, the authors obtained the existence and uniqueness of strong solutions for all $u_0 \in L^2(\Omega)$ and also studied the asymptotic behavior of these solutions as t tends to ∞. We also refer to [28] for more details.

The idea of studying the fractional problem (1.1) inspired in the classical case,

$$\frac{\partial u}{\partial t} - \operatorname{div} \sigma(x,t,u,Du) + g(x,t,u,Du) = f, \tag{1.2}$$

where the source term f is assumed to belong to $L^{p'}\left(0,T;W^{-1,p'}(\Omega;\mathbb{R}^m)\right)$. The problem (1.2) has been studied in [6]. The authors obtained the existence of a weak solution using the theory of Young measures. In the theory of nonlinear partial differential equations and in the calculus of variations, Young measures have recently become an increasingly important tool to establish the existence of solutions; see [13, 26] for more information on this theory. To the best of our knowledge, the parabolic problem (1.1) has never been studied by the theory of Young measures. We suggest readers to consult [5, 6, 7], which have been applied to some elliptic and parabolic problems. Balaadich and Azroul [8] proved the existence of weak solutions to the problem of the form $(-\Delta)_p^s u = f(x,u)$ equipped with a Dirichlet boundary condition. They used the Galerkin approximation and Young measures to get the needed result.

Motivated by all of the results above, especially the elliptical case in [8], we will investigate the existence of weak solutions to the parabolic problem (1.1) by using the theory of Young measures and the Galerkin approximation.

This article is organized into three sections. In section 2, we give some background information on fractional Sobolev spaces and a review of the theory of the Young measures. The last section is devoted to the Galerkin approximation and necessary a priori estimates, and we finish by passing to the limit and proving the main results.

2. Preliminaries and notations

In this part, we present some notations, definitions, and properties of functional spaces, which will be used in what follows.

Let s, p be two real numbers such that $0 < s < 1$ and $1 < p < \infty$. We define p_s^* the fractional critical exponent as follows:

$$p_s^* = \begin{cases} \infty \text{ if } ps \geq n, \\ np/(n-ps) \text{ if } ps < n. \end{cases}$$

Let Ω be an open set in $\mathbb{R}^n$, let $Q_\Omega = (\mathbb{R}^n \times \mathbb{R}^n)\setminus(\mathbb{R}^n\setminus\Omega \times \mathbb{R}^n\setminus\Omega)$, and let $Q_\tau = \Omega \times (0,\tau)$ for all $\tau \in (0,T]$. It is clear that $\Omega \times \Omega$ is strictly contained in Q_Ω. Also, W is a linear space of Lebesgue measurable functions from $\mathbb{R}^n$ to $\mathbb{R}^m$ that satisfies two conditions: Their restriction on Ω belongs to $L^p(\Omega;\mathbb{R}^m)$ and they satisfy the following inequality:

$$\iint_{Q_\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}} dydx < \infty.$$

In this context, W has a norm defined as follows:

$$\|u\|_W = \|u\|_{L^p(\Omega;\mathbb{R}^m)} + \left(\iint_{Q_\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}} dydx\right)^{\frac{1}{p}}.$$

Moreover, there exists a closed linear subspace of W denoted by W_0, consisting of the elements u of W that satisfy the following condition

$$u = 0 \text{ almost everywhere in } \mathbb{R}^n\setminus\Omega.$$

In the subspace W_0, a norm can also be defined as follows:

$$\|u\|_{W_0} = \left(\iint_{Q_\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}} dydx\right)^{\frac{1}{p}}.$$

It is well known that the pair $(W_0, \|\cdot\|_{W_0})$ is a uniformly convex reflexive Banach space (see [30]).

In what follows, let $2 \leq p < \frac{n}{s}$ and let C_i, $i = 1,2,\dots$ be positive constants that vary from line to line and are independent of the terms involved in any limit process. We note the following functional space $L^p(0,T;W_0)$, is a Banach space endowed with the norm

$$\|u\|_{L^p(0,T;W_0)} = \left(\int_0^T \|u\|_{W_0}^p dt\right)^{\frac{1}{p}}.$$

Lemma 2.1. [14] *$\mathcal{C}_0^\infty(\Omega;\mathbb{R}^m)$ is a space of infinitely differentiable functions with compact support on Ω which is dense in W_0.*

Lemma 2.2. [20] *The following embedding* $W_0 \hookrightarrow L^\theta(\Omega;\mathbb{R}^m)$ *is compact for all* $\theta \in [1, p_s^*)$ *and continuous for all* $\theta \in [1, p_s^*]$.

