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COMPUTATION OF SOME PROPERTIES OF THE LAGUERRE TYPE MATRIX POLYNOMIALS $L_n^{(\mathcal{H},q)}(z)$

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Abstract. In this paper, we introduce the Laguerre type matrix polynomials $L_n^{(\mathcal{H},q)}(z)$. Also, we discuss various properties of the Laguerre type matrix polynomials $L_n^{(\mathcal{H},q)}(z)$ including hypergeometric matrix series representation, generating matrix relations, integral images, fractional calculus operators, differential recurrence relations, and summation formulas. These results are very helpful in theory of special functions, operator theory, and mathematical physics.

1. Introduction and preliminaries

Particularly with regard to orthogonal polynomials and special functions, matrix theory is crucial to every branch of mathematics. In addition to the matrix version of Laguerre, Hermite, Gegenbauer, and Chebyshev matrix polynomials [13, 7, 8, 12], the special matrix functions also appear in statistics, group representation theory, number theory, and Lie theory in [3, 4, 11, 14]. Let $\mathbf{C}^{\mathbf{r}\times\mathbf{r}}$ be the complex space of all $\mathbf{r}$-square complex matrices. For $\mathcal{H} \in \mathbf{C}^{\mathbf{r}\times\mathbf{r}}$, its spectrum $\lambda(\mathcal{H})$ denotes the set of all eigenvalues of $\mathcal{H}$ and $\alpha(\mathcal{H}) = \max[\Re(\rho) : \rho \in \lambda(\mathcal{H})]$, $\beta(\mathcal{H}) = \min[\Re(\rho) : \rho \in \lambda(\mathcal{H})]$. Any square matrix $\mathcal{H} \in \mathbf{C}^{\mathbf{r}\times\mathbf{r}}$ is a positive stable if $\Re(\rho) > 0$ for all $\rho \in \lambda(\mathcal{H})$. The identity matrix and zero matrix in $\mathbf{C}^{\mathbf{r}\times\mathbf{r}}$ are denoted by I and $\mathbf{0}$, respectively.

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If $\mathcal{H} \in \mathbf{C}^{\mathbf{r}\times\mathbf{r}}$ and $\mathcal{H}+nI$ is an invertible matrix for all $n \in \mathcal{Z}^+ \cup \{\mathbf{0}\}$, then the matrix version of the pochhammer symbol $(\mathcal{H})_n$ is defined by [9]

$$\begin{aligned}(\mathcal{H})_n &= \Gamma(\mathcal{H}+nI)\Gamma^{-1}(\mathcal{H}) \\ &= \begin{cases} \mathcal{H}(\mathcal{H}+I)(\mathcal{H}+2I)\dots(\mathcal{H}+(n-1)I) & (n \in \mathbf{N}), \\ I & (n=0). \end{cases}\end{aligned} \tag{1.1}$$

Let $\mathcal{H}$ and $\mathcal{K}$ be commuting matrices in $\mathbf{C}^{\mathbf{r}\times\mathbf{r}}$ such that for $n \in \mathcal{Z}^+ \cup \{\mathbf{0}\}$, $\mathcal{H}+nI$, $\mathcal{K}+nI$, and $\mathcal{H}+\mathcal{K}+nI$ are invertible. Then by [9],

$$B(\mathcal{H},\mathcal{K}) = \Gamma(\mathcal{H})\Gamma(\mathcal{K})\Gamma^{-1}(\mathcal{H}+\mathcal{K}). \tag{1.2}$$

For positive stable matrices $\mathcal{H},\mathcal{K} \in \mathbf{C}^{\mathbf{r}\times\mathbf{r}}$, Jódar and Cortés [10] defined the Beta matrix function as

$$B(\mathcal{H},\mathcal{K}) = \int_0^1 x^{\mathcal{H}-I}(1-x)^{\mathcal{K}-I}dx. \tag{1.3}$$

