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CERTAIN SUBGROUPS OF FUNDAMENTAL GROUP OF GRAPHS

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Abstract. In this article, we introduce three families of subgroups of the fundamental group of graphs that indicate the presence of cycles of a specific length. We examine the interconnection between these families, such as their normality and their behavior with respect to the common homomorphisms between the fundamental groups of graphs. Ultimately we demonstrate that there is a finite length depended on the diameter of graph, after which all of these subgroups are equal to the fundamental group of graph.

1. Introduction and Motivation

The application of graphs in distributed systems [1] and the importance of graph coverings and fibrations in this theory [3] as well as distribution networks have caused serious studies to be carried out in the theory of graph coverings, graph fibrations, and their relationship with the fundamental group [11, 12].

Due to the importance of cycles in the graph models of distribution networks and also the role of the subgroups of the fundamental group in the classification of graph coverings [9, 10], here we study subgroups of the fundamental group of a graph.

We start by summarizing graph-theoretical definitions and basic properties, which will be used in Section 2. Then, in Section 3, we discuss three families of subgroups of the fundamental group of a given graph G.

- The first family contains subgroups $\pi_1^k(G,u)$, generated by all the equivalence classes of cycles with the length k at the vertex u of G. These

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subgroups are not normal and depend to the base vertex u, and therefore we do not care about it either.

- The second family contains subgroups $\pi_{1,k}(G,u)$, generated by all the equivalence classes of cycles with the length k that can be located anywhere on the graph (of course, by going back and forth along a certain path to reach said cycle). These subgroups are normal and independent of the base vertex u, but in Section 4, it will be proven that they are not invariant under group homomorphism induced on fundamental groups by a graph homomorphism.
- The third family contains subgroups $\pi_{1,\overline{k}}(G,u)$, generated by all the equivalence classes of cycles with the length at most k that can be located anywhere on the graph. These subgroups are normal, independent of the base vertex u and invariant under group homomorphism induced on fundamental groups by a graph homomorphism.

Also, we prove as a main result that for a connected graph G with diameter d and $u_0 \in V_G$, $\pi_1(G,u_0) = \pi_{1,\overline{2d+1}}(G,u_0)$.

Investigating the behavior of these families of subgroups when faced with partial covers is our final step in this paper.

2. Notations and preliminaries

In this section, we define notation and terminologies of this paper and mainly follow [4, 6]. We work in the category Gph of simple graphs: undirected graphs without multiple edges. Moreover, throughout this paper, we will assume that graphs are connected and finite.

A **simple graph** G is defined by a finite set V_G of vertices and a set E_G of edges connecting vertices. Each edge is given by an unordered pair of vertices. Any pair of vertices has at most one edge connecting them. Loops and isolated vertices are not allowed: each vertex must be connected to at least one other. A connecting edge will be notated by $v_1 \sim v_2$, and we say v_1 is adjacent with v_2.

A **graph homomorphism** $f : G \longrightarrow H$ is a map $f : V_G \longrightarrow V_H$ such that for $v_1, v_2 \in V_G$, if $v_1 \sim v_2 \in E_G$, then either $f(v_1) = f(v_2)$ or $f(v_1) \sim f(v_2) \in E_H$ (we shall remove subscripts whenever there is no confusion). In other words, a graph homomorphism maps vertices to vertices and edges to edges in a way that preserves the incidence relationship.

A graph with $n+1$ vertices $\{0,1,2,\dots,n\}$ such that $i \sim i+1$ for $i = 0,1,\dots,n-1$, is called **path graph** and denoted by P_n. The **cycle graph** C_n has n vertices $\{0,1,2,\dots,n-1\}$ such that i adjacent to j if and only if $i-j \equiv \pm 1 \ mod\ n$, for $i = 0,1,\dots,n-1$.

A **walk** in G of length n from $x \in V_G$ to $y \in V_G$ is a morphism $\alpha : P_n \longrightarrow G$ with $s(\alpha) = \alpha(0) = x$ and $t(\alpha) = \alpha(n) = y$. We shall usually describe a walk by a list of image vertices $(v_0 v_1 v_2 \dots v_n)$ such that $v_i \sim v_{i+1}$. The symbol $\overleftarrow{\alpha}$ denotes the walk $(v_n v_{n-1} \dots v_1 v_0)$, and we call it the inverse of α. A graph G is called **connected** if for every choice of x and y there is a walk from x to y.

