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ASYMPTOTIC ANALYSIS AND BLOW-UP RATE OF RADIAL SOLUTIONS OF A NONLINEAR ELLIPTIC EQUATION INCLUDING THE P-LAPLACIAN OPERATOR

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Abstract. The aim of this study is to examine a nonlinear elliptic equation subjected to singular boundary conditions. By employing the techniques of sub- and super-solutions, we demonstrate that the problem under discussion has only one positive solution and explore the asymptotic behavior of the solution obtained. In addition, we illustrate our theoretical results with some useful examples.

1. Introduction

Consider $E = \mathcal{B}_\rho(0)$ the ball of radius $\rho > 0$ centered at 0. We attempt to study the problem:

$$\begin{cases} -\Delta_p w = \lambda w^{l_1} - a(x) w^{l_2} & \text{in } \ E, \\ w \geq 0 & \text{in } \ E, \\ w = \infty & \text{on } \ \partial E, \end{cases} \tag{1.1}$$

with $\lambda > 0$, $1 \leq l_1 \leq p-1 < l_2$, a is a continuous and strictly positive function on $\mathcal{B}_\rho(0)$, and $\Delta_p w := div(|\nabla w|^{p-2}\nabla w)$ is the p-Laplacian operator. Boundary value problems involving the p-Laplacian operator appear in the study of several physical models as the reaction-diffusion systems, nonlinear elasticity, astronomy, glaciology, propagation through porous media, and so on. For example, problem (1.1) arises when modeling non-Newtonian fluids (see, e.g., [1, 17]) and non-Newtonian filtration [9]. The real number p in problem (1.1) characterizes the

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medium. When $p > 2$, it is called dilatant fluids and when $p = 2$, they represent Newtonian fluid. If $p = 2$, then (1.1) becomes

$$\begin{cases} -\Delta w = \lambda w^{l_1} - a(x) w^{l_2} & \text{in } E, \\ w \geq 0 & \text{in } E, \\ w = \infty & \text{on } \partial E, \end{cases} \tag{1.2}$$

and is known as the logistic equation. If $\lambda = 0$ and $a \equiv 1$, then the problem (1.2) was studied in [12, 15]. The case where $\lambda = 0$ and a is a positive continuous function such that a and $\frac{1}{a}$ are bounded, was studied in [2, 14]. If $\lambda = 0$ and $a(x) > 0$, then the problem (1.2) was concerned by [11]. In [3], the authors examined the problem (1.2) when $\lambda = 1$ and $a \equiv 1$. The paper [22] considered the following problem:

$$\begin{cases} -\Delta w = \lambda w^{\frac{1}{m}} - a(x) w^{\frac{p}{m}} & \text{in } E, \\ w \geq 0 & \text{in } E, \\ w = \infty & \text{on } \partial E, \end{cases} \tag{1.3}$$

where $p > m \geq 1$, $\lambda \geq 0$, and $a : E \to \mathbb{R}^+$ is a continuous function. The author of [22] showed under some conditions on the function a, that problem (1.3) admits a unique solution, and established the blow-up rate of this solution. When $\lambda \in \mathbb{R}$, $l_1 = 1$ and $a(x) > 0$ in E, problem (1.2) was discussed in [21, 20, 16, 24, 8, 19]. For example, in [8] the author considered the semilinear equation:

$$-\Delta w = \lambda w - a(x) w^p. \tag{1.4}$$

They studied the existence, nonexistence, and uniqueness of nonnegative solutions to (1.4) on both compact Riemannian manifolds and bounded smooth domains in $\mathbb{R}^N$ under homogeneous Dirichlet boundary conditions. In the case when $p > 2$, $\lambda = 0$, and $a \equiv 1$, problem (1.1) was investigated in [10]. The work [18] dealt with the following problem:

$$\begin{cases} \Delta_p w = a(x) g(w) & \text{in } E, \\ w \geq 0 & \text{in } E, \\ w = \infty & \text{on } \partial E. \end{cases} \tag{1.5}$$

He proved boundary asymptotics to solutions of (1.5) under some conditions on a and the nonlinearity g. Also, the uniqueness of solutions of problem (1.5) would follow as a consequence. If $p > 2$, $\lambda > 0$, $l_1 = p - 1$, and $a(x) > 0$ in E, problem (1.1) was discussed by [25].

Here, we focus our interesting into problem (1.1) under the assumptions:

$(\mathcal{H}_1)$ There is a real number $\mathcal{M}_0$ such that

$$\lim_{r \to \rho} \frac{(\mathcal{K}(r))^2}{\mathcal{K}^*(r) a(r)} = \mathcal{M}_0 \geq 1,$$

where $\mathcal{K}(r) = \int_r^\rho a(s)\, ds$ and $\mathcal{K}^*(r) = \int_r^\rho \mathcal{K}(s)\, ds$.

($\mathcal{H}_2$) The function $\dfrac{\mathcal{K}(r)}{a(r)}$ belongs to $\mathcal{C}^1([0,\rho])$.

We will establish that (1.1) has only one positive solution by employing the techniques of sub- and super-solutions. We investigate also the asymptotic behavior of the obtained solution via the use of a suitable couple of sub-solution and super-solution. In addition, we will give some examples of applications for our theoretical results. Note that the novelty and the originality of our work are presented not only on the consideration of the p-Laplacian operator but also can be viewed in the complexity of the behavior of the nonlinear source term in the right-hand side of the equation in (1.1). Another difficulty comeback from the considerations of the sub- and super-solutions methods. Since this method requires the knowledge of a suitable couple of functions that are often more complicated to be built. We mention that we will validate our main result via the construction of sub- and super-solution to (1.1). This later will give to the readers a feasible approach to build the so-called sub- and super-solutions not only to problem (1.1) but also to a more general class of nonlinear elliptic partial differential equations.