For a positive stable matrix $\mathcal{H} \in \mathbf{C}^{\mathbf{r}\times\mathbf{r}}$, Jódar and Cortés [10] defined the Gamma matrix function [10] as

$$\Gamma(\mathcal{H}) = \int_0^\infty e^{-x}x^{\mathcal{H}-I}dx. \tag{1.4}$$

Dwivedi and Sahai [5] discussed the generalized hypergeometric matrix function as

$$\begin{aligned}&{}_pF_q\left[\begin{matrix}\mathcal{H}_1,\mathcal{H}_2,\dots,\mathcal{H}_p \\ \mathcal{K}_1,\mathcal{K}_2,\dots,\mathcal{K}_q\end{matrix} ;\ z\right] \\ &= \sum_{n\geq 0} \frac{(\mathcal{H}_1)_n(\mathcal{H}_2)_n\dots(\mathcal{H}_p)_n[(\mathcal{K}_1)_n]^{-1}[(\mathcal{K}_2)_n]^{-1}\dots[(\mathcal{K}_q)_n]^{-1}z^n}{n!},\end{aligned} \tag{1.5}$$

where $\mathcal{H}_i,\mathcal{K}_j \in \mathbf{C}^{\mathbf{r}\times\mathbf{r}}$, $|z|<1$, $1 \leq i \leq p$, $1 \leq j \leq q$, $p,q \in \mathcal{K}$ and $\mathcal{K}_j+kI$ are invertible for all $k \in \mathbf{Z}^+ \cup \{\mathbf{0}\}$.

The Laguerre matrix polynomial $L_n^{(\mathcal{H},\lambda)}(z)$ is defined by Jódar [8] as

$$L_n^{(\mathcal{H},\lambda)}(z) = \sum_{k=0}^n \frac{(-1)^k\lambda^k}{k!(n-k)!}(\mathcal{H}+I)_n[(\mathcal{H}+I)_k]^{-1}z^k, \tag{1.6}$$

where λ is a complex number with $\Re(\lambda)>0$ and $\mathcal{H}$ is a matrix in $\mathbf{C}^{\mathbf{r}\times\mathbf{r}}$ with $\mathcal{H}+nI$ invertible for all $n \geq 1$.

Setting $\lambda=1$ in (1.6), this gives [1]

$$L_n^{\mathcal{H}}(z) = \frac{(\mathcal{H}+I)_n}{n!}\,{}_1F_1\left[\begin{matrix}-nI \\ \mathcal{H}+I\end{matrix} ;\ z\right]. \tag{1.7}$$

Lemma 1.1. *([6]) Let $\mathcal{P}(p,n)$, $\mathcal{Q}(p,n)$ be matrices in $\mathbf{C}^{\mathbf{r}\times\mathbf{r}}$. Then the following series relations are satisfied:*

$$\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\mathcal{P}(p,n)=\sum_{n=0}^{\infty}\sum_{p=0}^{n}\mathcal{P}(p,n-p),$$
$$\sum_{n=0}^{\infty}\sum_{p=0}^{n}\mathcal{Q}(p,n)=\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\mathcal{Q}(p,n+p).$$

2. The Laguerre type matrix polynomials $L_n^{(\mathcal{H},q)}(z)$

Definition 2.1. Let $\mathcal{H}\in\mathbf{C}^{\mathbf{r}\times\mathbf{r}}$ be a matrix satisfying the spectral condition,

$$\Re(\rho)>-1,\quad \text{for all } \rho\in\lambda(\mathcal{H}), \tag{2.1}$$

and $q\in\mathbf{Z}^{+}$. Then the matrix polynomials $L_n^{(\mathcal{H},q)}(z)$ are defined as

$$L_n^{(\mathcal{H},q)}(z)=\frac{(\mathcal{H}+I)_{qn}}{n!}\sum_{k=0}^{n}\frac{(-nI)_k z^k}{k!}(\mathcal{H}+I)_{qk}^{-1}. \tag{2.2}$$

Remark 2.2. For $q=1$, this reduces to the Laguerre matrix polynomial $L_n^{\mathcal{H}}(z)$ defined in [1].

