Khayyam J. Math. 11 (2025), no. 1, 38-50
DOI: 10.22034/kjm.2024.484764.3351

Khayyam Journal of Mathematics
emis.de/journals/KJM
kjm-math.org

WEAK SOLUTIONS TO QUASILINEAR ELLIPTIC OBSTACLE PROBLEMS

HASNAE EL HAMMAR[1], MOUAD ALLALOU[1], MOHAMED EL OUAARABI[2*] AND ABDERRAHMANE RAJI[1]

Communicated by A. Jiménez-Vargas

ABSTRACT. We study a class of obstacle problems in Sobolev spaces of the form

$$\begin{cases} \displaystyle\int_\Omega a(|D\varpi|)D\varpi) : D(\mathcal{U}-\varpi) + \left\langle \varpi|\varpi|^{r-2}, \mathcal{U}-\varpi \right\rangle \, \mathrm{d}y \geq 0, \\ \mathcal{U} \in \wp_{\ell,g}. \end{cases}$$

We prove the existence of a weak solution via Young measure theory and a theorem of Kinderlehrer and Stampacchia. Since our operator does not satisfy the property of monotonicity which is necessary in the proof, we suppose another condition to overcome this situation.

1. Introduction

Several areas of research and engineering, including fluid mechanics, material science, and image processing, interact with nonlinear elliptic obstacle problems. One example of an obstacle problem related to membranes is the challenge of finding the equilibrium shape of a flexible membrane constrained by an obstacle. In this problem, the membrane is represented as a two-dimensional surface with a given tension, and an obstacle is placed on this surface. The goal is to find the shape of the membrane that minimizes the total energy of the system, subject to the constraint that the membrane cannot penetrate the obstacle. This problem arises in a variety of applications, including the creation of inflatable structures, the study of biological membranes, and the materials science analysis of elastic

Date: Received: 21 October 2024; Revised: 05 December 2024; Accepted: 11 December 2024.
* Corresponding author.
2020 *Mathematics Subject Classification.* Primary: 35J88; Secondary: 28A33, 46E35.
Key words and phrases. Obstacle problem,Theorem of Kinderlehrer and Stampacchia, Young measures.

sheets. The aim of this paper is to investigate the existence of a weak solution for the obstacle problem of the following form:

$$\begin{cases} \displaystyle\int_\Omega a(|D\varpi|)D\varpi) : D(\mathcal{U} - \varpi) + \left\langle \varpi|\varpi|^{r-2}, \mathcal{U} - \varpi \right\rangle \, \mathrm{d}y \geq 0, \\ \mathcal{U} \in \wp_{\ell,g}, \end{cases} \tag{1.1}$$

where

$$\wp_{\ell,g} = \{\mathcal{U} \in W^{1,r}(\Omega;\mathbb{R}^m) : \mathcal{U} - g \in W_0^{1,r}(\Omega;\mathbb{R}^m), \qquad \mathcal{U} \geq \ell \text{ a.e in } \Omega\}. \tag{1.2}$$

Here Ω is a bounded open domain in $\mathbb{R}^n$ $(n \geq 2)$, and $\varpi : \Omega \to \mathbb{R}^m$ is a vector-valued function, and the function $a : \mathbb{R}^+ \to \mathbb{R}^+$ is of class C^1 and satisfies

$$-1 < t_a =: \inf_{t>0} \frac{a'(t)t}{a(t)} \leq \sup_{t>0} \frac{a'(t)t}{a(t)} =: h_a < \infty. \tag{1.3}$$

One physical example of our problem can be found in the modeling of liquid crystal elastomers (LCEs). LCEs are materials that exhibit both mechanical and optical properties, and they have important applications in microelectronics, photonics, and robotics. The behavior of LCEs is strongly influenced by the orientation of the liquid crystal molecules within the material, which can be described by a vector field ϖ. The elastic energy of the material is a function of the gradient of ϖ, which can be represented by the function $a(|D\varpi|)$. The objective of this problem is to find the vector field ϖ that minimizes the elastic energy of the material subject to the boundary conditions, and the constraint that ϖ must satisfy the equations of motion of the material. The constraint can be represented by a vector-valued function $\ell(\varpi)$ that describes the dynamics of the material, and the function g can represent the initial conditions for the problem. There are several numerical methods that can be used to solve obstacle problems, depending on the specific problem and the properties of the nonlinear operator and the obstacle.

First of all, the initial findings on obstacle problems come from Stampacchia's groundbreaking work in the 1960s. After the fundamental work of Lions and Stampacchia [20], the generally recognized notion that solutions to the obstacle problems cannot be of class $\mathcal{C}^2$ independent of how regular the obstacle is gave rise to the idea of a weak solution and the theory of variational inequalities. These problems can be resolved using recent classical methods of functional analysis (see, e.g., [2, 21, 22] and reference therein); however, it is crucial to create the conditions necessary for weak solutions to actually be classical ones, with the additional requirement that the solution remains above a given function, the obstacle (see [8]). We cite the following monographs for detailed details [6, 9, 19].

In [24], the authors considered the obstacle control problem

$$\int_\Omega A(y, \nabla y)\nabla(\mathcal{U} - y)dy \geqslant 0, \qquad \text{for all } \mathcal{U} \in \wp(\varphi), \tag{1.4}$$

where $y \in \wp(\varphi) =: \{\mathcal{U} \in H_0^1(\Omega) \mid \mathcal{U} \geqslant \varphi \; a.e. \; y \in \Omega\}$ and proved the existence of an optimal control and approximated the variational inequality (1.4) by a family of quasilinear elliptic equations.

Junxia and Yuming [18] have examined the boundary regularity of weak solutions to a nonlinear obstacle problem with $C^{1,\beta}$-obstacle function, and obtained the $C^{1,\alpha}_{loc}$ boundary regularity. The obstacle problems treated in [10] are related to elliptic equations of the type $-\operatorname{div} a(y, D\varpi) = \mu$ on $\Omega \subset \mathbb{R}^n$ with μ is a bounded Radon measure on Ω. The authors have investigated the role played by Wolff and Riesz potentials in regularity questions for solutions to obstacle problems.