We have structured our paper as follows. In section 2, we present some auxiliary results that will be utilized in the proof of primary findings. We will introduce a nonlinear elliptic problem with nonhomogeneous Dirichlet boundary conditions. We define the notions of the solution, sub- and super-solutions. After that, we show the existence of a large solution. Section 3 is devoted to presenting the main result of this work. We start by enunciating our main theorem. Thereafter, we give the demonstration of the main result into three subsections. In the first one, we give a technical construction of an ordered couple of sub- and super-solutions. In the second, we establish the existence of a solution to the considered problem, and we examine the asymptotic behavior of the obtained solution. The third subsection will be reserved to show the uniqueness result. In section 4, we will give some applications to illustrate our results. In section 5, we give a conclusion concerning the main results obtained.

2. Auxiliary results

Here, we give some interesting results used in the demonstration of our main result. Let φ be a positive function in $\mathcal{C}^1(\partial E)$, and assume the ($\mathcal{H}_1$) and ($\mathcal{H}_2$) hold true. Consider

$$\begin{cases} -\Delta_p w = \lambda w^{l_1} - a(x) w^{l_2} & \text{in } \ E, \\ w = \varphi & \text{on } \ \partial E. \end{cases} \tag{2.1}$$

The primary outcome of this section will be based on the sub- and super-solution methods. To the reader's convenience, we give in the following definitions the notion of weak solutions sub- and super-solution to be considered in the resolution of problem (2.1).

Definition 2.1. A function w defined on E and taking values in $\mathbb{R}$ is a solution to (2.1) if it belongs to $\mathcal{C}^1(E)$ and satisfies

$$\begin{cases} \int\limits_E |\nabla w|^{p-2}\nabla w \nabla \phi \,\mathrm{d}x = \lambda \int\limits_E w^{l_1}\phi \,\mathrm{d}x - \int\limits_E a(x) w^{l_2}\phi \,\mathrm{d}x, \\ w = \varphi \text{ on } \partial E, \end{cases}$$

for every test function $\phi \in \mathcal{C}_c^\infty(E)$.

Definition 2.2.
- A function $\underline{w}$ defined on E and taking values in $\mathbb{R}$ is a sub-solution to problem (2.1), if it belongs to $\mathcal{C}^1(E)$ and satisfies

$$\begin{cases} \int\limits_E |\nabla \underline{w}|^{p-2}\nabla \underline{w} \nabla \phi \,\mathrm{d}x \le \lambda \int\limits_E \underline{w}^{l_1}\phi \,\mathrm{d}x - \int\limits_E a(x) \underline{w}^{l_2}\phi \,\mathrm{d}x, \\ \underline{w} \le \varphi \text{ on } \partial E, \end{cases} \tag{2.2}$$

  for all nonnegative test function $\phi \in \mathcal{C}_c^\infty(E)$.
- A function $\overline{w}$ defined on E and taking values in $\mathbb{R}$ is a super-solution to (2.1) if it belongs to $\mathcal{C}^1(E)$ and satisfies the inequality (2.2) in which $(\le)$ is replaced by $(\ge)$.

Before enunciating the main result of this section, we recall an interesting lemma, which is often known as the comparison principle.

Lemma 2.3 ([25])**.** *Let $w_1, w_2 \in \mathcal{C}^1(E)$ be both positive in E such that*

$$\begin{cases} -\Delta_p w_1 \le \lambda w_1^{l_1} - a(x) w_1^{l_2} & \text{in } E, \\ -\Delta_p w_2 \ge \lambda w_2^{l_1} - a(x) w_2^{l_2} & \text{in } E, \end{cases} \tag{2.3}$$

satisfied in the sense of distribution, and $w_1 \le \varphi \le w_2$ on ∂E. Then$w_1 \le w_2$ on $\overline{E}$.

Lemma 2.4. *Let $(\underline{z}, \overline{z})$ be a couple of positive function belonging in $\mathcal{C}^1(E)$ such that*

(i)
$$\begin{cases} \int\limits_\Omega |\nabla \underline{z}|^{p-2}\nabla \underline{z} \nabla \phi \,\mathrm{d}x \le \lambda \int\limits_\Omega \underline{z}^{l_1}\phi \,\mathrm{d}x - \int\limits_\Omega a(x) \underline{z}^{l_2}\phi \,\mathrm{d}x, \\ \int\limits_\Omega |\nabla \overline{z}|^{p-2}\nabla \overline{z} \nabla \phi \,\mathrm{d}x \ge \lambda \int\limits_\Omega \overline{z}^{l_1}\phi \,\mathrm{d}x - \int\limits_\Omega a(x) \overline{z}^{l_2}\phi \,\mathrm{d}x, \end{cases} \tag{2.4}$$

for all nonnegative test functions $\phi \in \mathcal{C}_c^\infty(E)$.

(ii) $\lim\limits_{d(x)\to 0} \underline{z}(x) = \lim\limits_{d(x)\to 0} \overline{z}(x) = \infty.$

Then, there is at least one solution $z \in \mathcal{C}^1(E)$ to problem (1.2) that satisfies $\underline{z} \le z \le \overline{z}$, where $d(x) := dist(x, \partial E)$ is the distance between x and ∂E.

Proof. To prove the result of Lemma 2.4, we will follow the ideas presented in the works [26, 27]. We first of all start by considering for all $n \in \mathbb{N}$ the domains

$$E_n = \{x \in E \mid d(x) > \frac{1}{n+1}\}.$$

For any fixed $n \in \mathbb{N}$, we introduce the following problem:

$$(\mathcal{P}_n) \begin{cases} -\Delta_p z = \zeta(x,z) & \textit{in } E_n, \\ z = \underline{z} & \textit{in } \partial E_n, \end{cases}$$

where $\zeta(x,z) = \lambda z^{l_1} - a(x) z^{l_2}$. To assure the well-posedness of problem $(\mathcal{P}_n)$, we consider the following energy functional:

$$\mathcal{J}_n(z) = \frac{1}{p} \int\limits_{E_n} |\nabla z|^p \, \mathrm{d}x + \int\limits_{E_n} \Psi(x,w) \, \mathrm{d}x, \tag{2.5}$$

where $\Psi(x,z) = \int\limits_0^z \zeta(x,s) \, \mathrm{d}s$. We give the demonstration in four steps.