For $u, v \in V_G$, $d(u,v)$ denotes the distance between u and v, that is, the length of a shortest walk connecting u and v and k-neighborhood of $u \in V_G$ is

$$N_k(u) = \{v \in V_G | d(u,v) \leq k\}.$$

The diameter of a given graph G is the maximal distance between any two vertices in G.

A walk is called a **path** if $v_{i+2} \neq v_i$. A closed path is called a **cycle** if $v_0 = v_n$. By u-based n-cycle $\alpha : P_n \longrightarrow G$, we mean a closed path α of length n in which $\alpha(0) = \alpha(n) = u$. The (u) is considered as a closed walk of length 0 and also as a u-based cycle. A **tree** is a connected graph that contains no cycles.

The **concatenation** $\alpha * \beta$ of walks α and β is defined to be the walk α followed by the walk β when $t(\alpha) = s(\beta)$, that is, if $\alpha = (v_0, \ldots, v_m)$, $\beta = (u_0, \ldots, u_n)$, and $v_m = u_0$, then $\alpha * \beta = (v_0, \ldots, v_m, u_1, \ldots, u_n)$. The walk of the form $\alpha * \overleftarrow{\alpha}$ is called a backtrack.

Two walks α and β are called equivalent and denoted by $\alpha \sim \beta$ if $s(\alpha) = t(\alpha) = s(\beta) = t(\beta)$ and there exists a sequence

$$\alpha = \alpha_0,\ \alpha_1,\ \ldots,\ \alpha_n = \beta,$$

such that α_i and α_{i+1} differ form either the insertion or the removal of backtracks. The equivalence class of α is usually denoted by $[\alpha]$.

For a given $u \in V_G$, the **fundamental group** of G at u denoted by $\pi_1(G,u)$, is the set of all the equivalence classes of u-based cycles with the product operation: $[\alpha] * [\beta] = [\alpha * \beta]$. The trivial element of this group is $[(u)]$, and the inverse element of $[\alpha]$ is $[\overleftarrow{\alpha}]$. For a connected graph G, this group is independent of the choice of the base vertex u, up to isomorphism.

It can be easily checked by induction that every u-based closed walk α is equivalent to a unique u-based cycle with minimal length, and therefore we will use the equivalence class of this unique u-based cycle to display the group members.

3. Subgroups of the fundamental group

For the fundamental group of a given connected graph G, we have the following theorem. Let $u \in V_G$ and let T be a maximal tree in G (contains all the vertices) containing u. Let $\{e_\lambda : \lambda \in \Lambda\}$ denote the set of edges of G not contained in T. There are unique paths β_λ from u to $s(e_\lambda)$ and β'_λ from $t(e_\lambda)$ to u. Define $\alpha_\lambda := \beta_\lambda * e_\lambda * \beta'_\lambda$, which is a u-based closed path.

Theorem 3.1 ([9, 7])**.** *The fundamental group of a given graph G is a free group on the set of generators $\{\alpha_\lambda | \lambda \in \Lambda\}$.*

In this section, subgroups of the fundamental group of graph G will be introduced, which are generated by all the equivalence classes of cycles of specific length.

Definition 3.2. For a graph G and $u \in V_G$, the subgroup of $\pi_1(G,u)$ generated by all the equivalence classes of u-based k-cycle is denoted by $\pi_1^k(G,u)$.

Proposition 3.3. *Let G be a graph with $u \in V_G$. Then*

(i) For $k = 1, 2$, the subgroup $\pi_1^k(G,u)$ is trivial.

(ii) $\pi_1^k(G,u) \cap \pi_1^m(G,u)$ *is not necessarily* $\{[(u)]\}$, *for every* $k \neq m$.
(iii) The subgroup $\pi_1^k(G,u) \leq \pi_1(G,u)$ *is not necessarily normal.*
(iv) The subgroup $\pi_1^k(G,u) \leq \pi_1(G,u)$ *depends on the base vertex* u.

