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ASYMPTOTIC ANALYSIS OF BOUNDARY-VALUE-TRANSMISSION PROBLEMS WITH RETARDED ARGUMENT

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Abstract. We consider an eigenvalue problem for the Sturm–Liouville equation with retarded argument, subject to the mixed type boundary conditions and an eigenvalue parameter in transmission conditions. We investigate the spectrum, asymptotics of characteristic function and eigenvalues, oscillation of eigenfunctions, solution of an inverse nodal problem and trace of the differential operator of the related problem.

1. Introduction and preliminaries

The problem under consideration is defined by the differential expression

$$u''(t)+q(t)u(t-\Delta(t))+\lambda^2u(t)=0 \tag{1.1}$$

on the interval $\Lambda=\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]$ subject to the general boundary conditions

$$u(0)\cos\alpha+u'(0)\sin\alpha=0, \tag{1.2}$$

$$u(\pi)\cos\beta+u'(\pi)\sin\beta=0, \tag{1.3}$$

and eigenvalue-dependent transmission conditions

$$\frac{1}{\delta}u\left(\frac{\pi}{2}-0\right)=\lambda u(\frac{\pi}{2}+0), \tag{1.4}$$

$$\frac{1}{\delta}u'\left(\frac{\pi}{2}-0\right)=\lambda u'(\frac{\pi}{2}+0), \tag{1.5}$$

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where the real-valued function $q(t)$ is continuous in Λ and has finite limits $q\left(\frac{\pi}{2} \pm 0\right) = \lim_{t \to \frac{\pi}{2} \pm 0} q(t)$, the real-valued function $\Delta(t) \geq 0$ is continuous in Λ and has finite limits $\Delta\left(\frac{\pi}{2} \pm 0\right) = \lim_{t \to \frac{\pi}{2} \pm 0} \Delta(t)$, $t - \Delta(t) \geq 0$ if $t \in \Lambda$, λ is a spectral parameter; α, β, and $\delta \neq 0$ are arbitrary real numbers.

Transmission conditions for boundary-value-transmission problems arise in thermal conduction problems for a thin laminated plate (i.e., a plate composed by materials with different characteristics piled in the thickness). In this class of problems, transmission conditions across the interfaces should be added since the plate is laminated [20, 24]. The study of the structure of the solution in the matching region of the layer with the basis solution in the plate leads to consideration of an eigenvalue problem for a second-order differential operator with piecewise continuous coefficients and transmission conditions [21, 36]. It must be also noted that recently boundary value problems with transmission conditions attracted much attention in connection with the inverse acoustic scattering problem (see, e.g., [2, 6, 9]).

As for the differential equations with retarded argument, it can be stated that these kinds of equations describe processes with aftereffect, and thus they find many applications, particularly in the theory of automatic control, in the theory of self-oscillatory systems, in the study of problems connected with combustion in rocket engines (e.g., combustion in the chamber of a liquid propellant rocket engine and vibrations of the hammer in an electromagnetic circuit breaker), in a number of problems in economics, biophysics, and many other fields (see [10, 15, 18, 25, 34]).

The articles [1, 29, 30, 28, 37] are devoted to the study of asymptotics of eigenvalues and eigenfunctions of the Sturm–Liouville problems with retarded argument and with the potential having discontinuity of the first kind in the domain of definition of the solution and classical (local) type boundaru conditions. However, in the paper [31], equivalent formulas were obtained for non-local boundary conditions for a continuous Sturm–Liouville problem with retarded argument. Asymptotic formulas for eigenvalues and eigenfunctions of boundary-value-transmission problems without retarded argument (the case $\Delta \equiv 0$) are obtained in [3, 12, 17, 19, 23, 32, 35, 39].

In papers [28, 37], oscillation of eigenfunctions, solution of an inverse nodal problem, and trace of a discontinuous differential operator with retarded argument were obtained. The papers [7, 8, 14, 33, 37, 38] are devoted to study of inverse nodal problems of Sturm–Liouville type boundary-value-transmission problems. The papers [4, 16, 26] only deals with trace of a differential operator with retarded argument. It must be noted that studies on trace formulas for the Sturm–Liouville operators began with the work of Gelfand and Levitan [13]. The literature is voluminous, and we only refer to [5, 11, 22, 27] and the references therein. However, within this paper, spectrum, asymptotics of characteristic function and eigenvalues, oscillation of eigenfunctions, solution of inverse nodal problem, and regularized trace of the boundary-value-transmission problem (1.1)–(1.5) are investigated.