2.1. **Hypergeometric Matrix Series Representation.** Let $\mathcal{H}\in\mathbf{C}^{\mathbf{r}\times\mathbf{r}}$ be a matrix satisfying the spectral condition (2.1), $q\in\mathbf{Z}^{+}$ and $|\frac{z}{q^q}|<1$. Then hypergeometric matrix representation of the Laguerre type matrix polynomials (2.1) is given by

$$L_n^{(\mathcal{H},q)}(z)=\frac{(\mathcal{H}+I)_{qn}}{n!}\,{}_1F_q\left[\begin{matrix}-nI\\ \Delta(q;\mathcal{H}+I)\end{matrix}\;;\;\frac{z}{q^q}\right], \tag{2.3}$$

where

$$\Delta(q;\mathcal{H}+I)=\left(\frac{\mathcal{H}+I}{q}\right),\left(\frac{\mathcal{H}+2I}{q}\right),\left(\frac{\mathcal{H}+3I}{q}\right),\ldots,\left(\frac{\mathcal{H}+qI}{q}\right).$$

3. Generating matrix functions

Theorem 3.1. *Let $\mathcal{H}\in\mathbf{C}^{\mathbf{r}\times\mathbf{r}}$ be a matrix satisfying the spectral condition (2.1), let $\omega\in\mathbf{C}$, $q\in\mathbf{Z}^{+}$, let $\Re(\omega)>0$, and let $|\frac{-\omega z}{q^q}|<1$. Then the following matrix generating function holds:*

$$\sum_{n=0}^{\infty}\omega^n\left[(\mathcal{H}+I)_{qn}\right]^{-1}L_n^{(\mathcal{H},q)}(z)=e^{\omega}\,{}_0F_q\left[\begin{matrix}-\\ \Delta(q;\mathcal{H}+I)\end{matrix}\;;\;\frac{-\omega z}{q^q}\right]. \tag{3.1}$$

Proof. From the left-hand side of (3.1)

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\omega^n\left[(\mathcal{H}+I)_{qn}\right]^{-1}L_n^{(\mathcal{H},q)}(z)\\
&=\sum_{n=0}^{\infty}\left[(\mathcal{H}+I)_{qn}\right]^{-1}\left(\frac{(\mathcal{H}+I)_{qn}}{n!}\sum_{k=0}^{n}\frac{(-nI)_k z^k}{k!}\left[(\mathcal{H}+I)_{qk}\right]^{-1}\right)\omega^n\\
&=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^k z^k}{(n-k)!k!}\left[(\mathcal{H}+I)_{qk}\right]^{-1}\omega^n. \qquad (3.2)
\end{aligned}
$$

Applying Lemma 1.1, we get

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\omega^n\left[(\mathcal{H}+I)_{qn}\right]^{-1}L_n^{(\mathcal{H},q)}(z)\\
&=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-z)^k}{n!k!}\left[(\mathcal{H}+I)_{qk}\right]^{-1}\omega^{n+k}\\
&=\sum_{n=0}^{\infty}\frac{\omega^n}{n!}\sum_{k=0}^{\infty}\frac{(-\omega z)^k}{k!}\left[(\mathcal{H}+I)_{qk}\right]^{-1}\\
&=e^{\omega}\sum_{k=0}^{\infty}\frac{\left(\frac{-\omega z}{q^q}\right)^k}{k!}\prod_{m=1}^{q}\left[\left(\frac{\mathcal{H}+mI}{\delta}\right)_k\right]^{-1}\\
&=e^{\omega}\,{}_0F_q\left[\begin{matrix}-\\ \Delta(q;\mathcal{H}+I)\end{matrix};\frac{-\omega z}{q^q}\right].
\end{aligned}
$$

