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A NOVEL APPROACH FOR VECTOR LYAPUNOV FUNCTIONS AND PRACTICAL STABILITY OF CAPUTO FRACTIONAL DYNAMIC EQUATIONS ON TIME SCALE IN TERMS OF TWO MEASURES

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Abstract. This paper presents a novel approach to analyze the practical stability of Caputo fractional dynamic equations on time scales in terms of two measures (m, m_0), utilizing a new generalized derivative known as the Caputo fractional delta derivative, and the Caputo fractional delta Dini derivative of order $\alpha \in (0,1)$. This generalized concept not only provides a unified framework for analyzing dynamic systems across both continuous and discrete time domains, but also includes and unifies several known concepts of stability in a single setup making it suitable for hybrid systems exhibiting both gradual and abrupt changes. The established two measures practical stability results are demonstrated through an illustrative example.

1. Introduction

The theory of Lyapunov stability is fundamental to analyze and control the behavior of dynamic systems and has been widely applied across fields to ensure system reliability. Despite its foundational role, asymptotic stability is often more critical in practical applications, where the ability to approximate stability within a specific region provides insights into a system's resilience and operational efficacy. However, there are scenarios where a system might exhibit persistent oscillations or deviations close to an equilibrium point, yet maintain satisfactory performance without strict asymptotic convergence. Classical Lyapunov stability measures fall short in such cases, as they do not address system stability within

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practical bounds of acceptable performance. Instead, a more adaptable approach, known as practical stability, is employed. This concept assesses whether the system state remains within a specified range, which is often more realistic and less restrictive, accommodating the inevitable uncertainties and disturbances encountered in real-world systems. This framework allows systems to remain practically stable, operating reliably within certain boundaries.

Previous research, such as [1, 4, 5, 6, 7, 11, 12, 13, 14], has primarily focused on stability, uniform, asymptotic stability, and variational stability for delay, and impulsive, all on continuous time systems. On the other hand, studies like [17, 20] have considered stability in discrete domains. This work aims to address some gaps by developing a practical stability framework that is suitable for both continuous and discrete time scales, thereby providing a more comprehensive and realistic assessment of the behavior of fractional dynamic systems.

Practical stability, particularly when formulated in terms of two measures, is valuable for real-world applications where exact stability conditions may be impractically stringent. By allowing the system state to stay within a bounded region over time, practical stability frameworks provide a flexible approach that acknowledges minor deviations from equilibrium as acceptable, particularly in uncertain environments. This more accommodating approach has profound implications in engineering, control systems, and applied sciences, where minor instabilities may not compromise overall system function but instead reflect an inherent, acceptable level of fluctuation.

With the introduction of time scale calculus in [9], a unified approach for analyzing systems evolving in both discrete and continuous time domains has emerged, extending traditional continuous analysis techniques to discrete cases. Integrating this calculus with fractional dynamics enables a more sophisticated framework for understanding systems that undergo both gradual and abrupt changes–common in hybrid systems. When combined with fractional calculus, time scale calculus provides a versatile platform for analyzing dynamic behaviors in both continuous and discrete settings.

Building on these advancements, this paper extends the results in [15, 19, 21, 22, 23], presenting a novel methodology for examining the (m_0, m)-practical stability and (m_0, m)-practical asymptotic stability of Caputo fractional dynamic equations on time scales. By leveraging two measures, m and m_0, this approach incorporates a new generalized derivative, the Caputo fractional delta derivative and the Caputo fractional delta Dini derivative of order $\alpha \in (0, 1)$ to provide a robust framework for stability analysis across different time domains. This generalized concept unifies several known stability concepts, rendering it well-suited for systems that exhibit both continuous and discrete dynamics. The notion of practical stability in terms of two measures offers significant flexibility, ensuring system behavior remains within acceptable bounds across both gradual and abrupt transitions. Through the establishment of comparison results and practical stability conditions, this framework not only enriches existing stability theory but also provides an essential tool for ensuring that dynamic systems operate reliably under practical conditions. By considering two measures in the stability analysis, the results contribute a realistic and flexible framework for stability, that is, highly applicable across diverse fields such as engineering and control theory, where managing fluctuations within acceptable bounds is crucial for real world application [19].

