Khayyam J. Math. 10 (2024), no. 2, 394-402
DOI: 10.22034/kjm.2024.472214.3277

Khayyam Journal of Mathematics
emis.de/journals/KJM
kjm-math.org

TRANS-PARA-SASAKIAN MANIFOLDS SATISFYING CERTAIN CURVATURE CONDITIONS

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Communicated by T. Ridel

ABSTRACT. The main aim of this paper is to study $(2n+1)$-dimensional trans-para-Sasakian manifolds satisfying the curvature conditions $P.Q = 0$, $Q.P = 0$, $P.R = 0$, where P is the projective curvature tensor, Q is the Ricci operator, and R is the Riemannian curvature tensor. Finally, a 3-dimensional trans-para-Sasakian manifold example that satisfies our results is constructed.

1. INTRODUCTION

On certain types of almost Hermitian manifolds, various authors have worked to generalize Kaehler geometry. Kotô [12] established inclusion relations among various classes. Gray and Hervella [9] have tried to organize all these classes into a general system.

Oubina [15] has introduced two new classes of almost contact structures, called trans-Sasakian and almost trans-Sasakian structures, which are obtained from certain classes of Hermitian manifolds. Also, Oubina proved that an almost metric structure (ϕ, ξ, η, g) is a trans-Sasakian structure if and only if it is normal and

$$d\Phi = 2\beta\eta \wedge \Phi, d\eta = \alpha\Phi,$$

where $\alpha = \frac{1}{2n}\delta\Phi(\xi)$ and $\beta = \frac{1}{2n}div(\xi)$. This may be expressed as a condition [2]:

$$(\nabla_E \phi)F = \alpha[g(E, F)\xi - \eta(F)E] + \beta[g(\phi E, F)\xi - \eta(F)\phi E].$$

Date: Received: 07 August 2024; Revised: 28 November 2024; Accepted: 04 December 2024.
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2020 *Mathematics Subject Classification.* Primary: 53B30; Secondary: 53C25, 53D10.
Key words and phrases. Trans-para-Sasakian manifolds, Projectively curvature tensor, Einstein manifold.

Trans-Sasakian manifolds have arisen naturally out of the classification of almost contact metric structures by Chinea and Gonzales [3]. Trans-Sasakian manifolds represent a significant class of almost contact metric manifolds since they encompass Sasakian, Kenmotsu, and cosymplectic manifolds as specific instances. Marrero [14] completely characterized the local structure of trans-Sasakian manifolds of dimension $n \geq 5$. Many authors have studied some properties of trans-Sasakian structure [1, 4, 6, 5, 10, 11].

Projective curvature tensor is one of the most important tensor in differential geometry. Provided that there exists a bijective mapping between every coordinate neighborhood of a semi-Riemannian manifold M^{2n+1} and a corresponding field in Euclidean space, in such a way that each geodesic of the semi-Riemannian manifold is linked to a straight line in the Euclidean space, then the manifold M^{2n+1} is termed as *locally projectively flat*. The projective curvature tensor P is defined by [17]

$$P(E,F)U = R(E,F)U - \frac{1}{2n}[S(F,U)E - S(E,U)F] \tag{1.1}$$

for all $E, F, U \in \chi(M)$, where R is the curvature tensor and S is the Ricci tensor. For $n \geq 1$, M^{2n+1} is locally projectively flat if and only if the projective curvature tensor P vanishes. Actually, M^{2n+1} is projectively flat as long as it has constant curvature. Therefore, the projective curvature tensor is a measure of whether a semi-Riemannian manifold has nonconstant curvature. One of the main points of interest in semi-Riemannian geometry is the curvature properties and the extent to which they determine the manifold itself.

Zamkovoy [19] introduced the trans-para-Sasakian manifolds and studied some curvature properties. A trans-para-Sasakian manifold has a trans-para-Sasakian structure of type (α, β), where α and β are smooth functions. The trans-para-Sasakian manifolds of types (α, β) are respectively the para-cosympletic, para-Sasakian and para-Kenmotsu for $\alpha = \beta = 0; \alpha = 1, \beta = 0$ and $\alpha = 0, \beta = 1$. If α and β are constants, then trans-papa-Sasakian manifold of types $(\alpha, 0)$ and $(0, \beta)$ is called α-para-Sasakian and β-para-Kenmotsu, respectively. Also, trans-para-Sasakian manifolds are studied in [13, 16].

