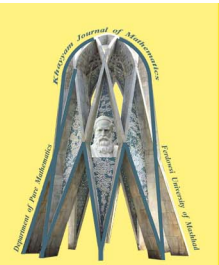
Khayyam J. Math. 10 (2024), no. 2, 384-393
DOI: 10.22034/kjm.2024.382530.2750



Khayyam Journal of Mathematics
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ON $AB5^*$ MODULES WITH NOETHERIAN DIMENSION

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Communicated by B. Torrecillas

Abstract. In this paper, we study the Noetherian dimension of the sum of certain modules. It is proved that for any module M that is an irredundant sum of submodules, each of which has Noetherian dimension $\leq \alpha$, if M has a finite spanning dimension (fsd-module, for short) or it is a weakly atomic module, then $n\text{-dim}\, M \leq \alpha$. Here, by a weakly atomic module, we mean a module M for which every proper nonsmall submodule N, has Noetherian dimension strictly less than that of M. Also, it is proved that if M is an $AB5^*$ module with Noetherian dimension and $\{N_i\}$ is a family of submodules of M such that $n\text{-dim}\, \frac{M}{N_i} \leq \alpha$, for each i, then $n\text{-dim}\, \frac{M}{\cap N_i} \leq \alpha$. Using this, we give a structure theorem for α-short modules in the category of $AB5^*$, and finally, we classify α-short modules in this category.

1. Introduction

Throughout, all rings are associative with identity, and unless stated otherwise, all modules are unital right modules. Let R be a ring and let M be an R-module. Then $N \ll M$ will denote N is a small submodule of M. Also, $k\text{-dim}\, M$ and $n\text{-dim}\, M$ denote the Krull dimension and the Noetherian dimension of M, respectively. The existence of one of these two dimensions necessitates the existence of the other; see [14]. The latter dimension is also called the dual Krull dimension in many other articles. For more details on the Krull and the Noetherian dimensions, the reader is referred to [1, 2, 9, 12, 14]. It is convenient, when we are dealing with these dimensions, to begin our list of ordinals with -1.

The purpose of this paper is to study some natural unquestioned results and properties concerning the Noetherian dimension of rings and modules. Although

Date: Received: 21 January 2024; Revised: 22 November 2024; Accepted: 1 December 2024.
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2020 *Mathematics Subject Classification.* Primary: 16P60; Secondary: 16P20, 16P40.
Key words and phrases. Noetherian dimension, Krull dimension, $AB5^*$ Modules, α-short modules.

many studies and advances have been made in this field, some trivial and natural questions in this area remain still unanswered. For example, one of the basic results concerning a module with Krull dimension due to Gordon, Robson, and Lenagan is as follows and in order to honor them, we call it GRL-theorem; see [9, 15].

Theorem 1.1. *[GRL] Let an R-module M have Krull dimension and also be the sum of submodules each of which has Krull dimension $\leq \alpha$. Then $k\text{-dim}\, M \leq \alpha$.*

The natural dual of this fact is as follows: Let M be an R-module with Noetherian dimension and $\{N_i\}$ a family of submodules of M such that $n\text{-dim}\, \frac{M}{N_i} \leq \alpha$, for each i. Then $n\text{-dim}\, \frac{M}{\cap N_i} \leq \alpha$. Here, it is natural to be wondered about the statements of Theorem 1.1 and its dual for Noetherian dimension, and we are led to ask the following questions.

Question 1.2. Is the statement of Theorem 1.1 is also valid for Noetherian dimension?

Question 1.3. Is the dual of Theorem 1.1 holds for Noetherian dimension?

The question that whether the above fact holds for Noetherian dimension is considered before. However, the fact that Artinian modules over commutative rings have finite Noetherian dimension show immediately that this fact does not hold for Noetherian dimension in general. Note that if M is an Artinian module, which is not Noetherian, over a Noetherian commutative ring R, then M as the sum of its cyclic submodules is a sum of some Noetherian submodules, however M is not Noetherian. Indeed, it is shown that if $M = \sum_{i\in I} M_i$ is an Artinian uniserial module, then $n\text{-dim}\, M \leq \sup\{n\text{-dim}\, M_i : i \in I\} + 1$; see [6, Proposition 4.6]. Also, it is proved that if $M = \sum_{i\in I} M_i$ is a uniserial module not necessarily Artinian, then $n\text{-dim}\, M \leq \sup\{n\text{-dim}\, M_i : i \in I\} + 1$; see [3, Proposition 3.1].