Consider the Caputo fractional dynamic system of order α with $0 < \alpha < 1$

$$\begin{aligned} {}^{C}\Delta^{\alpha} x &= f(t, x), \qquad t \in \mathbb{T}, \\ x(t_0) &= x_0, \qquad t_0 \geq 0, \end{aligned} \tag{1.1}$$

where $f \in C_{rd}[\mathbb{T} \times \mathbb{R}^N, \mathbb{R}^N]$, $f(t, 0) \equiv 0$, and ${}^{C}\Delta^{\alpha} x$ is the Caputo fractional delta derivative of $x \in \mathbb{R}^N$ of order α with respect to $t \in \mathbb{T}$. Let $x(t) = x(t, t_0, x_0) \in C^{\alpha}_{rd}[\mathbb{T}, \mathbb{R}^N]$ be a solution of (1.1), and suppose that the function f is smooth enough to guarantee existence, uniqueness and continuous dependence of solutions (results on existence and uniqueness of (1.1) are contained in [3, 11, 18]). This work aims to investigate the (m_0, m)-practical stability and (m_0, m)-practical asymptotic stability of the system (1.1).

The content of this research is organized as follows: In section 2, we introduce essential terminologies, remarks, and fundamental lemmas that form the foundation for subsequent developments. We present key definitions and important remarks relevant to the study. In section 3, we present the main results of our research, including the theoretical advancements and findings. In section 4, we give a practical example to demonstrate the relevance and application of our proposed approach. Then, In section 5, we summarize the key findings of this study and discuss their implication in the conclusion.

2. Preliminaries

Definition 2.1 ([8])**.** For $t \in \mathbb{T}$, the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined as

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\},$$

while the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined as

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\}.$$

(i) If $\sigma(t) > t$, then t is right scattered,
(ii) If $\rho(t) < t$, then t is left scattered,
(iii) If $t < \max \mathbb{T}$ and $\sigma(t) = t$, then t is called right dense,
(iv) If $t > \min \mathbb{T}$ and $\rho(t) = t$, then t is called left dense.

Definition 2.2 ([8])**.** The graininess function $\mu : \mathbb{T} \to [0, \infty)$ for $t \in \mathbb{T}$ is defined as

$$\mu(t) = \sigma(t) - t.$$

The derivative uses the set $\mathbb{T}^k$, which is derived from the time scale $\mathbb{T}$ as follows. If $\mathbb{T}$ has a left scattered maximum M, then $\mathbb{T}^k = \mathbb{T} \setminus \{M\}$. Otherwise, $\mathbb{T}^k = \mathbb{T}$.

Definition 2.3 ([8])**.** Let $h : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$. We define the delta derivative h^{Δ} also known as the Hilger derivative as

$$h^{\Delta}(t) = \lim_{s \to t} \frac{h(\sigma(t)) - h(s)}{\sigma(t) - s}, \quad s \neq \sigma(t),$$

provided the limit exists.

The function $h^\Delta : \mathbb{T} \to \mathbb{R}$ is called the (Delta) derivative of h on $\mathbb{T}^k$.

If t is right dense, then the delta derivative of $h : \mathbb{T} \to \mathbb{R}$, becomes

$$h^\Delta(t) = \lim_{s \to t} \frac{h(t) - h(s)}{t - s},$$

and if t is right scattered, then the Delta derivative becomes

$$h^\Delta(t) = \frac{h^\sigma(t) - h(t)}{\mu(t)}.$$

For a function $h : \mathbb{T} \to \mathbb{R}$, h^σ denotes $h(\sigma(t))$.