□

Corollary 3.2. *Let $\mathcal{H}\in\mathbf{C}^{\mathbf{r}\times\mathbf{r}}$ be a matrix satisfying the spectral condition (2.1), let $\omega\in\mathbf{C}$, $q\in\mathbf{Z}^+$, let $\Re(\omega)>0$, and let $|\frac{-\omega z}{q^q}|<1$. Then the following matrix generating relation holds:*

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\omega^n\Gamma^{-1}(\mathcal{H}+(qn+1)I)L_n^{(\mathcal{H},q)}(z)\\
&=e^{\omega}\Gamma^{-1}(\mathcal{H}+I)\,{}_0F_q\left[\begin{matrix}-\\ \Delta(q;\mathcal{H}+I)\end{matrix};\frac{-\omega z}{q^q}\right]. \qquad (3.3)
\end{aligned}
$$

Theorem 3.3. *Let $\mathcal{H},\mathcal{K}\in\mathbf{C}^{\mathbf{r}\times\mathbf{r}}$ be commuting matrices satisfying the spectral condition (2.1), let $\omega\in\mathbf{C}$, $q\in\mathbf{Z}^+$, let $\Re(\omega)>0$, and let $|\omega q^q|<1$. Then the* *following matrix generating relation holds:*

$$\sum_{n=0}^{\infty}\Gamma(\mathcal{K}+qnI)\Gamma^{-1}(\mathcal{H}+(qn+1)I)L_n^{(\mathcal{H},q)}(z)\omega^n$$
$$=\sum_{k=0}^{\infty}\frac{(-\omega z)^k(\mathcal{K})_{qk}}{k!}\Gamma^{-1}(\mathcal{H}+(qk+1)I)\Gamma(\mathcal{K})$$
$$\times\ {}_qF_0\left[\begin{matrix}\Delta(q;\mathcal{K}+qkI)\\ -\end{matrix}\ ;\ \omega q^q\right]. \tag{3.4}$$

Proof. From the left-hand side of (3.4),

$$\sum_{n=0}^{\infty}\Gamma(\mathcal{K}+qnI)\Gamma^{-1}(\mathcal{H}+(qn+1)I)L_n^{(\mathcal{H},q)}(z)\omega^n$$
$$=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-z)^k}{(n-k)!k!}\Gamma(\mathcal{K}+qnI)\Gamma^{-1}(\mathcal{H}+(qk+1)I)\omega^n. \tag{3.5}$$

Applying Lemma 1.1, we obtain

$$\sum_{n=0}^{\infty}\Gamma(\mathcal{K}+qnI)\Gamma^{-1}(\mathcal{H}+(qn+1)I)L_n^{(\mathcal{H},q)}(z)\omega^n$$
$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-\omega z)^k}{n!k!}\Gamma^{-1}(\mathcal{H}+(qk+1)I)\Gamma(\mathcal{K}+qnI+qkI)\omega^n$$
$$=\sum_{k=0}^{\infty}\frac{(-\omega z)^k(\mathcal{K})_{qk}}{k!}\Gamma^{-1}(\mathcal{H}+(qk+1)I)\Gamma(\mathcal{K})$$
$$\times\sum_{n=0}^{\infty}\frac{(\omega q^q)^n}{n!}\prod_{m=1}^{q}\left(\frac{\mathcal{K}+(qk+m-1)I}{q}\right)_n$$
$$=\sum_{k=0}^{\infty}\frac{(-\omega z)^k(\mathcal{K})_{qk}}{k!}\Gamma^{-1}(\mathcal{H}+(qk+1)I)\Gamma(\mathcal{K})$$
$$\times\ {}_qF_0\left[\begin{matrix}\Delta(q;\mathcal{K}+qkI)\\ -\end{matrix}\ ;\ \omega q^q\right].$$

□

Setting $\mathcal{K}=\mathcal{H}+I$ in (3.4), we deduce the following corollary.