Step 1: We show that, for any $n \geq 1$, problem $(\mathcal{P}_n)$ admits a solution.
Let $\underline{z}_n(x) = \underline{z}(x)_{|\partial E_n}$. We can verify easily that $\mathcal{J}_n$ is coercive and weakly lower semi-continuous in $\mathcal{Y}_n := \underline{z}_n + W_0^{1,p}(E)$. Thus $\mathcal{J}_n$ has a nontrivial minimum point z_n. Consequently, one may deduce that z_n is the solution of $(\mathcal{P}_n)$.
Step 2: Our aim in this steps is to show that every solution z_n of problem $(\mathcal{P}_n)$ satisfies the inequality $\underline{z} \leq z_n \leq \overline{z}$ in E_n. To do this, let us observe that $\underline{z}$ is a sub-solution to problem $(\mathcal{P}_n)$. Then, by the result of Lemma 2.3, we get that $\underline{z}(x) \leq z_n(x)$ a.e in E_n, for every $n \in \mathbb{N}$. On the other hand, as $z_n(x) = \underline{z}(x)$ on ∂E_n, we have $z_n(x) \leq z_{n+1}(x)$ on ∂E_n, for every $n \in \mathbb{N}$. Using again the result of Lemma 2.3, we obtain $z_n(x) \leq z_{n+1}(x)$ a.e in E_n, for all $n \in \mathbb{N}$. Now, starting the fact that $\overline{z}(x)$ is a super-solution with $\overline{z}(x) \geq \underline{z}(x)$ for any $x \in E$, we have $\overline{z}(x) \geq \underline{z}(x)$ for $x \in \partial E_n$. Since $\underline{z}(x) = z_n(x)$ on ∂E_n, we have $z_n(x) \leq \overline{z}(x)$ on ∂E_n. Thanks to the result of Lemma 2.3, we get $z_n(x) \leq \overline{z}(x)$ a.e in E_n, for all $n \in \mathbb{N}$. Then, we obtain $\underline{z} \leq z_n \leq \overline{z}$ a.e in E_n.
Step 3: On E_2, the set $\{z_n : n \geq 2\}$ is uniformly bounded by $\underline{z}(x)$ and $\overline{z}(x)$. Let us use the first equation in problem $(\mathcal{P}_n)$, we get

$$\int\limits_{E_2} |\nabla z_n|^p \, \mathrm{d}x = \int\limits_{E_2} \zeta z_n \, \mathrm{d}x. \tag{2.6}$$

As $\underline{z}(x)$ and $\overline{z}(x)$ are bounded on E_2, there are $K_1 > 0$ and $K_2 > 0$ such that

$$|| \, z_n \, ||_{L^\infty(E_2)} \leq K_1 \quad \textit{and} \quad || \, \zeta(x,z_n) \, ||_{L^\infty(E_2)} \leq K_2, \qquad \text{for all } n \geq 2. \tag{2.7}$$

Consequently, using (2.6), (2.7), and the Hölder's inequality, we get

$$\begin{aligned} \int\limits_{E_2} |\nabla z_n|^p \, \mathrm{d}x &= \int\limits_{E_2} \zeta(x,z_n) z_n \, \mathrm{d}x \\ &\leq || \, \zeta(x,z_n) \, ||_{L^\infty(E_2)} \int\limits_{E_2} |z_n| \, \mathrm{d}x \\ &\leq K_2 \int\limits_{E_2} |z_n| \, \mathrm{d}x \\ &\leq K_2 \, (meas \; E_2)^{\frac{p}{p-1}} \, || \, z_n \, ||_{L^p(E_2)} \, . \end{aligned} \tag{2.8}$$

By the Poincaré inequality, we know that there exists $C_{E_2} > 0$ such that

$$\| z_n \|_{L^p(E_2)} \leq C_{E_2} \| \nabla z_n \|_{L^p(E_2)} . \tag{2.9}$$

By employing (2.8) and (2.9), there exists $M_2 > 0$ such that $\| z_n \|_{W_0^{1,p}(E_2)} \leq M_2$. Now, we have two cases to discuss:

- If $1 < p < N$, then we know that $W_0^{1,p}(E_2)$ is continuously embedded in $L^{\frac{Np}{N-p}}(E_2)$. Then, it result that z_n is also bounded in $L^{\frac{Np}{N-p}}(E_2)$. Therefore from [13, Theorem 7.1], there exists $M_3 > 0$ such that

$$sup\{|z_n| \,:\, x \in E_2\} \leq M_3. \tag{2.10}$$

- If $p \geq N$, then we use the Sobolev embedding theorem, and therefore we deduce that (2.10) holds true.

Consequently by [13, Theorem 1.1], we get $z_n \in \mathcal{C}^\alpha(\overline{E_2})$ and there exists M_4 that satisfies $\| z_n \|_{\mathcal{C}^\alpha(E_2)} \leq M_4$, where α is a real number such that $0 < \alpha < 1$. From [23, Proposition 3.7], we see that $z_n \in C^{1,\alpha}(\overline{E_2})$, and there exists M_5 such that $\| z_n \|_{\mathcal{C}^{1,\alpha}} \leq M_5$. According to the above, we conclude the existence of a real number $L > 0$ such that $\|z_n\|_{\mathcal{C}^{1+\alpha}(E_1)} \leq L$, for all $n \geq 2$. Furthermore, it is known that $\mathcal{C}^{1+\alpha}(E_1) \hookrightarrow \mathcal{C}^1(E_1)$ is compact. Hence, there is a subsequence of $\{z_n\}_n$ named by $\{z_{n_{1k}}\}_k$, which converges in $\mathcal{C}^1(E_1)$. We put $z_1(x) = \lim\limits_{k\to+\infty} z_{n_{1k}}$ for any $x \in E_1$. Thus z_1 is a solution to

$$-\Delta_p z = \lambda z^{l_1} - a(x) z^{l_2}, \tag{2.11}$$

with $\underline{z}(x) \leqslant z_1(x) \leqslant \overline{z}(x)$.