2. The spectrum

Let $\phi_1(t,\lambda)$ be a solution of (1.1) on $\Lambda^- = \left[0, \frac{\pi}{2}\right]$, satisfying the initial conditions

$$\phi_1(0,\lambda) = \sin\alpha \ \textit{and} \ \phi_1'(0,\lambda) = -\cos\alpha. \tag{2.1}$$

The conditions (2.1) define a unique solution to (1.1) on Λ^- (see [5]).

After defining the above solution, then we will define the solution $\phi_2(t,\lambda)$ of (1.1) on $\Lambda^+ = \left[\frac{\pi}{2}, \pi\right]$ by means of the solution $\phi_1(t,\lambda)$ using the initial conditions

$$\delta\phi_2\left(\frac{\pi}{2},\lambda\right) = \frac{1}{\lambda}\phi_1\left(\frac{\pi}{2},\lambda\right) \ \textit{and} \quad \delta\phi_2'(\frac{\pi}{2},\ \lambda) = \frac{1}{\lambda}\phi_1'(\frac{\pi}{2},\lambda). \tag{2.2}$$

The conditions (2.2) define a unique solution to (1.1) on Λ^+.

Consequently, the function $\phi(t,\lambda)$ is defined on Λ by the equality

$$\phi(t,\lambda) = \begin{cases} \phi_1(t,\lambda), & t \in [0, \frac{\pi}{2}), \\ \phi_2(t,\lambda), & t \in \left(\frac{\pi}{2}, \pi\right] \end{cases}$$

is a solution to (1.1) on Λ, which satisfies one of the boundary conditions and the transmission conditions (1.4)–(1.5). Then the following integral equations hold:

$$\begin{aligned}\phi_1(t,\lambda) =& \sin\alpha\cos\lambda t - \frac{\cos\alpha}{\lambda}\sin\lambda t \\ & - \frac{1}{\lambda}\int_0^t q(\tau)\sin\lambda(t-\tau)\,\phi_1(\tau-\Delta(\tau),\lambda)\,d\tau,\end{aligned} \tag{2.3}$$

$$\begin{aligned}\phi_2(t,\lambda) =& \frac{1}{\lambda\delta}\phi_1\left(\frac{\pi}{2},\lambda\right)\cos\lambda\left(t-\frac{\pi}{2}\right) + \frac{\phi_1'\left(\frac{\pi}{2},\lambda\right)}{\lambda^2\delta}\sin\lambda\left(t-\frac{\pi}{2}\right) \\ & - \frac{1}{\lambda}\int_{\frac{\pi}{2}}^t q(\tau)\sin\lambda(t-\tau)\,\phi_2(\tau-\Delta(\tau),\lambda)\,d\tau.\end{aligned} \tag{2.4}$$

Solving (2.3)–(2.4) by the method of successive approximation, we obtain the following asymptotic expansions for $|\lambda| \to \infty$:

$$\begin{aligned}\phi_1(t,\lambda) =& \sin\alpha\cos\lambda t - \frac{\cos\alpha}{\lambda}\sin\lambda t - \frac{\sin\alpha}{2\lambda}\int_0^t q(\tau)\sin\lambda(t-\Delta(\tau))\,d\tau \\ & - \frac{\sin\alpha}{2\lambda}\int_0^t q(\tau)\sin\lambda(t-(2\tau-\Delta(\tau)))\,d\tau + O\left(\frac{1}{\lambda^2}\right).\end{aligned} \tag{2.5}$$

Differentiating (2.5) with respect to t, we get

$$\phi_1'(t,\lambda) = -\lambda\sin\alpha\sin\lambda t - \cos\alpha\cos\lambda t - \frac{\sin\alpha}{2}\int_0^t q(\tau)\cos\lambda(t-\Delta(\tau))\,d\tau$$

$$-\frac{\sin\alpha}{2}\int_{0}^{t} q(\tau)\cos\lambda\left(t-\left(2\tau-\Delta\left(\tau\right)\right)\right)d\tau+O\left(\frac{1}{\lambda}\right). \tag{2.6}$$