Corollary 3.4. *Let $\mathcal{H}\in\mathbf{C}^{\mathbf{r}\times\mathbf{r}}$ be a matrix satisfying the spectral condition (2.1), let $\omega\in\mathbf{C}$, $q\in\mathbf{Z}^+$, let $\Re(\omega)>0$, and let $|\omega q^q|<1$. Then the following matrix generating relation holds:*

$$\sum_{n=0}^{\infty}L_n^{(\mathcal{H},q)}(z)\omega^n=\sum_{k=0}^{\infty}\frac{(-\omega z)^k}{k!}\ {}_qF_0\left[\begin{matrix}\Delta(q;\mathcal{H}+(qk+1)I)\\ -\end{matrix}\ ;\ \omega q^q\right]. \tag{3.6}$$

4. Integral images

Theorem 4.1. *Let $\mathcal{H} \in \mathbf{C}^{\mathbf{r}\times\mathbf{r}}$ be a matrix satisfying the spectral condition (2.1), let $q \in \mathbf{Z}^+$, and let $|\frac{1}{q^q}| < 1$. Then*

$$\int_0^\infty z^{\mathcal{H}} e^{-z} L_n^{(\mathcal{H},q)}(z)dz = \frac{\Gamma(\mathcal{H}+(qn+1)I)}{n!}\, {}_2F_q\left[\begin{matrix}-nI, \mathcal{H}+I \\ \Delta(q;\mathcal{H}+I)\end{matrix} ;\ \frac{1}{q^q}\right]. \tag{4.1}$$

Proof. From left-hand side of (4.1), we find

$$\begin{aligned}
&\int_0^\infty z^{\mathcal{H}} e^{-z} L_n^{(\mathcal{H},q)}(z)dz \\
&= \frac{(\mathcal{H}+I)_{qn}}{n!}\sum_{k=0}^{n}\frac{(-nI)_k}{k!q^{qk}}\prod_{m=1}^{q}\left[\left(\frac{\mathcal{H}+kI}{q}\right)_n\right]^{-1} \\
&\quad\times\left(\int_0^\infty z^{\mathcal{H}+kI}e^{-z}dz\right) \\
&= \frac{\Gamma(\mathcal{H}+(qn+1)I)}{n!}\sum_{k=0}^{n}\frac{(-nI)_k(\mathcal{H}+I)_k}{k!q^{qk}} \\
&\quad\times\prod_{m=1}^{q}\left[\left(\frac{\mathcal{H}+kI}{q}\right)_n\right]^{-1} \\
&= \frac{\Gamma(\mathcal{H}+(qn+1)I)}{n!}\, {}_2F_q\left[\begin{matrix}-nI, \mathcal{H}+I \\ \Delta(q;\mathcal{H}+I)\end{matrix} ;\ \frac{1}{q^q}\right].
\end{aligned}$$

□

Setting $q = 1$ in (4.1), we deduce the following result.

Corollary 4.2. *Let $\mathcal{H} \in \mathbf{C}^{\mathbf{r}\times\mathbf{r}}$ be a matrix satisfying the spectral condition (2.1). Then*

$$\int_0^\infty z^{\mathcal{H}} e^{-z} L_n^{\mathcal{H}}(z)dz = \frac{\Gamma(\mathcal{H}+(n+1)I)}{n!}\, {}_1F_0\left[\begin{matrix}-nI \\ -\end{matrix} ;\ 1\right]. \tag{4.2}$$

