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PERTURBED FORM FOR SOME NONLINEAR PARABOLIC SYSTEMS

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ABSTRACT. This paper addresses a quasilinear parabolic system in divergence form given by $\frac{\partial u}{\partial t} - \operatorname{div} \sigma(x,t,Du - \Upsilon(u)) = f$, which features a gradient perturbation. The source term f belongs to $L^{p'}\big(0,T;W^{-1,p'}(\Omega;\mathbb{R}^m)\big)$. The existence of weak solutions $u : Q_T \to \mathbb{R}^m$, where $m \in \mathbb{N}^*$ and $Q_T = \Omega \times (0,T))$, is established through the use of Young measures and Galerkin approximations.

1. Introduction

In this paper, we focus on a class of nonlinear parabolic systems with a gradient perturbation, from which we obtain weak solutions $u : Q_T \to \mathbb{R}^m$ $(m \in \mathbb{N}^*)$.

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div} \sigma(x,t,Du - \Upsilon(u)) = & f \quad \text{in } Q_T, \\ u(x,t) = & 0 \quad \text{on } \partial Q_T, \\ u(x,0) = & u_0(x) \quad \text{in } \Omega, \end{cases} \tag{1.1}$$

with $\Omega \subset \mathbb{R}^n$ assumed to be open and bounded. For given $T > 0$, let $Q_T = \Omega \times (0,T)$ whose boundary is denoted by $\partial Q_T = \partial\Omega \times (0,T)$. The data $f \in L^{p'}\big(0,T;W^{-1,p'}(\Omega;\mathbb{R}^m)\big)$, with $p \in (\frac{2n}{n+2}, \infty)$ and p' its conjugate $(p' = \frac{p}{p-1})$. The functionals σ and Υ satisfy certain conditions, which we set below.

The field of application of the nonlinear partial differential equations that address problem (1.1) is diverse, and this problem models several natural phenomena (for example, in the fields of turbulent fluid flows, electrochemical problems, oceanography, and induction).

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An example of a model, as cited in [10], is the flow of a fluid in a porous medium governed by the following equation:

$$\frac{\partial\theta}{\partial t} - \operatorname{div}\Big(|\nabla\varphi(\theta) - K(\theta)e|^{p-2}\big(\nabla\varphi(\theta) - K(\theta)e\big)\Big) = 0,$$

where θ, $\varphi(\theta)$, $K(\theta)$, and e are the volumetric water content, the hydrostatic potential, the hydraulic conductivity, and the unit vector in the vertical direction, respectively.
Norbert Hungerbühler [15] treated the following system:

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}\sigma(x,t,u(x,t),Du(x,t)) = & f \quad \text{on } \Omega\times(0,T), \\ u(x,t) = & 0 \quad \text{on } \partial\Omega\times(0,T), \\ u(x,0) = & u_0(x) \quad \text{on } \Omega, \end{cases} \tag{1.2}$$

where $f \in L^{p'}\big(0,T;W^{-1,p'}(\Omega;\mathbb{R}^m)\big)$ $\big(p \in (\frac{2n}{n+2},\infty)$ and $u_0 \in L^2(\Omega;\mathbb{R}^m)\big)$.

The author used very mild monotonicity assumptions to establish the existence of a weak solution while upholding the classical conditions of regularity, growth, and coercivity for σ and using Young measures.

Azroul and Balaadich [3] addressed the same problem as Hungerbühler (cited in [15]) but in a more generalized space, namely the Orlicz–Sobolev space, using a similar approach and under comparable conditions.

In the following quasilinear elliptic problem, Azroul and Balaadich [2] introduced a gradient perturbation

$$\begin{cases} -\operatorname{div} A\big(x, Du - \Theta(u)\big) = & f \quad \text{in } \Omega, \\ u = & 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where the solution is showed within the framework of Sobolev spaces. The proof is based on the application of Young's measures and is confined to conditions of weak monotonicity.

When $\Upsilon = 0$, problem (1.1) becomes a special case of problem (1.2), which, prior to Hungerbühler's work in (1.2), was treated using classical Leray–Lions methods. These classical techniques emphasized the need for a specific type of monotonicity, such as strict monotonicity, to guarantee weak solutions, as indicated by works such as [5, 17, 18, 19], for example.

However, the added value of (1.2) lies in the fact that it weakens the monotonicity assumption established in previous studies by introducing an effective concept, namely quasimonotonicity — an integrated form of monotonicity — offering greater flexibility concerning compactness (see [8]). This approach uses Young measures and Galerkin approximations. Young's measures provide an effective solution to the challenges associated with weak convergence, overcoming the difficulties encountered when manipulating nonlinear functions and operators in contemporary nonlinear analysis.

Our aim is to extend problem (1.3) to the parabolic setting using Young measures with the support of Galerkin approximations.

The paper is structured into five sections. Section two examines the spaces used, exploring their main characteristics and reviewing the results and properties associated with Young's measures. In Section three, we outline the conditions and principal outcomes regarding the weak solutions of our problem. Section four focuses on Galerkin approximations and some auxiliary lemmas that provide crucial results concerning the Young measure generated by gradient sequences $(Du_j)_j$. Lastly, Section five is dedicated to proving the main theorem, Theorem 3.2.

2. Preliminary results and notations

In the first subsection, we recall some useful properties of the functional spaces used throughout this work. In the second subsection, we briefly review some results related to Young measures, highlighting their properties that will be useful in what follows.

For more details, the reader may consult the references [1, 7, 21, 4, 11, 13, 14].

2.1. **Functional spaces.** Let $(n \geq 2)$, let Ω be an open subset of $\mathbb{R}^n$, let $1 \leq p \leq +\infty$, and let $m \geq 1$.

For $a \in \mathbb{R}^m$, $|a|$ denotes its Euclidean norm, and for $A \in \mathbb{M}^{m\times n}$, $|A|$ denotes its norm when regarded as a vector in $\mathbb{R}^{mn}$ (where $\mathbb{M}^{m\times n}$ denotes the space of $m \times n$ matrices endowed with the reduced topology of $\mathbb{R}^{mn}$).

If $u = (u^1, \ldots, u^m)^t : \Omega \to \mathbb{R}^m$, then the Jacobian matrix Du is an $m \times n$ matrix defined by $Du_{lk} = \frac{\partial u^l}{\partial x_k}$.

For $m \geq 1$, the Lebesgue space $L^p(\Omega; \mathbb{R}^m)$ is defined as

$$L^p(\Omega; \mathbb{R}^m) = \Big\{\ u = (u^1, \ldots, u^m) :\ u^l \in L^p(\Omega) \text{ for all } 1 \leq l \leq m\Big\},$$

equipped with the norm

$$\|u\|_p = \left(\sum_{l=1}^{m} \int_\Omega |u^l(x)|^p dx\right)^{\frac{1}{p}}.$$

The Sobolev space $W^{1,p}(\Omega; \mathbb{R}^m)$ is defined as

$$W^{1,p}(\Omega; \mathbb{R}^m) = \Big\{u \in L^p(\Omega; \mathbb{R}^m) :\ Du_{lk} \in L^p(\Omega) \text{ for all } 1 \leq l \leq m,\ 1 \leq k \leq n\Big\},$$

endowed with the norm

$$\|u\|_{1,p} = \Big(\|u\|_p^p + \|Du\|_p^p\Big)^{\frac{1}{p}},$$

with

$$\|Du\|_p = \Big(\sum_{l=1}^{m}\sum_{k=1}^{n} \int_\Omega |Du_{lk}(x)|^p dx\Big)^{\frac{1}{p}}.$$

The Banach space $W_0^{1,p}(\Omega; \mathbb{R}^m)$ is the closure of $C_0^1(\Omega; \mathbb{R}^m)$ (or $C_0^\infty(\Omega; \mathbb{R}^m)$) in $W^{1,p}(\Omega; \mathbb{R}^m)$ with respect to the norm $\|u\|_{1,p}$. Moreover, the dual space of $W_0^{1,p}(\Omega; \mathbb{R}^m)$ can be identified with $W^{-1,p'}(\Omega; \mathbb{R}^m)$.

Given the Banach spaces $\mathbf{A}$ and $\mathbf{B}$, where $\mathbf{A}'$ denotes the dual space of $\mathbf{A}$ and $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $\mathbf{A}$ and $\mathbf{A}'$. For a given $0 < T < \infty$ and $1 \leq p \leq \infty$, the space $L^p(0, T; \mathbf{A})$ is defined as the set of measurable functions

$u : [0,T] \to \mathbf{A}$, identified modulo the equivalence relation of almost everywhere equality, satisfying

$$\int_0^T |u(t)|^p_{\mathbf{A}} dt < +\infty.$$

This space is equipped with the norm:

$$\|u\|_{L^p(0,T;\mathbf{A})} := \left(\int_0^T \|u(t)\|^p_{\mathbf{A}} dt \right)^{\frac{1}{p}}.$$

In the case where $p = \infty$, the norm is defined as

$$\|u\|_{L^\infty(0,T;\mathbf{A})} := \sup_{0<t<T} ess\ \|u\|_{\mathbf{A}} < +\infty.$$

When $p \leq q \leq \infty$, the following Hölder's inequality holds for all $u \in L^p(0,T;\mathbf{A})$ and $v \in L^{p'}(0,T;\mathbf{A}')$:

$$\int_0^T |\langle v, u\rangle| dt \leq \|u\|_{L^p(0,T;\mathbf{A})}\ \|v\|_{L^{p'}(0,T;\mathbf{A}')}.$$

If $\mathbf{A} \hookrightarrow \mathbf{B}$, then $L^q(0,T;\mathbf{A}) \hookrightarrow L^p(0,T;\mathbf{B})$, and for every $j \in \mathbb{N}$, $1 \leq p \leq \infty$, the space $C^j([0,T], \mathbf{A})$ is dense in $L^p(0,T;\mathbf{A})$ with a continuous embedding.