Step 4: We proceed as in Step 1. We derive the existence of a subsequence $\{z_{n_{2k}}\}_k$, which converges in $\mathcal{C}^1(E_2)$ to w_2. Hence likewise z_2 is a solution of (2.11) and $z_2 \mid_{E_1} = z_1$. We repeat Step 1 again on E_3, E_3, and so on. We get a sequence $\{z_{n_{ik}}\}_k$, which converges in $\mathcal{C}^1(E_i)$ and is a subsequence $\{z_{n_{(i-1)k}}\}_k$. Let $z_i = \lim\limits_{k\to+\infty} z_{n_{ik}}$. Consequently, z_i is a solution to (2.11) in $\mathcal{C}^1(E_i)$ and $z_{i|_{E_{i-1}}} = z_{i-1}$.

Step 5: By a diagonal process, $\{z_{n_{jj}}\}_{j=1,2,\ldots}$ is a subsequence of $\{z_{n_{ik}}\}_k$ for all k. Consequently, for any i, we have

$$\lim_{j\to+\infty} z_{n_{jj}}(x) = z_i(x), \qquad \text{for all } x \in E_i.$$

Consequently, if we define $z(x) = \lim\limits_{j\to+\infty} z_{n_{jj}}(x)$, then $z(x)$ is a solution to (2.11) and $\underline{z}(x) \leqslant z(x) \leqslant \overline{z}(x)$ because $\underline{z}(x) \leqslant z_i(x) \leqslant \overline{z}(x)$, for all i. □

3. Main result

In this theorem, we articulate the main result presented in our paper.

Theorem 3.1. *Under the assumptions* $(\mathcal{H}_1)$ *and* $(\mathcal{H}_2)$, (1.1) *has only one radially solution* $w(r)$. *Also,*

$$\lim_{r\to\rho} \frac{w(r)}{(\mathcal{K}^*(r))^{-\gamma}} = L, \tag{3.1}$$

where

$$L = \left[(p-1)\gamma^{p-1}((\gamma+1)\mathcal{M}_0 - 1)(\mathcal{M}_0 a(\rho))^{\frac{p}{2}-1}\right]^{\frac{1}{l_2-p+1}} \tag{3.2}$$

and

$$\gamma = \frac{p}{2(l_2 + 1 - p)}. \tag{3.3}$$

To show the results of Theorem 3.1, we will consider the problem:

$$\begin{cases} -\left[(|w'|^{p-2}w')' + \frac{N-1}{r}|w'|^{p-2}w'\right] = \lambda w^{l_1} - a(r)w^{l_2} \text{ in } (0,\rho), \\ \lim\limits_{r\to\rho} w(r) = \infty, \\ w'(0) = 0. \end{cases} \tag{3.4}$$

As the first step of our proof, we will need construction of sub- and super-solutions to problem (3.4). This will be done in the following subsection.

3.1. **Construction of super- and sub-solutions.**

The main objective of this part is to build a suitable sub- and super-solutions to (3.4). In the first lemma, we will give a typical construction of a super-solution to (3.4). Afterward, we will build a sub-solution to (3.4) in the second lemma.

Lemma 3.2. *For every $\sigma > 0$,*

$$\overline{w}_\sigma(r) = \alpha + \beta\left(\frac{r}{\rho}\right)^2 (\mathcal{K}^*(r))^{-\gamma}$$

is a super-solution of problem (3.4), where $\alpha > 0$ is a sufficiently large real number and

$$\beta = (1+\sigma)\left[(p-1)\gamma^{p-1}((\gamma+1)\mathcal{M}_0 - 1)(\mathcal{M}_0 a(\rho))^{\frac{p}{2}-1}\right]^{\frac{1}{l_2-p+1}}.$$

Proof. By a direct computation, we have

$$\begin{aligned} \overline{w}'_\sigma(r) &= 2\beta\frac{r}{\rho^2}(\mathcal{K}^*(r))^{-\gamma} + \gamma\beta\left(\frac{r}{\rho}\right)^2 \mathcal{K}(r)(\mathcal{K}^*(r))^{-\gamma-1}. \\ \overline{w}''_\sigma(r) &= 2\beta\frac{1}{\rho^2}(\mathcal{K}^*(r))^{-\gamma} + 4\beta\gamma\frac{r}{\rho^2}\mathcal{K}(r)(\mathcal{K}^*(r))^{-\gamma-1} \\ &\quad + \gamma(\gamma+1)\beta\left(\frac{r}{\rho}\right)^2 \mathcal{K}^2(r)(\mathcal{K}^*(r))^{-\gamma-2} - \gamma\beta\left(\frac{r}{\rho}\right)^2 a(r)(\mathcal{K}^*(r))^{-\gamma-1}. \end{aligned} \tag{3.5}$$

As $\lim\limits_{r\to\rho}\mathcal{K}^*(r) = 0$, then $\lim\limits_{r\to\rho}\overline{w}_\sigma(r) = \infty$, and as $\overline{w}'_\sigma(0) = 0$, then $\overline{w}_\sigma(r)$ is a super-solution of (3.4) if and only if

$$-\left[(|\overline{w}'_\sigma(r)|^{p-2}\overline{w}'_\sigma(r))' + \frac{N-1}{r}|\overline{w}'_\sigma(r)|^{p-2}\overline{w}'_\sigma(r)\right] \geqslant \lambda\overline{w}^{l_1}_\sigma(r) - a(r)\overline{w}^{l_2}_\sigma(r). \tag{3.6}$$

Consequently,

$$-(p-1)\left(\overline{w}'_\sigma(r)\right)^{p-2}\overline{w}''_\sigma(r) - \frac{N-1}{r}\left(\overline{w}'_\sigma(r)\right)^{p-1} \geqslant \lambda\overline{w}^{l_1}_\sigma(r) - a(r)\overline{w}^{l_2}_\sigma(r). \tag{3.7}$$