The paper is organized in the following way. In section 2, we present some properties of $(2n+1)$-dimensional trans-para-Sasakian manifolds. In section 3, we prove that a $(2n+1)$-dimensional trans-para-Sasakian manifold that satisfies the condition $P.Q = 0$ is a partially Einstein manifold. In section 4, we show that if a $(2n+1)$-dimensional trans-para-Sasakian manifold satisfies the condition $Q.P = 0$, then the trace of square of the Ricci operator of a trans-para-Sasakian manifold is equal to $-2n(\alpha^2 + \beta^2)$ times trace of the Ricci operator. In section 5, we prove that a $(2n+1)$-dimensional trans-para-Sasakian manifold that satisfies the condition $P.R = 0$ is an Einstein manifold. Finally, a three-dimensional trans-para-Sasakian manifold example that satisfies our results is constructed.

2. Preliminaries

A $(2n+1)$-dimensional manifold M is called *almost paracontact manifold* if it admits a triple (ϕ, ξ, η) satisfying the following condition:

$$\eta(\xi) = 1, \quad \phi^2 = I - \eta \otimes \xi,$$

and ϕ induces on almost paracomplex structure on each fiber of $\mathcal{D} = ker(\eta)$, where ϕ, ξ and η are $(1,1)$-tensor field, vector field and 1-form, respectively. As a natural consequence, the tensor field ϕ has $rank 2n$, $\phi\xi = 0$, and $\eta \circ \phi = 0$. Here, ξ denotes a certain vector field (referred to as the *Reeb* or *characteristic vector field*) ,which is dual to η and satisfying $d\eta(\xi, X) = 0$ for all $X \in \chi(M)$. Within the framework of almost paracontact manifolds, if the tensor field $N_\phi := [\phi, \phi] - 2d\eta \otimes \xi$ vanishes identically, then the almost paracontact manifold is said to be *normal* [18]. If the structure (M, ϕ, ξ, η) admits a pseudo-Riemannian metric such that

$$g(\phi E, \phi F) = -g(E, F) + \eta(E)\eta(F),$$

then we say that (M, ϕ, ξ, η, g) is an *almost paracontact metric manifold*. Note that any pseudo-Riemannian metric with a given almost paracontact metric manifold structure is necessarily of signature $(n+1, n)$. An orthogonal basis for an almost paracontact metric manifold may always be found $\{E_1, \ldots, E_n, F_1, \ldots, F_n, \xi\}$, namely ϕ-basis, such that $g(E_i, E_j) = -g(F_i, F_j) = \delta_{ij}$ and $F_i = \phi E_i$, for any $i, j \in \{1, \ldots, n\}$. Moreover, it is possible to establish the definition of a skew-symmetric tensor field (a 2-form), commonly referred to as the fundamental form, denoted as Φ, by using the equation

$$\Phi(E, F) = g(E, \phi F).$$

An almost paracontact metric manifold is said to be *η-Einstein* if its Ricci tensor S is of the form

$$S = \lambda g + \mu \eta \otimes \eta,$$

where λ and μ are smooth functions on the manifold.

A semi-Riemannian manifold is said to be *partially Einstein manifold* [7] if its Ricci tensor S is of the form

$$S^2 = \lambda S + \mu g,$$

where λ and μ are smooth functions on the manifold.

For the sake of the shortness of our calculations for a $(2n+1)$-dimensional trans-para-Sasakian manifold, we define the following tensors:

$$\begin{aligned}
\omega(E, F, U) &= g(F, U)E - g(E, U)F, \\
\Omega(E, F, U) &= g(F, U)E + g(E, U)F, \\
\omega(E, F, U, V) &= g(F, U)g(E, V) - g(E, U)g(F, V), \\
\Omega(E, F, U, V) &= g(F, U)g(E, V) + g(E, U)g(F, V), \\
\gamma_n(\alpha, \beta) &= \phi(grad\alpha) + (2n-1)grad\beta,
\end{aligned}$$

and

$$\gamma_n(\alpha, \beta, E) = -\phi E(\alpha) + (2n-1)E(\beta)$$

for all vector fields $E, F, U, V \in \chi(M)$, and α, β are smooth functions.