When the Banach space $\mathbf{A}$ is separable and $1 \leq p < \infty$, $L^p(0,T;\mathbf{A})$ is also separable. Moreover, if the Banach space $\mathbf{A}$ is reflexive and $1 < p < \infty$, then $L^p(0,T;\mathbf{A})$ becomes reflexive.

2.2. **Brief reminder on Young measures.** In this subsection, we delve into the fundamentals of Young measures, elucidating their properties and established results.

Let $C_0(\mathbb{R}^m)$ denote the set of real-valued functions φ in $C(\mathbb{R}^m)$ with compact support. This space is a Banach space when equipped with the L^∞-norm. Its dual space is identified with $\mathcal{M}(\mathbb{R}^m)$, which consists of signed Radon measures with finite mass.

For all $\varphi \in C_0(\mathbb{R}^m)$ and $\nu : \Omega \to \mathbb{R}$,

$$\langle \nu, \varphi\rangle = \int_{\mathbb{R}^m} \varphi(\xi) d\nu(\xi), \text{ denotes the duality pairing between these spaces.}$$

The support of $\nu \in \mathcal{M}(\mathbb{R}^m)$ is defined as

$$\operatorname{supp} \nu = \Big\{ \xi \in \mathbb{R}^m :\ \nu\big(B(\xi, r)\big) > 0 \text{ for any } r > 0 \Big\},$$

where $B(\xi, r)$ denotes the ball centered at ξ with radius $r > 0$.

The following theorem presents the fundamental result concerning Young measures.

Theorem 2.1 (Young, Tartar, Ball [8])**.** *Let $\Omega \subset \mathbb{R}^n$ be Lebesgue measurable (not necessarily bounded) and let $z_j : \Omega \to \mathbb{R}^m$, $j = 1, 2, \ldots$, be a sequence of Lebesgue measurable functions. Then there exist a subsequence z_k and a family $\{\nu_x\}_{x\in\Omega}$ of nonnegative Radon measures on $\mathbb{R}^m$, such that*

(i) $\|\nu_x\| := \int d\nu_x(\xi) \leq 1$ *for almost every $x \in \Omega$;*
(ii) $\varphi(z_k) \rightharpoonup^* \overline{\varphi}$ *weakly* in $L^\infty(\Omega)$ for all $\varphi \in C^0_0(\mathbb{R}^n)$, where $\overline{\varphi}(x) = \langle \nu_x, \varphi\rangle$;*

(iii) *If for all $R > 0$*

$$\lim_{L\to\infty} \sup_{k\in\mathbb{N}} \left|\left\{x \in \Omega \cap B(0,R) : \ |z_k(x)| \geq L\right\}\right| = 0, \tag{2.1}$$

then

$$\|\nu_x\| = 1 \quad \textit{for almost every } x \in \Omega, \tag{2.2}$$

and for all measurable $A \subset \Omega$, there holds

$$\begin{cases} \varphi(z_k) \rightharpoonup \overline{\varphi} = \langle \nu_x, \varphi\rangle \text{ weakly in } L^1(A) \text{ for a continuous function} \\ \varphi : \mathbb{R}^m \to \mathbb{R} \text{ provided the sequence } \varphi(z_k) \text{ is weakly precompact in } L^1(A). \end{cases} \tag{2.3}$$

Here, $C_0^0(\mathbb{R}^m) = C_0(\mathbb{R}^m)$, and $\Big|\cdot\Big|$ denotes the Lebesgue measure restricted to Ω.

Remark 2.2. The proof in [14, Theorem 1.2] establishes that the validity of (2.2) and (2.3) requires the condition (2.1). Consequently, it establishes the equivalence of (2.1), (2.2), and (2.3).

Lemma 2.3 ([11])**.** *Assume that the sequence $(z_j)_j$ is bounded in $L^\infty(\Omega;\mathbb{R}^m)$. Then, there exists a subsequence $(z_k)_k$ of $(z_j)_j$ and for a.e.$x \in \Omega$ a Borel probability measure ν_x on $\mathbb{R}^m$ such that for each $\varphi \in C(\mathbb{R}^m)$ we have*

$$\varphi(z_k) \overset{*}{\rightharpoonup} \overline{\varphi} \quad \textit{weakly in } L^\infty(\Omega), \quad \textit{where} \quad \overline{\varphi}(x) = \int_{\mathbb{R}^m} \varphi(\xi)d\nu_x(\xi) \quad (a.e.\ x \in \Omega).$$

Definition 2.4 ([11])**.** $(\nu_x)_{x\in\Omega}$ is called the family of Young measures associated with the subsequence $(z_k)_k$ or equivalently generated by the subsequence $(z_k)_k$.

In Ball's paper [4], it is proven that, assuming (2.1), the convergence $\varphi(\cdot, z_k) \rightharpoonup \langle \nu_x, \varphi(x,\cdot)\rangle = \int_{\mathbb{R}^m} \varphi(x,\xi)d\nu_x(\xi)$ holds in $L^1(\Omega')$ for any measurable subset Ω' of Ω. This convergence applies to every Carathéodory function $\varphi : \Omega' \times \mathbb{R}^m \to \mathbb{R}$, under the condition that the sequence $\varphi(\cdot, z_k)$ is sequentially weakly relatively compact in $L^1(\Omega')$. This result is thus equivalent to (2.1), (2.2), and (2.3).

Ball also showed that if $\varphi \in L^1(\Omega; C_0(\mathbb{R}^m))$, then

$$\lim_{k\to\infty} \int_\Omega \varphi(x, z_k(x))dx = \int_\Omega \langle \nu_x, \varphi(x,\cdot)\rangle dx,$$

whenever z_k generates the Young measure ν_x.

The significance of Theorem 2.1 lies in its crucial applications, serving as a foundational tool throughout the subsequent content. In this paper, several results are derived from these applications, with a few of them highlighted below.

Proposition 2.5 ([13])**.** *Suppose $|\Omega| < \infty$.*
(i) If ν_x is the Young measure generated by the (whole) sequence u_j, then the following equivalence holds:

$$u_j \to u \quad \textit{in measure} \quad \textit{if and only if} \quad \nu_x = \delta_{u(x)} \textit{ for almost every } x \in \Omega.$$

(ii) If the sequences $u_j : \Omega \to \mathbb{R}^m$ and $v_j : \Omega \to \mathbb{R}^d$ generate the Young measure $\delta_{u(x)}$ and ν_x respectively, then (u_j, v_j) generates the Young measure $\delta_{u(x)} \otimes \nu_x$.

Lemma 2.6 (Fatou lemma [8]). *Let $F : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{R}$ be a Carathéodory function and let $u_j : \Omega \to \mathbb{R}^m$ be a sequence of measurable functions such that $u_j \to u$ in measure and such that Du_j generates the Young measure ν_x. Then*

$$\liminf_{j\to\infty} \int_\Omega F(x, u_j(x), Du_j(x))dx \geq \int_\Omega \int_{\mathbb{M}^{m\times n}} F(x, u, \xi)d\nu_x(\xi)dx$$

provided that the negative part $F^-(x, u_j(x), Du_j(x))$ is equi-integrable.

3. Main result

The functions Υ and σ satisfy the conditions outlined below:
(C1) The function Υ is continuous, and there exists a constant $C_\Upsilon \geq 0$ depending on the diameter of Ω (denoted as $diam(\Omega)$) and the exponent p, such that

$$C_\Upsilon < \frac{1}{diam(\Omega)}\Big(\frac{1}{2}\Big)^{\frac{1}{p}},$$

satisfying the following for any $u, v \in \mathbb{R}^m$:

$$|\Upsilon(u) - \Upsilon(v)| \leq C_\Upsilon|u - v| \quad \text{and} \quad \Upsilon(0) = 0.$$

(C2) (Continuity) $\sigma : Q_T \times \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$ is a Carathéodory function. That is, $(x, t) \mapsto \sigma(x, t, \xi)$ is measurable for every $\xi \in \mathbb{M}^{m\times n}$ and $\xi \mapsto \sigma(x, t, \xi)$ is continuous for almost every $(x, t) \in Q_T$.
(C3) (Growth and coercivity) There exist $a_1 \in L^{p'}(Q_T)$, $b \in L^1(Q_T)$, $c_1 \geq 0$, and $c_2 > 0$ such that the following inequalities hold:

$$|\sigma(x, t, \xi - \Upsilon(\eta))| \leq a_1(x, t) + c_1|\xi - \Upsilon(\eta)|^{p-1},$$

$$\sigma(x, t, \xi - \Upsilon(\eta)) : \xi \geq c_2|\xi - \Upsilon(\eta)|^p - b(x, t),$$

where ' : ' denotes the inner product in $\mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$.
(C4) (Monotonicity) One of the following conditions is satisfied by σ:

(a) for any $((x, t), \eta) \in Q_T \times \mathbb{R}^m$, the map $\xi \mapsto \sigma(x, t, \xi - \Upsilon(\eta))$ is a C^1 function and is monotone. Here monotonicity means that

$$\big(\sigma(x, t, \xi - \Upsilon(\eta)) - \sigma(x, t, \xi' - \Upsilon(\eta))\big) : (\xi - \xi') \geq 0 \text{ for any } \xi, \xi' \in \mathbb{M}^{m\times n}.$$

(b) There is a function $W : Q_T \times \mathbb{M}^{m\times n} \to \mathbb{R}$ so that for any $((x, t), \eta) \in Q_T \times \mathbb{R}^m$, the following two properties hold:
   (i) $\xi \mapsto W(x, t, \xi - \Upsilon(\eta))$ is $\mathcal{C}^1$ and convex.
   (ii) $\sigma\big(x, t, \xi - \Upsilon(\eta)\big) = \frac{\partial W}{\partial \xi}\big(x, t, \xi - \Upsilon(\eta)\big)$.