This paper is organized as follows. In section 2, we answer the above questions as the main aim. Although Example 2.1 shows that the answer to Question 1.2 is not "Yes" in general. However, we shall see that it is "Yes" for certain modules that are an irredundant sum of their submodules. We show that if M is an irredundant sum of submodules with Noetherian dimension $\leq \alpha$ and $n\text{-dim}\, M > \alpha$, then M necessarily has an infinite strictly descending chain of nonsmall submodules with Noetherian dimension equal to $n\text{-dim}\, M$. From this, we see that if such a module M has finite spanning dimension or it is weakly-atomic (i.e., every nonsmall submodule of M has Noetherian dimension strictly less than $n\text{-dim}\, M$), then $n\text{-dim}\, M \leq \alpha$. Also by an example, we show that the answer to Question 1.3 is not "Yes" in general and we shall see that if M satisfies the property of $AB5^*$, then the answer is "Yes". Using these facts, in section 3, we give a structure theorem for α-short modules in the category of $AB5^*$, and then we classify these modules. We show that for any module M in the category of $AB5^*$, $M \in \mathcal{A}$ (that is, the α-shortness and the Noetherian dimension of M are equal) if and only if $n\text{-dim}\, M$ is a limit ordinal or $n\text{-dim}\, A \leq n\text{-dim}\, \frac{M}{A}$, for any atomic submodule A of M.

2. GRL theorem and its dual for Noetherian dimension

We begin by focus on Question 1.2. First, we give a counterexample to show that the answer to Question 1.2 is not "Yes" in general.

Example 2.1. The p-prufer group $M = \mathbb{Z}_{p^\infty}$ is a uniserial $\mathbb{Z}$-module and

$$H_0 \subseteq H_1 \subseteq H_2 \subseteq \dots H_n \subseteq H_{n+1} \subseteq \dots$$

is the unique chain of its submodule, where $H_n =< \frac{1}{p^n} + \mathbb{Z} >$, for each integer $n \geq 0$. Note that $\mathbb{Z}_{p^\infty} = \sum_{n\geq 0} H_n$. It is easy to verify that $n\text{-dim}\, H_n = 0$, that is, every H_n is Noetherian . However $n\text{-dim}\, M = 1$.

If $M = \sum_\Delta M_\delta$, then this sum is called irredundant if, for every δ_0, $M \neq \sum_{\delta\neq\delta_0} M_\delta$. Note that to represent a module as an irredundant sum of its submodules is not the case in general. For example uniserial modules and local modules are not representable as an irredundant sum of their submodules.

Proposition 2.2. *Let M be an irredundant sum of submodules each of which has Noetherian dimension $\leq \alpha$. If $n\text{-dim}\, M > \alpha$, then M has an infinite descending chain $A_1 \supsetneq A_2 \supsetneq A_3 \supsetneq \dots$ of nonsmall submodules such that $n\text{-dim}\, A_i = n\text{-dim}\, M$.*

Proof. Let $M = \sum_{i\in\Delta} N_i$ be an irredundant sum with $n\text{-dim}\, N_i \leq \alpha$. If $n\text{-dim}\, M > \alpha$, then $n\text{-dim}\, \frac{M}{N_i} > \alpha$ for each i. Set $A_1 = \Sigma_{j\neq 1} N_j$, then $N_1 \nsubseteq A_1$ (note, if $N_1 \subseteq A_1$, then N_1 may be omitted) and so $N_1 \cap A_1 \subsetneq N_1$. This implies that $0 \neq \frac{N_1}{N_1\cap A_1} \cong \frac{N_1+A_1}{A_1} = \frac{M}{A_1}$. So $n\text{-dim}\, \frac{M}{A_1} \leq \alpha$, and consequently, $n\text{-dim}\, A_1 > \alpha$. Continue this manner, and by way of induction, we obtain an infinite descending chain $A_1 \supsetneq A_2 \supsetneq A_3 \supsetneq \dots$ of nonsmall submodules, such that $n\text{-dim}\, \frac{M}{A_i} \leq \alpha$. Hence, $n\text{-dim}\, A_i = n\text{-dim}\, M$ and we are done. □