Definition 2.1. [19] If

$$(\nabla_E \phi)F = \alpha\omega(E, \xi, F) + \beta\omega(\phi E, \xi, F), \tag{2.1}$$

then the manifold $(M^{2n+1}, \phi, \eta, \xi, g)$ is said to be a *trans-para-Sasakian* manifold.

From (2.1), we have

$$\nabla_E \xi = -\alpha\phi E - \beta(E - \eta(E)\xi).$$

Furthermore, in a $(2n+1)$-dimensional trans-para-Sasakian manifold, the following identities hold [19]:

$$\begin{aligned} R(E,F)\xi = &-(\alpha^2+\beta^2)\omega(E,F,\xi) - 2\alpha\beta(\omega(\phi E,F,\xi)+\omega(E,\phi F,\xi)) \\ &+ \phi(\omega(E,F,grad\alpha)) + \phi^2(\omega(E,F,grad\beta)), \\ R(\xi,E)\xi =&(\alpha^2+\beta^2-\xi(\beta))\omega(E,\xi,\xi), \\ 2\alpha\beta - \xi(\alpha) =&0, \qquad\qquad (2.2) \\ R(\xi,E)F =&(\alpha^2+\beta^2)\omega(E,\xi,F) - 2\alpha\beta\omega(\phi E,\xi,F) \\ &+ \omega(\phi E, grad\alpha, F) + \omega(grad\beta, \phi^2 E, F), \qquad\qquad (2.3) \\ S(E,\xi) =&-(2n(\alpha^2+\beta^2)-\xi(\beta))\eta(E) + \gamma_n(\alpha,\beta,E), \\ Q\xi =&-(2n(\alpha^2+\beta^2)-\xi(\beta))\xi + \gamma_n(\alpha,\beta) \end{aligned}$$

where S is the Ricci tensor and Q is the Ricci operator given by

$$S(E,F) = g(QE,F),$$

for all $E, F \in \chi(M)$.

Corollary 2.2. [19] *If for a $(2n+1)$-dimensional trans-para-Sasakian manifold, we have $\gamma_n(\alpha,\beta) = 0$, then*

$$\xi(\beta) = g(\xi, grad\beta) = -\frac{1}{2n-1} g(\xi, \phi(grad\alpha)) = 0,$$

and hence

$$S(E,\xi) = -2n(\alpha^2+\beta^2)\eta(E), \tag{2.4}$$

$$Q\xi = -2n(\alpha^2+\beta^2)\xi, \tag{2.5}$$

for all $E \in \chi(M)$.

From here on, we assume that $\gamma_n(\alpha,\beta) = 0$.

3. Trans-para-Sasakian manifolds satisfying $P.Q = 0$

In this section, we will consider $(2n+1)$-dimensional trans-para-Sasakian manifolds that satisfy the condition $P.Q = 0$.

Theorem 3.1. *A $(2n+1)$-dimensional trans-para-Sasakian manifold that satisfies the condition $P.Q = 0$ is a partially Einstein manifold.*

Proof. Assume that a trans-para-Sasakian manifold fulfills the equation $P.Q = 0$, which means

$$P(E,F)QU - Q(P(E,F)U) = 0, \tag{3.1}$$

for any vector fields E, F, and U. Putting $F = \xi$ in (3.1), we have

$$P(E,\xi)QU - Q(P(E,\xi)U) = 0. \tag{3.2}$$

Letting $F = \xi$ and $U = QU$ in (1.1) and using (2.3) and (2.4), we obtain

$$\begin{aligned} P(E,\xi)QU =&(\alpha^2+\beta^2)g(E,QU)\xi + 2\alpha\beta\omega(\phi E,\xi,QU) - \omega(\phi E, grad\alpha, QU) \\ &+ \omega(\phi^2 E, grad\beta, QU) + \frac{1}{2n}S(E,QU)\xi. \end{aligned} \tag{3.3}$$

Again, putting $F = \xi$ and using (2.3), (2.4), and (2.5) in (1.1), we have

$$\begin{aligned} Q(P(E,\xi)U) =& -(\alpha^2+\beta^2)\{2n(\alpha^2+\beta^2)g(E,U) + 4n\alpha\beta g(\phi U, E) + S(E,U)\}\xi \\ &+ 2\alpha\beta\eta(U)Q\phi E + Q(\omega(\phi^2 E, grad\beta, U)) + Q(\omega(grad\alpha, \phi E, U)). \end{aligned} \tag{3.4}$$