Theorem 4.3. *Let $\mathcal{H}, \mathcal{K} \in \mathbf{C}^{\mathbf{r}\times\mathbf{r}}$ be matrices such that $\mathcal{HK} = \mathcal{KH}$, satisfying the spectral condition (2.1) and $q \in \mathbf{Z}^+$. Then*

$$\begin{aligned}
&L_n^{(\mathcal{H}+\mathcal{K},q)}(z) \\
&= \Gamma^{-1}(\mathcal{K})\Gamma(\mathcal{H}+\mathcal{K}+(qn+1)I)\Gamma^{-1}(\mathcal{H}+(qn+1)I) \\
&\quad\times\int_0^1 u^{\mathcal{H}}(1-u)^{\mathcal{K}-I}L_n^{(\mathcal{H},q)}(zu^q)du.
\end{aligned} \tag{4.3}$$

Proof. Consider the left-hand side of (4.3),

$$\begin{aligned}\Psi &= \int_0^1 u^{\mathcal{H}}(1-u)^{\mathcal{K}-I} L_n^{(\mathcal{H},q)}(zu^q)du \\ &= \frac{\Gamma(\mathcal{H}+(qn+1)I)}{n!}\sum_{k=0}^{n}\frac{(-nI)_k z^k}{k!}\Gamma^{-1}(\mathcal{H}+qnI+I) \\ &\quad \times \int_0^1 u^{\mathcal{H}+qkI}(1-u)^{\mathcal{K}-I}du. \end{aligned} \tag{4.4}$$

Using (1.2) and (1.3), we obtain

$$\begin{aligned}\Psi &= \frac{\Gamma(\mathcal{K})\Gamma(\mathcal{H}+(qn+1)I))}{n!}\sum_{k=0}^{n}\frac{(-nI)_k z^k}{k!}\Gamma^{-1}(\mathcal{H}+\mathcal{K}+(qk+1)I) \\ &= \Gamma(\mathcal{K})\Gamma(\mathcal{H}+(qn+1)I))\Gamma^{-1}(\mathcal{H}+\mathcal{K}+(q+1)I)L_n^{(\mathcal{H}+\mathcal{K},q)}(z). \end{aligned} \tag{4.5}$$

Using (4.5) in right-hand side of (4.3), we get

$$\begin{aligned}&L_n^{(\mathcal{H}+\mathcal{K},q)}(z) \\ &= \Gamma^{-1}(\mathcal{K})\Gamma(\mathcal{H}+\mathcal{K}+(qn+1)I)\Gamma^{-1}(\mathcal{H}+(qn+1)I) \\ &\quad \times \int_0^1 u^{\mathcal{H}}(1-u)^{\mathcal{K}-I}L_n^{(\mathcal{H},q)}(zu^q)du.\end{aligned}$$

□

Setting $q=1$ in (4.3), we find the following result.

Corollary 4.4. *Let $\mathcal{H},\mathcal{K}\in\mathbf{C}^{\mathbf{r}\times\mathbf{r}}$ be matrices such that $\mathcal{H}\mathcal{K}=\mathcal{K}\mathcal{H}$, satisfying the spectral condition (2.1). Then*

$$\begin{aligned}&L_n^{\mathcal{H}+\mathcal{K}}(z) \\ &= \Gamma^{-1}(\mathcal{K})\Gamma(\mathcal{H}+\mathcal{K}+(n+1)I)\Gamma^{-1}(\mathcal{H}+(n+1)I) \\ &\quad \times \int_0^1 u^{\mathcal{H}}(1-u)^{\mathcal{K}-I}L_n^{\mathcal{H}}(zu)du.\end{aligned} \tag{4.6}$$

5. Fractional calculus approach

Definition 5.1. ([2]) Let $\mathcal{H}\in\mathbf{C}^{\mathbf{r}\times\mathbf{r}}$ be a positive stable matrix and $\mu\in\mathbf{C}$ with $\Re(\mu)>0$. Then the Riemann–Liouville fractional integral of order μ is

$$I^{\mu}[z^{\mathcal{H}}] = \frac{1}{\Gamma(\mu)}\int_0^z (z-t)^{\mu-1}t^{\mathcal{H}}dt. \tag{5.1}$$