Grimaldi [15] proved some regularity properties of solutions to variational inequalities of the form

$$\int_\Omega \langle \mathcal{A}(y, \varpi, D\varpi), D(\mathcal{U} - \varpi)\rangle dy \geq \int_\Omega \mathcal{B}(y, \varpi, D\varpi)(\mathcal{U} - \varpi)dy, \quad \text{for all } \mathcal{U} \in \wp_\psi(\Omega),$$

and the class $\wp_\psi(\Omega)$ is defined as follows: $\wp_\psi(\Omega) = \{w \in W^{1,r}(\Omega) : w \geq \psi \text{ a.e. in } \Omega\}$. They obtained the existence of solutions to elliptic obstacle problems with measure data and derived pointwise and oscillation estimates for the gradients of solutions to elliptic obstacle problems via the nonlinear Wolff potentials.

We note that our operator $t \to a(|t|)t$ is monotone if it verifies the condition (1.3) (see Lemma 4.2). More precisely, if we assume that $\sigma : \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$ given by

$$\sigma(\xi) = \begin{cases} a(|\xi|)\xi & \text{if } \xi \neq 0, \\ 0 & \text{if } \xi = 0, \end{cases}$$

is a C^1-function and monotone, that is,

$$(\sigma(\xi) - \sigma(\eta)) : (\xi - \eta) \geq 0 \quad \text{for all } \xi, \eta \in \mathbb{M}^{m\times n},$$

then we deduce that we cannot use the proof of the case $(H_2)(a)$ in [17], because the following identity

$$\sigma(\lambda) : \xi = \sigma(D\varpi) : \xi + (\nabla_\xi \sigma(D\varpi)\xi) : (D\varpi - \lambda)$$

does not hold on the support of ν_y (see Definition 3.2 for the definition of ν_y). Instead, we will assume (1.3) to prove the monotonicity of the function a, which will serve us to prove the existence of weak solution for the obstacle problem (1.1)–(1.2) via the concept of Young measure combined with the theorem of Kinderlehrer and Stampacchia. For more applications of the theory of Young measures, we refer the readers to see [3, 1, 11, 16, 17, 4, 13, 5, 12].

2. Assumptions and main result

Let Ω be a bounded open domain in $\mathbb{R}^n$ $(n \geq 2)$, let $1 < r < \infty$, and let $m \in \mathbb{N}$. By $\mathbb{M}^{m\times n}$ we denote the set of $m \times n$ matrices with reduced $\mathbb{R}^{mn}$ topology, that is, if $\gamma \in \mathbb{M}^{m\times n}$, then $|\gamma|$ is the norm of γ when regarded as a vector of $\mathbb{R}^{mn}$. We endow $\mathbb{M}^{m\times n}$ with the product

$$\gamma : \mu = \sum_{i,j} \gamma_{ij}\mu_{ij}.$$

In addition to (1.3), we assume the following conditions:

(A_0): There exist $\alpha_0,\ \alpha_1 > 0$, and $d_1 \in L^1(\Omega)$ such that

$$|a(|\xi|)\xi| \leq \alpha_0 |\xi|^{r-1},$$
$$a(|\xi|)\xi : \xi \geq \alpha_1 |\xi|^r - d_1(y), \quad \text{for all } \xi \in \mathbb{M}^{m\times n},$$

for almost all $y \in \Omega$.

(A_1): The function a satisfies one of the following conditions:

i): There exists a convex and C^1-function $b : \mathbb{M}^{m\times n} \to \mathbb{R}$ such that

$$a(|\xi|)\xi = \frac{\partial b(\xi)}{\partial \xi} := D_\xi b(\xi).$$

ii): a is strictly quasimonotone, that is, there exist constants $c > 0$ and $p > 0$ such that

$$\int_\Omega (a(|\eta|)\eta - a(|\xi|)\xi) : (\eta - \xi) dy \geqslant c \int_\Omega |\eta - \xi|^r dy,$$

for all $\xi,\ \eta \in \mathbb{M}^{m\times n}$.

iii): a is strictly p-quasimonotone, that is,

$$\int_{\mathbb{M}^{m\times n}} (a(|\lambda|)\lambda - a(|\overline{\lambda}|)\overline{\lambda}) : (\lambda - \overline{\lambda}) d\nu_y(\lambda) > 0,$$

for $\overline{\lambda} = \langle \nu_y,\ id\rangle$, where $\nu = \{\nu_y\}_{y\in\Omega}$ is any family of Young measures generated by a sequence in $L^r(\Omega)$ and not a Dirac measure for a.e. $y \in \Omega$.

We will prove the following existence result for the obstacle problem (1.1)–(1.2):

Theorem 2.1. *Suppose $\wp_{\psi,g} \neq \emptyset$. Assume that (1.2) holds and that a satisfies the conditions $(A_0) - (A_1)$. Then there exists a weak solution $\varpi \in \wp_{\psi,g}$ to the obstacle problem (1.1)–(1.2). In other word, there exists a function $\varpi \in \wp_{\psi,g}$ satisfying*

$$\int_\Omega a(|D\varpi|)D\varpi : D(\mathcal{U} - \varpi) + \left\langle \varpi|\varpi|^{r-2}, \mathcal{U} - \varpi \right\rangle \,\mathrm{d}y \geq 0$$

for each $\mathcal{U} \in \wp_{\psi,g}$.

3. Preliminaries

In this section, we first introduce a theorem of Kinderlehrer and Stampacchia. To this end, a review on Young measures will be presented and some of its properties needed later.