Combining (2.5) and (2.6), we have

$$\begin{aligned}\phi_2(t,\lambda)=&\frac{\sin\alpha\cos\lambda t}{\lambda\delta}-\frac{\cos\alpha\sin\lambda t}{\lambda^2\delta}-\frac{\sin\alpha}{2\lambda^2\delta}\int_{0}^{\frac{\pi}{2}} q\left(\tau\right)\sin\lambda\left(t-\left(2\tau-\Delta\left(\tau\right)\right)\right)d\tau\\&-\frac{\sin\alpha}{2\lambda^2\delta}\int_{0}^{\frac{\pi}{2}} q\left(\tau\right)\sin\lambda\left(t-\Delta\left(\tau\right)\right)d\tau\\&-\frac{\sin\alpha}{2\lambda^2\delta}\int_{\frac{\pi}{2}}^{t} q\left(\tau\right)\sin\lambda\left(t-\left(2\tau-\Delta\left(\tau\right)\right)\right)d\tau\\&-\frac{\sin\alpha}{2\lambda^2\delta}\int_{\frac{\pi}{2}}^{t} q\left(\tau\right)\sin\lambda\left(t-\Delta\left(\tau\right)\right)d\tau+O\left(\frac{1}{\lambda^3}\right),\end{aligned}$$

$$\begin{aligned}\phi_2(t,\lambda)=&\frac{\sin\alpha\cos\lambda t}{\lambda\delta}-\frac{\cos\alpha\sin\lambda t}{\lambda^2\delta}-\frac{\sin\alpha}{2\lambda^2\delta}\int_{0}^{t} q\left(\tau\right)\sin\lambda\left(t-\left(2\tau-\Delta\left(\tau\right)\right)\right)d\tau\\&-\frac{\sin\alpha}{2\lambda^2\delta}\int_{0}^{t} q\left(\tau\right)\sin\lambda\left(t-\Delta\left(\tau\right)\right)d\tau+O\left(\frac{1}{\lambda^3}\right).\end{aligned} \tag{2.7}$$

Differentiating (2.7) with respect to t, we get

$$\begin{aligned}\phi_2'(t,\lambda)=&-\frac{\sin\alpha\sin\lambda t}{\delta}-\frac{\cos\alpha\cos\lambda t}{\lambda\delta}-\frac{\sin\alpha}{2\lambda\delta}\int_{0}^{t} q\left(\tau\right)\cos\lambda\left(t-\left(2\tau-\Delta\left(\tau\right)\right)\right)d\tau\\&-\frac{\sin\alpha}{2\lambda\delta}\int_{0}^{t} q\left(\tau\right)\cos\lambda\left(t-\Delta\left(\tau\right)\right)d\tau+O\left(\frac{1}{\lambda^2}\right).\end{aligned} \tag{2.8}$$

The solution $\phi(t,\lambda)$ defined above is a nontrivial solution of (1.1) satisfying conditions (1.2) and (1.4)–(1.5). Putting $\phi(t,\lambda)$ into (1.3), we get the characteristic equation

$$\Theta(\lambda)\equiv\phi'(\pi,\lambda)\sin\beta+\phi(\pi,\lambda)\cos\beta=0. \tag{2.9}$$

The set of eigenvalues of boundary-value-transmission problem (1.1)–(1.5) coincides with the set of the squares of roots of (2.9), and eigenvalues are simple (see

[5, 19, 21, 22]). From (2.7), (2.8) and (2.9), using the fact

$$O\left(\frac{1}{\lambda}\right)=\begin{cases}\int\limits_0^t q(\tau)\sin\lambda\left(2\tau-\Delta\left(\tau\right)\right)d\tau, & t\in\left[0,\frac{\pi}{2}\right];\\ \int\limits_0^t q(\tau)\cos\lambda\left(2\tau-\Delta\left(\tau\right)\right)d\tau, & t\in\left[0,\frac{\pi}{2}\right];\\ \int\limits_{\frac{\pi}{2}}^t q(\tau)\sin\lambda\left(2\tau-\Delta\left(\tau\right)\right)d\tau, & t\in\left[\frac{\pi}{2},\pi\right];\\ \int\limits_{\frac{\pi}{2}}^t q(\tau)\cos\lambda\left(2\tau-\Delta\left(\tau\right)\right)d\tau, & t\in\left[\frac{\pi}{2},\pi\right]\end{cases}$$