Using (3.5), the inequality (3.7) becomes

$$
\begin{aligned}
&- (p-1)\left(2\beta\frac{r}{\rho^2}\left(\mathcal{K}^*(r)\right)^{-\gamma}+\gamma\beta\left(\frac{r}{\rho}\right)^2\mathcal{K}(r)\left(\mathcal{K}^*(r)\right)^{-\gamma-1}\right)^{p-2}\\
&\qquad\times\Big[2\beta\frac{1}{\rho^2}\left(\mathcal{K}^*(r)\right)^{-\gamma}+4\beta\gamma\frac{r}{\rho^2}\mathcal{K}(r)\left(\mathcal{K}^*(r)\right)^{-\gamma-1}\\
&\quad+\gamma(\gamma+1)\beta\left(\frac{r}{\rho}\right)^2\mathcal{K}^2(r)\left(\mathcal{K}^*(r)\right)^{-\gamma-2}-\gamma\beta\left(\frac{r}{\rho}\right)^2a(r)\left(\mathcal{K}^*(r)\right)^{-\gamma-1}\Big]\\
&\quad-\frac{N-1}{r}\left(2\beta\frac{r}{\rho^2}\left(\mathcal{K}^*(r)\right)^{-\gamma}+\gamma\beta\left(\frac{r}{\rho}\right)^2\mathcal{K}(r)\left(\mathcal{K}^*(r)\right)^{-\gamma-1}\right)^{p-1}\\
&\quad\geqslant\lambda\left(\alpha+\beta\left(\frac{r}{\rho}\right)^2\left(\mathcal{K}^*(r)\right)^{-\gamma}\right)^{l_1}-a(r)\left(\alpha+\beta\left(\frac{r}{\rho}\right)^2\left(\mathcal{K}^*(r)\right)^{-\gamma}\right)^{l_2}. \qquad (3.8)
\end{aligned}
$$

Multiplying the two members of inequality (3.8) by $\frac{1}{(\mathcal{K}(r))^p}(\mathcal{K}^*(r))^{(\gamma+1)(p-1)+1}$ and replacing γ by $\frac{p}{2(l_2+1-p)}$, we obtain

$$
\begin{aligned}
&-(p-1)\left(2\beta\frac{r}{\rho^2}\frac{\mathcal{K}^*(r)}{\mathcal{K}(r)}+\gamma\beta\left(\frac{r}{\rho}\right)^2\right)^{p-2}\\
&\times\Bigg[2\beta\frac{1}{\rho^2}\left(\frac{\mathcal{K}^*(r)}{\mathcal{K}(r)}\right)^2+4\beta\gamma\frac{r}{\rho^2}\frac{\mathcal{K}^*(r)}{\mathcal{K}(r)}+\gamma(\gamma+1)\beta\left(\frac{r}{\rho}\right)^2\\
&-\gamma\beta\left(\frac{r}{\rho}\right)^2\frac{a(r)\mathcal{K}^*(r)}{(\mathcal{K}(r))^2}\Bigg]\\
&-\frac{N-1}{r}\frac{\mathcal{K}^*(r)}{\mathcal{K}(r)}\left(2\beta\frac{r}{\rho^2}\frac{\mathcal{K}^*(r)}{\mathcal{K}(r)}+\gamma\beta\left(\frac{r}{\rho}\right)^2\right)^{p-1}\\
&\quad\geq\lambda\left(\frac{\mathcal{K}^*(r)}{\mathcal{K}(r)}\right)^p(\mathcal{K}^*(r))^{\gamma(p-1-l_1)}\left(\alpha(\mathcal{K}^*(r))^\gamma+\beta\left(\frac{r}{\rho}\right)^2\right)^{l_1}\\
&\qquad-(a(r))^{1-\frac{p}{2}}\left(\frac{a(r)\mathcal{K}^*(r)}{(\mathcal{K}(r))^2}\right)^{\frac{p}{2}}\left(\alpha(\mathcal{K}^*(r))^\gamma+\beta\left(\frac{r}{\rho}\right)^2\right)^{l_2}. \qquad (3.9)
\end{aligned}
$$

Letting $r\to\rho$ in (3.9) and taking into account that $\lim_{r\to\rho}\mathcal{K}^*(r)=0$, $\lim_{r\to\rho}\frac{\mathcal{K}^*(r)}{\mathcal{K}(r)}=\lim_{r\to\rho}\frac{\mathcal{K}(r)}{a(r)}=0$ (by the Hospital rule), and $\lim_{r\to\rho}\frac{(\mathcal{K}(r))^2}{\mathcal{K}^*(r)a(r)}=\mathcal{M}_0\geqslant 1$, we get

$$
\beta\geqslant\left[(p-1)\gamma^{p-1}((\gamma+1)\mathcal{M}_0-1)(\mathcal{M}_0a(\rho))^{\frac{p}{2}-1}\right]^{\frac{1}{l_2-p+1}}.
$$

Consequently, by choosing

$$
\beta=(1+\sigma)\left[(p-1)\gamma^{p-1}((\gamma+1)\mathcal{M}_0-1)(\mathcal{M}_0a(\rho))^{\frac{p}{2}-1}\right]^{\frac{1}{l_2-p+1}},
$$

inequality (3.9) holsd in $(\rho-\eta, \rho]$ for some $\eta > 0$. Then, by the choice of α large enough, (3.9) is verified in the whole interval $[0, \rho]$ because $l_2 > l_1$ and the fact that $\mathcal{K}^*(r)$ is bounded away from zero in $[0, \rho-\eta]$. $\square$

Lemma 3.3. *For $\sigma > 0$ sufficiently small, we have*

$$\underline{w}_\sigma(r) = \max\left\{0, P + Q\left(\frac{r}{\rho}\right)^2 (\mathcal{K}^*(r))^{-\gamma}\right\}$$

is a sub-solution of problem (3.4), where $P < 0$ and

$$Q = (1-\sigma)\left[(p-1)\gamma^{p-1}((\gamma+1)\mathcal{M}_0 - 1)(\mathcal{M}_0 a(\rho))^{\frac{p}{2}-1}\right]^{\frac{1}{l_2-p+1}}.$$

Proof. Let us consider

$$h_P(r) = P + Q\left(\frac{r}{\rho}\right)^2 (\mathcal{K}^*(r))^{-\gamma}.$$

Hence,

$$h_P'(r) = 2Q\frac{r}{\rho^2}(\mathcal{K}^*(r))^{-\gamma} + \gamma Q\left(\frac{r}{\rho}\right)^2 \mathcal{K}(r)(\mathcal{K}^*(r))^{-\gamma-1} > 0 \quad in\ (0,\rho). \tag{3.10}$$

Consequently, the function $h_P(\cdot)$ is strictly increasing and

$$\lim_{r\to\rho} h_P(r) = +\infty \quad and \quad \lim_{r\to 0} h_P(r) = P < 0.$$