Let X be a reflexive Banach space and let X' be its dual. By $\langle \cdot, \cdot \rangle$ we denote the duality pairing between X and X' given by

$$\langle f, h \rangle = \int_\Omega fh \,\mathrm{d}y, \qquad h \in X, f \in X'.$$

Recalling the following theorem of Kinderlehrer and Stampacchia:

Theorem 3.1. *([19]). Let $\wp$ be a nonempty closed convex subset of X and let $L : \wp \to X'$ be monotone, coercive and strong-weakly continuous on $\wp$. Then there exists an element $\varpi \in \wp$ such that*

$$\langle L(\varpi), \mathcal{U} - \varpi \rangle \geq 0 \qquad \text{for all } \mathcal{U} \in \wp.$$

As stated in the introduction, we use the tool of Young measures to prove the existence result. This concept of Young measures is an amazing tool to understand and control difficulties that arise when weak convergence does not behave as one desires with respect to nonlinear functionals and operators. For convenience of the readers not familiar with this concept, we give an overview needed in this paper. See [7, 14, 16] for more details.

Definition 3.2. We call $\nu = \{\nu_y\}_{y \in \Omega}$ the family of Young measures associated to (z_k).

In [7], it is shown that if for all $R > 0$

$$\lim_{L \to \infty} \sup_{k \in \mathbb{N}} |\{y \in \Omega \cap B_R(0) \ : \ |z_k(y)| \geq L\}| = 0,$$

then the Young measure ν_y generated by z_k is a probability measure, that is, $\|\nu_y\|_{\mathcal{M}} = 1$ for a.e.$y \in \Omega$. The following properties build the basic tools used in what follows. If $|\Omega| < \infty$, then there holds

$$z_k \to z \text{ in measure } \Leftrightarrow \nu_y = \delta_{z(y)} \text{ for a.e. } y \in \Omega. \tag{3.1}$$

If we choose $z_k = Dw_k$ for $w_k : \Omega \to \mathbb{R}^m$, then the above results remain valid.

Lemma 3.3. *([11, 16, 17]). Assume that $D\varpi_k$ is bounded in $L^r(\Omega; \mathbb{M}^{m \times n})$. Then the Young measure ν_y generated by $D\varpi_k$ satisfies the following conditions:*

(1) *ν_y is a probability measure.*
(2) *The weak L^1- limit of $D\varpi_k$ is given by $\langle \nu_y, \ id \rangle$.*
(3) *The identification $\langle \nu_y, \ id \rangle = D\varpi(y)$ holds for a.e. $y \in \Omega$.*

The following Fatou-type lemma is useful for us.

Lemma 3.4. *([11, 16, 17]). Let $\varphi : \mathbb{M}^{m \times n} \to \mathbb{R}$ be a continuous function and let $\varpi_k : \Omega \to \mathbb{R}^m$ be a sequence of measurable functions such that Dw_k generates the Young measure ν_y, with $\|\nu_y\|_{\mathcal{M}(\mathbb{M}^{m \times n})} = 1$. Then*

$$\liminf_{k \to \ \infty} \int_\Omega \varphi(Dw_k) \ \mathrm{d}y \geq \int_\Omega \int_{\mathbb{M}^{m \times n}} \varphi(\lambda) d\nu_y(\lambda) \ \mathrm{d}y$$

provided that the negative part of $\varphi(Dw_k)$ is equi-integrable.

4. Weak solution of obstacle problem

In this section, we shall obtain the existence of the solution for obstacle problem (1.1)–(1.2). We define a mapping $L : \wp_{\ell,g} \to W^{-1,r}(\Omega; \mathbb{R}^m)$ by

$$\langle L(\varpi), \mathcal{U} \rangle = \int_\Omega a(|D\varpi|) D\varpi) : D\mathcal{U} + \left\langle \varpi |\varpi|^{r-2}, \mathcal{U} \right\rangle \ \mathrm{d}y, \quad \text{for all } \mathcal{U} \in W_0^{1,r}(\Omega; \mathbb{R}^m).$$

It satisfies the hypothesis of Theorem 3.1.

Lemma 4.1. *Let ϖ be a function in $\wp_{\ell,g}$. Then the following conditions hold:*
a) $L(\varpi) \in W^{-1,r'}(\Omega;\mathbb{R}^m)$.
b) *L is monotone and coercive on $\wp_{\ell,g}$.*
c) *L is strongly-weakly continuous.*

Proof. ***a)*** Using Hölder's inequality and the growth condition in A_0, we have

$$\begin{aligned}|\langle L(\varpi),\ \mathcal{U}\rangle| &= \left|\int_\Omega a(|D\varpi|)D\varpi : D\mathcal{U} + \left\langle \varpi|\varpi|^{r-2}, \mathcal{U}\right\rangle\ \mathrm{d}y\right| \\ &\leq \Big(\int_\Omega |a(|D\varpi|)D\varpi|^{r'}\ \mathrm{d}y\Big)^{\frac{1}{r'}}\Big(\int_\Omega |D\mathcal{U}|^r\ \mathrm{d}y\Big)^{\frac{1}{r}} + c\|\varpi\|_r\|\mathcal{U}\|_r \\ &\leq c(\int_\Omega |D\varpi|^r dy)^{\frac{1}{r'}}\|D\mathcal{U}\|_r + c'\|\varpi\|_r\|D\mathcal{U}\|_r \\ &\leq Const\|D\mathcal{U}\|_r.\end{aligned}$$

Thus $L(\varpi) \in W^{-1,r'}(\Omega;\mathbb{R}^m)$.

b) We prove that L is monotone and coercive on $\wp_{\ell,g}$. For fixed $\mathcal{U} \in \wp_{\ell,g}$, we have

$$\begin{aligned}&\langle L(\varpi) - L(\mathcal{U}), \varpi - \mathcal{U}\rangle \\ &\qquad= \int_\Omega (a(|D\varpi|)D\varpi - a(|D\mathcal{U}|)D\mathcal{U}) : (D\varpi - D\mathcal{U})\ \mathrm{d}y \\ &\qquad\quad+ \int_\Omega \left\langle \varpi|\varpi|^{r-2} - \mathcal{U}|\mathcal{U}|^{r-2}, \varpi - \mathcal{U}\right\rangle\ \mathrm{d}y \geq 0,\end{aligned}$$

by Lemma 4.2. Hence L is monotone by the monotonicity of a.