Proof. (i) This is obvious because there is neither 1-cycle nor 2-cycle in a simple graph.
(ii) Assume that $\beta = (uv_1v_2u)$ and $\beta' = (uv_2v_3u)$ are two 3-cycles. Then

$$\beta * \beta' = (uv_1v_2u) * (uv_2v_3u) = (uv_1v_2uv_2v_3u) \sim (uv_1v_2v_3u),$$

which is a 4-cycle and so $\pi_1^3(G,u) \cap \pi_1^4(G,u) \neq \{[(u)]\}$.
(iii) Assume that in a graph G, there exist two u-based cycles $\alpha = (uv_1v_2u) \in \pi_1^3(G,u)$ and $\beta = (uv_3v_4v_5u) \in \pi_1(G,u)$ such that $\{v_1,v_2\} \cap \{v_3,v_4,v_5\} = \emptyset$. Obviously, $[\beta] * [\alpha] * [\beta]^{-1} \notin \pi_1^3(G,u)$ since

$$[\beta] * [\alpha] * [\beta]^{-1} = [(uv_3v_4v_5uv_1v_2uv_5v_4v_3u)] \notin \pi_1^3(G,u).$$

(iv) This is also obvious because in some vertices we may have k-cycles and in some vertices we may not have them. For example, consider the graph H with vertices set $\{v_0,v_1,v_2,v_3,v_4\}$ and edges set $E = \{v_0v_1, v_1v_2, v_0v_2, v_2v_3, v_3v_4, v_2v_4\}$, as in Figure 1. Then $\pi_1^3(H,v_2)$ is a free group with two generators, while $\pi_1^3(H,v_0)$ is a free group with one generator.

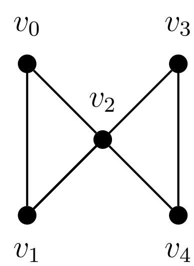

FIGURE 1. The graph H

□

The two main defects of $\pi_1^k(G,u)$, that is, nonnormality and its dependence on the base vertex, lead us to the following definition.

Definition 3.4. For a graph G and $u \in V_G$, the k-subgroup of $\pi_1(G,u)$ is denoted by $\pi_{1,k}(G,u)$ and is generated by the subgroups of cycles of the following type:

$$\prod_{j=1}^{n} [\alpha_j] * [\beta_j] * [\overleftarrow{\alpha_j}],$$

where the α_j's are arbitrary paths starting at v and each β_j is a $t(\alpha_j)$-based k-cycle.

In fact, $\pi_{1,k}(G,u)$ is generated by all the equivalence classes of cycles of length k, which may be located in any part of the G.

Proposition 3.5. *Let* G *be a graph with* $u \in V_G$. *Then*
(i) For $k = 1, 2$, *the subgroup* $\pi_{1,k}(G,u)$ *is trivial.*

(ii) $\pi_{1,k}(G,u) \cap \pi_{1,m}(G,u)$ *is not necessarily* $\{[(u)]\}$*, for every* $k \neq m$*.*
(iii) The subgroup $\pi_{1,k}(G,u) \leq \pi_1(G,u)$ *is normal.*

Proof. (i) and (ii) are similar to the Proposition 3.3.
(iii) For any $[\gamma] \in \pi_1(G,u)$ and any $\prod_{j=1}^{n}[\alpha_j] * [\beta_j] * [\overleftarrow{\alpha_j}] \in \pi_{1,k}(G,u)$, we have

$$\begin{aligned}
&[\gamma] * \Big(\prod_{j=1}^{n}[\alpha_j] * [\beta_j] * [\overleftarrow{\alpha_j}]\Big) * [\gamma]^{-1} \\
&= [\gamma] * \big([\alpha_1] * [\beta_1] * [\overleftarrow{\alpha_1}]\big) * \cdots * \big([\alpha_n] * [\beta_n] * [\overleftarrow{\alpha_n}]\big) * [\gamma]^{-1} \\
&= \big([\gamma] * [\alpha_1] * [\beta_1] * [\overleftarrow{\alpha_1}] * [\gamma]^{-1}\big) * \cdots * \big([\gamma] * [\alpha_n] * [\beta_n] * [\overleftarrow{\alpha_n}] * [\gamma]^{-1}\big) \\
&= \prod_{j=1}^{n}[\gamma * \alpha_j] * [\beta_j] * [\overleftarrow{\alpha_j} * \overleftarrow{\gamma}] \in \pi_{1,k}(G,u).
\end{aligned}$$

□

The second part of Proposition 3.5 makes there is no clear structure for the intersection of the two subgroups $\pi_{1,k}(G,u)$ and $\pi_{1,m}(G,u)$. Another defect of these subgroups, which is specified in Proposition 4.1 and the Example 4.2, is that they are not invariant under the induced homomorphisms on the fundamental groups. These lead to the following definition.