(c) The map $\xi \mapsto \sigma(x, t, \xi - \Upsilon(\eta))$ is strictly monotone for any $((x, t), \eta) \in Q_T \times \mathbb{R}^m$. This means that $\xi \mapsto \sigma(x, t, \xi - \Upsilon(\eta))$ is monotone and satisfies the condition:

$$\big(\sigma(x, t, \xi - \Upsilon(\eta)) - \sigma(x, t, \xi' - \Upsilon(\eta))\big) : (\xi - \xi') = 0 \text{ if and only if } \xi = \xi'.$$

(d) σ is strictly p-quasimonotone with respect to the second variable, that is, for all $((x, t), \eta) \in Q_T \times \mathbb{R}^m$

$$\int_{Q_T} \int_{\mathbb{M}^{m\times n}} \big(\sigma(x, t, \xi - \Upsilon(\eta)) - \sigma(x, t, \overline{\xi} - \Upsilon(\eta))\big) : (\xi - \overline{\xi})d\nu(\xi)dxdt > 0.$$

Here $\nu = \{\nu_{(x,t)}\}_{(x,t)\in Q_T}$ represents any family of Young measures that does not restrict to a Dirac measure and is generated by a sequence in $L^p(Q_T)$ for a.e. $(x,t) \in Q_T$ and $\overline{\xi} = \langle \nu_{(x,t)}, id\rangle$.

(e) σ is strictly quasimonotone with respect to the second variable.
That is, there exists $\alpha_0 > 0$ such that for any $(x,t) \in Q_T$ and $\eta, \eta' \in L^p(0,T; W_0^{1,p}(\Omega;\mathbb{R}^m))$, we have

$$\int_{Q_T} \big(\sigma(x,t,D\eta - \Upsilon(\eta)) - \sigma(x,t,D\eta' - \Upsilon(\eta))\big) : (D\eta - D\eta')dxdt \geq \alpha_0 \int_{Q_T} |D\eta - D\eta'|^p dxdt.$$

Definition 3.1. We say that the function

$$u \in L^p(0,T; W_0^{1,p}(\Omega;\mathbb{R}^m)) \cap L^\infty(0,T; L^2(\Omega;\mathbb{R}^m))$$

is a weak solution to the problem (1.1), if the following equality holds for any $\eta \in L^p(0,T; W_0^{1,p}(\Omega;\mathbb{R}^m)) \cap L^\infty(Q;\mathbb{R}^m)$:

$$-\int_{Q_T} u\frac{\partial \eta}{\partial t}dxdt + \int_\Omega u\eta dx\Big|_0^T + \int_{Q_T} \sigma\big(x,t,Du - \Upsilon(u)\big) : D\eta dxdt = \langle f, \eta\rangle,$$

where $\langle\cdot,\cdot\rangle$ represents the duality pairing between $L^p\big(0,T; W_0^{1,p}(\Omega;\mathbb{R}^m)\big)$ and $L^{p'}\big(0,T; W^{-1,p'}(\Omega;\mathbb{R}^m)\big)$.

The theorem presented below stands as the primary outcome of this paper.

Theorem 3.2. *If σ satisfies conditions **(C2)**–**(C4)**, Υ satisfies condition **(C1)**, and, in addition, $u_0 \in L^2(\Omega;\mathbb{R}^m)$ and $f \in L^{p'}(0,T; W^{-1,p'}(\Omega;\mathbb{R}^m))$, then the Dirichlet problem (1.1) admits a weak solution u as defined in Definition 3.1.*

4. Constructing approximate solutions for problem (1.1)

In this section, problem (1.1) will be approximated by a series of problems involving suitable systems of ordinary differential equations in finite-dimensional spaces. The solutions obtained from these problems are referred to as Galerkin approximations. The objective is to demonstrate the possibility of identifying, from the sequence of Galerkin approximations, a subsequence that converges weakly to a solution of the main problem. References [9, 20, 22] provide more information.

4.1. Galerkin approximation.

We select an orthonormal Galerkin basis $\{\psi_i\}_{i\geq 1}$ in the space $L^2(\Omega;\mathbb{R}^m)$ and define $E_j = \operatorname{span}\{\psi_1, \dots, \psi_j\}$. Here, for $i = 1, \dots, j$, each ψ_i belongs to the space $\mathcal{C}_0^\infty(\Omega;\mathbb{R}^m)$, and $\mathcal{C}_0^\infty(\Omega;\mathbb{R}^m) \subset \overline{\bigcup_{j\geq 1} E_j}^{\mathcal{C}^1(\overline{\Omega};\mathbb{R}^m)}$. To construct the approximate solutions for the problem described in (1.1), we apply the following approach

$$u_j(x,t) = \sum_{i=1}^{j} a_{ji}(t)\psi_i(x),$$

where a_{ji} represents real functions defined on $[0,T]$ that are measurable and bounded. Suppose $u_j \in L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m))$. Then u_j satisfies the boundary condition of problem (1.1) by construction. Regarding the initial condition for the same problem, we assign the initial coefficients $aji(0) = (u_0,\psi_i)_{L^2}$ in such a way that

$$u_j(\cdot,0) = \sum_{i=1}^{j} a_{ji}(0)\psi_i(\cdot) \to u_0 \quad \text{in } L^2(\Omega),$$

where $(\cdot,\cdot)_{L^2}$ denotes the inner product in L^2.

The task remaining is to find the coefficients $a_{ji}(t)$ that satisfy the following system of ordinary differential equations in the sense of distributions for all $j \in \mathbb{N}$

$$\big(\frac{\partial u_j}{\partial t},\psi_l\big)_{L^2} + \int_\Omega \sigma(x,t,Du_j - \Upsilon(u_j)) : D\psi_l dx = \langle f(t),\psi_l\rangle, \tag{4.1}$$

where $1 \le l \le j$.

To accomplish this, we begin by fixing $j \in \mathbb{N}$, choosing $0 < \tau < T$, and denoting I as the interval $[0,\tau]$. We select $r > 0$ to be sufficiently large so that, for any $1 \le l \le j$, the condition $a_{jl}(0) \in B(0,r) \subset \mathbb{R}^j$ is satisfied.

Let the function

$$\Phi : I \times \overline{B(0,r)} \to \mathbb{R}^j$$

$$(t,a_1,\dots,a_j) \mapsto \Big(\langle f(t),\psi_l\rangle - \int_\Omega \sigma\big(x,t,\sum_{i=1}^{j} a_i D\psi_i - \Upsilon(\sum_{i=1}^{j} a_i\psi_i)\big) : D\psi_l dx\Big)^t_{l=1,\dots,j},$$

which is a Carathéodory function according to conditions **(C1)** and **(C2)**.

For all $1 \le l \le j$, we can estimate Φ_l uniformly over $I \times \overline{B(0,r)}$ by

$$|\Phi_l(t,a_1,\dots,a_j)| \le c(r,j)h(t). \tag{4.2}$$

Here, $c(r,j)$ is a constant dependent on both r and j, while $h \in L^1(I)$ depends solely on t, independent of r and j.

Indeed, we initially establish that for $s \ge 1 + n(\frac{1}{2} - \frac{1}{p})$, we have $W_0^{s,2}(\Omega) \subset W_0^{1,p}(\Omega)$, and $D\psi_l \in W_0^{s-1,2}(\Omega) \subset L^\infty(\Omega)$ whenever $\psi_l \in W_0^{s,2}(\Omega)$.