On the other hand, by the strict quasimonotone condition, we get

$$\begin{aligned}\langle L(\varpi) - L(\mathcal{U}), \varpi - \mathcal{U}\rangle &= \int_\Omega (a(|D\varpi|)D\varpi - a(|D\mathcal{U}|)D\mathcal{U}) : (D\varpi - D\mathcal{U})\ \mathrm{d}y \\ &\quad+ \int_\Omega \left\langle \varpi|\varpi|^{r-2} - \mathcal{U}|\mathcal{U}|^{r-2}, \varpi - \mathcal{U}\right\rangle\ \mathrm{d}y \\ &\geq c_2\int_\Omega |D\varpi - D\mathcal{U}|^r dy + 2^{2-r}\int_\Omega |\varpi - \mathcal{U}|^r\ \mathrm{d}y\end{aligned}$$

from which we deduce

$$\frac{\langle L(\varpi) - L(\mathcal{U}), \varpi - \mathcal{U}\rangle}{\|\varpi - \mathcal{U}\|_{1,r}} \geq c\|\varpi - \mathcal{U}\|_{1,r}^{r-1} \to \infty$$

as $\|\varpi - \mathcal{U}\|_{1,r} \to \infty$. Hence L is coercive.

c) We show that L is strongly-weakly continuous.We pick out a sequence $\varpi_k \in \wp_{\ell,g}$ such that $\varpi_k \to \varpi \in \wp_{\ell,g}$ in $W^{1,r}(\Omega;\mathbb{R}^m)$. Then $\|\varpi_k\|_{1,r} \leq C$ for some constant C.

According to Lemma 3.3, there exists a Young measure ν_y generated by $D\varpi_k$ such that $\|\nu_y\|_{\mathcal{M}(\mathbb{M}^{m\times n})} = 1$ and

$$D\varpi_k \rightharpoonup \langle \nu_y, id\rangle = \int_{\mathbb{M}^{m\times n}} \lambda d\nu_y(\lambda) \quad \text{in } L^1(\Omega). \tag{4.1}$$

Since $L^r(\Omega;\mathbb{M}^{m\times n})$ is reflexive, then $D\varpi_k \rightarrow D\varpi$ in $L^r(\Omega;\mathbb{M}^{m\times n}) \subset L^1(\Omega;\mathbb{M}^{m\times n})$. Thus $D\varpi(y) = \langle \nu_y, id\rangle$ for a.e $y \in \Omega$ (by uniqueness of limit).

We will demonstrate ***c)*** using the following lemmas.

Lemma 4.2. *Assume that 1.3 holds. Then a is monotone, that is,*

$$(a(|\xi|)\xi - a(|\eta|)\eta) : (\xi - \eta) \geq 0 \quad \textit{for all } \xi,\ \eta \in \mathbb{M}^{m\times n}.$$

Proof. Let $t \in [0,1]$, and take $\theta_t = t\xi + (1-t)\eta$ for all $\xi,\ \eta \in \mathbb{M}^{m\times n}$. We have

$$\begin{aligned}
(a(|\xi|)\xi - a(|\eta|)\eta) : (\xi - \eta) &= (\int_0^1 \frac{d}{dt}[a(|\theta_t|)\theta_t]dt) : (\xi - \eta) \\
&= (\int_0^1 [a'(|\theta_t|)|\theta_t| + a(|\theta_t|)]\ \mathrm{d}y) : (\xi - \eta)^2 \\
&= (\int_0^1 a(|\theta_t|)[\frac{a'(|\theta_t|)|\theta_t|}{a(|\theta_t|)} + 1]\ \mathrm{d}y) : (\xi - \eta)^2 \geq 0
\end{aligned}$$

by using (1.3). Thus a is monotone. □

Lemma 4.3 (div-curl inequality)**.** *Assume that $(A_0) - (A_1)$ hold and that $D\varpi_k$ generates the Young measure ν_y. Then*

$$\int_\Omega \int_{\mathbb{M}^{m\times n}} (a(|\lambda|)\lambda - a(|D\varpi|)D\varpi) : (\lambda - D\varpi) d\nu_y(\lambda)\ \mathrm{d}y \leq 0.$$

Proof. Consider the sequence

$$\begin{aligned}
C_k &:= (a(|D\varpi_k|)D\varpi_k - a(|D\varpi|)D\varpi)\ :\ (D\varpi_k - D\varpi) \\
&= a(|D\varpi_k|)D\varpi_k : (D\varpi_k - D\varpi) - a(|D\varpi|)D\varpi : (D\varpi_k - D\varpi) \\
&=: C_{k,1} + C_{k,2}.
\end{aligned}$$

On the one hand, by the growth condition in (A_1),

$$\int_\Omega |a(|D\varpi|)D\varpi|^{r'}\ \mathrm{d}y \leq c\int_\Omega |D\varpi|^r\ \mathrm{d}y < \infty, \tag{4.2}$$

for arbitrary $\varpi \in W_0^{1,r}(\Omega;\mathbb{R}^m)$. Thus $a(|D\varpi|)D\varpi \in L^{r'}(\Omega;\mathbb{M}^{m\times n})$. According to a weak convergence described in (4.1), it follows that

$$\liminf_{k\to\ \infty} \int_\Omega C_{k,2}\ \mathrm{d}y = \int_\Omega a(|D\varpi|)D\varpi\ :\ \left(\ \int_{\mathbb{M}^{m\times n}} (\lambda - D\varpi)d\upsilon_y(\lambda)\ \mathrm{d}y\right) = 0. \tag{4.3}$$

Let $\Omega' \subset \Omega$ be an arbitrary measurable subset. By the growth condition in (A_0) together with Hölder inequality, we have

$$\int_{\Omega'} |a(|D\varpi_k|)D\varpi_k : D\varpi|\ \mathrm{d}y \leq \big(\underbrace{c\|D\varpi_k\|_r^{r-1}}_{\leq C}\big)\big(\int_{\Omega'} |D\varpi|^p\ \mathrm{d}y\big)^{\frac{1}{r}}.$$

Since $\displaystyle\int_{\Omega'} |D\varpi|^r\ \mathrm{d}y$ is arbitrary small if we choose the measure of Ω' small enough, it follows that the negative part $\big(a(|D\varpi_k|)D\varpi_k : D\varpi\big)^-$ is equi-integrable.