(see [5, 21, 22]), we obtain

$$\begin{aligned}\Theta(\lambda)\equiv&\left\{\frac{\sin\alpha\cos\lambda\pi}{\lambda\delta}-\frac{\cos\alpha\sin\lambda\pi}{\lambda^2\delta}-\frac{\sin\alpha}{2\lambda^2\delta}\int\limits_0^\pi q\left(\tau\right)\sin\lambda\left(\pi-\left(2\tau-\Delta\left(\tau\right)\right)\right)d\tau\right.\\ &\left.-\frac{\sin\alpha}{2\lambda^2\delta}\int\limits_0^\pi q\left(\tau\right)\sin\lambda\left(\pi-\Delta\left(\tau\right)\right)d\tau+O\left(\frac{1}{\lambda^3}\right)\right\}\cos\beta\\ &+\left\{-\frac{\sin\alpha\sin\lambda\pi}{\delta}-\frac{\cos\alpha\cos\lambda\pi}{\lambda\delta}\right.\\ &-\frac{\sin\alpha}{2\lambda\delta}\int\limits_0^\pi q\left(\tau\right)\cos\lambda\left(\pi-\left(2\tau-\Delta\left(\tau\right)\right)\right)d\tau\\ &\left.-\frac{\sin\alpha}{2\lambda\delta}\int\limits_0^\pi q\left(\tau\right)\cos\lambda\left(\pi-\Delta\left(\tau\right)\right)d\tau+O\left(\frac{1}{\lambda^2}\right)\right\}\sin\beta\\ =&\frac{\sin\alpha\cos\beta\cos\lambda\pi}{\lambda\delta}-\frac{\sin\alpha\sin\beta\sin\lambda\pi}{\delta}-\frac{\cos\alpha\sin\beta\cos\lambda\pi}{\lambda\delta}\\ &-\frac{\sin\alpha\sin\beta\cos\lambda\pi}{\lambda\delta}B\left(\lambda,\pi\right)-\frac{\sin\alpha\sin\beta\sin\lambda\pi}{\lambda\delta}A\left(\lambda,\pi\right)+O\left(\frac{1}{\lambda^2}\right)=0.\end{aligned}$$

Here,

$$A\left(\lambda,t\right)=\frac{1}{2}\int\limits_0^t q(\tau)\sin\left(\lambda\Delta\left(\tau\right)\right)d\tau,\qquad B\left(\lambda,t\right)=\frac{1}{2}\int\limits_0^t q(\tau)\cos\left(\lambda\Delta\left(\tau\right)\right)d\tau.$$

Define

$$\Theta_0(\lambda)\equiv-\frac{\sin\alpha\sin\beta\sin\lambda\pi}{\delta}.$$

Now, we have four possible cases: 1. $\sin\alpha\neq0,\sin\beta\neq0$, 2. $\sin\alpha\neq0,\sin\beta=0$, 3. $\sin\alpha=0,\sin\beta\neq0$, 4. $\sin\alpha=0,\sin\beta=0$. Within this work, we will only consider case 1. The other cases may be considered analogically.

Denote by λ_n, zeros of the function $\Theta_0\left(\lambda\right)$. Thus, we have

$$\lambda_n\sim n+O\left(1\right). \tag{2.10}$$

Denote by C_n the circle of radius, $0<\varepsilon<\frac{1}{2}$, centered at the origin λ_n^0 and by Γ_{N_0} the counterclockwise square contours with four vertices

$$\begin{aligned} K &= N_0+\varepsilon+N_0 i, \quad L=-N_0-\varepsilon+N_0 i, \\ M &= -N_0-\varepsilon-N_0 i, \quad N=N_0+\varepsilon-N_0 i, \end{aligned}$$

where $i=\sqrt{-1}$ and N_0 is a natural number. Obviously, if $\lambda\in C_n$ or $\lambda\in\Gamma_{N_0}$, then $|\Theta_0(\lambda)|\geq M|\lambda|e^{|\operatorname{Im}\lambda|\pi}$ $(M>0)$ by using a similar method in [37]. Thus, on $\lambda\in C_n$ or $\lambda\in\Gamma_{N_0}$, we have

$$\frac{\Theta(\lambda)}{\Theta_0(\lambda)}=1+\frac{1}{\lambda}\left[(\cot\alpha-\cot\beta+B(\lambda,\pi))\cot\lambda\pi-A(\lambda,\pi)\right]+O\left(\frac{1}{\lambda^2}\right).$$