Recall that an R-module M has a finite spanning dimension (fsd-module, for short) if for every descending chain $N_1 \supseteq N_2 \supseteq N_3 \supseteq \dots$ of submodules M, there is a number n such that $N_i \ll M$ for all $i > n$, or equivalently M satisfies the DCC on nonsmall submodules; see [16]. Artinian and hollow modules are examples for fsd-modules. From [8, Theorem 3.1], it follows that every fsd-module M is an irredundant finite sum of hollow submodules. Hence $n\text{-dim}\, M = n\text{-dim}\, H$ for some hollow submodule H of M. In what follows, we give the generalization of this fact for every arbitrary representation of M as an irredundant sum of submodules.

Theorem 2.3. *Let an fsd-module M be an irredundant sum of submodules each of which has Noetherian dimension $\leq \alpha$. Then $n\text{-dim}\, M \leq \alpha$.*

Proof. Since M is a fsd-module, it does not have any infinite descending chain of nonsmall submodules. Hence we are done, by Proposition 2.2. □

Definition 2.4. An R-module M is called weakly atomic if $n\text{-dim}\, N < n\text{-dim}\, M$, for every proper nonsmall submodule N of M.

Remark 2.5. Since every submodule of an atomic module is small, we conclude that every atomic module is a weakly atomic module, but not in vice versa. For

example, let M be a local module (i.e., if it has exactly one maximal submodule that contains all proper submodules) with the unique maximal submodule K such that $n\text{-dim}\, M \geq 1$. Since all proper submodules of M are small, M is weakly atomic. Furthermore, since M has Noetherian dimension ≥ 1, $n\text{-dim}\, M = \sup\{n\text{-dim}\, K, n\text{-dim}\, \frac{M}{K}\} = n\text{-dim}\, K$, and this shows that M is not atomic.

Proposition 2.6. *Let a weakly atomic module M be an irredundant sum of submodules each of which has Noetherian dimension $\leq \alpha$. Then $n\text{-dim}\, M \leq \alpha$.*

Theorem 2.7. *Let M be a weakly atomic module with $J(M) \ll M$. If M is a sum of hollow submodules each of which has Noetherian dimension $\leq \alpha$, then $n\text{-dim}\, M \leq \alpha$.*

Proof. Since $J(M) \ll M$, by [17, 41.5] we may assume that M is an irredundant sum of submodules. From the previous theorem, we conclude $n\text{-dim}\, M \leq \alpha$ and we are done. □

Now we are going to focus on Question 1.3. First we give an example in order to shows that the answer to Question 1.3 is not "Yes" in general.

Example 2.8. Let $R = \Pi_{i\in\mathbb{N}} F_i = F_1 \times F_2 \times \ldots$, where each F_i is a filed. Then $J(R) = 0$ and clearly R is not Noetherian. Note that for every maximal ideal M of R, we have $n\text{-dim}\, \frac{R}{M} = 0$. On the other hand, $\frac{R}{J(R)} \cong R$ is not Noetherian. This shows that the dual of Theorem 1.1 does not hold. Here, we emphasize that R not only is non-Noetherian, but also does not have Krull (resp., Noetherian) dimension, too. For this, note that it has no finite Goldie dimension; see [9, Proposition 1.4].

We cite the following fact from [9, Proposition 4.5] or [15].

Lemma 2.9. *If an R-module M has infinite chains*

$$M = A_0 \supseteq A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots \quad \text{and} \quad 0 = B_0 \subseteq B_1 \subseteq B_2 \subseteq B_3 \subseteq \ldots$$

such that $A_n \cap B_{n+1} \nsubseteq A_{n+1} + B_n$, for each $n \geq 0$, then M does not have Krull dimension.

Definition 2.10. An R-module M is said to be an $AB5^*$ module, if for every submodule B and inverse system $\{A_i\}_{i\in I}$ of submodules of M,

$$B + \bigcap_{i\in I} A_i = \bigcap_{i\in I} (B + A_i).$$

Note that though most modules M do not have this property, but fortunately Artinian modules and linearly compact modules satisfy the property of $AB5^*$; see [17, 29.8]. In what follows, we show that the dual of Theorem 1.1 holds for Noetherian dimension of $AB5^*$ modules.