In the following, $\mathcal{C}_0(\mathbb{R}^m)$ stands for the space of continuous functions on $\mathbb{R}^m$ with compact support with regard to the $\|\cdot\|_\infty$-norm. The space of signed Radon measures with finite mass is denoted by $\mathcal{M}(\mathbb{R}^m)$. The corresponding duality is given by

$$\langle \mu, \rho \rangle = \int_{\mathbb{R}^m} \rho(\lambda) d\mu(\lambda).$$

Definition 2.3. [8] Let a bounded sequence be denoted by $\{z_j\}_{j\geq 1}$ in $L^\infty(\Omega;\mathbb{R}^m)$. Then there exist a subsequence $\{z_k\} \subset \{z_j\}$ and a Borel probability measure μ_x on $\mathbb{R}^m$ for almost every $x \in \Omega$, such that for a.e. $\rho \in \mathcal{C}(\mathbb{R}^m)$ we have $\rho(z_k) \rightharpoonup^* \bar{\rho}$ weakly in $L^\infty(\Omega)$, where $\bar{\rho}(x) = \langle \mu_x, \rho \rangle = \int_{\mathbb{R}^m} \rho(\lambda) d\mu_x(\lambda)$ for a.e. $x \in \Omega$.

Lemma 2.4. [13] *Let* $\Omega \subset \mathbb{R}^n$ *be Lebesgue measurable (not necessarily bounded) and let* z_j *from* Ω *to* $\mathbb{R}^m$*, with* $j \in \mathbb{N}$*, be a sequence of Lebesgue measurable functions. Then there exist a subsequence* z_k *and a family* $\{\mu_x\}_{x\in\Omega}$ *of nonnegative Radon measures on* $\mathbb{R}^m$*, such that*

- **i)** $\|\mu_x\|_{\mathcal{M}(\mathbb{R}^m)} := \int_{\mathbb{R}^m} d\mu_x(\lambda) \leq 1$ *for almost* $x \in \Omega$*,*
- **ii)** $\rho(z_k) \rightharpoonup^* \bar{\rho}$ *weakly in* $L^\infty(\Omega)$ *for all* $\mathcal{C}_0(\mathbb{R}^m)$*, where* $\bar{\rho} = \langle \mu_x, \rho \rangle$*,*
- **iii)** *If for all* $M > 0$

$$\lim_{N\to\infty} \sup_{k\in\mathbb{N}} |\{x \in \Omega \cap B_M(0) : |z_k(x)| \geq N\}| = 0, \tag{2.1}$$

then $\|\mu_x\| = 1$ *for a.e.* $x \in \Omega$*, and for any measurable* $\Omega' \subset \Omega$ *we have* $\rho(z_k) \rightharpoonup \bar{\rho} = \langle \mu_x, \rho \rangle$ *weakly in* $L^1(\Omega')$ *for continuous function* ρ *provided the sequence* $\rho(z_k)$ *is weakly precompact in* $L^1(\Omega')$.

3. Existence of weak solutions

In this part, we will define a weak solution for the problem (1.1).

Definition 3.1. A function $u \in L^p(0,T;W_0)$ with $\frac{\partial u}{\partial t} \in L^{p'}(0,T;W_0^*)$ is called a weak solution of (1.1) if

$$\begin{aligned} &\int_{Q_T} \frac{\partial u}{\partial t} \phi dx dt \\ &+ \int_0^T \iint_{Q_\Omega} \frac{|u(x,t)-u(y,t)|^{p-2}(u(x,t)-u(y,t))}{|x-y|^{n+ps}} (\phi(x,t)-\phi(y,t)) dx dy dt \\ &\quad = \int_{Q_T} f(x,t) \phi dx dt, \end{aligned}$$

holds for all $\phi \in L^p(0,T;W_0)$.

The aim of this work is to prove the Theorem 3.2 below. The technique is structured as follows: We use the Galerkin method to build the approximation solution and the tool of Young measures to pass through the limit. We will prove the following existence theorem:

Theorem 3.2. *Let $u_0 \in W_0$ and let f be assumed to belong to $L^{p'}(0,T;W_0^*)$. Then, the problem (1.1) has at least one weak solution in the sense of Definition 3.1.*

Proof. **Step 1**: Galerkin approximation:
Similar to that in [17, 23], we take $\{\psi_j\}_{j\geq 1}$ an orthonormal basis in $L^2(\Omega;\mathbb{R}^m)$ such that $\{\psi_j\}_{j\geq 1} \subset C_0^\infty(\Omega;\mathbb{R}^m) \subset \overline{\bigcup_{k\geq 1} V_k}^{C_1(\bar{\Omega})}$, where $V_k = \operatorname{span}\{\psi_1,\ldots,\psi_k\}$.