Consequently, in accordance with condition **(C1)** and the growth condition in **(C3)**, and for a certain constant c depending on c_Υ, c_1, j and p, one obtains, along with the Hölder inequality

$$\begin{aligned}
|\Phi_l(t,a_1,\dots,a_j)| &\le |\langle f(t),\psi_l\rangle| + \int_\Omega |\sigma\big(x,t,\sum_{i=1}^{j} a_i D\psi_i - \Upsilon(\sum_{i=1}^{j} a_i\psi_i)\big)|\, |D\psi_l| dx \\
&\le \|f(t)\|_{-1,p'}\|\psi_l\|_{1,p} \\
&\quad + \Big(\int_\Omega \big|\sigma\big(x,t,\sum_{i=1}^{j} a_i D\psi_i - \Upsilon(\sum_{i=1}^{j} a_i\psi_i)\big)\big|^{p'} dx\Big)^{\frac{1}{p'}} \Big(\int_\Omega |D\psi_l|^p dx\Big)^{\frac{1}{p}} \\
&\le \|f(t)\|_{-1,p'}\|\psi_l\|_{1,p} + (c\int_\Omega b_1(x,t)dx + c),
\end{aligned}$$

because

$$\begin{aligned}
&\int_\Omega |\sigma\big(x,t,\sum_{i=1}^{j} a_i D\psi_i - \Upsilon(\sum_{i=1}^{j} a_i\psi_i)\big)|^{p'}dx \\
&\le 2^{p'-1}\int_\Omega \Big(|b_1(x,t)|^{p'} \\
&\quad + c_1|\sum_{i=1}^{j} a_i D\psi_i - \Upsilon(\sum_{i=1}^{j} a_i\psi_i)|^{p}\Big)dx \\
&\le 2^{p'-1}\int_\Omega \Big(|b_1(x,t)|^{p'} + 2^{p-1}c_1(|\sum_{i=1}^{j} a_i D\psi_i|^p + C_\Upsilon^p|\sum_{i=1}^{j} a_i\psi_i|^p)\Big)dx.
\end{aligned}$$

Hence, by applying the Carathéodory's existence result for ordinary differential equations, as specified in [16], to the system

$$\begin{cases} a_l'(t) = \Phi_l(t, a_1(t), \ldots, a_j(t)), \\ a_l(0) = a_{jl}(0), \end{cases} \tag{4.3}$$

for each $l \in \{1, \ldots, j\}$, it guarantees the existence of a distributional, continuous solution a_l for the above system, which depends on j and is defined over a time interval $[0, \tau')$ (with $\tau' > 0$ depending *a priori* on j).

Additionally, the corresponding integral equation

$$a_l(t) = a_l(0) + \int_0^t \Phi_l(t, a_1(z), \ldots, a_j(z))dz \tag{4.4}$$

holds on the interval $[0, \tau')$.

Subsequently, $u_j(x,t) = \sum_{l=1}^{j} a_l(t)\psi_l(x)$ effectively represents the intended solution to the system of ordinary differential equations in (4.1), ensuring that the initial condition satisfies $u_j(\cdot, 0) = \sum_{l=1}^{j} a_l(0)\psi_l(\cdot) \to u_0$ in $L^2(\Omega)$ as $j \to \infty$.

Until now, our goal has been to extend the previously constructed local solution to encompass the entire interval $[0, T)$, independently of j. We will accomplish this by utilizing the lemma that follows.

Lemma 4.1.

If $\Gamma = \Big\{\tau \in [0, T) : \textit{there exists a weak solution of } (4.3) \textit{ on } [0, \tau)\Big\}$, *then* Γ *is nonempty and both an open and a closed set.*

Proof. The set Γ is not empty, given that we already have a local solution.

Below, we show that Γ is both open and closed. To establish that Γ is open, consider $\tau \in \Gamma$ and $t_1, t_2 \in]0, \tau]$ with $t_1 < t_2$.

Consequently, by using (4.4) and (4.2), we get

$$\begin{aligned}|a_{jl}(t_2)-a_{jl}(t_1)| &= \Big|\int_{t_1}^{t_2}\Phi_l\big(t,a_{j1}(t),\ldots,a_{jj}(t)\big)dt\Big|\\ &\le \int_{t_1}^{t_2}|\Phi_l\big(t,a_{j1}(t),\ldots,a_{jj}(t)\big)|dt\\ &\le c(r,j)\int_{t_1}^{t_2}|h(t)|dt.\end{aligned}$$

Given $h\in L^1(0,T)$, we deduce that $a_{jl}(\cdot)$ is uniformly continuous.

At time τ, we take $u_j(\tau)=\lim_{t\nearrow\tau}u_j(t)$ as the initial data, and we begin solving equation (4.1) again. Thus, there exists a neighborhood $[0,\tau+\epsilon)$ on which we obtain a solution of (4.3).

We still need to show that Γ is closed. Consider a sequence $(t_i)_i$ consisting of elements from Γ such that $t_i\nearrow\tau$. We denote by $a_{jl,i}$ the solution of (4.3) that we have constructed on $[0,t_i]$, and we define the function

$$\widetilde{a}_{jl,i}(t):=\begin{cases} a_{jl,i}(t) & \text{if } 0\le t\le t_i,\\ a_{jl,i}(t_i) & \text{if } t_i<t<\tau.\end{cases}$$

The sequence $(a_{jl,i})_i$ being both bounded and equicontinuous on $[0;\tau)$, implies, according to the Arzelà–Ascoli theorem, the existence of a subsequence, also denoted as $(\widetilde{a}_{jl,i})_i(t)$, which uniformly converges in t on $[0,\tau)$ to $(a_{jl})(t)$.

Consequently, through the application of the Lebesgue convergence theorem to (4.4), we establish that $(a_{jl})(t)$ satisfies (4.3) on $[0,\tau)$.

Therefore, $\tau\in\Gamma$. In other words, Γ is closed. Thus, $\Gamma=[0,T)$. □

Lemma 4.2. *The sequence $(u_j)_j$ constructed by Galerkin approximations is bounded in*
$L^p\big(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m)\big)\cap L^\infty\big(0,T;L^2(\Omega;\mathbb{R}^m)\big)$.

Proof. Testing equation (4.1) by u_j in place of ψ_l, since it is linear, we obtain

$$\Big(\frac{\partial u_j}{\partial t},u_j\Big)_{L^2}+\int_\Omega\sigma\big(x,t,Du_j-\Upsilon(u_j)\big):Du_jdx=\langle f(t),u_j\rangle.$$

For any arbitrary $\tau\in[0,T)$, we integrate this equality to get

$$\int_{Q_\tau}\frac{\partial u_j}{\partial t}u_jdxdt+\int_{Q_\tau}\sigma\big(x,t,Du_j-\Upsilon(u_j)\big):Du_jdxdt=\int_0^\tau\langle f(t),u_j\rangle dt. \tag{4.5}$$

The first term gives

$$\int_{Q_\tau}\frac{\partial u_j}{\partial t}u_jdxdt=\frac{1}{2}\|u_j(\cdot,\tau)\|^2_{L^2(\Omega)}-\frac{1}{2}\|u_j(\cdot,0)\|^2_{L^2(\Omega)}. \tag{4.6}$$

Due to the coercivity condition outlined in **(C3)**, we obtain

$$\begin{aligned}\int_{Q_\tau}\sigma\big(x,t,Du_j-\Upsilon(u_j)\big):Du_jdxdt \ge c_2\int_{Q_\tau}&|Du_j-\Upsilon(u_j)|^pdxdt\\ &-\int_{Q_\tau}b(x,t)dxdt.\end{aligned} \tag{4.7}$$

The right-hand term is bounded by Hölder's inequality, hence

$$\Big|\int_0^\tau \langle f(t), u_j\rangle dt\Big| \le \|f\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))}\|u_j\|_{L^p(0,T;W_0^{1,p}(\Omega))}. \tag{4.8}$$

The estimates in (4.6), (4.7), and (4.8) allow us to deduce

$$\frac{1}{2}\|u_j(\cdot,\tau)\|^2_{L^2(\Omega)} \le \frac{1}{2}\|u_j(\cdot,0)\|^2_{L^2(\Omega)} + \int_{Q_\tau} b(x,t)dxdt + \int_0^\tau \langle f(t), u_j\rangle dt.$$

Hence,

$$\|u_j(\cdot,\tau)\|^2_{L^2(\Omega)} = \big|\big(a_{ji}(\tau)\big)_{i=1,\dots,j}\big|^2_{\mathbb{R}^j} \le \widetilde{c},$$

where $\widetilde{c}$ represents a constant that is independent of both τ and j. Thus, thanks to Lemma 4.1, $(u_j)_j$ is bounded in $L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m))\cap L^\infty(0,T;L^2(\Omega;\mathbb{R}^m))$. □

The two spaces $L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m))$ and $L^\infty(0,T;L^2(\Omega;\mathbb{R}^m))$ are reflexive. Therefore, according to Lemma 4.2 and the Eberlein-Shmulyan theorem (see, e.g, [23]), we can extract a suitably chosen subsequence, still denoted by $(u_j)_j$, such that there exists an element $u \in L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m))\cap L^\infty(0,T;L^2(\Omega;\mathbb{R}^m))$ that satisfies

$$u_j \rightharpoonup u \text{ in } L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m)), \tag{4.9}$$

$$u_j \rightharpoonup^* u \text{ in } L^\infty(0,T;L^2(\Omega;\mathbb{R}^m)). \tag{4.10}$$

Below, we will show that u is the weak solution expected for the problem (1.1).

Using (4.9), (4.10), and the fact that conditions **(C1)** and **(C3)** are satisfied, we can extract an appropriately chosen subsequence, still denoted as $(u_j)_j$, such that

$$-\operatorname{div}\sigma(x,t,Du_j-\Upsilon(u_j)) \rightharpoonup \kappa \text{ in } L^{p'}(0,T;W^{-1,p'}(\Omega;\mathbb{R}^m)), \tag{4.11}$$

and

$$u_j(\cdot,T) \rightharpoonup \widetilde{u} \text{ in } L^2(\Omega;\mathbb{R}^m) \text{ as } j\to\infty,$$

with $\kappa \in L^{p'}(0,T;W^{-1,p'}(\Omega;\mathbb{R}^m))$ and $\widetilde{u}\in L^2(\Omega;\mathbb{R}^m)$.

We still need to show that $\widetilde{u} = u(\cdot,T)$ and $u(\cdot,0)=u_0(\cdot)$.

