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FERMAT'S LAST THEOREM AND THE BERGMAN SPACE

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ABSTRACT. In this article, we study the operator solutions of Fermat's equation on the Bergman space. Furthermore, the solutions of Fermat's equation are investigated over the class of 2×2 and 3×3 operator matrices on the vector valued Bergman spaces where each entries of the matrix is considered as a linear bounded operator on the Bergman space $A^2(\mathbb{D})$.

1. INTRODUCTION AND PRELIMINARIES

In 1637, Pierre de Fermat stated that there are no such natural numbers a, b, c that satisfy the equation $a^n + b^n = c^n, n \geq 3$. Later on, Fermat's last theorem has turned into a conjecture. The validity of the Fermat's last theorem has been investigated by many researchers. Finally, in 1995, Wiles proved this theorem [18, 15] to be true. After that, many attempts have been made to find the solutions of the Fermat's last theorem over different fields. In spite of the availability of the solutions of Fermat's equation over integers, rational numbers, integer matrices, and rational matrices, there are also many problems on the Fermat's equation over matrices of order 2×2 and 3×3, which are unsettled. The Fermat's matrix equation is given by

$$X^n + Y^n = Z^n, \qquad \text{for all} \quad n \in \mathbb{N}. \tag{1.1}$$

The Fermat's matrix equation (1.1) has been studied by numerous mathematicians over different family of 2×2 matrices. Domiaty [5] in the year 1966, stated that the Fermat's equation (1.1) has infinitely many solutions in $M_2[\mathbb{Z}]$, for $n = 4$.

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Verstein [17] studied over $GL_2[\mathbb{Z}]$. Khazanov [11] provided the sufficient and necessary conditions for solution of (1.1) over $SL_2(\mathbb{Z})$. Cao and Grytczuk [2] explored over $\overline{SL_2(\mathbb{Z})}$, which is a commuting group. Jarvis and Meekin [8] investigated the solution of Fermat's equation over $\mathbb{Q}$. Chien and Meng [3] have studied over a family of 2×2 positive semidefinite integral matrices.

The enormous literature on solutions of Fermat's equation over integers, rationals and integer matrices inspire us to study the the following problems:

(a): When does the Fermat's equation has an operator solution?
(b): When does the Fermat's equation has an operator matrix solution?

In this article, we study the solutions of Fermat's equation in different family of 2×2 and 3×3 matrices. We prove the Fermat's equation has operator solutions on the Bergman space, which is a reproducing kernel Hilbert space. Furthermore, we investigate the operator matrix solutions of Fermat's equation on the vector valued Bergman spaces. This article is organized as follows:
The section 1 is about the introduction and literature review of Fermat's equation over integers, rational numbers, and different family of 2×2 and 3×3 matrices. In section 2, we have introduced some notations, definitions and preliminaries regarding several operators on the Bergman space. In section 3, we shall find some sufficient conditions on Hankel, Toeplitz, and composition operators on the Bergman space so that these operators will satisfy the Fermat's equation $X^n + Y^n = Z^n, n \geq 2$, where $X, Y, Z \in \mathcal{L}(A^2(\mathbb{D}))$. In sections 4 and 5, the 2×2 and 3×3 operator matrix solutions of the Fermat's equation on vector valued Bergman spaces are investigated, respectively.

Let $A^2(\mathbb{D})$ be denoted as a Bergman space to be a Hilbert space of all analytic functions g on $\mathbb{D} = \{\omega \in \mathbb{C} : |\omega| < 1\}$ such that

$$\|g\|_{A^2(\mathbb{D})} = \sqrt{\left(\int (|g(\omega)|^2 dA(\omega))\right)} < \infty,$$

with $dA(\omega)$ is the normalized Lebesgue area measure on the open unit disk $\mathbb{D}$. Suppose that $\mathcal{L}(A^2(\mathbb{D}))$ denotes the algebra of linear, bounded operators from $A^2(\mathbb{D})$ into itself. If $h(\omega) = \sum_{n=0}^{\infty} c_n \omega^n$ and $k(\omega) = \sum_{n=0}^{\infty} d_n \omega^n$, where $h, k \in A^2(\mathbb{D})$, then $\langle h, k \rangle$ is the inner product of h and k, defined by

$$\langle h, k \rangle = \int_{\mathbb{D}} h(\omega)\overline{k(\omega)} dA(\omega) = \sum_{n=0}^{\infty} \frac{c_n \overline{d_n}}{n+1}.$$

The function $K_\omega \in A^2(\mathbb{D})$, $\omega \in \mathbb{D}$ is denoted as the Bergman reproducing kernel such that $g(\omega) = \langle g, K_\omega \rangle$, for all $g \in A^2(\mathbb{D})$ and $k_\omega = \frac{K_\omega}{\|K_\omega\|_2}$, is the normalized reproducing kernel. For $n \in \mathbb{N} \cup \{0\}$, let $e_n(\omega) = \sqrt{n+1}\omega^n$ be an orthonormal basis for $A^2(\mathbb{D})$. Let $L^\infty(\mathbb{D})$ be the space of all essentially bounded Lebesgue measurable functions on $\mathbb{D}$. Let $\phi \in L^\infty(\mathbb{D})$. Then the Toeplitz operator T_ϕ on $A^2(\mathbb{D})$ is defined by $T_\phi g = P(\phi g)$, where P is a Bergman projection from $L^2(\mathbb{D}, dA)$ onto $A^2(\mathbb{D})$. Again for $\phi \in L^\infty(\mathbb{D})$, the big Hankel operator $H_\phi : A^2(\mathbb{D}) \to (A^2(\mathbb{D}))^\perp$ is defined by $H_\phi g = (I - P)(\phi g)$. For $f \in L^\infty(\mathbb{D})$, the dual Toeplitz operator S_f

on $(A^2(\mathbb{D}))^{\perp}$ is defined by $S_f g = (I - P)(fg)$. The operator S_f is related with Hankel operator H_f by the following equations:

$S_f^* = S_{\bar{f}}, S_{\alpha f+\beta g} = \alpha S_f + \beta S_g, f, g \in L^{\infty}(\mathbb{D}), \alpha, \beta \in \mathbb{C}$.

Thus, $T_{fg} = T_f T_g + H_{\bar{f}}^* H_g$, $S_{fg} = S_f S_g + H_f H_{\bar{g}}^*$, $H_{fg} = H_f T_g + S_f H_g$.

The Lie group containing all automorphisms (biholomorphic mappings) of $\mathbb{D}$ is denoted by $Aut(\mathbb{D})$. For all $b \in \mathbb{D}$, an automorphism $\phi_b \in Aut(\mathbb{D})$ satisfies the following properties
(i) $(\phi_b \ o \ \phi_b)(\omega) \equiv \omega$;
(ii) $\phi_b(0) = b, \phi_b(b) = 0$;
(iii) ϕ_b has a at most one fixed point in $\mathbb{D}$.

Moreover, $\phi_b(z) = \frac{b-z}{1-\bar{b}z}$, for all b, $z \in \mathbb{D}$. One can easily calculate that the derivative of ϕ_b at ω is equal to $-k_b(\omega)$, and $J_{\phi_b(\omega)} = |k_b(\omega)|^2 = \frac{\left(1-|b|^2\right)^2}{\left|1-\bar{b}\omega\right|^4}$ denotes the real Jacobian determinant of ϕ_b at ω. Suppose that f is any measurable function on $\mathbb{D}$ and that $b \in \mathbb{D}$, then we can define the function $U_b f$ by $U_b f(w) = k_b(w) f(\phi_b(w))$ on $\mathbb{D}$. Here, U_b is a bounded linear operator on $A^2(\mathbb{D})$ as well as on $L^2(\mathbb{D}, dA)$, for all $b \in \mathbb{D}$. The operator U_b satisfies $U_b^2 = I$, where I is the identity operator, $U_b^* = U_b, U_b(A^2(\mathbb{D})) \subset A^2(\mathbb{D})$ and $U_b((A^2(\mathbb{D}))^{\perp}) \subset (A^2(\mathbb{D}))^{\perp}$, for all $b \in \mathbb{D}$. Therefore, $U_b P = P U_b$, for all $b \in \mathbb{D}$. For more details, see [20]. Let $\mathbb{T}$ be the unit circle in $\mathbb{C}$. For a function f on $\mathbb{D}$ and $a \in \mathbb{T}$, define $(R_a f)(z) = f(az)$. Then $R_a^{-1} = R_a^* = R_{\bar{a}}$ and R_a is a unitary operator on $A^2(\mathbb{D})$ and $L^{\infty}(\mathbb{D})$. For more details, see [4]. Let $H^{\infty}(\mathbb{D})$ be the space of all bounded and analytic functions on $\mathbb{D}$. Let $h^{\infty}(\mathbb{D})$ be the space of all bounded and harmonic functions on $\mathbb{D}$.