Definition 2.4 ([10]). A function $h : \mathbb{T} \to \mathbb{R}$ is right dense continuous if it is continuous at all right dense points of $\mathbb{T}$ and its left sided limits exist and are finite at left dense points of $\mathbb{T}$. The set of all right dense continuous functions is denoted by

$$C_{rd} = C_{rd}(\mathbb{T}).$$

Definition 2.5 ([10]). Assume that $[a, b]$ is a closed and bounded interval in $\mathbb{T}$. Then a function $H : [a, b] \to \mathbb{R}$ is called a delta antiderivative of $h : [a, b] \to \mathbb{R}$ provided H is continuous on $[a, b]$, delta differentiable on $[a, b)$, and $H^\Delta(t) = h(t)$ for all $t \in [a, b)$. Then, we define the Delta integral by

$$\int_a^b h(t) = H(b) - H(a), \qquad \text{for all } a, b \in \mathbb{T}.$$

Remark 2.6 ([10]). All right dense continuous functions are delta integrable.

Definition 2.7 ([10]). A function $\phi : [0, r] \to [0, \infty)$ is of class $\mathcal{K}$ if it is continuous, and strictly increasing on $[0, r]$ with $\phi(0) = 0$. In addition to the class $\mathcal{K}$ function, we define the following class of functions:

$\mathfrak{L} = \{\sigma \in C[\mathbb{R}_+, \mathbb{R}_+] : \sigma(\mathfrak{u})$ is strictly decreasing in $\mathfrak{u}$ and $\lim_{\mathfrak{u} \to \infty} \sigma(\mathfrak{u}) = 0\}$,

$\mathcal{CK} = \{\mathfrak{a} \in C_{rd}[\mathbb{T} \times \mathbb{R}_+, \mathbb{R}_+] : \mathfrak{a}(t, s) \in \mathcal{K}$ for each $t\}$,

$\mathfrak{M} = \{m \in C_{rd}[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}_+] : \inf_{(t,x)} m(t, x) = 0\}$.

Definition 2.8 ([10]). A continuous function $\mathcal{V} : \mathbb{R}^n \to \mathbb{R}$ with $\mathcal{V}(0) = 0$ is called positive definite (negative definite) on the domain D if there exists a function $\phi \in \mathcal{K}$ such that $\phi(|x|) \leq \mathcal{V}(x)$ ($\phi(|x|) \leq -\mathcal{V}(x)$) for $x \in D$.

Definition 2.9 ([10]). A continuous function $\mathcal{V} : \mathbb{R}^n \to \mathbb{R}$ with $\mathcal{V}(0) = 0$ is called positive semidefinite (negative semi-definite) on D if $\mathcal{V}(x) \geq 0$ ($\mathcal{V}(x) \leq 0$) for all $x \in D$ and it can also vanish for some $x \neq 0$.

Definition 2.10 ([8]). Let $a, b \in \mathbb{T}$ and $h \in C_{rd}$. Then we define the integration on a time scale $\mathbb{T}$ as follows:

(i) If $\mathbb{T} = \mathbb{R}$, then

$$\int_a^b h(t)\Delta t = \int_a^b h(t)dt,$$

where $\int_a^b h(t)dt$ is the usual Riemann integral from calculus.

(ii) If $[a, b]$ consists of only isolated points, then

$$\int_a^b h(t)\Delta t = \begin{cases} \sum_{t\in[a,b)} \mu(t)h(t) & \text{if} \quad a < b, \\ 0 & \text{if} \quad a = b, \\ -\sum_{t\in[b,a)} \mu(t)h(t) & \text{if} \quad a > b. \end{cases}$$

(iii) If there exists a point $\sigma(t) > t$, then

$$\int_t^{\sigma(t)} h(s)\Delta s = \mu(t)f(t).$$

Definition 2.11 ([16])**.** Assume that $V \in C[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}_+]$, that $h \in C_{rd}[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^n]$, and that $\mu(t)$ is the graininess function then we define the Dini derivative of $V(t, x)$ as:

$$D_{-}V^{\Delta}(t, x) = \liminf_{\mu(t)\to 0} \frac{V(t, x) - V(t - \mu(t), x - \mu(t)h(t, x))}{\mu(t)}, \tag{2.1}$$

$$D^{+}V^{\Delta}(t, x) = \limsup_{\mu(t)\to 0} \frac{V(t + \mu(t), x + \mu(t)h(t, x)) - V(t, x)}{\mu(t)}. \tag{2.2}$$

If V is differentiable, then $D_{-}V^{\Delta}(t, x) = D^{+}V^{\Delta}(t, x) = V^{\Delta}(t, x)$.