For $l\le j$, let us take any element φ from $\mathcal{C}^\infty([0,T])$ and v from E_l. Then

$$\int_{Q_T}\frac{\partial u_j}{\partial t}v\varphi dxdt + \int_{Q_T}\sigma\big(x,t,Du_j-\Upsilon(u_j)\big) : Dv\varphi dxdt = \int_{Q_T} f.v\varphi dxdt.$$

We integrate the first term by parts; hence,

$$\int_\Omega u_j\varphi v dx\Big|_0^T - \int_{Q_T} u_j v\varphi' dxdt = \int_{Q_T} f.v\varphi dxdt - \int_{Q_T}\sigma\big(x,t,Du_j-\Upsilon(u_j) : Dv\varphi dxdt.$$

Consequently,

$$\int_\Omega u_j(x,T)\varphi(T)vdx - \int_\Omega u_j(x,0)\varphi(0)vdx - \int_{Q_T} u_jv\varphi' dxdt = \int_{Q_T} f.v\varphi dxdt - \int_{Q_T}\sigma\big(x,t,Du_j-\Upsilon(u_j) : Dv\varphi dxdt.$$

Subsequently, we pass to the limit to obtain

$$\int_\Omega \widetilde{u}\varphi(T)vdx - \int_\Omega u_0\varphi(0)vdx - \int_{Q_T} uv\varphi' dxdt = \int_{Q_T} f.v\varphi dxdt - \int_{Q_T} \kappa.v\varphi dxdt. \tag{4.12}$$

Choose φ so that $\varphi(0) = \varphi(T) = 0$. Then

$$\int_{Q_T} f.v\varphi dxdt - \int_{Q_T} \kappa.v\varphi dxdt = -\int_{Q_T} uv\varphi' dxdt = \int_{Q_T} u'v\varphi dxdt.$$

Replacing in (4.12) gives

$$\int_\Omega \widetilde{u}\varphi(T)vdx - \int_\Omega u_0\varphi(0)vdx = \int_{Q_T} uv\varphi' dxdt + \int_{Q_T} u'v\varphi dxdt = \int_\Omega u\varphi v dx\Big|_0^T.$$

For a certain φ, where $\varphi(T) = 0$ and $\varphi(0) = 1$, we have $u(\cdot, 0) = u_0$. Similarly, if $\varphi(0) = 0$ and $\varphi(T) = 1$, then $u(\cdot, T) = \widetilde{u}$.

Following this, we show that $\kappa = -\sigma\big(x, t, Du - \Upsilon(u)\big)$.

4.2. **Some auxiliary lemmas.** The following lemmas provide crucial results concerning the Young measure generated by gradient sequences $(Du_j)_j$.

Lemma 4.3. *Assume that the sequence $(u_j)_j$ is bounded in $L^p(0, T; L^p(\Omega))$, then the following assertions are verified*

1. *There is a Young measure $\nu_{(x,t)}$ is associated with the sequence (possibly a subsequence) of gradients (Du_j), and it satisfies $\|\nu_{(x,t)}\| = 1$.*
2. *The weak L^1-limit of a certain subsequence of $(Du_j)_j$ is denoted as*
$$\langle \nu_{(x,t)}, id\rangle = \int_{\mathbb{M}^{m\times n}} \xi d\nu_{(x,t)}(\xi).$$
3. *For almost every $(x, t) \in Q_T$, $\langle \nu_{(x,t)}, id\rangle = Du(x, t)$.*

Proof. **(1)** By using the inequality

$$L^p\big|\{(x,t) :\ |Du_j(x,t)| \geq L\}\big| \leq \int_{\{(x,t):\ |Du_j(x,t)|\geq L\}} |Du_j|^p dxdt \leq \int_{Q_T} |Du_j|^p dxdt,$$

it follows from the fact that $(Du_j)_j$ is bounded in $L^p(0, T; L^p(\Omega))$ that there exists $c \geq 0$ such that

$$L^p \sup_{j\in\mathbb{N}} \big|\{(x,t) :\ |Du_j(x,t)| \geq L\}\big| \leq \int_{Q_T} |Du_j|^p dxdt \leq c.$$

Consequently,

$$\sup_{j\in\mathbb{N}} \big|\{(x,t) :\ |Du_j(x,t)| \geq L\}\big| \leq \frac{c}{L^p} \to 0 \quad \text{as } L \to \infty.$$

According to Theorem 2.1, there exists a Young measure $\nu_{(x,t)}$ generated by the sequence of gradients $(Du_j)_j$, and according to **(iii)** of the same theorem, we will have $\|\nu_{(x,t)}\| = 1$.

(2) Given that $(Du_j)_j$ is uniformly bounded in $L^p(0, T; L^p(\Omega))$, where $p > 1$, and this space is reflexive, there exists a subsequence (again denoted by $(Du_j)_j$) that weakly converges in $L^p\big(0, T; L^p(\Omega)\big)$.

As a consequence, $(Du_j)_j$ also converges weakly in $L^1\big(0,T;L^1(\Omega)\big)$. To obtain the weak L^1 limit of Du_j, we take $\varphi := \mathrm{id}$ in Theorem 2.1(iii), resulting in:

$$\langle \nu_{(x,t)}, id\rangle = \int_{\mathbb{M}^{m\times n}} \xi d\nu_{(x,t)}(\xi).$$

(3) In (4.9), we established the convergence

$$u_j \rightharpoonup u \text{ in } L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m)).$$

Following this, for a subsequence of $(u_j)_j$, once again denoted as $(u_j)_j$, the Rellich–Kondrachov theorem [1] guarantees the convergence $u_j \to u$ in $L^p(0,T;L^p(\Omega))$ and $Du_j \rightharpoonup Du$ in $L^p(0,T;L^p(\Omega))$.

The embedding $L^p(0,T;L^p(\Omega)) \subset L^1(0,T;L^1(\Omega))$ allows us to conclude that $Du_j \rightharpoonup Du$ in $L^1(0,T;L^1(\Omega))$.

The uniqueness of the limit implies, by **(2)**, that $\langle \nu_{(x,t)}, id\rangle = Du(x,t)$ for a.e. $(x,t) \in Q_T$. □

The subsequent lemma is considered a pivotal element in the passage to the limit within approximate equations.

Lemma 4.4. *Suppose that $\nu_{(x,t)}$ corresponds to Young measure associated with the sequence (or a subsequence) of gradients $(Du_j)_j$ of the Galerkin approximations $(u_j)_j$. Then*

$$\int_{Q_T}\int_{\mathbb{M}^{m\times n}} \sigma\big(x,t,\xi-\Upsilon(u)\big) : \xi \, d\nu_x(\xi)dxdt \\ \leq \int_{Q_T}\int_{\mathbb{M}^{m\times n}} \sigma\big(x,t,\xi-\Upsilon(u)\big) : Du \, d\nu_x(\xi)dxdt.$$

Proof. Let $\sigma_j(x,t) = \sigma(x,t,Du_j-\Upsilon(u_j)) : Du_j - \sigma(x,t,Du_j-\Upsilon(u_j)) : Du$.

Both $\Big(\sigma(x,t,Du_j-\Upsilon(u_j)) : Du_j\Big)^-$ and $\Big(\sigma(x,t,Du_j-\Upsilon(u_j)) : Du\Big)^-$ are equi-integrable. Indeed, when we consider the condition specified in **(C1)** along with the coercivity condition in **(C3)**, for a measurable subset Q'_T of Q_T, we obtain

$$\sigma(x,t,Du_j-\Upsilon(u_j)) : Du_j \geq c_2|Du_j-\Upsilon(u_j)|^p - b(x,t) \geq -b(x,t).$$

Whence,

$$\int_{Q'_T} \big|\min\big(\sigma(x,t,Du_j-\Upsilon(u_j)) : Du_j, 0\big)\big| dxdt \leq \int_{Q'_T} |b(x,t)|dxdt < \infty.$$

Hence, $\big(\sigma(x,t,Du_j-\Upsilon(u_j)) : Du_j\big)^-$ is equi-integrable.

By applying the Hölder's inequality, we obtain

$$\int_{Q'_T} \big|\big(\sigma(x,t,Du_j-\Upsilon(u_j)) : Du\big)\big| dxdt \\ \leq \left(\int_{Q'_T} |\sigma\big(x,t,Du_j-\Upsilon(u_j)\big)|^{p'} dxdt\right)^{\frac{1}{p'}} \left(\int_{Q'_T} |Du|^p dxdt\right)^{\frac{1}{p}}.$$

Due to the growth condition specified in **(C3)** and the boundedness of the sequence $(u_j)_j$ in $L^p\big(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m)\big)$, the first integral in the last inequality is uniformly bounded independently of j. Additionally, the second integral will be arbitrarily small if we choose Q'_T of sufficiently small measure.

Therefore, given that $\big(\sigma\big(x,t,Du_j-\Upsilon(u_j)\big):Du\big)^-$ and $\big(\sigma_j(x,t)\big)^-$ are equi-integrable, and considering the boundedness of $(u_j)_j$ in $L^p\big(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m)\big)$, we can conclude that $u_j \rightharpoonup u$ in $L^p\big(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m)\big)$ and $u_j \to u$ in measure on Q_T for a subsequence, also denoted as $(u_j)_j$.