Hence, by the intermediate value theorem and the continuity of $h_P(\cdot)$, there exists a unique $R_1 = R_1(P) \in (0,\rho)$ such that

$$\begin{cases} P + Q\left(\dfrac{r}{\rho}\right)^2 (\mathcal{K}^*(r))^{-\gamma} < 0 & in\ [0, R_1(P)), \\ P + Q\left(\dfrac{r}{\rho}\right)^2 (\mathcal{K}^*(r))^{-\gamma} > 0 & in\ [R_1(P), \rho]. \end{cases} \tag{3.11}$$

Furthermore, $R_1(P)$ is decreasing and

$$\lim_{P\to-\infty} R_1(P)) = \rho \quad and \quad \lim_{P\to 0} R_1(P) = 0,$$

Using the definition of $\underline{w}_\sigma(r)$ and $R_1(P)$, we deduce

$$\underline{w}_\sigma(r) \equiv 0 \quad in\ [0, R_1(P)).$$

Consequently, it is clear that the inequality

$$-\left[(|\underline{w}_\sigma'(r)|^{p-2}\underline{w}_\sigma'(r))' + \frac{N-1}{r}|\underline{w}_\sigma'(r)|^{p-2}\underline{w}_\sigma'(r)\right] \leqslant \lambda \underline{w}_\sigma^{l_1}(r) - a(r)\underline{w}_\sigma^{l_2}(r) \tag{3.12}$$

holds true in $[0, R_1(P))$. Now, it remains to show that (3.12) is it also satisfied in $[R_1(P), \rho]$. From (3.10), (3.11), and the definition of $\underline{w}_\sigma(r)$, we have

$$-\frac{N-1}{r}\left(\underline{w}_\sigma'(r)\right)^{p-1} \leqslant 0 \leqslant \lambda \underline{w}_\sigma^{l_1}(r) \quad \text{for all } r \in [R_1(P), \rho].$$

Hence (3.12) is satisfied in $[R_1(P),\rho]$ if

$$-(p-1)\left(\underline{w}_\sigma'(r)\right)^{p-2}\underline{w}_\sigma''(r)\leqslant -a(r)\underline{w}_\sigma^{l_2}(r)\quad \text{for all } r\in[R_1(P),\rho]. \tag{3.13}$$

The above inequality is equivalent to

$$-(p-1)\left(2Q\frac{r}{\rho^2}\left(\mathcal{K}^*(r)\right)^{-\gamma}+\gamma Q\left(\frac{r}{\rho}\right)^2\mathcal{K}(r)\left(\mathcal{K}^*(r)\right)^{-\gamma-1}\right)^{p-2} \tag{3.14}$$

$$\times\Big[2Q\frac{1}{\rho^2}\left(\mathcal{K}^*(r)\right)^{-\gamma}+4Q\gamma\frac{r}{\rho^2}\mathcal{K}(r)\left(\mathcal{K}^*(r)\right)^{-\gamma-1} \tag{3.15}$$

$$+\gamma(\gamma+1)Q\left(\frac{r}{\rho}\right)^2\mathcal{K}^2(r)\left(\mathcal{K}^*(r)\right)^{-\gamma-2}-\gamma Q\left(\frac{r}{\rho}\right)^2a(r)\left(\mathcal{K}^*(r)\right)^{-\gamma-1}\Big] \tag{3.16}$$

$$\leqslant -a(r)\left(P+Q\left(\frac{r}{\rho}\right)^2\left(\mathcal{K}^*(r)\right)^{-\gamma}\right)^{l_2}. \tag{3.17}$$

As in Lemma 3.2, multiplying the two members of inequality (3.14) by $\frac{1}{(\mathcal{K}(r))^p}(\mathcal{K}^*(r))^{(\gamma+1)(p-1)+1}$ and replacing γ by $\frac{p}{2(l_2+1-p)}$, we get

$$\begin{aligned}
&-(p-1)\left(2Q\frac{r}{\rho^2}\frac{\mathcal{K}^*(r)}{\mathcal{K}(r)}+\gamma Q\left(\frac{r}{\rho}\right)^2\right)^{p-2}\\
&\times\left[2Q\frac{1}{\rho^2}\left(\frac{\mathcal{K}^*(r)}{\mathcal{K}(r)}\right)^2+4Q\gamma\frac{r}{\rho^2}\frac{\mathcal{K}^*(r)}{\mathcal{K}(r)}+\gamma(\gamma+1)Q\left(\frac{r}{\rho}\right)^2\right.\\
&\left.-\gamma Q\left(\frac{r}{\rho}\right)^2\frac{a(r)\mathcal{K}^*(r)}{(\mathcal{K}(r))^2}\right]\\
&\qquad\leq-\left(a(r)\right)^{1-\frac{p}{2}}\left(\frac{a(r)\mathcal{K}^*(r)}{(\mathcal{K}(r))^2}\right)^{\frac{p}{2}}\left(P(\mathcal{K}^*(r))^\gamma+Q\left(\frac{r}{\rho}\right)^2\right)^{l_2}.
\end{aligned} \tag{3.18}$$

Taking $r\to\rho$ in (3.18), it becomes

$$Q\leqslant\left[(p-1)\gamma^{p-1}((\gamma+1)\mathcal{M}_0-1)(\mathcal{M}_0a(\rho))^{\frac{p}{2}-1}\right]^{\frac{1}{l_2-p+1}}.$$

Consequently, by the following choice

$$Q=(1-\sigma)\left[(p-1)\gamma^{p-1}((\gamma+1)\mathcal{M}_0-1)(\mathcal{M}_0a(\rho))^{\frac{p}{2}-1}\right]^{\frac{1}{l_2-p+1}},$$

inequality (3.18) hold in $(\rho-\eta,\rho]$ for some $\eta=\eta(\sigma)>0$. Hence, choosing P such that $R_1(P)=\rho-\eta$, thus $\underline{w}_\sigma(r)$ is a sub-solution of (3.4). □