Expanding $\ln\frac{\Theta(\lambda)}{\Theta_0(\lambda)}$ by the Maclaurin formula, we find

$$\begin{aligned} \ln\frac{\Theta(\lambda)}{\Theta_0(\lambda)} =& \frac{1}{\lambda}\left[(\cot\alpha-\cot\beta+B(\lambda,\pi))\cot\lambda\pi-A(\lambda,\pi)\right] \\ &-\frac{1}{2\lambda^2}\left[(\cot\alpha-\cot\beta+B(\lambda,\pi))^2\cot^2\lambda\pi+A^2(\lambda,\pi)\right. \\ &\left.-2(\cot\alpha-\cot\beta+B(\lambda,\pi))A(\lambda,\pi)\cot\lambda\pi\right]+O\left(\frac{1}{\lambda^3}\right). \end{aligned}$$

Using the well-known Rouche theorem, we get that $\Theta(\lambda)$ has the same number of zeros inside Γ_{N_0} as $\Theta_0(\lambda)$ (see [35]). Using the residue theorem, we have

$$\begin{aligned} &\lambda_n-n=-\frac{1}{2\pi i}\oint_{C_n}\ln\frac{\Theta(\lambda)}{\Theta_0(\lambda)}d\lambda \\ &=-\frac{1}{2\pi i}\oint_{C_n}\frac{(\cot\alpha-\cot\beta+B(\lambda,\pi))\cot\lambda\pi}{\lambda}d\lambda+\frac{1}{2\pi i}\oint_{C_n}\frac{A(\lambda,\pi)}{\lambda}d\lambda \\ &\quad-\frac{1}{2\pi i}\oint_{C_n}\frac{(\cot\alpha-\cot\beta+B(\lambda,\pi))A(\lambda,\pi)\cot\lambda\pi}{\lambda^2}d\lambda+\frac{1}{2\pi i}\oint_{C_n}\frac{A^2(\lambda,\pi)}{2\lambda^2}d\lambda \\ &\quad+\frac{1}{2\pi i}\oint_{C_n}\frac{(\cot\alpha-\cot\beta+B(\lambda,\pi))^2\cot^2\lambda\pi}{2\lambda^2}d\lambda+O\left(\frac{1}{n^3}\right). \end{aligned}$$

Thus, using (2.10) and residue calculation we have proven the following theorem.

Theorem 2.1. *The spectrum of the problem* (1.1)–(1.5) *has the asymptotic distribution*

$$\begin{aligned} \lambda_n =& n+\frac{\cot\beta-\cot\alpha-B(n,\pi)}{n\pi}-\frac{(\cot\alpha-\cot\beta+B(n,\pi))A(n,\pi)}{n^2\pi} \\ &-\frac{(\cot\alpha-\cot\beta+B(n,\pi))^2}{n^3}+O\left(\frac{1}{n^3}\right) \end{aligned}$$

for sufficiently large $|n|$ *.*

3. The oscillation

In this chapter, we will find nodal points of eigenfunctions of the problem (1.1)–(1.5). We want to point out that the results obtained in this chapter are extensions to those in [37, 38].

Let us rewrite the equation (2.5) and replace λ by λ_n

$$\phi_1(t,\lambda_n) = \sin\alpha\cos\lambda_n t - \frac{\cos\alpha}{\lambda_n}\sin\lambda_n t - \frac{\sin\alpha\sin\lambda_n t}{2\lambda_n}\int_0^t q(\tau)\cos\left(\lambda_n\Delta\left(\tau\right)\right)d\tau$$

$$+\frac{\sin\alpha\cos\lambda_n t}{2\lambda_n}\int_0^t q(\tau)\sin\left(\lambda_n\Delta\left(\tau\right)\right)d\tau + O\left(\frac{1}{\lambda_n^2}\right).$$