Definition 3.6. For a graph G and $u \in V_G$, the $\overline{k}$-subgroup of $\pi_1(G,u)$ is denoted by $\pi_{1,\overline{k}}(G,u)$ and is generated by the subgroups of cycles of the following type:

$$\prod_{j=1}^{n}[\alpha_j] * [\beta_j] * [\overleftarrow{\alpha_j}],$$

where α_j's are arbitrary paths starting at u and each β_j is a $t(\alpha_j)$-based i-cycle for $i \leq k$.

In fact, $\pi_{1,k}(G,u)$ is generated by all the equivalence classes of cycles of length at most k, which may be located in any part of the G.

Proposition 3.7. *Let G be a graph with $u \in V_G$. Then*
(i) For $k = 1, 2$*, the subgroup* $\pi_{1,\overline{k}}(G,u)$ *is trivial.*
(ii) For every $k > m$*,* $\pi_{1,\overline{k}}(G,u) \cap \pi_{1,\overline{m}}(G,u) = \pi_{1,\overline{m}}(G,u)$*.*
(iii) The subgroup $\pi_{1,\overline{k}}(G,u) \leq \pi_1(G,u)$ *is normal.*

Proof. Proofs of (i) and (ii) come from definitions, and the proof of (iii) is similar to the proof of the part (iii) of Proposition 3.5.

□

Note that Part (ii) of Proposition 3.5 states that some generators of the subgroup $\pi_{1,k}(G,u)$ may be generated by generators of other k-subgroups. The following definition names such generators.

Definition 3.8. A k-cycle $\alpha = (u_0u_1 \ldots u_n)$ in a graph G is called decomposable if it can be broken to concatenation of two smaller cycles, that is, there exist

$0 \leq i < j \leq n$ and a path β from u_i to u_j such that $(u_0u_1 \dots u_i)*\beta*(u_j, u_{j+1} \dots u_n)$ and $(u_iu_{i+1} \dots u_j) * \overleftarrow{\beta}$ are cycles with length less than k.

In fact, if we consider $\gamma = (u_0 = u_n, u_{n-1} \dots u_j)*\overleftarrow{\beta}*(u_i \dots u_j)*(u_ju_{j+1} \dots u_n = u_0)$, then

$$\begin{aligned}
&\big((u_0u_1 \dots u_i) * \beta * (u_j, u_{j+1} \dots u_n)\big) * \gamma \\
&= \big((u_0u_1 \dots u_i) * \beta * (u_j, u_{j+1} \dots u_n)\big) * \big((u_n, u_{n-1} \dots u_j) \\
&\quad * \overleftarrow{\beta} * (u_i \dots u_j) * (u_ju_{j+1} \dots u_n)\big) \\
&= \big((u_0u_1 \dots u_i) * \beta\big) * \big((u_j, u_{j+1} \dots u_n) * (u_n, u_{n-1} \dots u_j)\big) \\
&\quad * \big(\overleftarrow{\beta} * (u_i \dots u_j) * (u_ju_{j+1} \dots u_n)\big) \\
&= \big((u_0u_1 \dots u_i) * \beta\big) * \big(\overleftarrow{\beta} * (u_i \dots u_j) * (u_ju_{j+1} \dots u_n)\big) \\
&= (u_0u_1 \dots u_n) = \alpha.
\end{aligned}$$

Then according to the definition of a decomposable k-cycle, one can conclude the following proposition.

Proposition 3.9. *Let G be a connected graph and let α be a u-based decomposable k-cycle. Then $[\alpha] \in \pi_{1,\overline{k-1}}(G, u)$.*

We know that every subgroup of a free group is a free group, and hence $\pi_{1,k}(G, u)$ is a free group by Theorem 3.1. The generators of $\pi_{1,k}(G, u)$ are $[\alpha_j] * [\beta_j] * [\overleftarrow{\alpha_j}]$, where α_j's are arbitrary paths starting at u and each β_j is a $t(\alpha_j)$-based k-cycle. However, we cannot consider the free product of these subgroups because for $k \neq m$, $\pi_{1,k}(G, u) \cap \pi_{1,m}(G, u)$ is not the trivial subgroup. Also, if A is the free group generated by all the equivalence classes of all decomposable cycles, since A is not a subgroup of all $\pi_{1,k}(G, u)$, then it would not be possible to use the amalgamated free product.