Definition 2.12 ([3])**. (Fractional Integral on Time Scales)**. Let $\alpha \in (0, 1)$, $[a, b]$ be an interval on $\mathbb{T}$ and let h be an integrable function on $[a, b]$. Then the fractional integral of order α of h is defined by

$${}_a^{\mathbb{T}}I_t^{\alpha} h^{\Delta}(t) = \int_a^t \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\Delta s.$$

Definition 2.13 ([3])**. (Caputo Derivative on Time Scale)** Let $\mathbb{T}$ be a time scale, let $t \in \mathbb{T}$, $0 < \alpha < 1$, and let $h : \mathbb{T} \to \mathbb{R}$. The Caputo fractional derivative of order α of h is defined by

$${}_a\Delta_t^{\alpha} h(t) = \frac{1}{\Gamma(1 - \alpha)} \int_a^t (t - s)^{-\alpha} h^{\Delta^n}(s)\Delta s.$$

Definition 2.14. Let $\mathbb{T}$ be a time scale. A point $t_0 \in \mathbb{T}$ is said to be a minimal element of $\mathbb{T}$ if, for any $t \in \mathbb{T}$, $t > t_0$ whenever $t \neq t_0$.

Remark 2.15. The concept of minimal element is essential in studying dynamic equations because it establishes a starting point, a reference time from which the dynamics of the system evolve. In the study of difference equations (a discrete-time setting), t_0 represents the initial time step. Similarly, in differential equations (a continuous-time setting), t_0 represents the initial time instant.

Lemma 2.16 ([19])**.** *Let $\mathbb{T}$ be a time scale with the minimal element $t_0 \geq 0$. Suppose that for each $t \in \mathbb{T}$, there exists a statement $\mathbf{S}(t)$ such that the following conditions hold:*

(i) *$\mathbf{S}(t_0)$ is true;*
(ii) *if t is right-scattered and $\mathbf{S}(t)$ is true, then $\mathbf{S}(\sigma(t))$ is also true;*
(iii) *for every right-dense t, there exists a neighborhood $\mathcal{U}$ such that if $\mathbf{S}(t)$ is true, then $\mathbf{S}(t^*)$ is also true for all $t^* \in \mathcal{U}$ with $t^* \geq t$;*

(iv) *for left-dense t, if $\mathbf{S}(t^*)$ is true for all t^* in $[t_0, t)$, then $\mathbf{S}(t)$ is true.*

Then the statement $\mathbf{S}(t)$ is true for all $t \in \mathbb{T}$.

Remark 2.17. When $\mathbb{T} = \mathbb{N}$, then Lemma 2.16 reduces to the well-known principle of mathematical induction. That is,

(1) $\mathbf{S}(t_0)$ is true is equivalent to the statement is true for $n = 1$;
(2) If $\mathbf{S}(t)$ is true, then, $\mathbf{S}(\sigma(t))$ is true is equivalent to if the statement is true for $n = k$, then the statement is true for $n = k + 1$.

Definition 2.18. Let $h \in C_{rd}^{\alpha}[\mathbb{T}, \mathbb{R}^n]$. Then the Grunwald–Letnikov fractional delta derivative is given by

$$^{GL}\Delta_0^{\alpha} h(t) = \lim_{\mu \to 0+} \frac{1}{\mu^{\alpha}} \sum_{r=0}^{[\frac{(t-t_0)}{\mu}]} (-1)^{r\alpha} C_r [h(\sigma(t) - r\mu)], \quad t \geq t_0, \tag{2.3}$$

and the Grunwald–Letnikov fractional delta Dini derivative is given by

$$^{GL}\Delta_{0+}^{\alpha} h(t) = \limsup_{\mu \to 0+} \frac{1}{\mu^{\alpha}} \sum_{r=0}^{[\frac{(t-t_0)}{\mu}]} (-1)^{r\alpha} C_r [h(\sigma(t) - r\mu)], \quad t \geq t_0, \tag{2.4}$$

where $0 < \alpha < 1$, ${}^{\alpha}C_r = \frac{q(q-1)\cdots(q-r+1)}{r!}$, and $[\frac{(t-t_0)}{\mu}]$ denotes the integer part of the fraction $\frac{(t-t_0)}{\mu}$.