Remark that since $u_0 \in W_0$, we can find a sequence $\delta_k(x) \in V_k$ such that

$$\delta_k(x) \to u_0(x) \text{ in } W_0 \text{ as } k \to \infty.$$

Indeed, for $u_0 \in W_0$, there exists a sequence η_k in $C_0^\infty(\Omega;\mathbb{R}^m)$ such that verifies that

$$\eta_k \to u_0 \text{ in } W_0 \text{ as } k \to \infty.$$

Since $\{\eta_k\} \subset C_0^\infty(\Omega;\mathbb{R}^m) \subset \overline{\bigcup_{M\geq 1} V_M}^{C^1(\bar{\Omega})}$, there exists a sequence $\{\eta_k^i\} \subset \bigcup_{M\geq 1} V_M$ such that

$$\eta_k^i \to \eta_k \text{ in } C^1\left(\bar{\Omega};\mathbb{R}^m\right) \text{ as } i \to \infty.$$

For $\frac{1}{2^k}$, there exists $i_k \geq 1$ such that

$$\left\|\eta_k^{i_k} - \eta_k\right\|_{C^1(\bar{\Omega})} \leq \frac{1}{2^k}.$$

Therefore

$$\left\|\eta_k^{i_k} - u_0\right\|_{W_0} \leq C_1 \left\|\eta_k^{i_k} - \eta_k\right\|_{C^1(\bar{\Omega})} + \|\eta_k - u_0\|_{W_0}.$$

Hence

$$\eta_k^{i_k} \to u_0 \text{ in } W_0 \text{ as } k \to \infty.$$

We set $u_k = \eta_k^{i_k}$. Since $u_k \in \bigcup_{M\geq 1} V_M$, there exists V_{M_k} such that $u_k \in V_{M_k}$. Without loss of generality, we assume that $V_{M_1} \subset V_{M_2}$ as $M_1 \leq M_2$. We suppose that $M_1 > 1$ and define δ_k as follows:

$$\begin{cases} \delta_k(x) = 0, & k = 1,\ldots,M_1 - 1,\\ \delta_k(x) = u_1, & k = M_1,\ldots,M_2 - 1,\\ \delta_k(x) = u_2, & k = M_2,\ldots,M_3 - 1,\\ \vdots & \vdots \end{cases}$$

We obtain the desired sequence $\{\delta_k\}$ such that

$$\delta_k \to u_0 \text{ in } W_0 \text{ as } k \to \infty.$$

Let H_k be a function of $[0,T) \times \mathbb{R}^k$ with these values in $\mathbb{R}^k$, where k is fixed and $i = 1,\ldots,k$. We define H_k as

$$(H_k(t,\varsigma))_i = \iint_{Q_\Omega} \frac{\left|\sum_{j=1}^k (\varsigma_j(t))_j \psi_j(x) - \sum_{j=1}^k (\varsigma_j(t))_j \psi_j(y)\right|^{p-2}}{|x-y|^{n+ps}} \times \left(\sum_{j=1}^k (\varsigma_j(t))_j \psi_j(x) - \sum_{j=1}^k (\varsigma_j(t))_j \psi_j(y)\right)(\psi_i(x) - \psi_i(y))\, dxdy.$$

The function $H_k(t,\varsigma)$ is continuous in t and ς. Now, we shall construct the approximating solutions for (1.1) as follows:

$$u_k(x,t)=\sum_{j=1}^{k}\left(a_j(t)\right)_j\psi_j(x), \tag{3.1}$$

where unknown functions $(a(t))_j$ are determined by the following system of ordinary differential equation:

$$\begin{cases} a'(t)+H_k(t,a(t))=G_k\,, & 0<t<T,\\ a(0)=\mathcal{M}_k(0), \end{cases} \tag{3.2}$$

where

$$(G_k)_i=\int_\Omega f_k\psi_i dx,\ (\mathcal{M}_k(0))_i=\int_\Omega \delta_k(x)\psi_i dx,$$

and

$$\delta_k(x)\to u_0 \text{ in } W_0 \text{ as } k\to\infty,\ \text{ where } \delta_k(x)\in V_k,$$

and

$$f_k\to f \text{ in } L^{p'}(0,T;W_0^*).$$

Multiplying (3.2) by $a(t)$, we arrive at the equality

$$a'a+H_k(t,a)a=G_k a. \tag{3.3}$$

Here

$$\begin{aligned} H_k(t,a)a=\iint_{Q_\Omega}&\frac{\left|\sum_{j=1}^{k}\left(a_j(t)\right)_j\psi_j(x)-\sum_{j=1}^{k}\left(a_j(t)\right)_j\psi_j(y)\right|^{p-2}}{|x-y|^{n+ps}}\\ &\times\left(\sum_{j=1}^{k}\left(a_j(t)\right)_j\psi_j(x)-\sum_{j=1}^{k}\left(a_j(t)\right)_j\psi_j(y)\right)\\ &\times\left(\sum_{i=1}^{k}\left(a_i(t)\right)_i\psi_i(x)-\sum_{i=1}^{k}\left(a_i(t)\right)_i\psi_i(y)\right)dxdy\geqslant 0. \end{aligned} \tag{3.4}$$