Theorem 2.11. *Let M be an $AB5^*$ module with Noetherian dimension and let $\{N_i\}_{i\in\Delta}$ a family of submodules of M such that $n\text{-dim}\, \frac{M}{N_i} \leq \alpha$, for each i. Then $n\text{-dim}\, \frac{M}{\cap N_i} \leq \alpha$.*

Proof. Set $N = \cap_{i\in\Delta} N_i$, and let $0 \neq \frac{A}{N}$ be a nonzero submodule of $\frac{M}{N}$. Then $N \subsetneq A$, so there exists an indexed submodule, N_1 say, such that $A \nsubseteq N_1$. Thus $N \subseteq A \cap N_1 \subsetneq A$ and hence $0 \neq \frac{A}{A\cap N_1} \cong \frac{A+N_1}{N_1}$ and n-dim $\frac{A}{A\cap N_1} \leq \alpha$. This shows that each submodule A of M, properly containing N, contains a proper submodule A' containing N and n-dim $\frac{A}{A'} \leq \alpha$. Thus, there exists a descending chain $M = A_0 \supsetneq A_1 \supsetneq A_2 \supsetneq \cdots \supseteq N$ with ordinal indexes such that for each ordinal λ, n-dim $\frac{A_\lambda}{A_{\lambda+1}} \leq \alpha$. For limit ordinal λ, define $A_\lambda = \cap_{\delta<\lambda} A_\delta$. Since M is a set, there exists an ordinal number λ such that $A_\lambda = N$. Now we proceed by induction on λ to show that

$$P(\lambda) : n\text{-dim}\,\frac{M}{A_\lambda} \leq \alpha.$$

Clearly $P(0)$ and also $P(\lambda)$, for every nonlimit ordinal number λ, since n-dim $\frac{M}{A_\lambda} = \sup\{n\text{-dim}\,\frac{M}{A_{\lambda-1}}, n\text{-dim}\,\frac{A_{\lambda-1}}{A_\lambda}\}$.

Now let λ be a limit ordinal number. If n-dim $\frac{M}{A_\lambda} \nleq \alpha$, then there exists an ascending chain $A_\lambda = N = B_0 \subsetneq B_1 \subsetneq B_2 \subsetneq \ldots$ such that n-dim $\frac{B_{i+1}}{B_i} \geq \alpha$. Note that $B_1 \nsubseteq N$, so there exists $\delta < \lambda$ such that $B_1 \nsubseteq A_\delta$. Set $A_\delta = A_1$; then $B_1 \nsubseteq A_1$ and so $A_0 \cap B_1 \nsubseteq A_1 + B_0$. Suppose that $A_i \cap B_{i+1} \nsubseteq A_{i+1} + B_i$, for each $0 \leq i \leq n-1 < \lambda$. In particular, $A_{n-1} \cap B_n \nsubseteq A_n + B_{n-1}$. We seek A_{n+1} and B_{n+1} such that $A_n \cap B_{n+1} \nsubseteq A_{n+1} + B_n$. If for each $k > 0$, $A_n \cap B_{n+k} \subseteq B_n$, then $A_n \cap B_{n+k} \subseteq B_{n+k-1}$. Then by modular laws, it follows that $B_{n+k-1} = (A_n \cap B_{n+k}) + B_{n+k-1} = (A_n + B_{n+k-1}) \cap B_{n+k}$. Therefore, $\frac{B_{n+k}}{B_{n+k-1}} \cong \frac{B_{n+k}}{B_{n+k}\cap(A_n+B_{n+k-1})} \cong \frac{A_n+B_{n+k}}{A_n+B_{n+k-1}} \cong \frac{(A_n+B_{n+k})/A_n}{(A_n+B_{n+k-1})/A_n)}$. This shows that n-dim $\frac{(A_n+B_{n+k})/A_n}{(A_n+B_{n+k-1})/A_n)} \geq \alpha$. Hence $\frac{A_n+B_n}{A_n} \subseteq \frac{A_n+B_{n+1}}{A_n} \subseteq \frac{A_n+B_{n+2}}{A_n} \subseteq \cdots$ is an ascending chain in $\frac{M}{A_n}$ such that every quotient has Noetherian dimension at least α, contradiction to n-dim $\frac{M}{A_n} \leq \alpha$. Consequently, there exists $k > 0$ such that $A_n \cap B_{n+k} \nsubseteq B_n$. Set $B_{n+k} = B_{n+1}$ and conclude $A_n \cap B_{n+1} \nsubseteq B_n$. Indeed $B_n = B_n + N = B_n + A_\lambda$, so $A_n \cap B_{n+1} \nsubseteq B_n + \cap_{\delta<\lambda} A_\delta$. Since M has the property of $AB5^*$, we have $B_n + \cap_{\delta<\lambda} A_\delta = \cap_{\delta<\lambda}(B_n + A_\delta)$. Consequently there exists $\delta > n$ that $A_n \cap B_{n+1} \nsubseteq A_\delta + B_n$. Set $A_\delta = A_{n+1}$ and conclude that $A_n \cap B_{n+1} \nsubseteq A_{n+1} + B_n$. Therefore, there exist chains