Theorem 3.10. *Let G be a graph with $u \in V_G$. Then there exists $n \in \mathbb{N}$ such that $\pi_1(G, u) \cong \pi_{1,\overline{n}}(G, u)$.*

Proof. Let n be the largest length of cycles. Then for every $k > n$ there is no u-based k-cycles to make $\pi_{1,k}(G, u)$ nontrivial. Also, for an n-cycle γ which is not u-based, there exists a path λ from u to a vertex $v \in \gamma$, and hence $[\lambda * \gamma * \overleftarrow{\lambda}] \in \pi_{1,n}(G, u)$. □

Theorem 3.11. *Let G be a connected graph with diameter d and let $u_0 \in V_G$. Then $\pi_1(G, u_0) = \pi_{1,\overline{2d+1}}(G, u_0)$.*

Proof. We prove that for every $k > 2d + 1$, $\pi_{1,k}(G, u_0) \leq \pi_{1,\overline{2d+1}}(G, u_0)$.
Let $[\alpha * \beta * \overleftarrow{\alpha}]$ be a generator of $\pi_{1,2d+2}(G, u_0)$, where $\alpha = (u_0u_1 \dots u_m)$ is a path, $m \in \mathbb{N}$, and $\beta = (v_0v_1 \dots v_{2d+1}v_0)$ is a $(2d + 2)$-cycle such that $u_m = v_0$. Since the distance between v_0 and v_{d+1} is at most d, the path $(v_0v_1 \dots v_{d+1})$ is not the shortest path connecting v_0 and v_{d+1}. Hence there exists a path λ from v_0 to v_{d+1} whose length is at most d and also is equal to $d(v_0, v_{d+1})$.

Now, we have

$$\alpha * \beta * \overleftarrow{\alpha} \sim \big(\alpha * \big((v_0 v_1 \dots v_{d+1}) * \overleftarrow{\lambda}\big) * \overleftarrow{\alpha}\big) * \big(\alpha * \big(\lambda * (v_{d+1} v_{d+2} \dots v_{2d+1} v_0)\big) * \overleftarrow{\alpha}\big),$$

which implies that $[\alpha * \beta * \overleftarrow{\alpha}] \in \pi_{1,\overline{2d+1}}(G,u)$ because the length of $\big((v_0 v_1 \dots v_{d+1}) * \overleftarrow{\lambda}\big)$ and $\big(\lambda * (v_{d+1} v_{d+2} \dots v_{2d+1} v_0)\big)$ is at most $2d+1$. □

4. k-subgroups and $\overline{k}$-subgroups in induced group homomorphisms

We know that a base point preserving graph homomorphism $f : (G,u) \longrightarrow (H,v)$ induces a group homomorphism $f_* : \pi_1(G,u) \longrightarrow \pi_1(H,v)$ defined by $f_*([\alpha]) = [f \circ \alpha]$.

Proposition 4.1. *Let $f : (G,u) \longrightarrow (H,v)$ be a base point preserving graph homomorphism, for $u \in V_G$ and $v \in V_H$. Then for every $2 < k \in \mathbb{N}$, we have*
(i) $f_*\big(\pi_{1,k}(G,u)\big) \leq \pi_{1,\overline{k}}(H,v)$.
(ii) $f_*\big(\pi_{1,\overline{k}}(G,u)\big) \leq \pi_{1,\overline{k}}(H,v)$.

Proof. (i) Let α be a u-based cycle with length k. Of course, $f_*(\alpha)$ has a length less than k because f may take more than one vertex in α to one vertex in V_H. This follows (i) and (ii). □

In the following example, we show that by the assumption of Proposition 4.1, $f_*\big(\pi_{1,k}(G,u)\big)$ is not necessarily a subgroup of $\pi_{1,k}(H,v)$.