From (3.3), (3.4), and Young's inequality, we conclude that

$$\frac{1}{2}\frac{d}{dt}|a(t)|^2\le|G_k a|\le\frac{1}{2}|G_k|^2+\frac{1}{2}|a|^2.$$

By integrating the above inequality, we get

$$|a(t)|\le C_k+\int_0^t|a(s)|^2ds.$$

According to Gronwall's inequality, we can infer that $|a(t)|\le C_k(T)$. Put

$$\mathcal{F}_k=\max_{t\in[0,T]}|G-H_k(t,a)| \qquad\text{and}\qquad s_k=\min\left\{T,\frac{C_k(T)}{\mathcal{F}_k}\right\}.$$

Since $H_k(t,a)$ is continuous in t and a, using [12, Peano theorem], we get that problem (3.2) has a C^1 solution on $[0,s_k]$. We put $l_1=s_k$, and let $a(l_1)$ be an initial value; then we can repeat the above process and get C^1 solution on $[l_1,l_2]$, with $l_2=l_1+s_k$. which means the existence of an interval $[l_{i-1},l_{i-2}]\subset[0,T]$, such that (3.2) has a solution on $[l_{i-1}, l_{i-2}]$, where $l_i = l_{i-1}+s_k, i = 1, \dots, N-1, l_N = T$. Therefore, we get a solution $a_k(t) \in C^1([0,T])$. Consequently, we arrive at the Galerkin approximation solution mentioned in (3.1).

Step 2: A priori estimates

By (3.2), for each $t \in [0,T]$ and $1 \le i \le k$, we have

$$\begin{aligned}&\int_\Omega \frac{\partial u_k}{\partial t}\psi_i dx\\&+\iint_{Q_\Omega} \frac{|u_k(x,t)-u_k(y,t)|^{p-2}(u_k(x,t)-u_k(y,t))}{|x-y|^{n+ps}}(\psi_i(x)-\psi_i(y))dxdy\\&\quad=\int_\Omega f(x,t)\psi_i dx.\end{aligned} \tag{3.5}$$

Remark 3.3. By (3.5), the following equality holds for all $\phi \in C^1(0,T;V_N)$ $(N \le n)$

$$\begin{aligned}&\int_0^T\int_\Omega \frac{\partial u_k}{\partial t}\phi dxdt\\&+\int_0^T\iint_{Q_\Omega} \frac{|u_k(x,t)-u_k(y,t)|^{p-2}(u_k(x,t)-u_k(y,t))}{|x-y|^{n+ps}}(\phi(x,t)-\phi(y,t))dxdydt\\&\quad=\int_0^T\int_\Omega f(x,t)\phi dxdt.\end{aligned}$$

Multiplying (3.5) by $(a(t))_i$ and summing with respect to i and integrating with respect to t from 0 to τ ($\tau \in (0,T]$), we arrive at

$$\begin{aligned}&\int_{Q_\tau} \frac{\partial u_k}{\partial t}u_k dxdt\\&+\int_0^\tau\iint_{Q_\Omega} \frac{|u_k(x,t)-u_k(y,t)|^{p-2}(u_k(x,t)-u_k(y,t))}{|x-y|^{n+ps}}(u_k(x,t)-u_k(y,t))dxdydt\\&\quad\le \|f\|_{L^{p'}(0,\tau;W_0^*)}\|u_k\|_{L^p(0,\tau;W_0)}.\end{aligned} \tag{3.6}$$

Since $u_k(x,0) \to u_0$ in W_0, in particular $u_k(x,0) \to u_0$ in $L^2(\Omega,\mathbb{R}^m)$, then $\int_\Omega u_k^2(x,0)dx \le C_2$. Moreover $\|f\|_{L^{p'}(0,T;W_0^*)} \le C_3$; we deduce that

$$\begin{aligned}\frac{1}{2}\int_\Omega |u_k(x,\tau)|^2dx+&\int_0^\tau \|u_k\|_{W_0}^p dt\\&\le C_4\left(\|u_k\|_{L^p(0,\tau;W_0)}+1\right).\end{aligned} \tag{3.7}$$