$$N = B_0 \subseteq B_1 \subseteq B_2 \subseteq \cdots \quad \text{and} \quad M = A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots$$

such that $A_n \cap B_{n+1} \nsubseteq A_{n+1} + B_n$. Lemma 2.9 shows that M has no Krull dimension and so it has no Noetherian dimension too, a contradiction. Therefore, n-dim $\frac{M}{N} \leq \alpha$. □

It is well known that every Artinian module M with $J(M) = 0$ is also Noetherian. In the next corollary which is now immediate, we generalize this fact to $AB5^*$ modules.

Corollary 2.12. *Let M be an $AB5^*$ module with Noetherian dimension. Then n-dim $\frac{M}{J(M)} \leq 0$. Moreover, if $J(M) = 0$, then M is Noetherian.*

3. Classification of α-short $AB5^*$ modules

The concepts of α-short and α-Krull modules have been studied and investigated, respectively, in [7] and [5]. It was proved that if M is an α-short module, then M has Noetherian dimension and either $n\text{-dim}\, M = \alpha$ or $n\text{-dim}\, M = \alpha + 1$. In particular, a semiprime (nondivision) ring R is α-short if and only if $n\text{-dim}\, R = \alpha$, where $\alpha \geq 0$; see [7, Proposition 2.18]. Here, we should emphasize that the statement of [7, Proposition 2.18] is not valid for $\alpha = -1$ and it may simply be corrected as the statement of Proposition 3.5 in this paper; see also the comment preceding [10, Theorem 5.1]. However from these, it naturally raised the question of characterization of the R-modules M that are α-short if and only if $n\text{-dim}\, M = \alpha$. We recall that [10, Theorem 5.2] partially settled this question. It asserts that a semiprime FQS-module M is α-short if and only if $n\text{-dim}\, M = \alpha$, where $\alpha \geq 0$. Here, by an FQS-module, we mean a module Mthat is finitely generated, quasi-projective and self-generator. Also, a module M is called semiprime if 0 is a semiprime submodule of M in the sense of [10]. Motivated by what is stated and similar to our work in [11], in the continuation of this paper, we will classify the α-short modules in the category of $AB5^*$.

We cite the following definition and results from [7].

Definition 3.1. An R-module M is said to be α-short if for every submodule N of M either $n\text{-dim}\, N \leq \alpha$ or $n\text{-dim}\, \frac{M}{N} \leq \alpha$ and α is the least ordinal with this property.

It is easy to see that (-1)-short modules are just simple modules and that 0-short modules are just short modules, in the sense of [4].

Lemma 3.2. *If M is an α-short R-module, then any submodule and any factor module of M is β-short for some ordinal $\beta \leq \alpha$.*

Theorem 3.3. *If M is an α-short R-module, then either $n\text{-dim}\, M = \alpha$ or $n\text{-dim}\, M = \alpha + 1$.*

Theorem 3.4. *Let N be a submodule of an R-module M.*

(1) *If N is α-short and $n\text{-dim}\, \frac{M}{N} \leq \alpha$, then M is α-short.*
(2) *If $\frac{M}{N}$ is α-short and $n\text{-dim}\, N \leq \alpha$, then M is α-short.*

The following result is the corrected statement of [7, Proposition 2.18], which we mentioned before.