Let us assume that t_n^j are the nodal points of the eigenfunction $\phi\left(t,\lambda_n\right)$. Taking $\sin\left(\lambda_n t\right)\neq 0$ into account for sufficiently large n, we get

$$\cot\left(\lambda_n t\right)\left[1+\frac{A\left(\lambda_n,t\right)}{\lambda_n}\right] = \frac{\cot\alpha + B\left(\lambda_n,t\right)}{\lambda_n} + O\left(\frac{1}{\lambda_n^2}\right).$$

It follows easily that

$$\tan\left(\lambda_n t + \frac{\pi}{2}\right) = -\frac{\cot\alpha + B\left(\lambda_n,t\right)}{\lambda_n} + O\left(\frac{1}{\lambda_n^2}\right). \tag{3.1}$$

Thus, solving (3.1), one obtains

$$t_n^j = \frac{\left(j-\frac{1}{2}\right)\pi}{\lambda_n} - \frac{\cot\alpha + B\left(\lambda_n,t_n^j\right)}{\lambda_n^2} + O\left(\frac{1}{\lambda_n^3}\right). \tag{3.2}$$

Note that

$$\lambda_n^{-1} = \frac{1}{n} + \frac{\cot\beta - \cot\alpha - B\left(n,\pi\right)}{n^3\pi} + O\left(\frac{1}{n^3}\right) \tag{3.3}$$

and

$$\lambda_n^{-2} = \frac{1}{n^2} + O\left(\frac{1}{n^4}\right). \tag{3.4}$$

Substituting (3.3) and (3.4) into (3.2), we have

$$\begin{aligned} t_n^j =& \frac{\left(j-\frac{1}{2}\right)\pi}{n} + \frac{\left(\cot\beta - \cot\alpha - B\left(n,\pi\right)\right)\left(j-\frac{1}{2}\right)}{n^3} \\ &- \frac{\cot\alpha + B\left(n,\frac{j\pi}{n}\right)}{n^2} + O\left(\frac{1}{n^3}\right), \qquad j=\overline{1,\left[\frac{n}{2}\right]}. \end{aligned} \tag{3.5}$$

Similarly, from (2.7), we get

$$\delta\phi_2(t,\lambda_n) = \frac{\sin\alpha\cos\lambda_n t}{\lambda_n} - \frac{\cos\alpha\sin\lambda_n t}{\lambda_n^2} - \frac{\sin\alpha\sin\lambda_n t}{2\lambda_n^2}\int_0^t q(\tau)\cos\left(\lambda_n\Delta\left(\tau\right)\right)d\tau$$

$$+\frac{\sin\alpha\cos\lambda_n t}{2\lambda_n^2}\int_0^t q(\tau)\sin\left(\lambda_n\Delta\left(\tau\right)\right)d\tau + O\left(\frac{1}{\lambda_n^3}\right).$$

For nodal points of $\phi_2(t,\lambda_n)$, again, taking $\sin(\lambda_n t) \neq 0$ into account for sufficiently large n, we get

$$\cot(\lambda_n t)\left[1+\frac{A(\lambda_n,t)}{\lambda_n}\right]=\frac{\cot\alpha+B(\lambda_n,t)}{\lambda_n}+O\left(\frac{1}{\lambda_n^2}\right),$$

and thus

$$\tan\left(\lambda_n t+\frac{\pi}{2}\right)=-\frac{\cot\alpha+B(\lambda_n,t)}{\lambda_n}+O\left(\frac{1}{\lambda_n^2}\right). \tag{3.6}$$

Thus, solving (3.6), one obtains

$$t_n^j=\frac{\left(j-\frac{1}{2}\right)\pi}{\lambda_n}-\frac{\cot\alpha}{\lambda_n^2}+\frac{1}{2\lambda_n^2}\int_0^{t_n^j} q(\tau)\cos(\lambda_n\Delta(\tau))\,d\tau+O\left(\frac{1}{\lambda_n^3}\right). \tag{3.7}$$

Substituting (3.3) and (3.4) into (3.7) we have

$$\begin{aligned} t_n^j=&\frac{\left(j-\frac{1}{2}\right)\pi}{n}+\frac{\left(\cot\beta-\cot\alpha-B(n,\pi)\right)\left(j-\frac{1}{2}\right)}{n^3}-\frac{\cot\alpha+B\left(n,\frac{j\pi}{n}\right)}{n^2}\\ &+\frac{\left(\cot\alpha+B\left(n,\frac{j\pi}{n}\right)\right)A\left(n,\frac{j\pi}{n}\right)}{n^3}+O\left(\frac{1}{n^3}\right),\qquad j=\overline{\left[\frac{n}{2}\right]+1,n}. \end{aligned} \tag{3.8}$$

Thus we have proven the following theorem.