According to (3.7), we get

$$\|u_k\|_{L^p(0,T;W_0)} \le C_5 \tag{3.8}$$

and

$$\int_\Omega |u_k(x,T)|^2dx \le C_6. \tag{3.9}$$

Now, we are in the position to pass to the limit

Assertion 3 : Passage to the limit

From the estimations in (3.7), the sequence $(u_k)_k$ is bounded in $L^p(0,T;W_0)$ and $L^\infty(0,T;L^2(\Omega;\mathbb{R}^m))$. Therefore, there exists a subsequence (still denoted by $(u_k)_k$) such that

$$u_k \rightharpoonup u \text{ in } L^p(0,T;W_0),$$
$$u_k \rightharpoonup^* u \text{ in } L^\infty(0,T;L^2(\Omega;\mathbb{R}^m)).$$

Let $\phi \in L^p(0,T;W_0)$; then there exists a sequence $\phi_k \in C^1(0,T;V_k)$ such that $\phi_k \to \phi$ in $L^p(0,T;W_0)$. By virtue of Remark 3.3 and Hölder's inequality, we get

$$\begin{aligned}&\left|\int_0^T \int_\Omega \frac{\partial u_k}{\partial t}\phi_k dxdt\right|\\ &=\left|\int_0^T \iint_{Q_\Omega} \frac{|u_k(x,t)-u_k(y,t)|^{p-2}(u_k(x,t)-u_k(y,t))}{|x-y|^{n+ps}}(\phi_k(x,t)-\phi_k(y,t))dxdydt\right.\\ &\left.-\int_0^T\int_\Omega f(x,t)\phi_k dxdt\right|\\ &\leq C_7\|\phi_k\|_{L^p(0,T;W_0)}.\end{aligned}$$

Therefore, $\left\|\frac{\partial u_k}{\partial t}\right\|_{L^{p'}\left(0,T;W_0^*\right)} \leq C_8$. Consequently

$$\frac{\partial u_k}{\partial t} \rightharpoonup \chi \quad \text{ in } L^{p'}\left(0,T;W_0^*\right) \text{ (for a subsequence).}$$

Let $\varphi \in C_0^\infty\left(Q_T;\mathbb{R}^m\right)$, by letting $k\to\infty$ in $\displaystyle\int_{Q_T}\frac{\partial u_k}{\partial t}\varphi dxdt = -\int_{Q_T} u_k\frac{\partial \varphi}{\partial t}dxdt.$ We arrive at

$$\int_{Q_T}\chi\varphi dxdt = -\int_{Q_T} u\frac{\partial\varphi}{\partial t}dxdt.$$

This implies $\chi = \frac{\partial u}{\partial t}$. Furthermore, since $\displaystyle\int_\Omega u_k(x,T)^2dx \leq C_6$, then $u_k(x,T)\to\hat{u}$ in $L^2\left(\Omega;\mathbb{R}^m\right)$ (for a subsequence), where $\hat{u}$ in $L^2\left(\Omega;\mathbb{R}^m\right)$. Moreover, for any $a(t)\in C^1([0,T])$ and $\phi\in C_0^\infty(\Omega)$, such that

$$\int_{Q_T}\frac{\partial u_k}{\partial t}a\phi dxdt = \int_\Omega u_k(x,T)a(T)\phi dx - \int_\Omega u_k(x,0)a(0)\phi dx - \int_{Q_T} u_k\frac{\partial a}{\partial t}\phi dxdt.$$

Tending k to ∞, we get

$$\int_\Omega\left(\hat{u}-u(x,T)\right)a(T)\phi dx - \int_\Omega\left(u_0(x)-u(x,0)\right)a(0)\phi dx = 0.$$

We choosing $a(T)=1$, $a(0)=0$ or $a(T)=0$, $a(0)=1$, we have $\hat{u}=u(x,T)$ and $u_0(x)=u(x,0)$.

The Young measure is a tool used for fixing problems that may arise when weak convergence does not behave as desired. We consider the following lemma, which gives the weak convergence of approximation solution.

Lemma 3.4. *Assume that (3.8) is holds. Then, there exists a Young measure $\mu_{(x,y,t)}$ generated by $\frac{u_k(x,t)-u_k(y,t)}{|x-y|^{\frac{n}{p}+s}} \in L^p(Q_\Omega \times (0,T))$ such that*

1) $\left\|\mu_{(x,y,t)}\right\|_{\mathcal{M}(\mathbb{R}^m)} = 1$ *for a.e.* $(x,y,t) \in Q_\Omega \times (0,T)$*, that is,* $\mu_{(x,y,t)}$ *is a probability measure.*

2) $\left\langle \mu_{(x,y,t)}, id \right\rangle = \int_{\mathbb{R}^m} \lambda d\mu_{(x,y,t)}(\lambda)$ *is the weak* L^1*-limit of* $\frac{u_k(x,t)-u_k(y,t)}{|x-y|^{\frac{n}{p}+s}}$.