Proposition 3.5. *A ring R is α-short if and only if it is a division ring, where $\alpha = -1$, and when R is a semiprime nondivision ring, it is α-short if and only if $n\text{-dim}\, R = \alpha$, where $\alpha \geq 0$.*

Proof. The proof of the first part follows from the definition and that of the second part is exactly the proof of [7, Proposition 2.18], without even changing a single word. □

Definition 3.6. Let M be an R-module and let α be an ordinal number. We set

$$\mathcal{X}_\alpha(M) = \{A \subseteq M : n\text{-dim}\, \frac{M}{A} \leq \alpha\}, \quad X_\alpha(M) = \bigcap \mathcal{X}_\alpha(M)$$

$$\mathcal{Y}_\alpha(M) = \{B \subseteq M : n\text{-dim}\, B > \alpha\}, \quad Y_\alpha(M) = \bigcap \mathcal{Y}_\alpha(M).$$

Note that in the case that n-dim $M = \alpha$, we have $X_\alpha(M) = 0$ and $Y_\alpha(M) = M$, and if M is an $AB5^*$ module, then it follows from Theorem 2.11 that n-dim $\frac{M}{X_\alpha(M)} \leq \alpha$. The next result is now immediate.

Lemma 3.7. *Let M be an $AB5^*$ module. If n-dim $M > \alpha$, then $Y_\alpha(M) \subseteq X_\alpha(M)$.*

Proof. It suffices to show that $\mathcal{X}_\alpha(M) \subseteq \mathcal{Y}_\alpha(M)$. To show this, let $A \in \mathcal{X}_\alpha(M)$. Since n-dim $\frac{M}{A} \leq \alpha$ and n-dim $M = \sup\{n\text{-dim}\, A, n\text{-dim}\, \frac{M}{A}\}$, we have n-dim $A > \alpha$. This shows that $A \in \mathcal{Y}_\alpha(M)$. So $\mathcal{X}_\alpha(M) \subseteq \mathcal{Y}_\alpha(M)$ and we are done. □

In view of the previous lemma and Theorem 3.3, we have the following result.

Corollary 3.8. *Let M be an α-short $AB5^*$ module. If $Y_\alpha(M) \nsubseteq X_\alpha(M)$, then n-dim $M = \alpha$.*

The following important result is our structure theorem for α-short $AB5^*$ modules and plays the major role in achieving what we seek.

Theorem 3.9. *Let M be an $AB5^*$ module and let α be an ordinal number. The following statements are equivalent:*

(1) *M is an α-short module.*
(2) *M has a submodule A such that n-dim $\frac{M}{A} \leq \alpha$ and n-dim $N \leq \alpha$, for any submodule $N \not\supseteq A$ and α is the least ordinal with this property.*

Proof. Let M be α-short and let $A = X_\alpha(M)$. Then n-dim $\frac{M}{A} \leq \alpha$ by Theorem 2.11. Now, let N be a submodule of M such that $A \nsubseteq N$. Then n-dim $\frac{M}{N} \nleq \alpha$ and since M is α-short, we have n-dim $N \leq \alpha$. In order to show that α is the least ordinal with this property, it suffices to prove the converse. Conversely, let M has a submodule A with mentioned conditions. For any submodule N of M, if $A \subseteq N$, then n-dim $\frac{M}{N} \leq n$-dim $\frac{M}{A} \leq \alpha$, and if $A \nsubseteq N$, then n-dim $N \leq \alpha$. Since α is the least ordinal with this property, M is α-short. □

Notation 3.10. We use the following notation in the continue of the paper.

(1) $\mathcal{M}$ = the set of all modules with Noetherian dimension.
(2) $\mathcal{M}_\alpha$ = the set of all α-short modules in $\mathcal{M}$.
(3) $\mathcal{A}_\alpha$ = the set of all $M \in \mathcal{M}_\alpha$ with n-dim $M = \alpha$ and $\mathcal{A} = \cup_\alpha \mathcal{A}_\alpha$.
(4) $\mathcal{B}_\alpha$ = The set of all $M \in \mathcal{M}_\alpha$ with n-dim $M = \alpha + 1$ and $\mathcal{B} = \cup_\alpha \mathcal{B}_\alpha$.