By virtue of (3.3) and (3.4), we get from (3.2) that

$$\begin{aligned} 0 =&(\alpha^2+\beta^2)g(E,QU)\xi + \frac{1}{2n}S(E,QU)\xi + 2n(\alpha^2+\beta^2)^2 g(E,U)\xi \\ &+ (\alpha^2+\beta^2)S(E,U)\xi - 2\alpha\beta g(QU,\phi E)\xi + 2\alpha\beta\eta(QU)\phi E - QU(\alpha)\phi E \\ &+ g(QU,\phi E)grad\alpha + QU(\beta)\phi^2 E - g(QU,\phi^2 E)grad\beta \\ &- 4n\alpha\beta(\alpha^2+\beta^2)g(U,\phi E)\xi - 2\alpha\beta\eta(U)Q\phi E + U(\alpha)Q\phi E \\ &- g(U,\phi E)Q(grad\alpha) - U(\beta)Q\phi^2 E + g(U,\phi^2 E)Q(grad\beta). \end{aligned} \tag{3.5}$$

The equation

$$S^2(E,U) = S(QE,U) = -4n(\alpha^2+\beta^2)S(E,U) - 4n^2(\alpha^2+\beta^2)^2 g(E,U)$$

follows from taking the inner product of (3.5) with ξ and using (2.2) and Corollary 2.2. □

From [8, Lemma 1] and Theorem 3.1, we can give the following corollary.

Corollary 3.2. *Let us consider a $(2n+1)$-dimensional trans-para-Sasakian manifold that satisfies the condition $P.Q = 0$. Then $K.K = \alpha Q(g,K)$, where $K = g\overline{\wedge}S$ and $\alpha = -4n(\alpha^2+\beta^2), \lambda = -4n^2(\alpha^2+\beta^2)^2$.*

4. Trans-para-Sasakian manifolds satisfying $Q.P = 0$

Now, we will consider $(2n+1)$-dimensional trans-para-Sasakian manifolds that satisfy the condition $Q.P = 0$.

Theorem 4.1. *If a $(2n+1)$-dimensional trans-para-Sasakian manifold satisfies the condition $Q.P = 0$, then the trace of square of the Ricci operator of a trans-para-Sasakian manifold is equal to $-2n(\alpha^2+\beta^2)$ times trace of the Ricci operator.*

Proof. Let us investigate a trans-para-Sasakian manifold that fulfills the condition $(Q.P)(E,F)U = 0$, namely,

$$Q(P(E,F)U) - P(QE,F)U - P(E,QF)U - P(E,F)QU = 0, \tag{4.1}$$

for any vector fields E, F and U.

By putting $F = \xi$ in (4.1), we get

$$Q(P(E,\xi)U) = P(QE,\xi)U + P(E,Q\xi)U + P(E,\xi)QU. \tag{4.2}$$

We have already calculated the left side of (4.2) in (3.4). Hence for calculating the right side of (4.2), we can write

$$\begin{aligned}
&-(\alpha^2+\beta^2)\{2n(\alpha^2+\beta^2)g(E,U) + 4n\alpha\beta g(\phi U,E) + S(E,U)\}\xi\\
&+ 2\alpha\beta\eta(U)Q\phi E + Q(\omega(\phi^2E, grad\beta, U)) + Q(\omega(grad\alpha, \phi E, U))\\
&= R(QE,\xi)U - \frac{1}{2n}\omega(QE,\xi,QU) + R(E,Q\xi)U - \frac{1}{2n}\omega(E,Q\xi,QU)\\
&+ R(E,\xi)QU - \frac{1}{2n}\omega(E,\xi,Q^2U).
\end{aligned}$$