Theorem 3.1. *For sufficiently large n, we have the formulas (3.5) and (3.8) of the nodal points for the problem (1.1)–(1.5).*

4. An inverse problem

We see that there exists N_0 such that for all $n > N_0$, the eigenfunction $\phi(t,\lambda_n)$ of the problem has exactly n simple nodes in the interval $(0,\pi)$. The set $t=\{t_n^j\}$ is called the nodal set of the problem (1.1)–(1.5). We also define the function $j_n(t)$ to be the largest index j such that $0 \le t_n^j \le t$. Thus, $j = j_n(t)$ if and only if $t \in [t_n^j, t_n^{j+1})$.

Theorem 4.1. *For each $t\in[0,\pi]$, let $\{t_n^j\}\subset t$ be chosen such that $\lim_{n\to\infty} t_n^j = t$. Then the following finite limit exists and corresponding equality holds:*

$$\lim_{n\to\infty} n^2\left[t_n^j-\frac{\left(j-\frac{1}{2}\right)\pi}{n}\right]=f(t) \tag{4.1}$$

and

$$f(t)=\begin{cases}\frac{(\cot\beta-\cot\alpha)t}{\pi}-\cot\alpha-\frac{1}{2}\int\limits_0^t q(\tau)\,d\tau, & \Delta(\tau)=0,\\ \frac{(\cot\beta-\cot\alpha-B(n,\pi))t}{\pi}-\cot\alpha-B(n,t), & \Delta(\tau)\neq 0.\end{cases} \tag{4.2}$$

Proof. Using the formulas (3.5) and (3.8) for nodal points and the fact that $\lim_{n\to\infty} t_n^j = t$, it follows that as $n\to\infty$ the limits of left-hand side in (4.1) exists, and (4.2) holds. Thus, the proof is completed. □

Now, we can construct the potential function $q(t)$ via the following theorem.

Theorem 4.2. *Let $T^0 = \left\{t^j_{n_k}\right\}$ and let $T^0 \subset T$ be a subset of nodal points such that $\left\{t^j_{n_k}\right\}$ is dense in $(0, \pi)$. Also, assume that $\lim_{n\to\infty} n^2 \left[t^j_n - \frac{(j-\frac{1}{2})\pi}{n}\right]$ exists for fixed j. Then, T^0 uniquely determines the parameters α, β in the boundary conditions and the function $q(t)$. Moreover, if $\Delta(\tau) = 0$, then the numbers α, β and the function $q(t)$ can be constructed via the following algorithm:*

* *For each $t \in [0, \pi]$, choose a sequence $\{t^j_n\} \subset T^0$ such that $\lim_{n\to\infty} t^j_n = t$.*

* *Find the function $f(t)$ via (4.2).*

* *Calculate the parameters $\alpha, \beta > 0$:*

$$\alpha = -\operatorname{arccot} f(0), \quad \beta = -\operatorname{arccot} f(\pi).$$

* *Calculate the function $q(t)$:*

$$q(t) - \frac{1}{\pi}\int_0^\pi q(\tau)\,d\tau = \frac{2}{\pi}\left(f(\pi) - f(0)\right) - 2f'(t).$$

5. The trace

Lastly, we will get trace formula for the problem (1.1)–(1.5).