Remark 3.11. It is easy to see that $\mathcal{A}_\alpha \cap \mathcal{B}_\alpha = \emptyset$ and $\mathcal{A}_\alpha \cup \mathcal{B}_\alpha = \mathcal{M}_\alpha$ and so $\{\mathcal{A}_\alpha, \mathcal{B}_\alpha\}$ is a partition for $\mathcal{M}_\alpha$. Similarly, $\mathcal{A} \cap \mathcal{B} = \emptyset$ and $\mathcal{M} = \mathcal{A} \cup \mathcal{B}$ so $\{\mathcal{A}, \mathcal{B}\}$ is a partition for $\mathcal{M}$.

Theorem 3.12. *Let $M \in \mathcal{B}_\alpha$ and let N be a submodule of M. Then either $N \in \mathcal{B}_\alpha$ or $\frac{M}{N} \in \mathcal{B}_\alpha$.*

Proof. By the above notation, M is α-short and n-dim $M = \alpha + 1$. We have the following cases:

(1) If $n\text{-dim}\, N > n\text{-dim}\, \frac{M}{N}$, then $n\text{-dim}\, N = n\text{-dim}\, M = \alpha + 1$. Also, N is β-short for some $\beta \leq \alpha$. If $\beta < \alpha$, by Theorem 3.3, then $n\text{-dim}\, N \leq \beta + 1 \leq \alpha$, a contradiction. Consequently, N is α-short with $n\text{-dim}\, N = \alpha + 1$. Therefore, $N \in \mathcal{B}_\alpha$.
(2) If $n\text{-dim}\, \frac{M}{N} > n\text{-dim}\, N$, then $n\text{-dim}\, \frac{M}{N} = n\text{-dim}\, M = \alpha + 1$. Also, $\frac{M}{N}$ is β-short for some $\beta \leq \alpha$. If $\beta < \alpha$, by Theorem 3.3, we have $n\text{-dim}\, \frac{M}{N} \leq \beta + 1 \leq \alpha$, a contradiction. Consequently, $\frac{M}{N}$ is α-short with $n\text{-dim}\, \frac{M}{N} = \alpha + 1$. Therefore, $\frac{M}{N} \in \mathcal{B}_\alpha$.
(3) If $n\text{-dim}\, N = n\text{-dim}\, \frac{M}{N}$, then the same argument shows that both N and $\frac{M}{N}$ belong to $\mathcal{B}_\alpha$. Hence, we are done.

□

In view of the proof of Theorem 3.12, the next result is now immediate.

Corollary 3.13. *Let $M \in \mathcal{B}_\alpha$ and let N be a submodule of M.*
(1) *If $n\text{-dim}\, N > n\text{-dim}\, \frac{M}{N}$, then $N \in \mathcal{B}_\alpha$.*
(2) *If $n\text{-dim}\, N < n\text{-dim}\, \frac{M}{N}$, then $\frac{M}{N} \in \mathcal{B}_\alpha$.*
(3) *If $n\text{-dim}\, N = n\text{-dim}\, \frac{M}{N}$, then $N, \frac{M}{N} \in \mathcal{B}_\alpha$.*

Corollary 3.14. *Let N be a submodule of M. If $N, \frac{M}{N} \in \mathcal{A}$, then $M \in \mathcal{A}$.*

Corollary 3.15. *If $M_1, M_2, \ldots, M_n \in \mathcal{A}$, then $M_1 \oplus M_2 \oplus \cdots \oplus M_n \in \mathcal{A}$.*

Remark 3.16. The converse of Corollary 3.15 is not true in general. For example, if $M = M_1 \oplus M_2$ where M_1 and M_2 are simple modules, then $M_1, M_2 \in \mathcal{B}_{-1}$. Indeed $M \in \mathcal{M}$ and it is also a (nonsimple) semisimple module. This shows that $M \in \mathcal{A}$. Note that any semisimple module $M \in \mathcal{M}$ is both a short and Noetherian module.