Hence, using Lemma 2.6, we obtain

$$\liminf_{j\to\infty}\int_{Q_T}\sigma_j(x,t)dxdt \geq \int_{Q_T}\int_{\mathbb{M}^{m\times n}}\big(\sigma(x,t,\xi-\Upsilon(u)\big):(\xi-Du)d\nu_x(\xi)dxdt. \tag{4.13}$$

There is still a need to show that $\liminf\limits_{j\to\infty}\int_{Q_T}\sigma_j(x,t)dxdt \leq 0$.

Indeed, by (4.5), we have

$$\int_{Q_T}\sigma(x,t,Du_j-\Upsilon(u_j)):Du_jdxdt = \int_0^T\langle f(t),u_j\rangle dt - \int_{Q_T}\frac{\partial u_j}{\partial t}u_jdxdt. \tag{4.14}$$

Given that

$$u_j(\cdot,0)\to u(\cdot,0) \text{ and } u_j(\cdot,T)\rightharpoonup u(\cdot,T) \text{ in } L^2(\Omega;\mathbb{R}^m) \text{ as } j\to\infty,$$

it follows, when we pass to the lim inf in (4.14), that

$$\begin{aligned}\liminf_{j\to\infty}\int_Q\sigma(x,t,Du_j-\Upsilon(u_j)):Du_jdxdt \leq &\int_0^T\langle f(t),u\rangle dt\\ &-\frac{1}{2}\|u(\cdot,T)\|^2_{L^2(\Omega)}+\frac{1}{2}\|u(\cdot,0)\|^2_{L^2(\Omega)}.\end{aligned} \tag{4.15}$$

On the other hand, by (4.11), we have

$$\begin{aligned}\liminf_{j\to\infty}-\int_{Q_T}\sigma(x,t,Du_j-\Upsilon(u_j)):Dudxdt &= -\int_0^T\langle\kappa,u\rangle dt\\ &= -\int_0^T\langle f(t),u\rangle dt+\frac{1}{2}\|u(\cdot,T)\|^2_{L^2(\Omega)}-\frac{1}{2}\|u(\cdot,0)\|^2_{L^2(\Omega)}.\end{aligned} \tag{4.16}$$

We conclude from (4.15) and (4.16) that $\liminf\limits_{j\to\infty}\int_{Q_T}\sigma_j(x,t)dxdt \leq 0$.

Taking into account (4.13), the desired result follows. □

The next lemma is another version of lemma 4.4, known as the div-curl inequality, which serves as a powerful tool to achieve the main theorem's objective.

Lemma 4.5. *Suppose that $\nu_{(x,t)}$ corresponds to Young measure associated with the sequence (or a subsequence) of gradients $(Du_j)_j$ of the Galerkin approximations $(u_j)_j$. Then*

$$\int_{Q_T}\int_{\mathbb{M}^{m\times n}}\Big(\sigma\big(x,t,\xi-\Upsilon(u)\big)-\sigma\big(x,t,Du-\Upsilon(u)\big)\Big):\Big(\xi-Du\Big)d\nu_{(x,t)}(\xi)dxdt \leq 0. \tag{4.17}$$

Proof. We maintain consistent notation as used in the proof of Lemma 4.4. By using the same procedure outlined in the proof of Lemma 4.4, we establish the equi-integrability for $\big(\sigma(x,t,Du-\Upsilon(u)) : Du_j\big)^-$ and $\big(\sigma(x,t,Du-\Upsilon(u)) : Du\big)^-$, as well as $\big(\sigma(x,t,Du-\Upsilon(u)) : (Du_j-Du)\big)^-$. Since Du_j weakly convergent to $\int_{\mathbb{M}^{m\times n}} \xi d\nu_{(x,t)}(\xi) = Du$ in $L^1(0,T;L^1(\Omega))$, we obtain

$$\begin{aligned}\liminf_{j\to\infty} \int_{Q_T} &\sigma(x,t,Du-\Upsilon(u)) : (Du_j - Du)dxdt \\ &= \int_{Q_T} \sigma(x,t,Du-\Upsilon(u)) : \Big(\int_{\mathbb{M}^{m\times n}} \xi d\nu_{(x,t)}(\xi) - Du\Big)dxdt = 0.\end{aligned}$$

Consequently,

$$\begin{aligned}\liminf_{j\to\infty} \int_{Q_T} &(\sigma_j(x,t) - \sigma(x,t,Du-\Upsilon(u)) : (Du_j - Du)dxdt \\ &= \liminf_{j\to\infty} \int_{Q_T} \sigma_j(x,t)dxdt.\end{aligned}$$

Given that $\Big(\sigma_j(x,t) - \sigma\big(x,t,Du-\Upsilon(u)\big) : \big(Du_j - Du\big)\Big)^-$ is equi-integrable, and with the help of Lemma 2.6, we obtain

$$\begin{aligned}\int_{Q_T}\int_{\mathbb{M}^{m\times n}} &\Big(\sigma\big(x,t,\xi-\Upsilon(u)\big) - \sigma\big(x,t,Du-\Upsilon(u)\big)\Big) : \Big(\xi - Du\Big)d\nu_{(x,t)}(\xi)dxdt \\ &\leq \liminf_{j\to\infty} \int_{Q_T} \Big(\sigma_j(x,t) - \sigma\big(x,t,Du-\Upsilon(u)\big) : \big(Du_j - Du\big)\Big)dxdt \\ &= \liminf_{j\to\infty} \int_{Q_T} \sigma_j(x,t)dxdt \\ &\leq 0.\end{aligned}$$

□

We also need to localize the support of $\nu_{(x,t)}$, as outlined in the following lemma.

Lemma 4.6. *If (4.17) is satisfied, then for almost every $(x,t) \in Q_T$, the subsequent equation holds:*

$$\Big(\sigma(x,t,\xi-\Upsilon(u)) - \sigma(x,t,Du-\Upsilon(u)\Big) : \Big(\xi - Du\Big) = 0 \quad \text{on } supp\, \nu_{(x,t)}. \tag{4.18}$$

Proof. First, from (4.17), we have

$$\int_{Q_T}\int_{\mathbb{M}^{m\times n}} \Big(\sigma\big(x,t,\xi-\Upsilon(u)\big) - \sigma\big(x,t,Du-\Upsilon(u)\big)\Big) : \Big(\xi - Du\Big)d\nu_{(x,t)}(\xi)dxdt \leq 0.$$

The integrand in the last inequality should hardly be negative due to the monotonicity of σ, and is therefore null concerning the product measure $d\nu_{(x,t)}(\xi)\otimes dxdt$. Consequently, equality (4.18) is established. □

5. Proof of Theorem 3.2

Hereinafter, $(u_j)_j$ denotes the sequence of Galerkin approximations. We will prove that its limit, as we pass to the limit in the Galerkin equations (for a subsequence), is indeed a weak solution of problem (1.1).

To achieve this, we consider five cases corresponding to those in **(C4)**. In cases **(a)** and **(b)**, we establish weak convergence in $L^{p'}(Q_T)$ of $\sigma\big(x,t,Du_j-\Upsilon(u_j)\big)$ to $\sigma\big(x,t,Du-\Upsilon(u)\big)$, and in the three remaining cases, we prove that $Du_j \to Du$ in measure (or for an unlabeled subsequence).

Before proceeding, we establish that $u_j \to u$ in measure on Q_T. Indeed, the sequence $(u_j)_j$ is a bounded sequence in $L^p(0,T;W_0^{1,p}(\Omega))$. According to [15, Lemma 3], it follows that there exists a subsequence, also denoted by $(u_j)_j$, for which $u_j \to u$ in $L^p(0,T;L^q(\Omega))$ for all $q < p^* = \frac{np}{n-p}$, and converges in measure on Q_T.

Case (a): We first establish that, for all $\big((x,t),\zeta\big) \in Q_T \times \mathbb{M}^{m\times n}$, the following equality is satisfied within $\operatorname{supp}\nu_{(x,t)}$:

$$\sigma\big(x,t,\xi-\Upsilon(u)\big) : \zeta = \sigma\big(x,t,Du-\Upsilon(u)\big) : \zeta + \big(\nabla_\xi\sigma\big(x,t,Du-\Upsilon(u)\big)\zeta\big) : (Du-\xi), \tag{5.1}$$

where ∇_ξ stands for the derivative of σ concerning its second variable.