On the other hand, by the coercivity condition in (A_0), we have

$$a(|D\varpi_k|)D\varpi_k : D\varpi_k \geq -d_1(y) + \alpha_1|D\varpi_k|^r \geq -d_1(y).$$

Thus

$$\int_{\Omega'} \big(a(|D\varpi_k|)D\varpi_k : D\varpi_k\big)^- \,\mathrm{d}y \le \int_{\Omega'} |d_1(y)| \,\mathrm{d}y.$$

Hence $\big(a(|D\varpi_k|)D\varpi_k : D\varpi_k\big)^-$ is equi-integrable. We infer from Lemma 3.4 that

$$\begin{aligned} C &= \liminf_{k\to\infty} \int_\Omega a(|D\varpi_k|)D\varpi_k : (D\varpi_k - D\varpi) \,\mathrm{d}y \\ &\ge \int_\Omega \int_{\mathbb{M}^{m\times n}} a(|\lambda|)\lambda : (\lambda - D\varpi) d\nu_y(\lambda) \,\mathrm{d}y. \end{aligned}$$

We prove that $C \le 0$. According to Mazur's theorem (see, e.g., [23]), there exists $\mathcal{U}_k \in W^{1,r}(\Omega;\mathbb{R}^m)$, where each $\mathcal{U}_k$ is a convex linear combination of $\{\varpi_1, \dots, \varpi_k\}$ such that $\mathcal{U}_k \to \varpi$ in $W^{1,r}(\Omega;\mathbb{R}^m)$. This implies that $\mathcal{U}_k$ belongs to the same space as ϖ_k. Hence

$$\begin{aligned} C &= \liminf_{k\to\infty} \int_\Omega a(|D\varpi_k|)D\varpi_k : (D\varpi_k - D\varpi) \,\mathrm{d}y \\ &= \liminf_{k\to\infty} \Big(\int_\Omega a(|D\varpi_k|)D\varpi_k : (D\varpi_k - D\mathcal{U}_k) \,\mathrm{d}y \\ &\quad + \int_\Omega a(|D\varpi_k|)D\varpi_k : (D\mathcal{U}_k - D\varpi) \,\mathrm{d}y \Big) \\ &\le \liminf_{k\to\infty} \big\| |a(|D\varpi_k|)D\varpi_k| \big\|_{r'} \|D\varpi_k - D\mathcal{U}_k\|_r \\ &\quad + \big\| |a(|D\varpi_k|)D\varpi_k| \big\|_{r'} \|D\mathcal{U}_k - D\varpi\|_r = 0, \end{aligned}$$

by the boundedness of $a(|D\varpi_k|)D\varpi_k$ in $L^{r'}(\Omega;\mathbb{M}^{m\times n})$, the construction of the sequence $\mathcal{U}_k$, and the following fact

$$\|\varpi_k - \mathcal{U}_k\|_{1,r} \le \|\varpi_k - \varpi\|_{1,r} + \|\mathcal{U}_k - \varpi\|_{1,r} \to 0 \text{ as } k \to \infty.$$

Since

$$\begin{aligned} &\int_\Omega \int_{\mathbb{M}^{m\times n}} a(|D\varpi|)D\varpi : (\lambda - D\varpi) \,\mathrm{d}\nu_y(\lambda) \,\mathrm{d}y \\ &= \int_\Omega a(|D\varpi|)D\varpi : \Big(\int_{\mathbb{M}^{m\times n}} \lambda \,\mathrm{d}\nu_y(\lambda) - D\varpi \Big) \,\mathrm{d}y = 0 \end{aligned}$$

together with $C \le 0$, the inequality of Lemma 4.3 follows. □

Remark 4.4. An intermediary result is the following inequality:

$$\liminf_{k\to\infty} \int_\Omega \big(a(|D\varpi_k|)D\varpi_k - a(|D\varpi|)D\varpi\big) : (D\varpi_k - D\varpi) \,\mathrm{d}y \le 0.$$

To see this it is sufficient to repeat the proof of Lemma 4.3.

Lemma 4.5. *For almost every $y \in \Omega$, we have*

$$\big(a(|\lambda|)\lambda - a(|D\varpi|)D\varpi\big) : (\lambda - D\varpi) = 0 \textit{ on } \operatorname{supp} \nu_y.$$

Proof. By Lemma 4.3, we have

$$\int_\Omega \int_{\mathbb{M}^{m\times n}} \big(a(|\lambda|)\lambda - a(|D\varpi|)D\varpi\big) : (\lambda - D\varpi)\, \mathrm{d}\nu_y(\lambda)\, \mathrm{d}y \le 0.$$

By the monotonicity of a, the above integrand is nonnegative, thus must vanish with respect to the product measure $d\nu_y(\lambda) \otimes\, \mathrm{d}y$. Therefore

$$\big(a(|\lambda|)\lambda - a(|D\varpi|)D\varpi\big) : (\lambda - D\varpi) = 0 \text{ on } \operatorname{supp} \nu_y.$$

□

We shall prove $c)$ for each case listed in (A_2). In a first step, suppose that a satisfies the condition $(A_1)(i)$. We argue as follows: We start by proving that for almost every $y \in \Omega$,

$$\operatorname{supp} \nu_y \subset E_y = \big\{\lambda \in \mathbb{M}^{m\times n} :\, b(\lambda) = b(D\varpi) + a(|D\varpi|)D\varpi : (\lambda - D\varpi)\big\}.$$

Let $\lambda \in \operatorname{supp} \nu_y$. Then by Lemma 4.5, we get

$$(1-\tau)\big(a(|\lambda|)\lambda - a(|D\varpi|)D\varpi\big) : (\lambda - D\varpi) = 0, \qquad \text{for all } \tau \in [0,1]. \tag{4.4}$$