3.2. **Existence results and asymptotic behavior at ρ.**

By Lemmas 3.2 and 3.3, we know that there is a couple of functions $(\underline{w}_\sigma, \overline{w}_\sigma)$ sub- and super-solutions to problem (3.4) such that $\underline{w}_\sigma(r) \leqslant \overline{w}_\sigma(r)$. As a consequence, the results of Lemma 2.4 means that (3.4) has a solution $w_\sigma(r)$ such that $\underline{w}_\sigma(r) \leqslant w_\sigma(r) \leqslant \overline{w}_\sigma(r)$. Furthermore, since

$$1-\sigma \leqslant \lim_{r\to\rho} \frac{\underline{w}_\sigma(r)}{L\left(\mathcal{K}^*(r)\right)^{-\gamma}} \leqslant \lim_{r\to\rho} \frac{\overline{w}_\sigma(r)}{L\left(\mathcal{K}^*(r)\right)^{-\gamma}} \leqslant 1+\sigma,$$

then

$$1-\sigma \leqslant \liminf_{r\to\rho} \frac{w_\sigma(r)}{L\left(\mathcal{K}^*(r)\right)^{-\gamma}} \leqslant \limsup_{r\to\rho} \frac{w_\sigma(r)}{L\left(\mathcal{K}^*(r)\right)^{-\gamma}} \leqslant 1+\sigma. \tag{3.19}$$

Letting $\sigma \to 0$ in (3.19), we get

$$\lim_{r\to\rho} \frac{w(r)}{L\left(\mathcal{K}^*(r)\right)^{-\gamma}} = 1. \tag{3.20}$$

3.3. **Uniqueness results.**

In this paragraph, we will establish that (3.4) has only one solution. To do this, we consider w_1 and w_2 two solutions to problem (3.4). From the result of (3.20), we have $\lim\limits_{d_{\mathcal{B}}(x)\to 0} \dfrac{w_1(x)}{w_2(x)} = 1$, where $d_{\mathcal{B}}(x) = dist(x, \partial\mathcal{B}_\rho(0))$. Consequently, for any $\sigma > 0$, there exists η such that

$$(1-\sigma)w_2(x) \leqslant w_1(x) \leqslant (1+\sigma)w_2(x), \quad \text{when} \quad 0 < d_{\mathcal{B}}(x) \leqslant \eta. \tag{3.21}$$

Let

$$(\mathcal{Q}) \begin{cases} -\Delta_p v = \lambda v^{l_1} - a(x)v^{l_2} & in\ \mathcal{B}_{\rho-\eta}(0), \\ v = w_1 & in\ \partial\mathcal{B}_{\rho-\eta}(0). \end{cases}$$

As w_1 is a solution of problem (1.1) and by using the fact that $t \to \delta^t$ is decreasing if $0 < \delta < 1$ and $l_1 \leqslant p-1 < l_2$, we have

$$\begin{aligned} -\Delta_p[(1-\sigma)w_2] = -(1-\sigma)^{p-1}\Delta_p w_2 &= \lambda(1-\sigma)^{p-1}w_2^{l_1} - (1-\sigma)^{p-1}a(x)w_2^{l_2} \\ &\leqslant \lambda[(1-\sigma)w_2]^{l_1} - a(x)[(1-\sigma)w_2]^{l_2}. \end{aligned}$$

Hence, $\underline{v} = (1-\sigma)w_2$ is a sub-solution of problem $(\mathcal{Q})$, and by a similar argument, we get that $\overline{v} = (1+\sigma)w_2$ provides us with a super-solution of problem $(\mathcal{Q})$. Consequently by Lemma 2.4, it follows that

$$(1-\sigma)w_2(x) \leqslant v(x) \leqslant (1+\sigma)w_2(x), \quad in\ \mathcal{B}_{\rho-\eta}(0).$$

The uniqueness of v implies that $v = w_1$ in $\mathcal{B}_{\rho-\eta}(0)$, and taking into account (3.21), we obtain

$$(1-\sigma)w_2(x) \leqslant w_1(x) \leqslant (1+\sigma)w_2(x), \quad in\ \mathcal{B}_\rho(0).$$

Then, by letting $\sigma \to 0$, we obtain $w_1(x) = w_2(x)$. This finishes the proof.

4. Applications of our results

In this section, we shall give some examples to validate the theoretical results of our work.

Example 4.1. We take $a(r) = \mathcal{C} > 0$. It is easy to see that $a(\cdot)$ satisfies the hypotheses $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$. On the other hand, a simple computation yields that

$$\begin{aligned}\mathcal{K}(r) &= \mathcal{C}(\rho - r),\\ \mathcal{K}^*(r) &= \frac{\mathcal{C}}{2}(\rho - r)^2,\\ \mathcal{M}_0 &= 2.\end{aligned}$$

Thus by employing our main result, Theorem 3.1, we conclude that problem (1.1) has a unique $w(r)$ radial solution such that

$$w(r) \underset{\rho}{\sim} \left[(p-1)\gamma^{p-1}(2\gamma+1)(2\mathcal{C})^{\frac{p}{2}-1}\right]^{\frac{1}{l_2-p+1}} \left[\frac{\mathcal{C}}{2}(\rho - r)^2\right]^{-\gamma}.$$

Example 4.2. We consider $a(r) = r + 1$. As a first step, we will determine the explicit expressions of the functions $\mathcal{K}(\cdot)$, $\mathcal{K}^*(\cdot)$, and again this of $\mathcal{M}_0$. Using direct computations, we derive that

$$\begin{aligned}\mathcal{K}(r) &= (\rho - r)\left[\frac{1}{2}(\rho + r) + 1\right],\\ \mathcal{K}^*(r) &= (\rho - r)^2\left(\frac{1}{3}\rho + \frac{1}{6}r + \frac{1}{2}\right),\\ \mathcal{M}_0 &= 2.\end{aligned}$$

Again, thanks to Theorem 3.1, we deduce that the problem (1.1) has only one radial solution $w(r)$, and

$$w(r) \underset{\rho}{\sim} \left[(p-1)\gamma^{p-1}(2\gamma+1)(2\rho+2)^{\frac{p}{2}-1}\right]^{\frac{1}{l_2-p+1}} \left(\mathcal{K}^*(r)\right)^{-\gamma}.$$

5. Conclusion

In the present paper, we studied the following singular problem:

$$\begin{cases} -\left[(|w'|^{p-2}w')' + \frac{N-1}{r}|w'|^{p-2}w'\right] = \lambda w^{l_1} - a(r)w^{l_2} \text{ in } (0,\rho), \qquad \rho > 0,\\ \lim\limits_{r\to\rho} w(r) = \infty,\\ w'(0) = 0, \end{cases} \tag{5.1}$$

with $\lambda > 0$, $1 \le l_1 \le p-1 < l_2$ and a is a continuous and strictly positive function on $[0,\rho]$.