The monotonicity of σ implies that, for any $\tau \in \mathbb{R}$,

$$\big(\sigma(x,t,\xi-\Upsilon(u)) - \sigma(x,t,Du-\Upsilon(u)+\tau\zeta)\big) : (\xi - Du - \tau\zeta) \geq 0.$$

Moreover, using equality (4.18), we deduce that

$$\begin{aligned}-\sigma\big(x,t,\xi-\Upsilon(u)\big) : \tau\zeta \geq & -\sigma\big(x,t,Du-\Upsilon(u)\big) : (\xi-Du)\\ & + \sigma\big(x,t,Du-\Upsilon(u)+\tau\zeta\big) : (\xi-Du-\tau\zeta).\end{aligned}$$

Given that the function $\xi \mapsto \sigma(x,t,\xi-\Upsilon(u))$ is C^1, we establish

$$\sigma\big(x,t,Du-\Upsilon(u)+\tau\zeta\big) = \sigma\big(x,t,Du-\Upsilon(u)\big) + \nabla_\xi\sigma\big(x,t,Du-\Upsilon(u)\big)\tau\zeta + o(\tau).$$

Consequently,

$$\begin{aligned}&-\sigma\big(x,t,\xi-\Upsilon(u)\big) : \tau\zeta\\ &\geq \tau\big[\nabla_\xi\sigma(x,t,Du-\Upsilon(u))\zeta : (\xi-Du) - \sigma(x,t,Du-\Upsilon(u)) : \zeta\big] + o(\tau).\end{aligned} \tag{5.2}$$

Therefore, inequality (5.2) naturally leads to (5.1) since the sign of τ is arbitrary. As the sequence $\sigma(x,t,Du_j-\Upsilon(u_j))$ is equi-integrable, Balls' theorem asserts that its weak L^1 limit is determined by

$$\overline{\sigma} = \int_{\mathbb{M}^{m\times n}} \sigma\big(x,t,\xi-\Upsilon(u)\big)d\nu_{(x,t)}(\xi) = \int_{\operatorname{supp}\nu_{(x,t)}} \sigma\big(x,t,\xi-\Upsilon(u)\big)d\nu_{(x,t)}(\xi).$$

Given (5.1), we obtain

$$\begin{aligned}\overline{\sigma} = &\int_{\operatorname{supp}\nu_{(x,t)}} \sigma(x,t,Du-\Upsilon(u))d\nu_{(x,t)}(\xi)\\ &\qquad + \big(\nabla_\xi \sigma(x,t,Du-\Upsilon(u))\big)^t \underbrace{\int_{\operatorname{supp}\nu_{(x,t)}} (Du-\xi)d\nu_{(x,t)}(\xi)}_{=:0}\\ &= \sigma(x,t,Du-\Upsilon(u)).\end{aligned}$$

Given the boundedness of the sequence $\big(\sigma(x,t,Du_j-\Upsilon(u_j))\big)_j$ and the reflexivity of $L^{P'}(Q_T;\mathcal{M}^{m\times n})$ (where $p'>1$), the sequence $\big(\sigma(x,t,Du_j-\Upsilon(u_j))\big)_j$ weakly converges to $\big(\sigma(x,t,Du-\Upsilon(u))\big)$ in $L^{P'}(Q_T;\mathcal{M}^{m\times n})$ due to the Eberlein–Šmulian theorem [6] and the uniqueness of the limit.

Case (b): For $(x,t)\in Q_T$ and $u\in\mathbb{R}^m$, we use $\Gamma_{(x,t)}$ to denote the following set

$$\begin{aligned}\Gamma_{(x,t)} = \Big\{&\xi\in\mathbb{M}^{m\times n}:\\ &W\big(x,t,\xi-\Upsilon(u)\big) = W(x,t,Du-\Upsilon(u)) + \sigma(x,t,Du-\Upsilon(u)):(\xi-Du)\Big\}.\end{aligned}$$

Hereafter, we will show that $\operatorname{supp}\nu_{(x,t)}\subset\Gamma_{(x,t)}$, for almost all $(x,t)\in Q_T$.

Indeed, let us consider $\xi\in\operatorname{supp}\nu_{(x,t)}$ and $\tau\in[0,1]$. The monotonicity of σ immediately gives the following two inequalities:

$$\big(\sigma(x,t,Du-\Upsilon(u)+\tau(\xi-Du))-\sigma(x,t,Du-\Upsilon(u))\big):\tau(\xi-Du)\geq 0, \quad (5.3)$$

and

$$(1-\tau)\big(\sigma(x,t,Du-\Upsilon(u)+\tau(\xi-Du))-\sigma\big(x,t,\xi-\Upsilon(u)\big)\big):(Du-\xi)\geq 0. \quad (5.4)$$

Since $\xi\in\operatorname{supp}\nu_{(x,t)}$, we deduce from (4.18)

$$(1-\tau)\big(\sigma\big(x,t,\xi-\Upsilon(u)\big)-\sigma(x,t,Du-\Upsilon(u))\big):(\xi-Du)=0. \quad (5.5)$$

Subtracting (5.5) from (5.4), we get

$$(1-\tau)\big(\sigma(x,t,Du-\Upsilon(u)+\tau(\xi-Du))-\sigma(x,t,Du-\Upsilon(u))\big):(Du-\xi)\geq 0. \quad (5.6)$$

Inequalities (5.3) and (5.6) together yield

$$\big(\sigma(x,t,Du-\Upsilon(u)+\tau(\xi-Du))-\sigma(x,t,Du-\Upsilon(u))\big):(\xi-Du)=0.$$

Furthermore, for any $\tau\in[0,1]$,

$$\sigma(x,t,Du-\Upsilon(u)):(\xi-Du)=\sigma(x,t,Du-\Upsilon(u)+\tau(\xi-Du)):(\xi-Du). \quad (5.7)$$

Observing that

$$W\big(x,t,\xi-\Upsilon(u)\big)=W\big(x,t,Du-\Upsilon(u)\big)+\Big(W\big(x,t,\xi-\Upsilon(u)\big)-W\big(x,t,Du-\Upsilon(u)\big)\Big), \quad (5.8)$$

by considering condition **(b)(i)** of **(C4)** and (5.7), the equality (5.8) becomes

$$\begin{aligned} W\big(x,t,\xi-\Upsilon(u)\big) &= W(x,t,Du-\Upsilon(u)) \\ &\quad + \int_0^1 \sigma(x,t,Du-\Upsilon(u)+\tau(\xi-Du)) : (\xi-Du)\,\mathrm{d}\tau \\ &= W(x,t,Du-\Upsilon(u)) + \sigma(x,t,Du-\Upsilon(u)) : (\xi-Du). \end{aligned}$$

Consequently, $\operatorname{supp}\nu_{(x,t)} \subset \Gamma_x$.

The convexity condition for W with respect to the second variable, as stated in **(b)(ii)** of **(C4)**, yields the following inequality for all $\xi \in \mathbb{M}^{m\times n}$

$$W\big(x,t,\xi-\Upsilon(u)\big) \geq W(x,t,Du-\Upsilon(u)) + \sigma(x,t,Du-\Upsilon(u)) : (\xi-Du).$$

For $\tau \in \mathbb{R}$ and $\xi \in \Gamma_x$, we put

$$\mathcal{A}(\xi) = W\big(x,t,\xi-\Upsilon(u)\big) \quad \text{and} \quad \mathcal{B}(\xi) = \mathcal{A}(\xi) + \sigma(x,t,Du-\Upsilon(u)) : (\xi-Du).$$

The function $\xi \mapsto W(x,t,\xi-\Upsilon(u))$ is $\mathcal{C}^1$, so for all $\zeta \in \mathbb{M}^{m\times n}$ it follows that

$$\begin{aligned} \frac{\mathcal{A}(\xi+\tau\zeta)-\mathcal{A}(\xi)}{\tau} &\geq \frac{\mathcal{B}(\xi+\tau\zeta)-\mathcal{B}(\xi)}{\tau} \quad \text{if } \tau > 0, \\ \frac{\mathcal{A}(\xi+\tau\zeta)-\mathcal{A}(\xi)}{\tau} &\leq \frac{\mathcal{B}(\xi+\tau\zeta)-\mathcal{B}(\xi)}{\tau} \quad \text{if } \tau < 0. \end{aligned}$$

As a consequence, $D_\xi\mathcal{A} = D_\xi\mathcal{B}$, implying that for all $\xi \in \Gamma_x \supset \operatorname{supp}\nu_{(x,t)}$, we obtain the equality

$$\sigma\big(x,t,\xi-\Upsilon(u)\big) = \sigma(x,t,Du-\Upsilon(u)). \tag{5.9}$$

Thus,

$$\begin{aligned} \overline{\sigma}(x,t) = \int_{\mathbb{M}^{m\times n}} \sigma\big(x,t,\xi-\Upsilon(u)\big) d\nu_{(x,t)}(\xi) &= \int_{\operatorname{supp}\nu_{(x,t)}} \sigma(x,t,Du-\Upsilon(u)) d\nu_{(x,t)}(\xi) \\ &= \sigma(x,t,Du-\Upsilon(u)). \end{aligned}$$

Which means that $\sigma(x,t,Du_j-\Upsilon(u_j)) \rightharpoonup \sigma(x,t,Du-\Upsilon(u))$ in $L^1(Q_T)$.

We have reached this result in **Case (a)**, but we aim to establish this convergence as strong. To achieve this, for $u \in \mathbb{R}^m$ and $\xi \in \mathbb{M}^{m\times n}$, we introduce the function $\Lambda(x,t,u,\xi) = |\sigma\big(x,t,\xi-\Upsilon(u)\big) - \overline{\sigma}(x,t)|$, which is a Carathéodory function. Additionally, we consider the sequence $\Lambda_j(x,t) = \Lambda(x,t,u_j,Du_j)$.

Moreover, $\Upsilon(u_j)$ converges in measure to $\Upsilon(u)$ due to the continuity condition in **C1** and the fact that u_j converges to u in measure. Then, $(\Upsilon(u_j), Du_j)$ generates the Young measure $\delta_{\Upsilon(u(x))} \otimes \nu_{(x,t)}$ according to Proposition 2.5.

Given that $\sigma(x,t,Du_j-\Upsilon(u_j))$ is equi-integrable due to its weak convergence in $L^{p'}(Q_T)$, it follows that $\Lambda_j(x,t)$ is indeed equi-integrable.