The algebra of all bounded and linear operators from a Hilbert space H into itself is denoted by $\mathcal{L}(H)$. For $T \in \mathcal{L}(H)$, the Berezin transform $\widetilde{T}$ defined by $\widetilde{T}(\omega) = \langle T k_\omega, k_\omega \rangle$ on $\mathcal{L}(H)$. For $\phi \in L^{\infty}(\mathbb{D})$, the Berezin transform of the Toeplitz operator T_ϕ is same as the Berezin transform $\tilde{\phi}$. In other words, $\tilde{\phi} = \widetilde{T_\phi}$. Moreover, $\tilde{\phi}(\omega) = \widetilde{T_\phi}(\omega) = \langle T_\phi k_\omega, k_\omega \rangle = \langle P(\phi k_\omega), k_\omega \rangle = \langle \phi k_\omega, k_\omega \rangle$ for all $\omega \in \mathbb{D}$. Let $E_{n,\phi} = \langle T_\phi \sqrt{n+1}\omega^n, \sqrt{n+1}\omega^n \rangle$. Let $\phi : \mathbb{D} \longrightarrow \mathbb{D}$ be an analytic function. Then the composition operator C_ϕ is defined from $A^2(\mathbb{D})$ into $A^2(\mathbb{D})$ by $C_\phi f = f o \phi$. Let g be any measurable function on $\mathbb{D}$; then for given $a \in \mathbb{D}$, the function C_a is defined by $C_a g = g o \phi_a$, where $\phi_a \in \mathrm{Aut}(\mathbb{D})$.

Let $T \in \mathcal{L}(H)$ and let T have the polar decomposition $U|T|$. Then the Aluthge transform of T [1] is denoted as $\Delta(T)$ and defined by $\Delta(T) = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$. Let $[S, T]$ denote the commutator operator of S and T and be defined by $[S, T] = ST - TS$, where $S, T \in \mathcal{L}(H)$. An operator $S \in \mathcal{L}(H)$ is called algebraically hyponormal if $q(S)$ is hyponormal, where q is a nonconstant complex polynomial. Similarly, an operator $S \in \mathcal{L}(H)$ is said to be quasinilpotent if its spectrum $\sigma(S) = \{0\}$. An operator S is called paranormal if $\|S^2 g\| \geq \|Sg\|^2$ for every unit vector $g \in H$. An operator $S \in \mathcal{L}(H)$ is called quasinormal if S commutes with $S^* S$.

2. Fermat's equation over operators on the Bergman space

In this section, we will provide some sufficient conditions on the operators on the Bergman space, which will satisfy the Fermat's equation $X^n + Y^n = Z^n$, for all $n \in \mathbb{N}$. In the present study, we are considering only Toeplitz, Hankel, and composition operator on the Bergman space.

Theorem 2.1. *Let $\phi, \psi, \chi, f \in H^\infty(\mathbb{D})$. If $\phi^n + \psi^n = \chi^n$ and f has absolute value 1, then $(T_f T_\phi T_f^*)^n + (T_f T_\psi T_f^*)^n = (T_f T_\chi T_f^*)^n$, for all $n \in \mathbb{N}$.*

Proof. Now,

$$\begin{aligned}(T_f T_\phi T_f^*)^2 = (T_f T_\phi T_f^*)(T_f T_\phi T_f^*) &= T_f T_\phi T_{\overline{f}} T_f T_\phi T_f^* \\ &= T_f T_\phi T_{|f|^2} T_\phi T_f^* \qquad (\because f \in H^\infty(\mathbb{D})) \\ &= T_f T_\phi^2 T_f^* \qquad (\because |f| = 1).\end{aligned}$$

Similarly, $(T_f T_\phi T_f^*)^n = T_f T_\phi^n T_f^* = T_f T_{\phi^n} T_f^*$. Thus, $(T_f T_\psi T_f^*)^n = (T_f T_{\psi^n} T_f^*)$. Therefore,

$$\begin{aligned}(T_f T_\phi T_f^*)^n + (T_f T_\psi T_f^*)^n &= T_f T_{\phi^n} T_f^* + T_f T_{\psi^n} T_f^* \\ &= T_f T_{\phi^n + \psi^n} T_f^* \\ &= (T_f T_{\chi^n} T_f^*) = (T_f T_\chi T_f^*)^n.\end{aligned}$$

□

Theorem 2.2. *Let f be a analytic function on $\mathbb{D}$. If f and $\overline{f}$ are harmonic functions on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$ and f has constant value with modulus 1, then $(U_z T_f^* T_f U_z)^n + (U_z (T_f^*, T_f] U_z)^n = I^n$, for all $n \in \mathbb{N}$.*

Proof. Since f is harmonic on $\mathbb{D}$ and f is constant of modulus 1, then by [12], $T_f^* T_f$ is a projection operator on the Bergman space and $(U_z T_f^* T_f U_z)^n = (U_z T_f^* T_f U_z)$. Again, $(U_z (T_f^*, T_f] U_z)^n = (U_z (T_f^*, T_f] U_z)$ ($\because (T_f^*, T_f]$ is a projection). Now,

$$\begin{aligned}U_z T_f^* T_f U_z)^n + (U_z (T_f^*, T_f] U_z)^n &= (U_z T_f^* T_f U_z) + (U_z (T_f^*, T_f] U_z) \\ &= (U_z T_f^* T_f U_z) + U_z T_{\overline{f} f} U_z - U_z T_f^* T_f U_z \\ &= U_z T_{|f|^2} U_z = U_z^2 = I \qquad (\because |f| = 1).\end{aligned}$$

□

Theorem 2.3. *Let $z \in \mathbb{D}$ and $\phi \in L^\infty(\mathbb{D})$ with $\|\phi\|_\infty \leq 1$. Assume that $\xi = \inf_{z \in \mathbb{D}} |\widetilde{\phi}(z)| > 0$ and there exists a sequence $\mu = \{\psi_n\}$ such that*

$$\lambda_\phi^\mu = \left(\sum_{n=0}^{\infty} (1 - 2Re\widetilde{\phi}(\overline{\psi_n}) E_{n,\phi}) + |\widetilde{\phi}(\psi_n)|^2) \right)^{\frac{1}{2}} < \infty.$$

If $\xi > \lambda_\phi^\mu$ and $T_\phi^{-1} = T_{\phi o \phi_z}$ for some $z \in \mathbb{D}$, then

$$(RT_z)^n + [T_\phi, T_{\overline{\phi}}]^n = I^n, \text{ for all } n \in \mathbb{N}.$$

Proof. Since $\xi > \lambda_\phi^\mu$ and $T_\phi^{-1} = T_{\phi o \phi_z}$ for some $z \in \mathbb{D}$, then it follows from [9] that T_ϕ is unitary. That implies $[T_\phi, T_{\overline{\phi}}] = [T_\phi, T_\phi^\star] = T_\phi T_\phi^\star - T_\phi^\star T_\phi = 0$. Hence

$[T_\phi, T_{\overline{\phi}}]^n = 0$. Now, $R : A^2(\mathbb{D}) \to A^2(\mathbb{D})$ be such that $Rf(z) = \frac{f(z)-f(0)}{z}$. Assume that $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Then $T_z f(z) = \sum_{n=0}^{\infty} a_n z^{n+1}$.