The study is based on the method of sub- and super-solutions. More precisely, we were able to construct a sub-solution and a super-solution of the problem (5.1), hence the existence of a unique positive solution w that converges to infinity in the neighborhood of ρ. In addition, we have specified the exact explosion rate of the solution w of problem (5.1). Afterwards, we have given two interesting examples, which depend on the expression of the function a and ensure the theoretical results obtained.

In the end, it should be noted that the case $l_1 > p-1 \geq l_2 > 1$ remains an open question, which will be raised in future research work.

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References

1. G. Astrita and G. Marrucci, *Principles of non-Newtonian fluid mechanics*, McGraw-Hill, 1974.
2. C. Bandle and M. Marcus, *Large solutions of semilinear elliptic equations: Existence, uniqueness and asymptotic behaviour*, J. Anal. Math. **58** (1992), 9–24.
3. C. Bandle and M. Marcus, *On second order effects in the boundary behaviour of large solutions of semilinear elliptic problems*, Differ. Integral Equ. **11** (1998), no. 1, 23–34.
4. A. Bouzelmate and M. El Hathout, *Existence and global behavior of decaying solutions of a nonlinear elliptic equation*, WSEAS Trans. Am. Math. **19** (2020), 662–675.
5. A. Bouzelmate and A. Gmira, *Singular solutions of an inhomogeneous elliptic equation*, Nonlinear Funct. Anal. Appl. **26** (2021), no. 2, 237–272.
6. A. Bouzelmate and A. Gmira, *Existence and asymptotic behavior of unbounded positive solutions of a nonlinear degenerate elliptic equation*, Nonlinear Dyn. **21** (2021), no. 1, 27–55.
7. A. Bouzelmate, A. Gmira and G. Reyes, *Radial selfsimilar solutions of a nonlinear Ornstein-Uhlenbeck equation*. Electron. J. Differ. Equ. 2007, Paper No. 67.
8. Y. Du and Q. Huang, *Blow-up solutions for a class of semilinear elliptic and parabolic equations*, SIAM J. Math. Anal. **31** (1999), no. 1, 1–18.
9. J.R. Esteban and J.L. Vazquez, *On the equation of turbulent filtration in one-dimensional porous media*, Nonlinear Anal. **10** (1982), 1303–1325.
10. F. Gladiali and G. Porru, *Estimates for explosive solutions to p-Laplace equations*, in: Progress in Partial Differential Equations, Pont-á-Mousson, 1997, vol. 1, in: Pitman Res. Notes Math. Ser. **383** (1998), 117–127.
11. S. Huang, Q. Tian, S. Zhang, J. Xi and Z. Fan, *The exact blow-up rates of large solutions for semilinear elliptic equations*, Nonlinear Anal. **73** (2010), 3489–3501.
12. J.B. Keller, *On solutions of* $\Delta u = f(u)$, Comm. Pure Appl. Math. **10** (1957), 503–510.
13. O.A. Ladyzhenskaya and N.N. Ural'tseva, *Linear and quasilinear elliptic equations*, Academic Press, New York, 1968.
14. A.C. Lazer and P.J. McKenna, *On a problem of Bieberbach and Rademacher*, Nonlinear Anal. **21** (1993), 327–335.
15. C. Loewner and L. Nirenberg, *Partial differential equations invariant under conformal or projective transformations*, Contrib. Anal., Academic Press, New York, 1974, 245–272.
16. J. López-Gómez, *Optimal uniqueness theorems and exact blow-up rates of large solutions*, J. Differ. Equ.**224** (2006), 385–439.
17. L.K. Martinson and K.B. Pavlov, *Unsteady shear flows of a conducting fluid with a theological power law*, Magnit. Gidrodinamika. **2** (1971), 50–58.
18. A. Mohammed, *Boundary asymptotic and uniqueness of solutions to the p-Laplacian with infinite boundary values*, J. Math. Anal. Appl. **325** (2007), 480–489.
19. T. Ouyang, *On the positive solutions of semilinear equations* $\Delta u + \lambda u - hu^p = 0$ *on compact manifolds*, Trans. Am. Math. Soc. **331** (1992), 503–527.
20. T. Ouyang and Z. Xie, *The uniqueness of blow-up solution for radially symmetric semilinear elliptic equation*, Nonlinear Anal. **64** (2006), 2129–2142.

21. T. Ouyang and Z. Xie, *The exact boundary blow-up rate of large solutions for semilinear elliptic problems*, Nonlinear Anal. **68** (2008), 2791–2800.
22. F. Peng, *On the blow-up rate of large solutions for a porous media logistic equation on radial domain*, J. Math. Anal. Appl. **329** (2007), 347–356.
23. P. Tolksdorf, *On the Dirichlet problem for quasilinear equations in domains with conical boundary points*, Commun. Partial Different. Equ. (7)**8** (1983), 773–817.
24. Z. Xie *Uniqueness and blow-up rate of large solutions for elliptic equation $-\Delta u = \lambda u - b(x)h(u)$*, J. Differ. Equ. **247** (2009), 344–363.
25. Z. Xie and C. Zhao, *Blow-up rate and uniqueness of singular radial solutions for a class of quasi-linear elliptic equations*, J. Differ. Equ. **252** (2012), 1776–1788.
26. Z. Yang, *Existence of positive bounded entire solutions for quasilinear elliptic equations*, Appl. Math. Comput. **156** (2004), 743–754.
27. Q. Zhang, *Existence, nonexistence and asymptotic behavior of boundary blow-up solutions to $p(x)$-Laplacian problems with singular coefficient*, Nonlinear Anal. **74** (2011), 2045–2061.

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