Lemma 5.2. *([2]) Let $\mathcal{H}\in\mathbf{C}^{\mathbf{r}\times\mathbf{r}}$ be a positive stable matrix and let $\mu\in\mathbf{C}$ with $\Re(\mu)>0$. Then the Riemann–Liouville fractional integral of order μ is*

$$I^{\mu}[z^{\mathcal{H}-I}] = \Gamma(\mathcal{H})\Gamma^{-1}(\mathcal{H}+\mu I)z^{\mathcal{H}+(\mu-1)I}. \tag{5.2}$$

Theorem 5.3. *Let $\mathcal{H} \in \mathbf{C}^{\mathbf{r}\times\mathbf{r}}$ be a matrix satisfying the spectral condition (2.1), let $\mu \in \mathbf{C}$, let $\Re(\mu) > 0$, and let $q \in \mathbf{Z}^+$. Then*

$$\begin{aligned} &I^{\mu}\left[z^{\mathcal{H}} L_n^{(\mathcal{H},q)}(z^q)\right] \\ &= z^{\mathcal{H}+\mu I}\Gamma(\mathcal{H}+qnI+I)\Gamma^{-1}(\mathcal{H}+qnI+\mu I+I)L_n^{(\mathcal{H}+\mu I,q)}(z^q). \end{aligned} \tag{5.3}$$

Proof. From left-hand side of (5.3) and applying Lemma 5.2, we get

$$\begin{aligned} &I^{\mu}\left[z^{\mathcal{H}} L_n^{(\mathcal{H},q)}(z^q)\right] \\ &= \frac{1}{\Gamma(\mu)}\int_0^z (z-t)^{\mu-1} t^{\mathcal{H}} L_n^{(\mathcal{H},q)}(z^q)dt \\ &= \frac{\Gamma(\mathcal{H}+qnI+I)}{n!}\sum_{k=0}^{n}\frac{(-nI)_k}{k!}\Gamma^{-1}(\mathcal{H}+qkI+I)I^{\mu}(z^{\mathcal{H}+qkI}) \\ &= \frac{z^{\mathcal{H}+\mu I}\Gamma(\mathcal{H}+qnI+I)}{n!}\sum_{k=0}^{n}\frac{(-nI)_k z^{qk}}{k!}\Gamma^{-1}(\mathcal{H}+qkI+\mu I+I), \end{aligned}$$

which yields the result in (5.3). □

Setting $q = 1$ in (5.3), we deduce the following result.

Corollary 5.4. *Let $\mathcal{H} \in \mathbf{C}^{\mathbf{r}\times\mathbf{r}}$ be a matrix satisfying the spectral condition (2.1), let $\mu \in \mathbf{C}$, let $\Re(\mu) > 0$, and let $q \in \mathbf{Z}^+$. Then*

$$I^{\mu}\left[z^{\mathcal{H}} L_n^{\mathcal{H}}(z)\right] = z^{\mathcal{H}+\mu I}\Gamma(\mathcal{H}+(n+1)I)\Gamma^{-1}(\mathcal{H}+(n+\mu+1)I)L_n^{\mathcal{H}+\mu I}(z). \tag{5.4}$$

6. Differential recurrence relations and summation formulas

Theorem 6.1. *Let $\mathcal{H} \in \mathbf{C}^{\mathbf{r}\times\mathbf{r}}$ be a matrix satisfying the spectral condition (2.1) and let $q \in \mathbf{Z}^+$. Then the following differential recurrence relations hold:*

$$\frac{d}{dz}\left(z^{\mathcal{H}} L_n^{(\mathcal{H},q)}(z^q)\right) = z^{(\mathcal{H}-I)}(\mathcal{H}+qnI)L_n^{(\mathcal{H}-I,q)}(z^q) \tag{6.1}$$

$$\mathcal{H}L_n^{(\mathcal{H},q)}(z) + zq\frac{d}{dz}L_n^{(\mathcal{H},q)}(z) = (\mathcal{H}+qnI)L_n^{(\mathcal{H}-I,q)}(z). \tag{6.2}$$