Therefore,

$$\begin{aligned}
(RT_z)f(z) &= R(T_z f(z)) \\
&= \frac{T_z f(z) - T_z f(0)}{z} \\
&= \frac{\sum_{n=0}^{\infty} a_n z^{n+1} - 0}{z} \\
&= \sum_{n=0}^{\infty} a_n z^n = f(z).
\end{aligned}$$

Hence, $RT_z = I$. Thus $(RT_z)^n + [T_\phi, T_{\overline{\phi}}]^n = I^n$.

$\square$

Theorem 2.4. *Let $\phi, \psi, \chi \in H^\infty(\mathbb{D})$. If $\phi^n + \psi^n = \chi^n$, then $(U_a T_\phi U_a)^n + (U_a T_\psi U_a)^n = (U_a T_\chi U_a)^n$, for all $n \in \mathbb{N}$ and $a \in \mathbb{D}$.*

Proof. Since ϕ, ψ and $\chi \in H^\infty(\mathbb{D}), T_\phi^n = T_{\phi^n}, T_\psi^n = T_{\psi^n}$ and $T_\chi^n = T_{\chi^n}, n \in \mathbb{N}$. Now, $T_\phi^n + T_\psi^n = T_{\phi^n} + T_{\psi^n} = T_{\phi^n+\psi^n} = T_{\chi^n} = T_\chi^n$. Thus,

$$\begin{aligned}
(U_a T_\phi U_a)^n + (U_a T_\psi U_a)^n &= U_a(T_{\phi^n} + T_{\psi^n})U_a \\
&= U_a T_{\phi^n+\psi^n} U_a \\
&= U_a T_{\chi^n} U_a = (U_a T_\chi U_a)^n.
\end{aligned}$$

$\square$

Theorem 2.5. *Let $\phi, \phi_j \in H^\infty(\mathbb{D}), j = 1, 2, \ldots k$. If $\phi_1^n + \phi_2^n + \cdots + \phi_k^n = (\phi_1 + \phi_2 + \cdots + \phi_k)^n = \phi^n$, where $\phi_1 + \phi_2 + \cdots + \phi_k = \phi$, then*
$(U_a T_{\phi_1} U_a)^n + (U_a T_{\phi_2} U_a)^n + \cdots + (U_a T_{\phi_k} U_a)^n = (U_a T_\phi U_a)^n$, for all $n \in \mathbb{N}$.

Proof. Now, $(U_a T_{\phi_j} U_a)^n = U_a T_{\phi_j}^n U_a = U_a T_{\phi_j^n} U_a$, for all $j = 1, 2 \ldots, k$. Then,

$$\begin{aligned}
&(U_a T_{\phi_1} U_a)^n + (U_a T_{\phi_2} U_a)^n + \cdots + (U_a T_{\phi_k} U_a)^n \\
&= (U_a T_{\phi_1^n} U_a) + (U_a T_{\phi_2^n} U_a) + \cdots + (U_a T_{\phi_k^n} U_a) \\
&= U_a(T_{\phi_1^n} + T_{\phi_2^n} + \cdots + T_{\phi_k^n})U_a \\
&= U_a T_{\phi_1^n+\phi_2^n+\cdots+\phi_k^n} U_a \\
&= U_a T_{\phi^n} U_a \qquad (\because \phi_1^n + \phi_2^n + \cdots + \phi_k^n = \phi^n) \\
&= (U_a T_\phi U_a)^n.
\end{aligned}$$

$\square$

Corollary 2.6. *Let $\phi \in H^\infty(\mathbb{D})$ and $z \in \mathbb{D}$. If $\phi^n + z^n = (\phi + z)^n$, then $T_{\phi o \phi_a}^n + T_{\phi_a}^n = T_{(\phi+z) o \phi_a}^n$, for all $n \in \mathbb{N}$.*

Proof. Since $U_aT_zU_a = T_{\phi_a}$ [13] for all $a, z \in \mathbb{D}$, then by Theorem 2.4, we obtained

$$\begin{aligned}
T^n_{(\phi+z)o\phi_a} &= (U_aT_{(\phi+z)}U_a)^n \\
&= U_aT^n_{(\phi+z)}U_a \\
&= U_aT_{(\phi+z)^n}U_a \\
&= U_aT_{\phi^n+z^n}U_a \\
&= U_a(T_{\phi^n} + T_{z^n})U_a \\
&= U_aT_{\phi^n}U_a + U_aT_{z^n}U_a \\
&= U_aT^n_{\phi}U_a + U_aT^n_{z}U_a \\
&= (U_aT_{\phi}U_a)^n + (U_aT_zU_a)^n \\
&= T^n_{\phi o\phi_a} + T^n_{\phi_a}.
\end{aligned}$$

□

Theorem 2.7. *Let $g, \overline{g} \in H^\infty(\mathbb{D})$ and $f \in L^\infty(\mathbb{D})$. Then $(T_f, T_g]^n + (S_g, S_g]^n = 0,$ for all $n \in \mathbb{N}$.*

Proof. Since $(T_f, T_g] = T_{fg} - T_fT_g$, then from the basic algebraic relation between the Toeplitz and Hankel operator on the Bergman space[14], it follows that

$$T_{fg} = T_fT_g + H^*_{\overline{f}}H_g \Rightarrow T_{fg} - T_fT_g = H^*_{\overline{f}}H_g,$$

$$S_{fg} = S_fS_g + H_fH^*_{\overline{g}} \Rightarrow S_{fg} - S_fS_g = H_fH^*_{\overline{g}}.$$

As $g \in H^\infty(\mathbb{D})$, then $H_g = 0$. Therefore, $(T_f, T_g] = H^*_{\overline{f}}H_g$ and $(S_f, S_g] = H_fH^*_{\overline{g}}$. Hence the result follows. □

Theorem 2.8. *Let $\phi \in h^\infty(\mathbb{D})$. If the numerical range of T_ϕ lies in upper-half plane and contains the origin, then $[T_\phi, T_{\overline{\phi}}]^n + [U_z, P]^n = 0,$ for all $n \in \mathbb{N}$, where P is the Bergman projection.*

Proof. Since ϕ is bounded harmonic, and the numerical range of T_ϕ lies in the upper half-plane and contains origin, then by [16], $T_\phi = T^\star_\phi$. Therefore, $[T_\phi, T_{\overline{\phi}}] = T_\phi T_{\overline{\phi}} - T_{\overline{\phi}}T_\phi = T_\phi T^*_\phi - T^*_\phi T_\phi = 0$. Again since $PU_z = U_zP, [U_z, P] = U_zP - PU_z = 0$. Hence the result follows. □

Let us denote $E(T_\phi) = R^*_\epsilon T_\phi R_\epsilon$ be the unitary transform of T_ϕ, where R_ϵ is a unitary operator on $A^2(\mathbb{D})$.