Corollary 3.17. *If $M \in \mathcal{B}_\alpha$, then $\frac{M}{E} \in \mathcal{B}_\alpha$ for some essential submodule E of M.*

Proof. Since $M \in \mathcal{B}_\alpha$, $n\text{-dim}\, M = \alpha + 1 > 0$, we conclude that M is not semisimple (note, every semisimple module with Noetherian dimension is both Artinian and Noetherian). Hence, M has a proper essential submodule. We have $n\text{-dim}\, M = sup\{n\text{-dim}\, \frac{M}{E} : E \quad \text{is an essential submodule of} \quad M\}$, so $n\text{-dim}\, \frac{M}{E} = \alpha + 1$ for some essential submodule E of M. By Theorem 3.4, $\frac{M}{E} \in \mathcal{B}_\alpha$. □

Recall that by an α-atomic module we mean a nonzero module M such that $n\text{-dim}\, M = \alpha$ and $n\text{-dim}\, N < \alpha$ for every proper submodule N of M; see [13].

Theorem 3.18. *Let M be an α-short $AB5^*$ module. The following statements are equivalent:*
(1) *$M \in \mathcal{B}_\alpha$.*
(2) *$X_\alpha(M)$ is $(\alpha + 1)$-atomic.*

Proof. (1) $\Rightarrow$ (2) Since $M \in \mathcal{B}_\alpha$, we have $n\text{-dim}\, M = \alpha + 1$. By Theorem 2.11, $n\text{-dim}\, \frac{M}{X_\alpha(M)} \leq \alpha$. This implies that $X_\alpha(M) \neq 0$ and $n\text{-dim}\, X_\alpha(M) = \alpha + 1$. Let $B \subsetneq X_\alpha(M)$. By Theorem 3.9, $n\text{-dim}\, B \leq \alpha$ and so $X_\alpha(M)$ is $(\alpha + 1)$-atomic.

(2) $\Rightarrow$ (1) Let $X_\alpha(M)$ be $(\alpha+1)$-atomic. Then $X_\alpha(M)$ is clearly α-short, and by Theorem 3.9, $n\text{-dim}\,\frac{M}{X_\alpha(M)} \leq \alpha$. Hence M is α-short, by Theorem 3.4. □

The next theorem determines $AB5^*$ modules, belonging to $\mathcal{B}$.

Theorem 3.19. *Let M be an $AB5^*$ module. The following statements are equivalent:*

(1) $M \in \mathcal{B}$.
(2) $n\text{-dim}\,M$ *is not a limit ordinal and M has an atomic submodule A with* $n\text{-dim}\,\frac{M}{A} < n\text{-dim}\,A$.

Proof. (1) $\Rightarrow$ (2) If $M \in \mathcal{B}$, then there exists an ordinal α such that $M \in \mathcal{B}_\alpha$, so $n\text{-dim}\,M = \alpha + 1$ is not a limit ordinal. Also, $A = X_\alpha(M)$ is an $(\alpha+1)$-atomic submodule of M by Theorem 3.18. Moreover, $n\text{-dim}\,\frac{M}{A} \leq \alpha < \alpha + 1 = n\text{-dim}\,A$, by Theorem 3.9.
(2) $\Rightarrow$ (1) Let $n\text{-dim}\,M = \alpha + 1$ and let A be an atomic submodule of M such that $n\text{-dim}\,\frac{M}{A} < n\text{-dim}\,A$. Then $n\text{-dim}\,A = \alpha + 1$. Indeed A is α-short and so is M, by Theorem 3.4. Therefore, $M \in \mathcal{B}$. □

The next important result is now immediate.

Theorem 3.20. *Let M be an $AB5^*$ module. The following statements are equivalent:*

(1) $M \in \mathcal{A}$.
(2) $n\text{-dim}\,M$ *is a limit ordinal or* $n\text{-dim}\,A \leq n\text{-dim}\,\frac{M}{A}$, *for any atomic submodule A of M.*

Acknowledgments

The authors would like to thank the referees for reading the article carefully and giving useful comments and efforts towards improving the manuscript.
The first author is grateful to the Research Council of Shahid Chamran University of Ahvaz for financial support (GN: SCU.MM1403.473).

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