Observe that if the domain is $\mathbb{R}$, then (2.4) becomes

$$^{GL}\Delta_{0+}^{\alpha} h(t) = \limsup_{d \to 0+} \frac{1}{d^{\alpha}} \sum_{r=0}^{[\frac{(t-t_0)}{d}]} (-1)^{r\alpha} C_r [h(t - rd)], \quad t \geq t_0.$$

Remark 2.19. It is necessary to note that the relationship between the Caputo fractional delta derivative and the Grunwald–Letnikov fractional delta derivative is given by

$$^{C}\Delta_0^{\alpha} h(t) = ^{GL}\Delta_0^{\alpha}[h(t) - h(t_0)]. \tag{2.5}$$

Substituting (2.3) into (2.5) we have that the Caputo fractional delta derivative becomes

$$^{C}\Delta_0^{\alpha} h(t) = \lim_{\mu \to 0+} \frac{1}{\mu^{\alpha}} \sum_{r=0}^{[\frac{(t-t_0)}{\mu}]} (-1)^{r\alpha} C_r [h(\sigma(t) - r\mu) - h(t_0)], \quad t \geq t_0,$$

$$^{C}\Delta_0^{\alpha} h(t) = \lim_{\mu \to 0+} \frac{1}{\mu^{\alpha}} \left\{ h(\sigma(t)) - h(t_0) + \sum_{r=1}^{[\frac{(t-t_0)}{\mu}]} (-1)^{r\alpha} C_r [h(\sigma(t) - r\mu) - h(t_0)] \right\}, \tag{2.6}$$

and the Caputo fractional delta Dini derivative becomes

$$^{C}\Delta_{0+}^{\alpha} h(t) = \limsup_{\mu \to 0+} \frac{1}{\mu^{\alpha}} \sum_{r=0}^{[\frac{(t-t_0)}{\mu}]} (-1)^{r\alpha} C_r [h(\sigma(t) - r\mu) - h(t_0)], \quad t \geq t_0, \tag{2.7}$$

which is equivalent to

$$ {}^{C}\Delta^{\alpha}_{0^+}h(t) = \limsup_{\mu\to 0+} \frac{1}{\mu^{\alpha}}\Big\{h(\sigma(t)) - h(t_0) + \sum_{r=1}^{[\frac{(t-t_0)}{\mu}]} (-1)^{r\alpha} C_r[h(\sigma(t)-r\mu) - h(t_0)]\Big\}. \tag{2.8} $$

For notation simplicity, we shall represent the Caputo fractional delta derivative of order α as ${}^{C}\Delta^{\alpha}$ and the Caputo fractional delta Dini derivative of order α as ${}^{C}\Delta^{\alpha}_{+}$.

Given that $\lim_{N\to\infty}\sum_{r=0}^{N}(-1)^{r\alpha}C_r = 0$ where $\alpha \in (0,1)$, and $\lim_{\mu\to 0^+}[\frac{(t-t_0)}{\mu}] = \infty$ then it is easy to see that

$$ \lim_{\mu\to 0^+} \sum_{r=1}^{[\frac{(t-t_0)}{\mu}]} (-1)^{r\alpha} C_r = -1. \tag{2.9} $$

Also, from (2.7) and since the Caputo and Riemann–Liouville formulations coincide when $h(t_0) = 0$, then we have

$$ \limsup_{\mu\to 0^+} \frac{1}{\mu^{\alpha}} \sum_{r=0}^{[\frac{(t-t_0)}{\mu}]} (-1)^{r\alpha} C_r = {}^{RL}\Delta^{\alpha}(1) = \frac{(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}, \quad t \geq t_0. \tag{2.10} $$

Now, we introduce the derivative of the Lyapunov function using the Caputo fractional delta Dini derivative of $h(t)$ given in (2.7).