Theorem 2.9. *Let $\phi \in L^\infty(\mathbb{D})$ and $\epsilon \in \mathbb{T}$, where $\mathbb{T}$ is a unit circle in $\mathbb{C}$. If $[T^n_\phi, |T_\phi|^{\frac{1}{2}}] = |T_\phi|^{\frac{1}{2}}T^n_\phi$ and $[R_\epsilon, T^n_\phi] = T^n_\phi R_\epsilon$, then $(\triangle(T_\phi))^n + (E(T_\phi))^n = T^n_\phi,$ for all $n \in \mathbb{N}$, where $\triangle(T_\phi)$ and $E(T_\phi)$ represent the Aluthge transform and unitary transform of T_ϕ, respectively.*

Proof. Since $\triangle(T_\phi) = |T_\phi|^{\frac{1}{2}}U|T_\phi|^{\frac{1}{2}}$, where $T_\phi = U|T_\phi|$ is a polar decomposition of T_ϕ, then

$$\begin{aligned}\triangle(T_\phi) &= |T_\phi|^{\frac{1}{2}}U|T_\phi|^{\frac{1}{2}}\\ &= |T_\phi|^{\frac{1}{2}}U|T_\phi||T_\phi|^{\frac{-1}{2}}\\ &= |T_\phi|^{\frac{1}{2}}T_\phi|T_\phi|^{\frac{-1}{2}}.\end{aligned}$$

It is easy to compute $(\triangle(T_\phi))^n = |T_\phi|^{\frac{1}{2}}T_\phi^n|T_\phi|^{\frac{-1}{2}}$. Now, $E(T_\phi) = R_\epsilon^* T_\phi R_\epsilon$, then $(E(T_\phi))^n = R_\epsilon^* T_\phi^n R_\epsilon$. Now,

$$(\triangle(T_\phi))^n + (E(T_\phi))^n = |T_\phi|^{\frac{1}{2}}T_\phi^n|T_\phi|^{\frac{-1}{2}} + R_\epsilon^* T_\phi^n R_\epsilon. \tag{2.1}$$

Since $[T_\phi^n, |T_\phi|^{\frac{1}{2}}] = |T_\phi|^{\frac{1}{2}}T_\phi^n$, then $T_\phi^n|T_\phi|^{\frac{1}{2}} - |T_\phi|^{\frac{1}{2}}T_\phi^n = |T_\phi|^{\frac{1}{2}}T_\phi^n$. Thus

$$|T_\phi|^{\frac{1}{2}}T_\phi^n = \frac{1}{2}T_\phi^n|T_\phi|^{\frac{1}{2}}. \tag{2.2}$$

Again, since $[R_\epsilon, T_\phi^n] = T_\phi^n R_\epsilon$, then $R_\epsilon T_\phi^n - T_\phi^n R_\epsilon = T_\phi^n R_\epsilon$. Therefore,

$$T_\phi^n R_\epsilon = \frac{1}{2}R_\epsilon T_\phi^n. \tag{2.3}$$

Using (2.2) and (2.3) in (2.1), we get $(\triangle(T_\phi))^n + (E(T_\phi)^n = T_\phi^n$. □

Let $m(T_\phi) = \inf\{\|T_\phi f\| : \|f\| = 1\}$ and $m(T_{\overline{\phi}}) = \inf\{\|T_\phi^\star f\| : \|f\| = 1\}$.

Theorem 2.10. *Let $\phi \in L^\infty(\mathbb{D})$. Assume that $m(T_\Phi) > 0$ and $m(T_{\overline{\phi}}) > 0$. If T_ϕ is partial isometry with $T_{\overline{\phi}}^2 T_\phi^2 + 2kT_{\overline{\phi}}T_\phi + k^2 \geq 0$ for all $k \in \mathbb{R}$, then*

$$(\triangle(T_\phi))^n + [U_z, U_z^\star]^n = T_\phi^n, \text{ for all } n \in \mathbb{N}.$$

Proof. Since $m(T_\phi), m(T_{\overline{\phi}}) > 0$, then by [19], T_ϕ is invertible. Furthermore, since $T_{\overline{\phi}}^2 T_\phi^2 + 2kT_{\overline{\phi}}T_\phi + k^2 \geq 0$ for any k, T_ϕ is paranormal. As partial isometry and paranormal operators are quasinormal, so T_ϕ is quasinormal. Thus by [10], $\triangle(T_\phi) = T_\phi$. Now,

$$\begin{aligned}[U_z, U_z^\star] &= U_z U_z^\star - U_z^\star U_z\\ &= U_z^2 - U_z^2 = 0 \qquad (\because U_z^\star = U_z).\end{aligned}$$

Therefore, $(\triangle(T_\phi))^n + [U_z, U_z^\star]^n = T_\phi^n + 0 = T_\phi^n$. □

Theorem 2.11. *Let $\phi, \psi, \chi \in H^\infty(\mathbb{D})$. If $\phi^n + \psi^n = \chi^n$, where $\chi = \phi + \psi$, then $(C_a T_\phi C_a)^n + (C_a T_\psi C_a)^n = (C_a T_\chi C_a)^n$, for all $n \in \mathbb{N}$ and $a \in \mathbb{D}$.*

Proof. Now, $(C_a T_\phi C_a)^n = (C_a T_\phi C_a)(C_a T_\phi C_a)\cdots n \textit{ times} = C_a T_\phi^n C_a = C_a T_{\phi^n} C_a$. Similarly, $(C_a T_\psi C_a)^n = C_a T_{\psi^n} C_a$. So

$$\begin{aligned}(C_a T_\phi C_a)^n + (C_a T_\psi C_a)^n &= C_a(T_{\phi^n} + T_{\psi^n})C_a\\ &= C_a T_{\phi^n + \psi^n} C_a\\ &= C_a T_{\chi^n} C_a = (C_a T_\chi C_a)^n.\end{aligned}$$

□

Theorem 2.12. *Let $\phi, \phi_j \in H^\infty(\mathbb{D}), j = 1, 2, \dots k$. If $\phi_1^n + \phi_2^n + \cdots + \phi_k^n = \phi^n$, where $\phi_1 + \phi_2 + \cdots + \phi_k = \phi$, then*
$$(C_aT_{\phi_1}C_a)^n + (C_aT_{\phi_2}C_a)^n + \cdots + (C_aT_{\phi_k}C_a)^n = (C_aT_\phi C_a)^n, \text{ for all } n \in \mathbb{N}.$$

Proof. Now, $(C_aT_{\phi_j}C_a)^n = C_aT^n_{\phi_j}C_a = C_aT_{\phi_j^n}C_a$, for all $j = 1, 2, \dots, k$. Then,

$$\begin{aligned}
&(C_aT_{\phi_1}C_a)^n + (C_aT_{\phi_2}C_a)^n + \cdots + (C_aT_{\phi_k}C_a)^n \\
&= (C_aT_{\phi_1^n}C_a) + (C_aT_{\phi_2^n}C_a) + \cdots + (C_aT_{\phi_k^n}C_a) \\
&= C_a(T_{\phi_1^n} + T_{\phi_2^n} + \cdots + T_{\phi_k^n})C_a \\
&= C_aT_{\phi_1^n+\phi_2^n+\cdots+\phi_k^n}C_a \\
&= C_aT_{\phi^n}C_a \qquad (\because \phi_1^n + \phi_2^n + \cdots + \phi_k^n = \phi^n) \\
&= (C_aT_\phi C_a)^n.
\end{aligned}$$

□

Theorem 2.13. *Let $\phi : \mathbb{D} \longrightarrow \mathbb{D}$ be holomorphic. If C_ϕ is algebraically hyponormal and quasinilpotent, then $(C_aC_\phi C_a)^n + (U_zC_\phi U_z)^n = 0$ for all $z, a \in \mathbb{D}$, for all $n \in \mathbb{N}$.*

Proof. Now, $(C_aC_\phi C_a)^2 = C_aC_\phi C_a^2C_\phi C_a = C_aC_\phi^2C_a$ as $C_a^2 = I$. Then, for each $n \in \mathbb{N}$, $(C_aC_\phi C_a)^n = C_aC_\phi^nC_a$. Again $(U_zC_\phi U_z)^2 = U_zC_\phi^2U_z$ as $U_z^2 = I$. Similarly, $(U_zC_\phi U_z)^n = U_zC_\phi^nU_z, n \in \mathbb{N}$. Since C_ϕ is algebraically hyponormal and quasinilpotent, then from [7], $C_\phi^n = 0$. Hence, $(C_aC_\phi C_a)^n + (U_zC_\phi U_z)^n = 0$. □