3) $\left\langle \mu_{(x,y,t)}, id \right\rangle = \frac{u_k(x,t)-u_k(y,t)}{|x-y|^{\frac{n}{p}+s}}$ *for a.e.* $(x,y,t) \in Q_\Omega \times (0,T)$.

Proof. **1)** We consider

$$v_k(x,y,t) = \frac{u_k(x,t)-u_k(y,t)}{|x-y|^{\frac{n}{p}+s}} \in L^p\left(Q_\Omega \times (0,T); \mathbb{R}^m\right). \tag{3.10}$$

Since $B_M = B(0,M)$ is the ball of radius M and of center 0, we know that for any $M > 0$, $(\Omega \cap B_M)^2 \subseteq \Omega \times \Omega \subsetneq Q_\Omega$. Let $N \in \mathbb{R}$ such that

$$Q_N \equiv \{(x,y,t) \in \Omega \cap B_M \times \Omega \cap B_M \times (0,T) : |u_k(x,y,t)| \geq N\}.$$

Using (3.8), we get

$$\|v_k\|^p_{L^p(Q_\Omega \times (0,T))} = \int_0^T \iint_{Q_\Omega} \frac{|u_k(x,t)-u_k(y,t)|^p}{|x-y|^{n+ps}} dxdydt = \|u_k\|^p_{L^p(0,T;W_0)} \leq M.$$

Consequently, there exists $C_9 \geq 0$ such that

$$C_9 \geq \iint_{Q_\Omega \times (0,T)} |v_k(x,y,t)|^p \, dxdy \geq \iint_{Q_N} |v_k(x,y,t)|^p \, dxdy \geq N^p \, |Q_N|, \tag{3.11}$$

where $|Q_N|$ is the Lebesgue measure of Q_N. According to (3.11), the sequence (v_k) satisfies (2.1). Hence, a Young measure denoted by $\mu_{(x,y,t)}$ is generated by v_k such that $\left\|\mu_{(x,y,t)}\right\|_{\mathcal{M}(\mathbb{R}^m)} = 1$ for a.e. $(x,y,t) \in Q_\Omega \times (0,T)$.

2) By (3.8), there exists a subsequence still denoted by (v_k) that convergences in $L^p\left(Q_\Omega \times (0,T); \mathbb{R}^m\right)$. Since $L^p\left(Q_\Omega \times (0,T); \mathbb{R}^m\right)$ is reflexive $(p > 1)$, then v_k is weakly convergent in $L^1\left(Q_\Omega \times (0,T); \mathbb{R}^m\right)$. By the third assertion in Lemma 2.4, we replace the function ρ by the identity function, and we then have

$$v_k \rightharpoonup \left\langle \mu_{(x,y,t)}, id \right\rangle = \int_{\mathbb{R}^m} \lambda d\mu_{(x,y,t)}(\lambda) \text{ weakly in } L^1\left(Q_\Omega \times (0,T); \mathbb{R}^m\right).$$

3) According to (3.8), v_k is bounded in $L^p\left(Q_\Omega \times (0,T); \mathbb{R}^m\right)$. Then there exists a subsequence such that $v_k \rightharpoonup v$ in $L^p\left(Q_\Omega \times (0,T); \mathbb{R}^m\right)$. Owing to the previous arguments, we get from the uniqueness of limits that

$$\left\langle \mu_{(x,y,t)}, id \right\rangle = v(x,y,t) = \frac{u(x,t)-u(y,t)}{|x-y|^{\frac{n}{p}+s}} \quad \text{for a.e. } (x,y,t) \in Q_\Omega \times (0,T).$$

□

Now, let $\{v_k\}$ be the sequence given in (3.10), that is,

$$v_k(x,y,t) = \frac{u_k(x,t) - u_k(y,t)}{|x-y|^{\frac{n+ps}{p}}}.$$

According to weak convergence given in Lemma 3.4, we have

$$\begin{aligned} |v_k(x,y,t)|^{p-2}v_k(x,y,t) &\rightharpoonup \int_{\mathbb{R}^m} |\lambda|^{p-2}\lambda d\mu_{(x,y,t)}(\lambda) \\ &= |v(x,y,t)|^{p-2}v(x,y,t) \\ &= \frac{|u(x,t)-u(y,t)|^{p-2}(u(x,t)-u(y,t))}{|x-y|^{\frac{n+ps}{p'}}} \end{aligned} \tag{3.12}$$