According to Lemma 4.2 and (4.4), for all $\tau \in [0,1]$,

$$\begin{aligned} 0 &\le (1-\tau)(a(|\lambda|)\lambda - a(|D\varpi + \tau(\lambda - D\varpi)|)(D\varpi + \tau(\lambda - D\varpi))) : (\lambda - D\varpi) \\ &= (1-\tau)(a(|D\varpi|)D\varpi - a(|D\varpi + \tau(\lambda - D\varpi)|)(D\varpi + \tau(\lambda - D\varpi))) : (\lambda - D\varpi). \end{aligned} \tag{4.5}$$

By monotonicity, we can write

$$(a(|D\varpi|)D\varpi - a(|D\varpi + \tau(\lambda - D\varpi)|)(D\varpi + \tau(\lambda - D\varpi))) \ : \ \tau(D\varpi - \lambda) \ge 0,$$

which implies, since $\tau \in [0,1]$, that

$$(a(|D\varpi|)D\varpi - a(|D\varpi + \tau(\lambda - D\varpi)|)(D\varpi + \tau(\lambda - D\varpi))) : (1-\tau)(D\varpi - \lambda) \ge 0. \tag{4.6}$$

From (4.5) and (4.6) we deduce

$$(a(|D\varpi|)D\varpi - a(|D\varpi + \tau(\lambda - D\varpi)|)(D\varpi + \tau(\lambda - D\varpi))) \ : \ (\lambda - D\varpi) = 0$$

for $\tau \in [0,1]$. It results from the above equation and $a(|\xi|)\xi = (\partial b(\xi))/(\partial \xi)$ that

$$\begin{aligned} b(\lambda) &= b(D\varpi) + \int_0^1 a(|D\varpi + \tau(\lambda - D\varpi)|)(D\varpi + \tau(\lambda - D\varpi)) : (\lambda - D\varpi)\, \mathrm{d}\tau \\ &= b(D\varpi) + a(|D\varpi|)D\varpi \ : \ (\lambda - D\varpi). \end{aligned}$$

Hence $\lambda \in E_y$, that is, $\operatorname{supp} \nu_y \subset E_y$. In view of the convexity of b, we have

$$b(\lambda) \ge b(D\varpi) + a(|D\varpi|)D\varpi : (\lambda - D\varpi) =: B(\lambda), \ \text{ for all } \lambda \in \mathbb{M}^{m\times n}.$$

Since b is a C^1-function, it follows for $\tau \in \mathbb{R}$ that

$$\frac{b(\lambda + \tau F) - b(\lambda)}{\tau} \ge \frac{B(\lambda + \tau F) - B(\lambda)}{\tau} \quad \text{if } \tau > 0,$$

$$\frac{b(\lambda + \tau F) - b(\lambda)}{\tau} \le \frac{B(\lambda + \tau F) - B(\lambda)}{\tau} \quad \text{if } \tau < 0,$$

where $F \in \mathbb{M}^{m\times n}$. Thus $Db = DB$, and then

$$a(|\lambda|)\lambda = a(|D\varpi|)D\varpi \quad \text{ for all } \lambda \in E_y \supset \operatorname{supp} \nu_y. \tag{4.7}$$

The equi-integrability of $a(|D\varpi_k|)D\varpi_k$ implies that its weak L^1-limit is given by

$$\begin{aligned}\overline{a} := \int_{\mathbb{M}^{m\times n}} a(|\lambda|)\lambda \, \mathrm{d}\nu_y(\lambda) &= \int_{\operatorname{supp}\nu_x} a(|\lambda|)\lambda \, \mathrm{d}\nu_y(\lambda) \\ &= \int_{\operatorname{supp}\nu_y} a(|D\varpi|)D\varpi \, \mathrm{d}\nu_y(\lambda) = a(|D\varpi|)D\varpi,\end{aligned} \tag{4.8}$$

where we have used (4.7) and $\|\nu_y\|_{\mathcal{M}} = 1$.

Now consider the Carathéodory function

$$g(\lambda) = |a(|\lambda|)\lambda - \overline{a}(y)|, \quad \lambda \in \mathbb{M}^{m\times n}.$$

The sequence $g_k(y) := g(D\varpi_k(y))$ is equi-integrable by that of $a(|D\varpi_k(y)|)D\varpi_k(y)$, hence its weak L^1-limit is given by

$$g_k \rightharpoonup \overline{g} \quad \text{in } L^1(\Omega),$$

where

$$\begin{aligned}\overline{g}(y) &= \int_{\mathbb{M}^{m\times n}} |a(|\lambda|)\lambda - \overline{a}(y)| \, \mathrm{d}\nu_y(\lambda) \\ &= \int_{\operatorname{supp}\nu_y} |a(|\lambda|)\lambda - \overline{a}(y)| \, \mathrm{d}\nu_y(\lambda) = 0 \qquad \text{(by (4.8) and (4.7))}.\end{aligned}$$

Since $g_k \geq 0$, we deduce that $g_k \to 0$ in $L^1(\Omega)$ as $k \to \infty$.

Hence

$$\int_\Omega a(|D\varpi_k|)D\varpi_k : D\mathcal{U} \, \mathrm{d}y \to \int_\Omega a(|D\varpi|)D\varpi : D\mathcal{U} \, \mathrm{d}y \quad as \quad k \to \infty, \text{ that is,}$$

$$\langle L(\varpi_k), v\rangle \to \langle L(\varpi), v\rangle.$$

For the case when $(A_1)(ii)$, we have

$$\int_\Omega |D\varpi_k - D\varpi|^r \, \mathrm{d}y \leq c \int_\Omega \big(a(|D\varpi_k|)D\varpi_k - a(|D\varpi|)D\varpi\big) : (D\varpi_k - D\varpi) \, \mathrm{d}y.$$