Theorem 2.14. *Let $\phi, \psi, \chi \in H^\infty(\mathbb{D})$. If $\phi^n + \psi^n = \chi^n$, then for $a \in \mathbb{T}$ and all $n \in \mathbb{N}$,*

$$(R_aT_\phi R_a^*)^n + (R_aT_\psi R_a^*)^n = (R_aT_\chi R_a^*)^n.$$

Proof. Now, $(R_aT_\phi R_a^*)^2 = R_aT_\phi^2R_a^* \qquad (\because R_a^*R_a = I)$. Similarly, $(R_aT_\phi R_a^*)^n = R_aT_\phi^nR_a^*$ and $(R_aT_\psi R_a^*)^n = R_aT_\psi^nR_a^*$. Hence,

$$\begin{aligned}
(R_aT_\phi R_a^*)^n + (R_aT_\psi R_a^*)^n &= R_aT_\phi^nR_a^* + R_aT_\psi^nR_a^* \\
&= R_a(T_\phi^n + T_\psi^n)R_a^* \\
&= R_aT_{\phi^n+\psi^n}R_a^* \\
&= R_aT_{\chi^n}R_a^* \\
&= R_aT_\chi^nR_a^* \\
&= (R_aT_\chi R_a^*)^n.
\end{aligned}$$

□

Theorem 2.15. *Let $\phi, \phi_j \in H^\infty(\mathbb{D}), j = 1(1)k$. If $\phi_1^n + \phi_2^n + \cdots + \phi_k^n = \phi^n$, then*
$$(R_aT_{\phi_1}R_a^*)^n + (R_aT_{\phi_2}R_a^*)^n + \cdots + (R_aT_{\phi_k}R_a^*)^n = (R_aT_\phi R_a^*)^n, \text{ for all } n \in \mathbb{N}.$$

Proof. Since $(R_aT_{\phi_j}R_a^*)^2 = (R_aT_{\phi_j}R_a^*)(R_aT_{\phi_j}R_a^*) = R_aT_{\phi_j}^2R_a^*$, then $(R_aT_{\phi_j}R_a^*)^n = R_aT_{\phi_j}^nR_a^* = R_aT_{\phi_j^n}R_a^*$, for all $j = 1, 2, \ldots, k$. Then,

$$\begin{aligned}
&(R_aT_{\phi_1}R_a^*)^n + (R_aT_{\phi_2}R_a^*)^n + \cdots + (R_aT_{\phi_k}R_a^*)^n \\
&= (R_aT_{\phi_1^n}R_a^*) + (R_aT_{\phi_2^n}R_a^*) + \cdots + (R_aT_{\phi_k^n}R_a^*) \\
&= R_a(T_{\phi_1^n} + T_{\phi_2^n} + \cdots + T_{\phi_k^n})R_a^* \\
&= R_aT_{\phi_1^n+\phi_2^n+\cdots+\phi_k^n}R_a^* \\
&= R_aT_{\phi^n}R_a^* \qquad (\because \phi_1^n + \phi_2^n + \cdots + \phi_k^n = \phi^n) \\
&= (R_aT_\phi R_a^*)^n.
\end{aligned}$$

□

Theorem 2.16. *Let $z, a \in \mathbb{D}$. Then $(U_zC_aU_z)^n + [R_\epsilon, R_{\bar{\epsilon}}]^n = I$, when $n = 2m, m \in \mathbb{N}$.*

Proof. Since $(U_zC_aU_z)^2 = U_zC_aU_z^2C_aU_z = I$ ($\because U_z^2 = I$ *and* $C_a^2 = I$), hence $(U_zC_aU_z)^{2m} = I, m \in \mathbb{N}$. Since R_ϵ is unitary and $R_\epsilon^{-1} = R_\epsilon = R_\epsilon^*$, then $[R_\epsilon, R_{\bar{\epsilon}}] = 0$. Furthermore, since $C_a^2 = I$, then $C_a^{2m} = I, m \in \mathbb{N}$. Hence $(U_zC_aU_z)^n + [R_\epsilon, R_{\bar{\epsilon}}]^n = C_a^n = I$, when $n = 2m, m \in \mathbb{N}$. □

Theorem 2.17. *Let $z, a \in \mathbb{D}$ and $\phi \in L^\infty(\mathbb{D})$ with $\phi \geq 0$. Then $(C_aU_zC_a)^n + [T_\phi, T_{\overline{\phi}}]^n = I$, when $n = 2m, m \in \mathbb{N}$.*

Proof. Since $(C_aU_zC_a)^2 = C_aU_zC_aC_aU_zC_a = I$. Then $(C_aU_zC_a)^{2m} = I$, where $m \in \mathbb{N}$. Furthermore, since $\phi \geq 0$, so $T_\phi \geq 0$. Thus T_ϕ is positive on $A^2(\mathbb{D})$. Since positive operators are self-adjoint, $T_\phi^* = T_\phi = T_{\overline{\phi}}$. Therefore, $[T_\phi, T_{\overline{\phi}}] = T_\phi T_{\overline{\phi}} - T_{\overline{\phi}}T_\phi = 0$. Moreover, $U_a^2 = I$ and $U_a^{2m} = I, m \in \mathbb{N}$. □

Theorem 2.18. *Let $f, f_j \in H^\infty(\mathbb{D})$, for $j = 1, 2 \ldots k$. If $f_1^n + f_2^n + \cdots + f_k^n = f^n$, where $f = f_1 + f_2 + \cdots + f_k$. Then $(U_zS_{f_1}U_z)^n + (U_zS_{f_2}U_z)^n + \cdots + (U_zS_{f_k}U_z)^n = (U_zS_fU_z)^n$, for all $z \in \mathbb{D}$, for all $n \in \mathbb{N}$.*

Proof. As $(U_zS_{f_j}U_z)^n = U_zS_{f_j}^nU_z = U_zS_{f_j^n}U_z$, for all $j = 1, 2, \ldots, k$, thus,

$$\begin{aligned}
&(U_zS_{f_1}U_z)^n + (U_zS_{f_2}U_z)^n + \cdots + (U_zS_{f_k}U_z)^n \\
&= (U_zS_{f_1^n}U_z) + (U_zS_{f_2^n}U_z) + \cdots + (U_zS_{f_k^n}U_z) \\
&= U_z(S_{f_1^n} + S_{f_2^n} + \cdots + S_{f_k^n})U_z \\
&= U_zS_{f_1^n+f_2^n+\cdots+f_k^n}U_z \\
&= U_zS_{f^n}U_z = (U_zS_fU_z)^n.
\end{aligned}$$

□

Theorem 2.19. *Let $\phi, \phi_j \in H^\infty(\mathbb{D}), j = 1, 2 \ldots, k, S = T_z$ be the Bergman shift operator on $A^2(\mathbb{D})$ and let $\phi_1^n + \phi_2^n + \cdots + \phi_k^n = \phi^n$. Then $(ST_{\phi_1}R)^n + (ST_{\phi_2}R)^n + \cdots + (ST_{\phi_k}R)^n = (ST_\phi R)^n$, for all $n \in \mathbb{N}$.*

Proof. Note that $(ST_{\phi_1}R)^n = ST_{\phi_1^n}R$ since $RS = I$. Again, $(ST_{\phi_2}R)^n = ST_{\phi_2^n}R$. Similarly, $(ST_{\phi_k}R)^n = ST_{\phi_k^n}R$. Therefore,

$$\begin{aligned}(ST_{\phi_1}R)^n + (ST_{\phi_2}R)^n + \cdots + (ST_{\phi_k}R)^n &= (ST_{\phi_1}R)^n + ST_{\phi_2}R)^n + \cdots + (ST_{\phi_k}R)^n \\ &= S(T_{\phi_1^n} + T_{\phi_2^n} + \cdots + T_{\phi_k^n})R \\ &= ST_{\phi_1^n+\phi_2^n+\cdots+\phi_k^n}R \\ &= ST_{\phi^n}R = (ST_\phi R)^n.\end{aligned}$$

□

3. Fermat's equation over 2×2 operator matrices

Throughout this section, we study 2×2 operator matrix solutions of the Fermat's equation (1.1) on $A^2(\mathbb{D}) \oplus A^2(\mathbb{D})$, and if $T \in \mathcal{L}(A^2(\mathbb{D}) \oplus A^2(\mathbb{D}))$, then T has the representation of the form $\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$ on $A^2(\mathbb{D}) \oplus A^2(\mathbb{D})$, where $T_{ij} \in \mathcal{L}(A^2(\mathcal{D}))$.