If we employ (2.3), (2.4), and (2.5) in the last equation, we have

$$\begin{aligned}
&-(\alpha^2+\beta^2)\{2n(\alpha^2+\beta^2)g(E,U) + 4n\alpha\beta g(\phi U,E) + S(E,U)\}\\
&+ 2\alpha\beta\eta(U)Q\phi E + Q(\omega(\phi^2E, grad\beta, U)) + Q(\omega(grad\alpha, \phi E, U))\\
&\quad = (\alpha^2+\beta^2)S(E,U)\xi + 2\alpha\beta(g(\phi U, QE)\xi + \eta(U)\phi(QE)) - U(\alpha)\phi(QE)\\
&\qquad - g(\phi U, QE)grad\alpha + U(\beta)\phi^2(QE) + g(\phi U, \phi(QE))grad\beta + \frac{1}{2n}g(Q^2E,U)\xi\\
&\qquad - 2n(\alpha^2+\beta^2)[(\alpha^2+\beta^2)g(E,U)\xi + 2\alpha\beta(g(\phi U,E)\xi + \eta(U)\phi E) \qquad (4.3)\\
&\qquad - U(\alpha)\phi E - g(\phi U,E)grad\alpha + U(\beta)\phi^2E + g(\phi U,\phi E)grad\beta - \frac{1}{2n}S(E,U)\xi]\\
&\qquad + (\alpha^2+\beta^2)g(E,QU)\xi + 2\alpha\beta(g(\phi(QU),E)\xi + \eta(QU)\phi E) - QU(\alpha)\phi E\\
&\qquad - g(\phi(QU),E)grad\alpha + QU(\beta)\phi^2E + g(\phi(QU),\phi E)grad\beta + \frac{1}{2n}g(Q^2E,U)\xi.
\end{aligned}$$

By taking the inner product of (4.3) with ξ and using the fact that $\xi(\beta) = 0$ and (2.2), we get

$$-2n(\alpha^2+\beta^2)g(QE,U) = g(Q^2E,U). \tag{4.4}$$

Replacing E, U by e_i in (4.4) and taking summation over i (Let $\{e_1, e_2, \ldots, e_{2n+1}\}$ be an ϕ-basis of the tangent space at any point of the manifold), we obtain

$$\sum_{i=1}^{2n+1} -2n(\alpha^2+\beta^2)\varepsilon_i g(Qe_i,e_i) = \sum_{i=1}^{2n+1} \varepsilon_i g(Q^2e_i,e_i). \tag{4.5}$$

From (4.5), the conclusion is simply obtained. $\square$

5. Trans-para-Sasakian manifolds satisfying $P.R = 0$

We will consider $(2n+1)$-dimensional trans-para-Sasakian manifolds that satisfy the condition $P.R = 0$.

Theorem 5.1. *A $(2n+1)$-dimensional trans-para-Sasakian manifold that satisfies the condition $P.R = 0$ is an Einstein manifold.*

Proof. Let us consider a trans-para-Sasakian manifold satisfies

$$(P(E,F).R)(\zeta,\vartheta)V = 0,$$

namely,

$$\begin{aligned} 0 =& P(E,F)R(\zeta,\vartheta)V - R(P(E,F)\zeta,\vartheta)V - R(\zeta,P(E,F)\vartheta)V \\ &- R(\zeta,\vartheta)P(E,F)V, \end{aligned} \tag{5.1}$$

for any vector fields E, F, ζ, ϑ and V.

By putting ξ instead of E and ζ in (5.1), we obtain

$$\begin{aligned} 0 =& - R(P(\xi,F)\xi,\vartheta)V - R(\xi,P(\xi,F)\vartheta)V - R(\xi,\vartheta)P(\xi,F)V \\ &+ P(\xi,F)R(\xi,\vartheta)V. \end{aligned} \tag{5.2}$$

Letting $E = U = \xi$ in (5.1) and using (2.3), (2.4), we calculate the first and second term of (5.2)

$$\begin{aligned} R(P(\xi,F)\xi,\vartheta)V =& 0, \\ R(\xi,P(\xi,F)\vartheta)V =& - 2\alpha\beta\eta(\vartheta)R(\xi,\phi F)V + \vartheta(\alpha)R(\xi,\phi F)V \\ &- \vartheta(\beta)R(\xi,F)V + g(\phi\vartheta,F)R(\xi,grad\alpha)V \\ &- g(\phi F,\phi\vartheta)R(\xi,grad\beta)V. \end{aligned}$$

With the same progress, one can compute the third and fourth term of (5.2). After some straightforward calculation, letting $V = \xi$ and taking the inner product of final equation with ξ, we obtain

$$S(F,\vartheta) = -2n(\alpha^2+\beta^2)g(F,\vartheta).$$

Hence, we complete the proof of the theorem. □

6. Example

Example 6.1. We consider the three-dimensional manifold M and the vector fields

$$e_1 = \frac{\partial}{\partial x}, \quad e_2 = \frac{\partial}{\partial y}, \quad e_3 = (x+y)\frac{\partial}{\partial x} + (x+y)\frac{\partial}{\partial y} + \frac{\partial}{\partial z},$$

where

$$g = \begin{pmatrix} 1 & 0 & -\frac{x+y}{2} \\ 0 & -1 & \frac{x+y}{2} \\ -\frac{x+y}{2} & \frac{x+y}{2} & 1 \end{pmatrix},$$

$$\phi = \begin{pmatrix} 0 & 1 & -(x+y) \\ 1 & 0 & -(x+y) \\ 0 & 0 & 0 \end{pmatrix}.$$