The asymptotic formula (2.10) for the eigenvalues implies that for all sufficiently large N_0, the numbers λ_n with $|n| \le N_0$ are inside Γ_{N_0}, and the numbers λ_n with $|n| > N_0$ are outside Γ_{N_0}. It follows that

$$\begin{aligned}
&\sum_{n=-N_0}^{N_0} \lambda_n^2 - n^2 = -\frac{1}{2\pi i}\oint_{\Gamma_n} 2\lambda \ln\frac{\Theta(\lambda)}{\Theta_0(\lambda)}d\lambda \\
&= -\frac{1}{2\pi i}\oint_{\Gamma_n} 2\left[\cot\alpha - \cot\beta + B(\lambda,\pi)\right]\cot(\lambda\pi)\,d\lambda \\
&\quad + \frac{1}{2\pi i}\oint_{\Gamma_n} 2A(\lambda,\pi)\,d\lambda - \frac{1}{2\pi i}\oint_{\Gamma_n}\frac{2\left[\cot\alpha - \cot\beta + B(\lambda,\pi)\right]A(\lambda,\pi)\cot(\lambda\pi)}{\lambda}d\lambda \\
&\quad + \frac{1}{2\pi i}\oint_{\Gamma_n}\frac{A^2(\lambda,\pi)}{\lambda}d\lambda + \frac{1}{2\pi i}\oint_{\Gamma_n}\frac{\left[\cot\alpha - \cot\beta + B(\lambda,\pi)\right]^2\cot^2\lambda\pi}{\lambda}d\lambda \\
&\quad + O\left(\frac{1}{N_0}\right),
\end{aligned}$$

by calculations, which implies that

$$\begin{aligned}
&\sum_{n=-N_0}^{N_0} \lambda_n^2 - n^2 \\
&= -\frac{2}{\pi}\sum_{0\neq n=-N_0}^{N_0}\left[\cot\alpha - \cot\beta + B(n,\pi)\right] - \frac{2}{\pi}\left[\cot\alpha - \cot\beta + B(0,\pi)\right]
\end{aligned}$$

$$- \sum_{0\neq n=-N_0}^{N_0} -2\left[\left(\cot\alpha - \cot\beta + B\left(n,\pi\right)\right)A\left(n,\pi\right)\right]\frac{1}{n\pi} + R$$
$$+ 2A^2\left(0,\pi\right) - \left(\cot\alpha - \cot\beta + B\left(0,\pi\right)\right)^2 + O\left(\frac{1}{N_0}\right), \tag{5.1}$$

where

$$R = Res_{\lambda=0}\left[-2\left(\left(\cot\alpha - \cot\beta + B\left(\lambda,\pi\right)\right)A\left(\lambda,\pi\right)\right)\right]\frac{\cot\lambda\pi}{\lambda}.$$

Passing to the limit as $N_0 \to \infty$ in (5.1), we have

$$\lambda_0^2 + \sum_{0\neq n=-\infty}^{\infty}\Bigg[\lambda_n^2 - n^2 - \frac{2}{\pi}\left(\cot\beta - \cot\alpha - B\left(n,\pi\right)\right)$$
$$-\frac{2}{n\pi}\left(\cot\beta - \cot\alpha - B\left(n,\pi\right)\right)A\left(n,\pi\right)\Bigg]$$
$$= \frac{2}{\pi}\Big[\cot\beta - \cot\alpha - B\left(0,\pi\right) + R + 2A^2\left(0,\pi\right)$$
$$- \left(\cot\alpha - \cot\beta + B\left(0,\pi\right)\right)\Big]^2. \tag{5.2}$$

The sum on the left side of (5.2) is called the regularized trace of the problem (1.1)–(1.5).

6. Conclusion

In this paper, spectrum, oscillations, regularized trace, and an inverse problem of a discontinuous Sturm–Liouville type boundary-value-transmission problem with retarted argument was investigated. The considered problem differs from the classical Sturm–Liouville problems with retarded argument in that it contains eigenvalue-dependent transmission conditions at an interior point. If we take $\Delta \equiv 0$, then the formulas obtained in this work coincide with those for the classical Sturm–Liouville problems. However, the retarded argument Δ reflects rather well the real process in a number of cases, for example, when the retarded argument is connected with the transmission of an audio signal, in hydraulic shock or in other wave processes. This work can be also considered as a continuation of the paper [30] in which under the conditions $q(x) \in C^n[0,\pi], n = 0, 1$ asymptotic formulas for eigenvalues and eigenfunctions for the problem (1.1)–(1.5) were obtained. We also want to note that the same differential equation but with different boundary conditions or a differential equation with a weight function with the same boundary conditions can be also investigated from similar aspects.

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