Theorem 3.1. *Let $\psi, \phi \in H^\infty(\mathbb{D})$. If $\phi(1-\phi)^{n-1} + \phi^{n-1}(1-\phi) = 0$ and $\phi^n + (1-\phi)^n = \psi^n$, then $\begin{pmatrix} T_\phi & T_{\phi-1} \\ 0 & T_\phi \end{pmatrix}^n + \begin{pmatrix} T_{1-\phi} & -T_\phi \\ 0 & T_{1-\phi} \end{pmatrix}^n = \begin{pmatrix} T_\psi & 0 \\ 0 & T_\psi \end{pmatrix}^n$, for all $n \in \mathbb{N}$ on $A^2(\mathbb{D}) \oplus A^2(\mathbb{D})$.*

Proof. Let A$=\begin{pmatrix} T_\phi & T_{\phi-1} \\ 0 & T_\phi \end{pmatrix}$. Then $A^n = \begin{pmatrix} T_\phi^n & nT_\phi^{n-1}T_{\phi-1} \\ 0 & T_\phi^n \end{pmatrix}$. Again assume that $B = \begin{pmatrix} T_{1-\phi} & -T_\phi \\ 0 & T_{1-\phi} \end{pmatrix}$. Then $B^n = \begin{pmatrix} T_{1-\phi}^n & -nT_\phi T_{1-\phi}^n \\ 0 & T_{1-\phi}^n \end{pmatrix}$. Now,

$$A^n + B^n = \begin{pmatrix} T_\phi^n + T_{1-\phi}^n & nT_\phi^{n-1}T_{\phi-1} - nT_\phi T_{1-\phi}^{n-1} \\ 0 & T_\phi^n + T_{1-\phi}^n \end{pmatrix}.$$

Since, $\phi \in H^\infty(\mathbb{D})$ and $\phi(1-\phi)^{n-1} + \phi^{n-1}(1-\phi) = 0$, then $T_{\phi(1-\phi)^{n-1}+\phi^{n-1}(1-\phi)} = 0$. That implies $T_\phi T_{(1-\phi)}^{n-1} + T_\phi^{n-1}T_{1-\phi} = 0$. Therefore,

$$A^n + B^n = \begin{pmatrix} T_{\phi^n} + T_{(1-\phi)^n} & 0 \\ 0 & T_{\phi^n} + T_{(1-\phi)^n} \end{pmatrix} = \begin{pmatrix} T_{\phi^n+(1-\phi)^n} & 0 \\ 0 & T_{\phi^n+(1-\phi)^n} \end{pmatrix} = \begin{pmatrix} T_{\psi^n} & 0 \\ 0 & T_{\psi^n} \end{pmatrix} = \begin{pmatrix} T_\psi & 0 \\ 0 & T_\psi \end{pmatrix}^n.$$

□

Theorem 3.2. *Let $\phi, \psi \in H^\infty(\mathbb{D})$. If $\phi(\phi-1) = 0$ and $\phi^n + (1-\phi)^n = \psi^n$, then for all $n \in \mathbb{N}$, $\begin{pmatrix} T_\phi & T_{1-\phi} \\ 0 & T_\phi \end{pmatrix}^n + \begin{pmatrix} T_{1-\phi} & 0 \\ T_\phi & T_{1-\phi} \end{pmatrix}^n = \begin{pmatrix} T_\psi & 0 \\ 0 & T_\psi \end{pmatrix}^n$ on $A^2(\mathbb{D}) \oplus A^2(\mathbb{D})$.*

Proof. Now, $\begin{pmatrix} T_\phi & T_{1-\phi} \\ 0 & T_\phi \end{pmatrix}^2 = \begin{pmatrix} T_\phi^2 & 2T_\phi T_{(1-\phi)} \\ 0 & T_\phi^2 \end{pmatrix}$.

Similarly one can verify that $\begin{pmatrix} T_\phi & T_{(1-\phi)} \\ 0 & T_\phi \end{pmatrix}^n = \begin{pmatrix} T_\phi^n & nT_\phi^{n-1}T_{(1-\phi)} \\ 0 & T_\phi^n \end{pmatrix}$. Again, $\begin{pmatrix} T_{(1-\phi)} & 0 \\ T_\phi & T_{(1-\phi)} \end{pmatrix}^n = \begin{pmatrix} T_{(1-\phi)}^n & 0 \\ nT_\phi T_{1-\phi}^{n-1} & T_{(1-\phi)}^n \end{pmatrix}$.

Thus $\begin{pmatrix} T_\phi & T_{(1-\phi)} \\ 0 & T_\phi \end{pmatrix}^n + \begin{pmatrix} T_{(1-\phi)} & 0 \\ T_\phi & T_{(1-\phi)} \end{pmatrix}^n = \begin{pmatrix} T_\phi^n + T_{(1-\phi)}^n & nT_\phi^{n-1}T_{(1-\phi)} \\ nT_\phi T_{(1-\phi)}^{n-1} & T_\phi^n + T_{(1-\phi)}^n \end{pmatrix}$.

Since $\phi \in H^\infty(\mathbb{D})$ and $\phi^n + (1-\phi)^n = \psi^n$, then

$$\begin{aligned} T_\phi^n + T_{(1-\phi)}^n &= T_{\phi^n} + T_{(1-\phi)^n} \\ &= T_{\phi^n + (1-\phi)^n} \\ &= T_{\psi^n} = T_\psi^n. \end{aligned}$$

Furthermore, since $\phi^2 - \phi = 0$, then $T_\phi^2 = T_\phi$, that implies $T_\phi(I - T_\phi) = 0$. Therefore, $T_\phi T_{(1-\phi)}^{n-1} = T_\phi(I - T_\phi)^{n-1} = 0$ and $T_\phi^{n-1}T_{(1-\phi)} = 0$. Hence the result follows. □

Theorem 3.3. *Let $\phi, \psi, \chi \in H^\infty(\mathbb{D}), S \in \mathcal{L}(A^2(\mathbb{D}))$ and $T_z = WS$ with $S \geq 0, W$ being power bounded, $S \leq Re(T_z)$. If $(\phi + z)^n + z^n = \psi^n$ and $(\phi - z)^n + z^n = \chi^n$, then $\left[U^* \begin{pmatrix} T_\phi & T_z \\ T_z & T_\phi \end{pmatrix} U\right]^n + \begin{pmatrix} T_z & T_z - S \\ 0 & S \end{pmatrix}^n = \begin{pmatrix} T_\psi & 0 \\ 0 & T_\chi \end{pmatrix}^n$ for all $n \in \mathbb{N}$, where $U = \begin{pmatrix} \frac{I}{\sqrt{2}} & \frac{-I}{\sqrt{2}} \\ \frac{I}{\sqrt{2}} & \frac{I}{\sqrt{2}} \end{pmatrix}$ on $A^2(\mathbb{D}) \oplus A^2(\mathbb{D})$.*

Proof. It is easy to compute

$$\begin{aligned} &\begin{pmatrix} \frac{I}{\sqrt{2}} & \frac{I}{\sqrt{2}} \\ \frac{-I}{\sqrt{2}} & \frac{I}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} T_\phi & T_z \\ T_z & T_\phi \end{pmatrix} \begin{pmatrix} \frac{I}{\sqrt{2}} & \frac{-I}{\sqrt{2}} \\ \frac{I}{\sqrt{2}} & \frac{I}{\sqrt{2}} \end{pmatrix} \\ &= \begin{pmatrix} T_\phi + T_z & 0 \\ 0 & T_\phi - T_z \end{pmatrix}. \end{aligned}$$