Definition 2.20. Let $[t_0, T)_{\mathbb{T}} = [t_0, T) \cap \mathbb{T}$, where $T > t_0$. Consider a function $V(t,x) \in C_{rd}[\mathbb{T}\times\mathbb{R}^N, \mathbb{R}^N_+]$, we define the generalized Caputo fractional Dini derivative relative to (1.1) as follows: Given $\epsilon > 0$, there exists a neighborhood $P(\epsilon)$ of $t \in \mathbb{T}$ such that

$$ \begin{aligned} &\frac{1}{\mu^{q}}\Big\{V(\sigma(t), x(\sigma(t)) - V(t_0, x_0) \\ &- \sum_{r=1}^{[\frac{t-t_0}{\mu}]} (-1)^{r+1}({}^{q}C_r)[V(\sigma(t)-r\mu, x(\sigma(t) - \mu^{q} f(t,x(t)) - V(t_0,x_0)]\Big\} \\ &\qquad < {}^{C}\Delta^{\alpha}_{+}V(t,x) + \epsilon \end{aligned} $$

for each $s \in P(\epsilon)$ and $s > t$, where $\mu = \sigma(t) - t$ and $x(t) = x(t, t_0, x_0)$ is any solution of (1.1).

Definition 2.21. Using Definition 2.20, we state the Caputo fractional delta derivative and Dini derivative of $V(t,x)$ as

$$ \begin{aligned} {}^{C}\Delta^{\alpha}_{+}V(t,x) \;=\; &\limsup_{\mu\to 0^+} \frac{1}{\mu^{\alpha}}\Bigg[\sum_{r=0}^{[\frac{t-t_0}{\mu}]} (-1)^{r}({}^{\alpha}C_r)[V(\sigma(t)-r\mu, x(\sigma(t)) - \mu^{\alpha} f(t,x(t)) \\ &-V(t_0, xi_0)]\Bigg], \end{aligned} \tag{2.11} $$

and can be expanded as

$$
\begin{aligned}
{}^{C}\Delta_{+}^{\alpha}V(t,x) = &\limsup_{\mu\to 0^{+}} \frac{1}{\mu^{\alpha}}\Big\{V(\sigma(t),x(\sigma(t)) - V(t_0,x_0) \\
&- \sum_{r=1}^{[\frac{t-t_0}{\mu}]} (-1)^{r+1}({}^{\alpha}C_r)[V(\sigma(t)-r\mu, x(\sigma(t)) - \mu^{\alpha} f(t,x(t)) - V(t_0,x_0)]\Big\},
\end{aligned}
$$

where $t \in \mathbb{T}$, and $x, x_0 \in \mathbb{R}^N$, $\mu = \sigma(t) - t$, and $x(\sigma(t)) - \mu^{\alpha} f(t,x) \in \mathbb{R}^N$.

If $\mathbb{T}$ is discrete and $V(t,x(t))$ is continuous at t, the Caputo fractional delta Dini derivative of the Lyapunov function in discrete times, is given by

$$
{}^{C}\Delta_{+}^{\alpha}V(t,x) = \frac{1}{\mu^{\alpha}}\Bigg[\sum_{r=0}^{[\frac{t-t_0}{\mu}]}(-1)^{r}({}^{\alpha}C_r)(V(\sigma(t),x(\sigma(t))) - V(t_0,x_0))\Bigg], \tag{2.12}
$$

and if $\mathbb{T}$ is continuous, that is $\mathbb{T} = \mathbb{R}$, and $V(t,x(t))$ is continuous at t, then

$$
\begin{aligned}
{}^{C}\Delta_{+}^{\alpha}V(t,x) &= {}^{C}D_{+}^{\alpha}V(t,x) \qquad (2.13)\\
&= \limsup_{\kappa\to 0^{+}} \frac{1}{\kappa^{\alpha}}\Big\{V(t,x(t)) - V(t_0,x_0) \\
&\quad - \sum_{r=1}^{[\frac{t-t_0}{\kappa}]} (-1)^{r+1}({}^{\alpha}C_r)[V(t-r\kappa, x(t)) - \kappa^{\alpha} f(t,x(t)) - V(t_0,x_0)]\Big\}.
\end{aligned}
$$

Note that (2.13) is the same in [2], where $\kappa > 0$.