Proof. From left-hand side of (6.1), we get

$$\begin{aligned} &\frac{d}{dz}\left(z^{\mathcal{H}} L_n^{(\mathcal{H},q)}(z^q)\right) \\ &= z^{(\mathcal{H}-I)}\frac{\Gamma(\mathcal{H}+(qn+1)I)}{n!} \\ &\times \sum_{k=0}^{n}\frac{(-nI)_k(\mathcal{H}+qkI)z^{qk}}{k!}\Gamma^{-1}(\mathcal{H}+(qk+1)I) \\ &= z^{(\mathcal{H}-I)}(\mathcal{H}+qnI)\frac{(\mathcal{H})_{qn}}{n!}\sum_{k=0}^{n}\frac{(-nI)_k z^{qk}}{k!}\left[(\mathcal{H})_{qk}\right]^{-1} \\ &= z^{(\mathcal{H}-I)}(\mathcal{H}+qnI)L_n^{(\mathcal{H}-I,q)}(z^q). \end{aligned}$$

This completes the proof of the result (6.1). Similarly, by making use of (2.2) in right-hand side of (6.2), we obtain the result (6.2). □

Theorem 6.2. *Let $\mathcal{H} \in \mathbf{C}^{\mathrm{r}\times\mathrm{r}}$ be a matrix satisfying the spectral condition (2.1), let $\mu \in \mathbf{C}$, let $\Re(\mu) > 0$, and let $q \in \mathbf{Z}^+$. Then the following summation formulas hold:*

$$\sum_{m=0}^{p} \frac{\begin{pmatrix} p \\ m \end{pmatrix} z^m}{(\mathcal{H}+I-kI)_m} D_z^m L_n^{(\mathcal{H},q)}(z^q) = \frac{(\mathcal{H}+I)_{qn}}{(\mathcal{H}+I-kI)_{qn}} L_n^{(\mathcal{H}-pI,q)}(z^q), \tag{6.3}$$

$$\sum_{m=0}^{p} \frac{(\mu)^m}{m!} D_z^m L_n^{(\mathcal{H},q)}(z) = L_n^{(\mathcal{H},q)}(z+\mu). \tag{6.4}$$

Proof. From left-hand side of (6.3), we get

$$\begin{aligned}
&\sum_{m=0}^{p} \frac{\begin{pmatrix} p \\ m \end{pmatrix} z^m}{(\mathcal{H}+I-kI)_m} D_z^m L_n^{(\mathcal{H},q)}(z^q) \\
&= \frac{(\mathcal{H}+I)_{qn}}{n!} \sum_{m=0}^{p} \frac{(-kI)_m}{m!(\mathcal{H}+I-kI)_m} \\
&\quad \times \left(\sum_{k=0}^{n} \frac{(-nI)_k(-qk)_m z^{qk}}{k!} \left[(\mathcal{H}+I)_{qk}\right]^{-1} \right) \\
&= \frac{(\mathcal{H}+I)_{qn}}{n!} \sum_{k=0}^{n} \frac{(-nI)_k z^{qk}}{k!} \left[(\mathcal{H}+I-pI)_{qk}\right]^{-1} \\
&= \frac{(\mathcal{H}+I)_{qn}}{(\mathcal{H}+I-kI)_{qn}} L_n^{(\mathcal{H}-pI,q)}(z^q).
\end{aligned}$$

This completes the proof of the result (6.3). Similarly, we can prove the result (6.4).

□

Concluding Remarks. In this article, we introduced Laguerre type matrix polynomials and obtained several properties, such as the hypergeometric matrix function representation, generating matrix functions, integral images, fractional integral operator, differential recurrence relations, and summation formulas. These obtained results may be useful in the theory of special matrix functions, operator theory, and mathematical physics.

Conflict of Interest. The authors have no conflict of interests related to this publication.

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