Therefore,

$$\begin{aligned} &\left[\begin{pmatrix} \frac{I}{\sqrt{2}} & \frac{I}{\sqrt{2}} \\ \frac{-I}{\sqrt{2}} & \frac{I}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} T_\phi & T_z \\ T_z & T_\phi \end{pmatrix} \begin{pmatrix} \frac{I}{\sqrt{2}} & \frac{-I}{\sqrt{2}} \\ \frac{I}{\sqrt{2}} & \frac{I}{\sqrt{2}} \end{pmatrix}\right]^n \\ &= \begin{pmatrix} T_\phi + T_z & 0 \\ 0 & T_\phi - T_z \end{pmatrix}^n = \begin{pmatrix} T_{(\phi+z)^n} & 0 \\ 0 & T_{(\phi-z)^n} \end{pmatrix}. \end{aligned}$$

Furthermore, $\begin{pmatrix} T_z & T_z - S \\ 0 & S \end{pmatrix}^n = \begin{pmatrix} T_z^n & T_z^n - S^n \\ 0 & S^n \end{pmatrix}$. Since $T_z = WS$ with $S \geq 0$ and W is a power bounded operator on $A^2(\mathbb{D})$ and $S \leq Re(T_z)$, then by Fong and Tsui [6] $S = T_z$. Now,

$$\left[U^*\begin{pmatrix} T_\phi & T_z \\ T_z & T_\phi \end{pmatrix}U\right]^n + \begin{pmatrix} T_z & T_z - S \\ 0 & S \end{pmatrix}^n$$
$$= \begin{pmatrix} T_{(\phi+z)^n} & 0 \\ 0 & T_{(\phi-z)^n} \end{pmatrix} + \begin{pmatrix} T_z^n & T_z^n - S^n \\ 0 & S^n \end{pmatrix}$$
$$= \begin{pmatrix} T_{(\phi+z)^n+z^n} & 0 \\ 0 & T_{(\phi-z)^n+z^n} \end{pmatrix} = \begin{pmatrix} T_{\psi^n} & 0 \\ 0 & T_{\chi^n} \end{pmatrix} = \begin{pmatrix} T_\psi & 0 \\ 0 & T_\chi \end{pmatrix}^n.$$

□

Let $Z_n[T]$ be the ring of polynomials generated by the operator T, where $T \in \mathcal{L}(H)$. Then we have the following results.

Theorem 3.4. *Let $\phi \in H^\infty(\mathbb{D})$ and $\phi^{n-1}(\phi - 1) = 0$. If $T_{\phi^n} \in \mathbb{Z}_{2^{n-1}}[T_\phi]$, then on $A^2(\mathbb{D}) \oplus A^2(\mathbb{D})$, $\begin{pmatrix} T_\phi & T_{\phi-1} \\ 0 & T_\phi \end{pmatrix}^n + \begin{pmatrix} T_\phi & T_\phi \\ T_\phi & T_\phi \end{pmatrix}^n = \begin{pmatrix} T_\phi & 0 \\ 0 & T_\phi \end{pmatrix}^n$, for all $n \in \mathbb{N}$.*

Proof. Now, $\begin{pmatrix} T_\phi & T_{\phi-1} \\ 0 & T_\phi \end{pmatrix}^n = \begin{pmatrix} T_\phi^n & nT_\phi^{n-1}T_{\phi-1} \\ 0 & T_\phi^n \end{pmatrix} = \begin{pmatrix} T_{\phi^n} & nT_{\phi^{n-1}(\phi-1)} \\ 0 & T_{\phi^n} \end{pmatrix}$. Again $\begin{pmatrix} T_\phi & T_\phi \\ T_\phi & T_\phi \end{pmatrix}^n = \begin{pmatrix} 2^{n-1}T_\phi^n & 2^{n-1}T_\phi^n \\ 2^{n-1}T_\phi^n & 2^{n-1}T_\phi^n \end{pmatrix} = \begin{pmatrix} 2^{n-1}T_{\phi^n} & 2^{n-1}T_{\phi^n} \\ 2^{n-1}T_{\phi^n} & 2^{n-1}T_{\phi^n} \end{pmatrix}$. Thus $\begin{pmatrix} T_\phi & T_{\phi-1} \\ 0 & T_\phi \end{pmatrix}^n + \begin{pmatrix} T_\phi & T_\phi \\ T_\phi & T_\phi \end{pmatrix}^n = \begin{pmatrix} (1+2^{n-1})T_\phi^n & nT_\phi^n - nT_\phi^{n-1} + 2^{n-1}T_\phi^n \\ 2^{n-1}T_\phi^n & (1+2^{n-1})T_\phi^n \end{pmatrix}$. Since $\mathbb{Z}_{2^{n-1}}[T_\phi]$ is a ring of polynomials in T_ϕ with characteristic 2^{n-1}, then $2^{n-1}T_\phi^n = 0$. Hence, $\begin{pmatrix} (1+2^{n-1})T_\phi^n & nT_\phi^n - nT_\phi^{n-1} + 2^{n-1}T_\phi^n \\ 2^{n-1}T_\phi^n & (1+2^{n-1})T_\phi^n \end{pmatrix}$ is reduced to $\begin{pmatrix} T_\phi^n & nT_\phi^n - nT_\phi^{n-1} \\ 0 & T_\phi^n \end{pmatrix}$. Furthermore, since $\phi^{n-1}(\phi - 1) = 0$, then $T_{\phi^{n-1}}(T_\phi - I) = 0$. That implies $T_\phi^{n-1}(T_\phi - I) = 0$. Thus $\begin{pmatrix} T_\phi^n & nT_\phi^n - nT_\phi^{n-1} \\ 0 & T_\phi^n \end{pmatrix} = \begin{pmatrix} T_\phi & 0 \\ 0 & T_\phi \end{pmatrix}^n$. Hence the result follows. □

4. Fermat's equation over 3×3 operator matrices

In this section, we investigate 3×3 operator matrix solutions of the Fermat's equation on vector valued Bergman spaces.

Theorem 4.1. *Let $\phi, \psi \in H^\infty(\mathbb{D})$and $\phi^n + \psi^n = 1, n \geq 1$. If $T_\phi \in \mathbb{Z}_n[T_\phi]$, then on $A^2(\mathbb{D}) \oplus A^2(\mathbb{D}) \oplus A^2(\mathbb{D})$,*

$$\begin{pmatrix} T_\phi & 0 & 0 \\ 0 & T_\phi & T_\phi \\ 0 & 0 & T_\phi \end{pmatrix}^n + \begin{pmatrix} T_\psi & 0 & 0 \\ 0 & T_\psi & 0 \\ 0 & 0 & T_\psi \end{pmatrix}^n = \begin{pmatrix} I & T_\phi & T_\phi \\ 0 & I & T_\phi \\ 0 & 0 & I \end{pmatrix}^n, \text{ for all } n \in \mathbb{N}.$$

Proof. Let

$$A = \begin{pmatrix} T_\phi & 0 & 0 \\ 0 & T_\phi & T_\phi \\ 0 & 0 & T_\phi \end{pmatrix}, B = \begin{pmatrix} I & T_\phi & T_\phi \\ 0 & I & T_\phi \\ 0 & 0 & I \end{pmatrix} \text{ and } D = \begin{pmatrix} T_\psi & 0 & 0 \\ 0 & T_\psi & 0 \\ 0 & 0 & T_\psi \end{pmatrix}.$$

On direct computation, one can get

$$A^n = \begin{pmatrix} T_\phi^n & 0 & 0 \\ 0 & T_\phi^n & nT_\phi^n \\ 0 & 0 & T_\phi^n \end{pmatrix} \text{ and } B^n = \begin{pmatrix} I & nT_\phi & \frac{n(n-1)}{2}T_\phi^2 + nT_\phi \\ 0 & I & nT_\phi \\ 0 & 0 & I \end{pmatrix}.$$

Since $\phi, \psi \in H^\infty(\mathbb{D})$ and $\psi^n = 1 - \phi^n$, then $I - T_\phi^n = T_{1-\phi^n} = T_{\psi^n} = T_\psi^n$. Furthermore, since $T_\phi \in \mathbb{Z}_n[T_\phi]$ and $\mathbb{Z}_n[T_\phi]$ is a ring of polynomials in T_ϕ with characteristic n, therefore

$$B^n - A^n = \begin{pmatrix} I - T_\phi^n & nT_\phi & \frac{n(n-1)}{2}T_\phi^2 + nT_\phi \\ 0 & I - T_\phi^n & nT_\phi - nT_\phi^n \\ 0 & 0 & I - T_\phi^n \end{pmatrix}$$

is reduced to $\begin{pmatrix} T_\psi & 0 & 0 \\ 0 & T_\psi & 0 \\ 0 & 0 & T_\psi \end{pmatrix}^n = D^n$. Thus $B^n - A^n = D^n$, as required. $\square$