Moreover,

$$\Lambda_j \rightharpoonup \overline{\Lambda} \quad \text{weakly in } L^1(Q),$$

with

$$\overline{\Lambda}(x) = \int_{\mathbb{R}^m\times\mathbb{M}^{m\times n}} \Lambda(x,t,\eta,\xi) d\delta_{\Upsilon(u(x))}(\eta) \otimes d\nu_{(x,t)}(\xi).$$

Whence,

$$\begin{aligned}\overline{\Lambda}(x) &= \int_{\mathbb{M}^{m\times n}} |\sigma\big(x,t,\xi-\Upsilon(u)\big) - \sigma(x,t,Du-\Upsilon(u))| d\nu_{(x,t)}(\xi) \\ &= \int_{\operatorname{supp}\nu_{(x,t)}} |\sigma\big(x,t,\xi-\Upsilon(u)\big) - \sigma(x,t,Du-\Upsilon(u))| d\nu_{(x,t)}(\xi).\end{aligned}$$

Taking (5.9) into account, this implies that $\overline{\Lambda}(x) = 0$. However, $\Lambda_j \geq 0$. Hence, $\Lambda_j \to 0$ strongly in $L^1(Q_T)$.

Since Λ_j is bounded in $L^{p'}(Q_T)$, it follows from Vitali's convergence theorem that

$$\sigma(x,t,Du_j-\Upsilon(u_j)) \rightharpoonup \sigma(x,t,Du-\Upsilon(u)) \text{ in } L^{p'}(Q_T). \tag{5.10}$$

Property (5.10) holds true in both cases **(a)** and **(b)**. Therefore, by passing to the limit in the Galerkin equations, we can prove our main theorem in this situation.

In the remaining three cases, our method aims to initially show, for each case, the convergence of the sequence Du_j to Du in measure (or for an unlabeled subsequence). Subsequently, we draw the conclusion.

Case (c): Since σ is strictly monotone, and by Lemma 4.6, we conclude that $\operatorname{supp}\nu_{(x,t)} = Du(x,t)$. Therefore, for almost all $(x,t) \in Q_T$, we have $\nu_{(x,t)} = \delta_{Du(x,t)}$. Consequently, Proposition 2.5(i) implies that Du_j converges to Du in measure as $j \to \infty$.

Case (d): Consider Q'_T as a subset of Q_T with a positive Lebesgue measure, and suppose that $\nu_{(x,t)}$ is not a Dirac measure on this subset.

Given $\overline{\xi} = \langle \nu_{(x,t)}, \mathrm{id}\rangle = Du$, we obtain

$$\begin{aligned}\int_{\mathbb{M}^{m\times n}} \sigma(x,\overline{\xi}-\Upsilon(u)) : \overline{\xi} d\nu_{(x,t)}(\xi) &= \sigma(x,Du-\Upsilon(u)) : \int_{\mathbb{M}^{m\times n}} \xi d\nu_{(x,t)}(\xi) \\ &= \int_{\mathbb{M}^{m\times n}} \sigma(x,\overline{\xi}-\Upsilon(u)) : \xi d\nu_{(x,t)}(\xi).\end{aligned} \tag{5.11}$$

Due to the strict p-quasi-monotonicity of σ with respect to the second variable, as stated in **(C4)(d)**, and in view of (5.11), we deduce that for almost every $(x,t) \in Q'_T$,

$$\int_{\mathbb{M}^{m\times n}} \sigma\big(x,t,\xi-\Upsilon(u)\big) : \xi, d\nu_{(x,t)}(\xi) > \int_{\mathbb{M}^{m\times n}} \sigma\big(x,t,\xi-\Upsilon(u)\big) : \overline{\xi}, d\nu_{(x,t)}(\xi).$$

After integrating the last inequality over Q_T, one obtains

$$\begin{aligned}&\int_{Q_T}\int_{\mathbb{M}^{m\times n}} \sigma\big(x,t,\xi-\Upsilon(u)\big) : \xi d\nu_{(x,t)}(\xi) dxdt \\ &\qquad > \int_{Q_T}\int_{\mathbb{M}^{m\times n}} \sigma\big(x,t,\xi-\Upsilon(u)\big) : \overline{\xi} d\nu_{(x,t)}(\xi) dxdt.\end{aligned}$$

Using Lemma 4.4 and the fact that $Du = \overline{\xi}$, we obtain the following inequality:

$$\int_{Q_T}\int_{\mathbb{M}^{m\times n}} \sigma\big(x,t,\xi-\Upsilon(u)\big) :\xi d\nu_{(x,t)}(\xi)dxdt \\ \leq \int_{Q_T}\int_{\mathbb{M}^{m\times n}} \sigma\big(x,t,\xi-\Upsilon(u)\big) : \overline{\xi} d\nu_{(x,t)}(\xi)dxdt.$$

Upon comparing these last two inequalities, a contradiction arises.

Therefore, there exists a function χ such that $\nu_{(x,t)} = \delta_{\chi(x,t)}$ for almost all $(x,t)\in Q_T$. This implies that

$$\chi(x,t) = \int_{\mathbb{M}^{m\times n}} \xi d\delta_{\chi(x,t)}(\xi) = \int_{\mathbb{M}^{m\times n}} \xi d\nu_{(x,t)}(\xi) = Du(x,t).$$

Consequently, $\nu_{(x,t)} = \delta_{Du(x)}$ for almost all $(x,t)\in Q_T$. Thus, from Proposition 2.5(i), we conclude that Du_j converges to Du in measure on Q_T as $j\to\infty$.

Case (e): According to condition **(C4)(e)** of strict quasimonotonicity of σ with respect to the second variable, we have

$$\begin{aligned}
&\int_{Q_T} |Du_j - Du|^p dxdt \\
&\quad \leq \frac{1}{\alpha_0}\int_{Q_T}\Big(\sigma\big(x,t,Du_j-\Upsilon(u_j)\big)-\sigma\big(x,t,Du-\Upsilon(u_j)\big)\Big) : \big(Du_j-Du\big)dxdt \\
&\quad \leq \frac{1}{\alpha_0}\int_{Q_T}\Big(\sigma\big(x,t,Du_j-\Upsilon(u_j)\big)-\sigma\big(x,t,Du-\Upsilon(u)\big)\Big) : \big(Du_j-Du\big)dxdt \\
&\quad + \frac{1}{\alpha_0}\int_{Q_T}\Big(\sigma\big(x,t,Du-\Upsilon(u)\big)\big)-\sigma\big(x,t,Du-\Upsilon(u_j)\big)\Big) : \big(Du_j-Du\big)dxdt.
\end{aligned}$$

The proof of Lemma 4.5 led us to the following result:

$$\liminf_{j\to\infty}\int_{Q_T}\big(\sigma(x,t,Du_j-\Upsilon(u_j))-\sigma(x,t,Du-\Upsilon(u))\big) : (Du_j-Du)dxdt \leq 0.$$

Since u_j converges in measure to u, we can once again extract a subsequence from $(u_j)_j$ that converges almost everywhere to u; refer to, for instance, [12, Theorem 2.30]. As a result, due to the continuity conditions in **C1** and **C2**, it follows that $\sigma(x,t,Du-\Upsilon(u_j)) \to \sigma(x,t,Du-\Upsilon(u))$ as $j\to\infty$, and also in $L^1(Q_T)$. Using the same approach as in the proof of **Case (a)**, the sequence $\sigma(x,t,Du-\Upsilon(u_j))$ converges weakly in $L^{P'}(Q_T,\mathbb{M}^{m\times n})$. Therefore, by virtue of the uniqueness of the limit, $\sigma(x,t,Du-\Upsilon(u))$ represents its weak $L^{P'}$-limit (for a subsequence). Consequently, Hölder inequality allows us to write

$$\liminf_{j\to\infty}\int_{Q_T}\big(\sigma(x,t,Du-\Upsilon(u_j)))-\sigma(x,t,Du-\Upsilon(u))\big) : (Du_j-Du)dxdt = 0.$$

Whence,

$$\liminf_{j\to\infty}\int_{Q_T} |Du_j-Du|^p dxdt = 0.$$

Thus, $Du_j \to Du$ in measure as $j\to\infty$ for a subsequence.

In the last three cases, we have established that $Du_j \to Du$ in measure. Moreover, since $u_j \to u$ in measure as well, we can accordingly extract a subsequence from $(u_j)_j$ if needed (see [12, Theorem 2.30]), ensuring that $u_j \to u$ and $Du_j \to Du$ for almost every $(x,t) \in Q_T$.

Hence, $\sigma(x,t,Du_j - \Upsilon(u_j))$ converges to $\sigma(x,t,Du - \Upsilon(u))$ as $j \to \infty$ almost everywhere in Q_T and also in $L^1(Q_T)$. On other hand, $\sigma(x,t,Du_j - \Upsilon(u_j))$ is bounded thanks to conditions **C1** and **C3**. Therefore, by applying the Vitali convergence theorem, we deduce that $\sigma(x,t,Du_j - \Upsilon(u_j))$ converges to $\sigma(x,t,Du - \Upsilon(u))$ in $L^r(Q_T)$ for all $r \in [1,p')$.

Consequently,

$$-\operatorname{div}\sigma(x,t,Du_j - \Upsilon(u_j)) \rightharpoonup -\operatorname{div}\sigma(x,t,Du - \Upsilon(u)).$$

Due to (4.11) and the uniqueness of the limit, we deduce

$$\kappa = -\operatorname{div}\sigma(x,t,Du - \Upsilon(u)).$$

Therefore, by passing to the limit in the Galerkin equations, we conclude the proof of Theorem 3.2 for these last three cases. Thus, the main theorem is established, confirming consequently that System (1.1) possesses a weak solution.

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