One can observe that

$$g(e_1,e_1)=g(e_3,e_3)=1, g(e_2,e_2)=-1, g(e_1,e_2)=g(e_1,e_3)=g(e_2,e_3)=0$$

and

$$\phi(e_1)=e_2, \quad \phi(e_2)=e_1, \quad \phi(e_3)=0.$$

We get

$$[e_1,e_3]=e_1+e_2, \quad [e_2,e_3]=e_1+e_2, \quad [e_1,e_2]=0.$$

Taking $e_3=\xi$ and using the Koszul formula, we can calculate

$$\begin{aligned} \nabla_{e_1}e_1&=-\xi, & \nabla_{e_2}e_1&=0, & \nabla_{e_3}e_1&=-e_2\\ \nabla_{e_1}e_2&=0, & \nabla_{e_2}e_2&=\xi, & \nabla_{e_3}e_2&=-e_1\\ \nabla_{e_1}e_3&=e_1, & \nabla_{e_2}e_3&=e_2, & \nabla_{e_3}e_3&=0. \end{aligned}$$

We also see that

$$\begin{aligned} (\nabla_{e_1}\phi)e_1 &= \nabla_{e_1}\phi(e_1)-\phi(\nabla_{e_1}e_1)=-0\\ &=0(-g(e_1,e_1)\xi+\eta(e_1)e_1)-1(g(e_1,\phi(e_1))\xi+\eta(e_1)\phi(e_1)), \end{aligned}$$

$$\begin{aligned} (\nabla_{e_1}\phi)e_2 &= \nabla_{e_1}\phi(e_2)-\phi(\nabla_{e_1}e_2)=-\xi\\ &=0(-g(e_1,e_2)\xi+\eta(e_2)e_1)-1(g(e_1,\phi(e_2))\xi+\eta(e_2)\phi(e_1)), \end{aligned}$$

$$\begin{aligned} (\nabla_{e_1}\phi)e_3 &= \nabla_{e_1}\phi(e_3)-\phi(\nabla_{e_1}e_3)=-e_2\\ &=0(-g(e_1,e_3)\xi+\eta(e_3)e_1)-1(g(e_1,\phi(e_3))\xi+\eta(e_3)\phi(e_1)). \end{aligned}$$

In the above equations, we see that the manifold satisfies the condition (2.1) for $X=e_1$, $\alpha=0, \beta=-1$, and $e_3=\xi$. Similarly, it is also true for $X=e_2$ and $X=e_3$.

The 1-form $\eta=dz$ and the fundamental 2-form $\Phi=dx\wedge dy-(x+y)dx\wedge dz+(x+y)dy\wedge dz$ defines a trans-para-Sasakian manifold, where $d\eta=\alpha\Phi$, $d\Phi=-2\beta\eta\wedge\Phi$. Hence, the manifold is a trans-para-Sasakian manifold of type $(0,-1)$.

Then the expressions of the curvature tensor is given by

$$\begin{aligned} &R(e_1,e_2)e_3=0, \quad R(e_2,e_3)e_3=-e_2, \quad R(e_1,e_3)e_3=-e_1,\\ &R(e_1,e_2)e_2=e_1, \quad R(e_2,e_3)e_2=-\xi, \quad R(e_1,e_3)e_2=0,\\ &R(e_1,e_2)e_1=e_2, \quad R(e_2,e_3)e_1=0, \quad R(e_1,e_3)e_1=\xi. \end{aligned} \tag{6.1}$$

Therefore, we have $S(e_1,e_1)=-2, S(e_2,e_2)=2$, and $S(e_3,e_3)=-2$. It implies that the scalar curvature $r=-6$. Also, using (3.1), (4.1), (5.1), and (6.1), we can see that this example satisfy our theorems.

Acknowledgement. The authors would like to acknowledge that this paper is submitted in partial fulfilment of the requirements for PhD degree at Bursa Technical University.

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