Example 4.2. Let $G = C_4$ with the vertices set $V = \{u_i | i = 0,1,2,3\}$, and let $H = C_3$ be the vertices set $V = \{v_i | i = 0,1,2\}$. Let $f : (G,u_0) \longrightarrow (H,v_0)$ be a base point preserving graph homomorphism defined by $f(u_i) = v_i$ for $i = 0,1,2$ and $f(u_3) = v_2$. Obviously, $\pi_{1,k}(G,u_0)$ is trivial for $k \neq 4$ and $\pi_{1,4}(G,u_0) = \pi_1(G,u_0) \cong \mathbb{Z}$. Also, $\pi_{1,k}(H,v_0)$ is trivial for $k \neq 3$ and $\pi_{1,3}(H,v_0) = \pi_1(H,v_0) \cong \mathbb{Z}$. Now, $f_*\big(\pi_{1,4}(G,u_0)\big) = \pi_{1,3}(H,v_0) \cong \mathbb{Z}$ while $\pi_{1,4}(H,v_0)$ is trivial.

For a connected graph G and $u, u' \in V_G$, consider a walk γ from u to u'. The map $h_\gamma : \pi_1(G,u) \longrightarrow \pi_1(G,u')$ defined by $h_\gamma([\alpha]) = [\overleftarrow{\gamma} * \alpha * \gamma]$ is a group isomorphism, which makes the fundamental group of a connected graph independent of the base point. Also, it can be easily checked that if two walks γ, γ' from u to u' are equivalent, then $h_\gamma = h_{\gamma'}$.

Proposition 4.3. *For every connected graph G and every $u \in V_G$, the k-subgroups and the $\overline{k}$-subgroups of $\pi_1(G,u)$ are invariant by each h_γ, for every walk γ from u to u', that is,*

$$h_\gamma\big(\pi_{1,k}(G,u)\big) = \pi_{1,k}(G,u'), \qquad h_\gamma\big(\pi_{1,\overline{k}}(G,u)\big) = \pi_{1,\overline{k}}(G,u').$$

Proof. For any walk γ from u to u' and any $\prod_{j=1}^{n}[\alpha_j]*[\beta_j]*[\overleftarrow{\alpha_j}] \in \pi_{1,k}(G,u)$, we have

$$\begin{aligned}
&h_\gamma\big(\prod_{j=1}^{n}[\alpha_j]*[\beta_j]*[\overleftarrow{\alpha_j}]\big)\\
&=[\overleftarrow{\gamma}]*\big(\prod_{j=1}^{n}[\alpha_j]*[\beta_j]*[\overleftarrow{\alpha_j}]\big)*[\gamma]\\
&=[\overleftarrow{\gamma}]*\big([\alpha_1]*[\beta_1]*[\overleftarrow{\alpha_1}]\big)*\big([\alpha_2]*[\beta_2]*[\overleftarrow{\alpha_2}]\big)*\cdots*\big([\alpha_n]*[\beta_n]*[\overleftarrow{\alpha_n}]\big)*[\gamma]\\
&=\big([\overleftarrow{\gamma}]*[\alpha_1]*[\beta_1]*[\overleftarrow{\alpha_1}]*[\gamma]\big)*\cdots*\big([\overleftarrow{\gamma}]*[\alpha_n]*[\beta_n]*[\overleftarrow{\alpha_n}]*[\gamma]\big)\\
&=\prod_{j=1}^{n}[\overleftarrow{\gamma}*\alpha_j]*[\beta_j]*[\overleftarrow{(\overleftarrow{\gamma}*\alpha_j)}] \in \pi_{1,k}(G,u').
\end{aligned}$$

The proof of the second part follows from the first part and the definition of the $\overline{k}$-subgroups of $\pi_1(G,u)$. $\square$

Corollary 4.4. *If G is connected, then the subgroups $\pi_{1,k}(G,u)$ and $\pi_{1,\overline{k}}(G,u)$ are independent of the base vertex u, up to isomorphism.*

Definition 4.5 ([8, 2, 5])**.** Given graphs G and H, a homomorphism $f : G \longrightarrow H$ is called a partial cover of graphs if it is locally injective that means $f|_{N_1(v)}$ is an injective map.