weakly in $L^1(Q_\Omega \times (0,T);\mathbb{R}^m)$. Since L^p is reflexive and $|v_k(x,y,t)|^{p-2}v_k(x,y,t)$ is bounded in $L^{p'}(Q_\Omega \times (0,T);\mathbb{R}^m)$, the sequence $|v_k(x,y,t)|^{p-2}v_k(x,y,t)$ converges in $L^{p'}(Q_\Omega \times (0,T);\mathbb{R}^m)$. Hence its weak $L^{p'}$-limit is also $|v(x,y,t)|^{p-2}v(x,y,t)$. Thus, for any $\varphi \in L^p(0,T;W_0)$ we have

$$\frac{\varphi(x,t)-\varphi(y,t)}{|x-y|^{\frac{n+ps}{p}}} \in L^p(Q_\Omega \times (0,T);\mathbb{R}^m).$$

According to weak limit in (3.12), we arrive at

$$\begin{aligned} &\lim_{k\to\infty} \int_0^T \iint_{Q_\Omega} \frac{|u_k(x,t)-u_k(y,t)|^{p-2}(u_k(x,t)-u_k(y,t))}{|x-y|^{n+ps}}(\varphi(x,t)-\varphi(y,t))dxdydt \\ &= \int_0^T \iint_{Q_\Omega} \frac{|u(x,t)-u(y,t)|^{p-2}(u(x,t)-u(y,t))}{|x-y|^{n+ps}}(\varphi(x,t)-\varphi(y,t))dxdydt \end{aligned}$$

for every $\varphi \in L^p(0,T;W_0)$.
According to Remark 3.3, for all $\varphi \in C^1(0,T;V_N)$, $N \le k$, we have

$$\begin{aligned} &\int_{Q_T} \frac{\partial u_k}{\partial t}\varphi dxdt \\ &\quad + \int_0^T \iint_{Q_\Omega} \frac{|u_k(x,t)-u_k(y,t)|^{p-2}(u_k(x,t)-u_k(y,t))}{|x-y|^{n+ps}}(\varphi(x,t)-\varphi(y,t))dxdydt \\ &\qquad = \int_{Q_T} f(x,t)\varphi dxdt. \end{aligned}$$

For N tend to ∞, from the discussion above, we get

$$\begin{aligned} &\int_{Q_T} \frac{\partial u}{\partial t}\varphi dxdt \\ &\quad + \int_0^T \iint_{Q_\Omega} \frac{|u(x,t)-u(y,t)|^{p-2}(u(x,t)-u(y,t))}{|x-y|^{n+ps}}(\varphi(x,t)-\varphi(y,t))dxdydt \\ &\qquad = \int_{Q_T} f(x,t)\varphi dxdt. \end{aligned} \tag{3.13}$$

Then, (3.13) holds for all $\varphi \in L^p(0,T;W_0)$, according to the density of the set $C^1(0,T;\cup_{M\geqslant 1}V_M)$ in $L^p(0,T;W_0)$. □