Theorem 4.2. *Let $\phi, \psi \in H^\infty(\mathbb{D})$. If $\phi^n + 1 = \psi^n, n \geq 1$ and $T_{(1-\phi)} \in \mathbb{Z}_n[T_\phi]$, then on $A^2(\mathbb{D}) \oplus A^2(\mathbb{D}) \oplus A^2(\mathbb{D})$, for all $n \in \mathbb{N}$,*

$$\begin{pmatrix} T_\phi & 0 & 0 \\ 0 & T_\phi & T_{(1-\phi)} \\ 0 & 0 & T_\phi \end{pmatrix}^n + \begin{pmatrix} I & T_{(1-\phi)} & T_{(1-\phi)} \\ 0 & I & T_{(1-\phi)} \\ 0 & 0 & I) \end{pmatrix}^n = \begin{pmatrix} T_\psi & 0 & 0 \\ 0 & T_\psi & 0 \\ 0 & 0 & T_\psi \end{pmatrix}^n.$$

Proof. Direct calculation leads to

$$\begin{pmatrix} T_\phi & 0 & 0 \\ 0 & T_\phi & T_{(1-\phi)} \\ 0 & 0 & T_\phi \end{pmatrix}^n = \begin{pmatrix} T_{\phi^n} & 0 & 0 \\ 0 & T_{\phi^n} & nT_{(1-\phi)}T_{\phi^{n-1}} \\ 0 & 0 & T_{\phi^n} \end{pmatrix}.$$

Similarly, $\begin{pmatrix} I & T_{(1-\phi)} & T_{(1-\phi)} \\ 0 & I & T_{(1-\phi)} \\ 0 & 0 & I) \end{pmatrix}^n = \begin{pmatrix} I & nT_{1-\phi} & \frac{n(n-1)}{2}T_{1-\phi}^2 + nT_{1-\phi} \\ 0 & I & nT_{(1-\phi)} \\ 0 & 0 & I \end{pmatrix}.$

As $\mathbb{Z}_n[T_\phi]$ has characteristic n and $T_{1-\phi} \in \mathbb{Z}_n[T_\phi]$, then $nT_{1-\phi} = 0$ and the condition $1 + \phi^n = \psi^n$ implies

$$\begin{aligned} &\begin{pmatrix} T_\phi & 0 & 0 \\ 0 & T_\phi & T_{(1-\phi)} \\ 0 & 0 & T_\phi \end{pmatrix}^n + \begin{pmatrix} I & T_{(1-\phi)} & T_{(1-\phi)} \\ 0 & I & T_{(1-\phi)} \\ 0 & 0 & I) \end{pmatrix}^n \\ &= \begin{pmatrix} T_{\phi^n} + I & nT_{1-\phi} & \frac{n(n-1)}{2}T_{1-\phi}^2 + nT_{1-\phi} \\ 0 & T_{\phi^n} + I & nT_{1-\phi}T_\phi^{n-1} + nT_{(1-\phi)} \\ 0 & 0 & T_{\phi^n} + I \end{pmatrix} \\ &= \begin{pmatrix} T_{\phi^n+1} & 0 & 0 \\ 0 & T_{\phi^n+1} & 0 \\ 0 & 0 & T_{\phi^n+1} \end{pmatrix} = \begin{pmatrix} T_{\psi^n} & 0 & 0 \\ 0 & T_{\psi^n} & 0 \\ 0 & 0 & T_{\psi^n} \end{pmatrix} = \begin{pmatrix} T_\psi & 0 & 0 \\ 0 & T_\psi & 0 \\ 0 & 0 & T_\psi \end{pmatrix}^n. \end{aligned}$$

$\square$

Theorem 4.3. *Let $\phi, \psi \in \mathbb{H}^\infty(\mathbb{D})$. If $\phi^n + (1-\phi)^n = \psi^n, \phi^{n-1}(1-\phi) = 0$ and $(1-\phi)^{n-1}\phi = 0$, then on $A^2(\mathbb{D}) \oplus A^2(\mathbb{D}) \oplus A^2(\mathbb{D})$, for all $n \in \mathbb{N}$,*

$$\begin{pmatrix} T_\phi & 0 & 0 \\ T_{1-\phi} & T_\phi & 0 \\ 0 & 0 & T_\phi \end{pmatrix}^n + \begin{pmatrix} T_{1-\phi} & 0 & 0 \\ 0 & T_{1-\phi} & T_\phi \\ 0 & 0 & T_{1-\phi} \end{pmatrix}^n = \begin{pmatrix} T_\psi & 0 & 0 \\ 0 & T_\psi & 0 \\ 0 & 0 & T_\psi \end{pmatrix}^n.$$

Proof. Let

$$A = \begin{pmatrix} T_\phi & 0 & 0 \\ T_{1-\phi} & T_\phi & 0 \\ 0 & 0 & T_\phi \end{pmatrix}, B = \begin{pmatrix} T_{1-\phi} & 0 & 0 \\ 0 & T_{1-\phi} & T_\phi \\ 0 & 0 & T_{1-\phi} \end{pmatrix} \text{ and } C = \begin{pmatrix} T_\psi & 0 & 0 \\ 0 & T_\psi & 0 \\ 0 & 0 & T_\psi \end{pmatrix}.$$

Simple calculation shows that

$$A^n = \begin{pmatrix} T_\phi^n & 0 & 0 \\ nT_\phi^{n-1}T_{1-\phi} & T_\phi^n & 0 \\ 0 & 0 & T_\phi^n \end{pmatrix} = \begin{pmatrix} T_{\phi^n} & 0 & 0 \\ nT_{\phi^n-1}T_{1-\phi} & T_{\phi^n} & 0 \\ 0 & 0 & T_{\phi^n} \end{pmatrix}, \text{ and}$$

$$B^n = \begin{pmatrix} T_{1-\phi}^n & 0 & 0 \\ 0 & T_{1-\phi}^n & nT_{1-\phi}^{n-1}T_\phi \\ 0 & 0 & T_{1-\phi}^n \end{pmatrix} = \begin{pmatrix} T_{(1-\phi)^n} & 0 & 0 \\ 0 & T_{(1-\phi)^n} & nT_{(1-\phi)^{n-1}}T_\phi \\ 0 & 0 & T_{(1-\phi)^n} \end{pmatrix}.$$

Now,

$$A^n + B^n = \begin{pmatrix} T_{\phi^n} + T_{(1-\phi)^n} & 0 & 0 \\ nT_{\phi^{n-1}}T_{1-\phi} & T_{\phi^n} + T_{(1-\phi)^n} & nT_{(1-\phi)^{n-1}}T_\phi \\ 0 & 0 & T_\phi^n + T_{(1-\phi)^n} \end{pmatrix}.$$

Since $\phi^{n-1}(1-\phi) = 0$, then $T_{\phi^{n-1}(1-\phi)} = 0$. Thus $T_{\phi^{n-1}}T_{1-\phi} = 0$. Similarly $(1-\phi)^{n-1}\phi = 0$ implies $T_{(1-\phi)^{n-1}}T_\phi = 0$. As $\phi^n + (1-\phi)^n = \psi^n$, we get

$$A^n + B^n = \begin{pmatrix} T_{\psi^n} & 0 & 0 \\ 0 & T_{\psi^n} & 0 \\ 0 & 0 & T_{\psi^n} \end{pmatrix} = \begin{pmatrix} T_\psi & 0 & 0 \\ 0 & T_\psi & 0 \\ 0 & 0 & T_\psi \end{pmatrix}^n = C^n.$$

□

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