Theorem 4.6. *Let $f : (G,u) \longrightarrow (H,v)$ be a partial cover of graphs, for $u \in V_G$. Then the following properties hold:*

(i) $f_*\big(\pi_{1,k}(G,u)\big) \cap \pi_{1,k-1}(H,v) = \{[(v)]\}$.

(ii) $f_*\big(\pi_{1,k}(G,u)\big) \cap \pi_{1,k-2}(H,v) = \{[(v)]\}$.

Proof. (i) By the definition of k-subgroups, it suffices to prove that for every k-cycle $\alpha = (u_0u_1 \dots u_k)$, there is no $(k-1)$-cycle $\beta = (v_0v_1 \dots v_{k-1})$ in H such that $f_*([\alpha]) = [\beta]$. By contrary assume that $f_*([\alpha]) = [\beta]$ and that β is a $(k-1)$-cycle $(v_0v_1 \dots v_{k-1})$ in H. Hence $\big(f(u_0)f(u_1)\dots f(u_k)\big) \sim (v_0v_1 \dots v_{k-1})$. This means that there exists $0 < i < k$ such that $f(u_i) = f(u_{i+1})$ or $\big(f(u_{i-1})f(u_i)f(u_{i+1})\big)$ is a backtrack. If $f(u_i) = f(u_{i+1})$, then $f|_{N_1(u_i)}$ is not an injective map, which is a contradiction. If $\big(f(u_{i-1})f(u_i)f(u_{i+1})\big)$ is a backtrack, then $f(u_{i-1}) = f(u_{i+1})$, which is a contradiction because $f|_{N_1(u_i)}$ is an injective map.
(ii) The proof is similar to part (i) and follows from the fact that we do not have any 2-cycle.

$\square$

It is remarkable that the Theorem 4.6 is not valid for $k-3$, that is, $f_*\big(\pi_{1,k}(G,u)\big) \cap \pi_{1,k-3}(H,v)$ is not necessarily the trivial subgroup $\{[(v)]\}$. We would confirm this, in the next example.

Example 4.7. Let $G = C_6 = (u_0u_1 \dots u_5)$ and let H be the graph as Figure 1. Define $f : G \longrightarrow H$ by

$$f(u_0) = v_0, \quad f(u_1) = v_2, \quad f(u_2) = v_3, \quad f(u_3) = v_4, \quad f(u_4) = v_2, \quad f(u_5) = v_1.$$

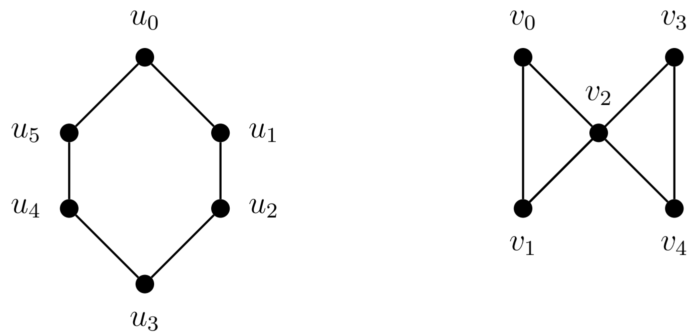

FIGURE 2. The graphs G and H

Intuitively, it can be said that the graph the H is obtained by joining vertices u_1 and u_4 in the graph G. Moreover, f is a partial cover and $f_*\big(\pi_{1,6}(G, u_0)\big) \cap \pi_{1,3}(H, v_0) \neq \{[(v)]\}$. For if, let $\alpha = (u_0 u_1 \dots u_5 u_0)$, let $\beta = (v_0 v_2 v_1 v_0)$, $\beta' = (v_2 v_3 v_4 v_2)$, and let $\gamma = (v_0 v_1 v_2)$. Then

$$\begin{aligned} f_*([\alpha]) &= [\big(f(u_0) f(u_1) \dots f(u_5) f(u_0)\big)] = [\big(v_0 v_2 v_3 v_4 v_2 v_1 v_0\big)] \\ &= [(v_0 v_2 v_1 v_0)] * [(v_0 v_1 v_2 v_3 v_4 v_2 v_1 v_0)] = [\beta] * [\gamma * \beta' * \overleftarrow{\gamma}] \in \pi_{1,3}(H, v_0), \end{aligned}$$

since β and β' are 3-cycles in H.

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