References

1. B. Abdellaoui, *On the behavior of entropy solutions for a fractional*, Rend. Mat. Appl., **43** (2022), no. 7, 103–132.
2. B. Abdellaoui, A. Attar, R. Bentifour and I. Peral, *On fractional p-Laplacian parabolic problem with general data*, Ann. Mat. Pura Appl., **197** (2018), 329–356.
3. N. Alibaud, B. Andreianov and M. Bendahmane, *Renormalized solutions of the fractional Laplace equation*, C. R. Math., **348** (2010), no. 13-14, 759–762.
4. D. Applebaum, *Lévy processes-from probability to finance and quantum groups*, Not. Am. Math. Soc., **51** (2004), no. 11, 1336–1347.
5. E. Azroul and F. Balaadich, *Weak solutions for generalized p-Laplacian systems via Young measures*, Moroc. J. Pure Appl. Anal., **4** (2018), no. 2, 77–84.
6. E. Azroul and F. Balaadich, *Strongly quasilinear parabolic systems in divergence form with weak monotonicity*, Khayyam J. Math., **6** (2020), no. 1, 57–72.
7. F. Balaadich, *Existence of solutions for parabolic variational inequalities*, Rend. Circ. Mat. Palermo (2), **73** (2023), no. 2, 731–745.
8. F. Balaadich and E. Azroul, *Existence results for fractional p-Laplacian systems via Young measures*, Math. Model. Anal., **27** (2022), no. 2, 232–241.
9. L.A. Caffarelli, *Nonlocal equations, drifts and games*, in: Non. Partial Diff. Eq.: The Abel Symposium 2010. Springer Berlin Heidelberg, 2012.
10. L. Caffarelli and L. Silvestre, *An extension problem related to the fractional Laplacian*, Comm. Partial Differential Equations, **32** (2007), no. 8, 1245–1260.
11. L. Caffarelli and E. Valdinoci, *Uniform estimates and limiting arguments for nonlocal minimal surfaces*, Calc. Var. Partial Differ. Equ., **41** (2011), no. 1-2, 203–240.
12. E.A. Coddington, N. Levinson and T. Teichmann, *Theory of ordinary diferential equations*, London, 1956.
13. G. Dolzmann, N. Hungerbühler and S. Müller, *Non-linear elliptic systems with measure-valued right hand side*, Math. Z., **226** (1997), 545–574.
14. A. Fiscella, R. Servadei, and E. Valdinoci, *Density properties for fractional Sobolev spaces*, Ann. Acad. Sci. Fenn. Math., **40** (2015), no. 1, 235–253.
15. J. Giacomon and S. Tiwari, *Existence and global behavior of solutions to fractional p-laplacian parabolic problems*, Electron. J. Differential Equations, **2018** (2018), no. 44, 1–20.
16. K.H. Karlsen, F. Petitta and S. Ulusoy, *A duality approach to the fractional Laplacian with measure data*, Publ. Mat., **55** (2011), no. 1, 151–161.
17. R. Landes, *On the existence of weak solutions for quasilinear parabolic initial-boundary value problems*, in: Proceedings of the Royal Society of Edinburgh Section A: Mathematics, **89** (1981), no 3-4, 217–237.
18. T. Leonori, I. Peral, A. Primo and F. Soria, *Basic estimates for solutions of a class of nonlocal elliptic and parabolic equations*, Disc. Contin. Dyn. Syst., **35** (2015), no. 12, 6031–6068.
19. M. Liao, Q. Liu and H. Ye, *Global existence and blow-up of weak solutions for a class of fractional p-Laplacian evolution equations*, Adv. Nonlinear Anal., **9** (2020), no. 1, 1569–1591.
20. Y. Liu, Z. Liu, C. F. Wen and J. C. Yao, *Existence of solutions for non-coercive variational-hemivariational inequalities involving the nonlocal fractional p-Laplacian*, Optimization, **71** (2022), no. 3, 485–503.
21. J.M. Mazón, J.D. Rossi and J. Toledo, *Fractional p-Laplacian evolution equations*, J. Math. Pures Appl., **105** (2016), no. 6, 810–844.
22. R. Metzler and J. Klafter, *The restaurant at the end of the random walk: recent developments in the description of anomalous transport by fractional dynamics*, J. Phys. A Math. Gen., **37** (2004), no. 31, R161.
23. X. Mingqi, V. D. Rădulescu and B. Zhang, *Nonlocal Kirchhoff diffusion problems: local existence and blow-up of solutions*, *Nonlinearity*, **31** (2018), no.7, 3228.

24. G. Molica Bisci and R. Servadei, *A bifurcation result for non-local fractional equations*, Anal. Appl., **13** (2015), no.4, 371–394.
25. N. Pan, B. Zhang and J. Cao, *Degenerate Kirchhoff-type diffusion problems involving the fractional p-Laplacian*, Nonlinear Anal. Real World Appl., **37** (2017), 56–70.
26. L. Tartar, *Compensated compactness and applications to partial differential equations*, In: Nonlinear analysis and mechanics: Heriot-Watt symposium, **4** (1997), 136–212.
27. K. Teng, C. Zhang and S. Zhou, *Renormalized and entropy solutions for the fractional p-Laplacian evolution equations*, J. Evol. Equ., **19** (2019), 559–584.
28. J.L. Vázquez, *The Dirichlet problem for the fractional p-Laplacian evolution equation*, J. Differ. Equ., **260** (2016), no. 7, 6038–6056.
29. J.L. Vázquez, *The evolution fractional p-Laplacian equation in $\mathbb{R}^N$. Fundamental solution and asymptotic behaviour*, Nonlinear Analysis, **199** (2020), 112034.
30. M. Xiang, B. Zhang, and M. Ferrara, *Existence of solutions for Kirchhoff type problem involving the non-local fractional p-Laplacian*, J. Math. Appl., **424** (2015), no. 2, 1021–1041.

[1] LABORATORY OF APPLIED MATHEMATICS AND SCIENTIFIC COMPUTING, SULTAN MOULAY SLIMANE UNIVERSITY, BENI MELLAL, MOROCCO

Email address: `talibiihya@gmail.com; balaadich.edp@gmail.com;`

Email address: `elboukaribrahim@yahoo.fr; elg_jalila@yahoo.fr`