We remark that the limit inferior of the right-hand side of the above inequality is less than or equal to zero by Remark 4.4. It follows that

$$\liminf_{k\to\infty} \int_\Omega |D\varpi_k - D\varpi|^r \, \mathrm{d}y = 0.$$

Let $E_{k,\epsilon} = \big\{y \in \Omega : |D\varpi_k - D\varpi| \geq \epsilon\big\}$. Then

$$\int_\Omega |D\varpi_k - D\varpi|^r \, \mathrm{d}y \geq \int_{E_{k,\epsilon}} |D\varpi_k - D\varpi|^r \, \mathrm{d}y \geq \epsilon^r \big|E_{k,\epsilon}\big|,$$

which gives

$$\big|E_{k\epsilon}\big| \leq \frac{1}{\epsilon^r} \int_\Omega |D\varpi_k - D\varpi|^r \, \mathrm{d}y \to 0 \quad \text{as } k \to \infty.$$

Hence

$$D\varpi_k \to D\varpi \quad \text{in measure on } \Omega \quad \text{(for a subsequence)}.$$

After extracting a suitable subsequence if necessary, we can infer that $D\varpi_k \to D\varpi$ for almost every $y \in \Omega$. Then $a(|D\varpi_k|)D\varpi_k \to a(|D\varpi|)D\varpi$ for almost every $y \in \Omega$, and in the measure. By the equi-integrability of $a(|D\varpi_k|)D\varpi_k : D\mathcal{U}$, the Vitali theorem implies

$$\int_\Omega a(|D\varpi_k|)D\varpi_k : D\mathcal{U} \text{ dy} \to \int_\Omega a(|D\varpi|)D\varpi : D\mathcal{U} \text{ d}y \text{ as } k \to \infty.$$

Hence $\langle L(\varpi_k),\mathcal{U}\rangle \to \langle L(\varpi),\mathcal{U}\rangle$, that is, to say L is strong-weakly continuous.

For the last case where $(A_2)(iii)$, we suppose that ν_y is not a Dirac measure on a set $y \in \Omega'$ of positive measure $|\Omega'| < \infty$. Since $\overline{\lambda} = \langle \nu_y,\ id\rangle = D\varpi(y)$ for a.e. $y \in \Omega$, we remark first that

$$\begin{aligned}&\int_{\mathbb{M}^{m\times n}} a(|\overline{\lambda}|)\overline{\lambda} : (\lambda - \overline{\lambda}) \text{ d}\nu_y(\lambda)\\ &= a(|\overline{\lambda}|)\overline{\lambda} : \int_{\mathbb{M}^{m\times n}} \lambda \text{ d}\nu_y(\lambda) - a(|\overline{\lambda}|)\overline{\lambda} : \overline{\lambda}\int_{\mathbb{M}^{m\times n}} \text{d}\nu_y(\lambda) = 0.\end{aligned}$$

Therefore, the strict r-quasimonotone implies

$$\int_{\mathbb{M}^{m\times n}} a(|\lambda|)\lambda : \lambda \text{ d}\nu_y(\lambda) > \int_{\mathbb{M}^{m\times n}} a(|\lambda|)\lambda : \overline{\lambda} \text{ d}\nu_y(\lambda).$$

After integrating the above inequality over Ω, we obtain by Lemma 4.3 the following contradiction:

$$\begin{aligned}\int_\Omega\int_{\mathbb{M}^{m\times n}} a(|\lambda|)\lambda : \lambda \text{ d}\nu_y(\lambda) &> \int_\Omega\int_{\mathbb{M}^{m\times n}} a(|\lambda|)\lambda : \overline{\lambda} \text{ d}\nu_y(\lambda)\\ &\geq \int_\Omega\int_{\mathbb{M}^{m\times n}} a(|\lambda|)\lambda : \lambda \text{ d}\nu_y(\lambda).\end{aligned}$$

Hence ν_y is a Dirac measure, that is, $\nu_y = \delta_{g(y)}$ for a.e. $y \in \Omega$. We have

$$g(y) = \int_{\mathbb{M}^{m\times n}} \lambda \text{ d}\delta_{g(y)}(\lambda) = \int_{\mathbb{M}^{m\times n}} \lambda \text{ d}\nu_y(\lambda) = D\varpi(y) \text{ for a.e. } y \in \Omega.$$

Therefore $\nu_y = \delta_{D\varpi(y)}$. By virtue of the (3.1), $D\varpi_k \to D\varpi$ in measure and almost everywhere (up to a subsequence). By the continuity of function a, $a(|D\varpi_k|)D\varpi_k \to a(|D\varpi|)D\varpi$ almost everywhere. Since, $a(|D\varpi_k|)D\varpi_k$ is equi-integrable, it follows from the Vitali convergence theorem that

$$a(|D\varpi_k|)D\varpi_k \to a(|D\varpi|)D\varpi \quad \text{in} \quad L^1(\Omega; \mathrm{M}^{m\times n}).$$

This is again the strong-weakly continuous of L on $\wp_{\psi,g}$ in this case.

- Let us now demonstrate that

$$\left\langle \varpi_k \left|\varpi_k\right|^{r-2}, \mathcal{U}\right\rangle \to \left\langle \varpi \left|\varpi\right|^{r-2}, \mathcal{U}\right\rangle \quad \text{as } k \to \infty.$$

Since $\varpi_k \to \varpi$ in $W^{1,r}(\Omega;\mathbb{R}^m)$, the continuity of the inner product implies

$$\langle \varpi_k, \mathcal{U}\rangle \to \langle \varpi, \mathcal{U}\rangle \quad \text{as } k \to \infty.$$

Thus, we can conclude that $\left\langle \varpi_k \left|\varpi_k\right|^{r-2}, \mathcal{U}\right\rangle$ remains bounded for all $\varpi_k \geq \psi$ almost everywhere in Ω, with $r \geq 1$.

Consequently, by the convergence $\varpi_k \to \varpi$, we obtain the following limit:

$$\left\langle \varpi_k \left|\varpi_k\right|^{r-2}, \mathcal{U}\right\rangle \to \left\langle \varpi \left|\varpi\right|^{r-2}, \mathcal{U}\right\rangle \quad \text{as } k \to \infty.$$

Next, we take the limit, and we obtain

$$\begin{aligned}(L(\varpi_k), \mathcal{U}) &= \int_\Omega a(|D\varpi_k|)D\varpi_k : D\mathcal{U} + \left\langle \varpi_k \left|\varpi_k\right|^{r-2}, \mathcal{U}\right\rangle \,\mathrm{d}z \\ &\to \int_\Omega a(|D\varpi|)D\varpi : D\mathcal{U} + \left\langle \varpi|\varpi|^{r-2}, \mathcal{U}\right\rangle \,\mathrm{d}z = (L(\varpi), \mathcal{U}).\end{aligned}$$

This shows the strong-weak continuity of L on $\varpi \in \wp_{\psi,g}$. This concludes the proof of Lemma 4.1.

□

We now have all the necessary elements to apply Theorem 3.1 to the mapping L. We conclude the existence of an element $\varpi \in \wp_{\psi,g}$ such that $\langle L(\varpi), \mathcal{U} - \varpi\rangle \geq 0$, that is,

$$\int_\Omega a(|D\varpi|)D\varpi : D(\mathcal{U} - \varpi) + \left\langle \varpi \left|\varpi\right|^{r-2}, \mathcal{U} - \varpi\right\rangle \,\mathrm{d}y \geq 0 \quad \text{for all } \mathcal{U} \in \wp_{\psi,g}.$$

Acknowledgement. The authors like to express their sincere thanks to the referee(s) for this paper.

References

1. M. Allalou, M. El Ouaarabi, H. El Hammar and A. Raji, *On a class of obstacle problem via Young measure in generalized Sobolev space*, Adv. Oper. Theory **9** (2024), no. 3, 1–23.
2. M. Allalou, M. El Ouaarabi and A. Raji, *Weak solution for obstacle problem with variable growth and weak monotonicity in Sobolev spaces with variable exponent*, Filomat, **38** (2024), no. 17, 6245–6257.
3. M. Allalou, M. El Ouaarabi and A. Raji, *Existence and uniqueness results for a class of obstacle problem via Young's measure theory*, Anal. Math. Phys. **14** (2024), no. 6, 123.
4. M. Allalou, M. El Ouaarabi and A. Raji, *Existence of weak solutions for nonlocal Dirichlet problems via Young measure theory*, Electron. J. Differ. Equ. **2024** (2024), no. 73, 1–12.
5. M. Allalou, A. Raji and K. Hilal, *Study of the existence and uniqueness of solutions for a class of Kirchhoff-type variational inequalities involving using Young measures*, Annali Dell'universita'di Ferrara, (2024) 1–20.
6. C. Baiocchi and A. Capelo, *Disequazioni variazionali e quasi variazionali. Applicazioni a problemi di frontiera libera*, Quaderni U.M.I. Pitagora, Bologna 1978. English transl. J. Wiley, Chichester-New York, 1984.
7. J.M. Ball, *A version of the fundamental theorem for Young measures*, In: PDEs and Continuum Models of Phase Transitions (Nice, 1988). Lect. Notes Phys. **344**, (1989) 207–215.
8. L. Caffarelli, *The regularity of elliptic and parabolic free boundaries*, Bull. Amer. Math. Soc. **82**, (1976) 616–618.
9. M. Chipot, *Variational inequalities and flow in porous media*, Springer-Verlag, Berlin and New York, 1984.
10. S. Christoph, *Gradient potential estimates in non-linear elliptic obstacle problems with measure data*, J. Func. Anal. **262** (2012), no. 6, 2777–2832.
11. G. Dolzmann, N, Hungerbühler and S. Muller, *Nonlinear elliptic systems with measure valued right hand side*, Math. **226**, (1997) 545–574.

12. H. El Hammar, S. Ait Temghart, C. Allalou and S. Melliani, *Existence results of quasilinear elliptic systems via Young measures*, International Journal of Nonlinear Analysis and Applications, **14** (2023), no. 1, 1111–1123.
13. H. El Hammar, C. Allalou and S. Melliani, *On strongly quasilinear degenerate elliptic systems with weak monotonicity and nonlinear physical data*, J. Math. Sci. (2022) 1–17.
14. L.C. Evans, *Weak convergence methods for nonlinear partial differential equations*, Amer. Math. Soc, **74**, 1990.
15. A.G. Grimaldi, *Regularity results for solutions to a class of obstacle problems*, Nonlinear Anal. Real World Appl. **62**, (2021) 103377.
16. N. Hungerbühler, *A refinement of Ball's theorem on Young measures*, N. Y. J. Math. **3**, (1997) 48–53.
17. N. Hungerbühler, *Quasilinear elliptic systems in divergence form with weak monotonicity*, N. Y. J. Math. **5**, (1999) 83–90.
18. M. Junxia and C. Yuming, *Boundary regularity of weak solutions to nonlinear elliptic obstacle problems*, Bound. Value Probl. **1**, (2006) 1–15.
19. D. Kinderlehrer and G. Stampacchia, *An introduction to variational inequalities and their applications*, Academic Press, NY, 1980.
20. J.L. Lions and G. Stampacchia, *Variational inequalities*, Commun. Pure Appl. Math. **20**, (1967) 493–519.
21. M.A. Ragusa, *On some local and nonlocal variational problems*, Trends Math. **5**, (2024) 321–331.
22. M.P. Tran, T.N. Nguyen and L.T.N. Pham, *Gradient bounds for non-uniformly quasilinear elliptic two-sided obstacle problems with variable exponents*, J. Math. Anal. Appl. **531** (2024), no. 1, 127776.
23. K. Yosida, *Functional analysis*, Springer, Berlin, 1980.
24. Ye. Yuquan and Q. Chen, *Optimal control of the obstacle in a quasilinear elliptic variational inequality*, J. Math. Anal. Appl. **294** (2004), no. 1, 258–272.

[1] Laboratory LMACS, Faculty of Science and Technics, Sultan Moulay Slimane University, BP 523, 23000, Beni Mellal, Morocco.
Email address: hasnaeelhammar11@gmail.com
Email address: mouadallalou@gmail.com
Email address: rajiabd2@gmail.com

[2] Fundamental and Applied Mathematics Laboratory, Faculty of Sciences Aïn Chock, Hassan II University, BP 5366, 20100, Casablanca, Morocco.
Email address: mohamedelouaarabi93@gmail.com