Definition 2.22. System (1.1) is said to be

- (P_1) (m_0,m)-practically stable if, given $(\lambda, A) \in \mathbb{R}_+$ with $0 < \lambda < A$, we have $m_0(t_0,x_0) < \lambda$ implies $m(t,x(t)) < A$, $t \geq t_0$ for some $t_0 \in \mathbb{T}$, where $x(t) = x(t;t_0,x_0)$ is any solution of (1.1);
- (P_2) (m_0,m)-practically quasi-stable if given $(\lambda, B, T) > 0$ and some $t_0 \in \mathbb{T}$, we have $m_0(t_0,x_0) < \lambda$ implies $m(t,x(t)) < B$, $t \geq t_0 + T$, where where $x(t) = x(t;t_0,x_0)$ is any solution of (1.1);
- (P_3) (m_0,m)-strongly practically stable if (P_1) and (P_3) hold simultaneously;
- (P_4) (m_0,m)-practically asymptotically stable if (P_1) holds and $\lim_{t\to\infty} m(t,x(t)) = 0$.

Consider the comparison system,

$$
{}^{C}\Delta^{\alpha}\kappa = \Theta(t,\kappa), \qquad \kappa(t_0) = \kappa_0 \geq 0, \tag{2.14}
$$

where $\Theta \in C_{rd}[\mathbb{T}\times\mathbb{R}^n, \mathbb{R}^n]$.

Definition 2.23. Corresponding to Definition 2.22, the solution $\kappa(t) = \kappa(t;,t_0,\kappa_0)$ of (2.14), is said to be practically stable if given $0 < \lambda < A$, we have

$$
\sum_{i=1}^{n}\kappa_0 < \lambda \implies \sum_{i=1}^{n}\kappa_i(t;t_0,\kappa_0) < A, \qquad t \geq t_0, \tag{2.15}
$$

for some $t_0 \in \mathbb{T}\cap\mathbb{R}_+$.

Other definitions of practical stability for the comparison system (2.14) follows from Definition 2.23.

Lemma 2.24. *Assume h and $m \in C_{rd}(\mathbb{T}, \mathbb{R}^n)$. Suppose there exists $t_1 > t_0$, where $t_1 \in \mathbb{T}$, such that $h(t_1) = m(t_1)$ and $h(t) < m(t)$ for $t_0 \le t < t_1$. Then, if the Caputo fractional delta Dini derivatives of h and m exist at t_1, the inequality ${}^C\Delta_+^\alpha h(t_1) >^C \Delta_+^\alpha m(t_1)$ holds.*

Proof. Applying (2.7), we have

$$\begin{aligned}{}^C\Delta_+^\alpha(h(t) - m(t)) &= \limsup_{\mu\to 0^+} \frac{1}{\mu^\alpha}\Bigg\{\sum_{r=0}^{[\frac{t-t_0}{\mu}]}(-1)^{r\alpha}C_r[h(\sigma(t)-r\mu) \\ &\quad -m(\sigma(t)-r\mu)] - [h(t_0) - m(t_0)]\Bigg\} \\ {}^C\Delta_+^\alpha h(t) -^C \Delta_+^\alpha m(t) &= \limsup_{\mu\to 0^+} \frac{1}{\mu^\alpha}\Bigg\{\sum_{r=0}^{[\frac{t-t_0}{\mu}]}(-1)^{r\alpha}C_r[h(\sigma(t)-r\mu) \\ &\quad -m(\sigma(t)-r\mu)] - [h(t_0) - m(t_0)]\Bigg\}.\end{aligned}$$

At t_1, we have

$$ {}^C\Delta_+^\alpha h(t_1) = -\limsup_{\mu\to 0^+} \frac{1}{\mu^\alpha}\Bigg\{\sum_{r=0}^{[\frac{t_1-t_0}{\mu}]}(-1)^{r\alpha}C_r[h(t_0) - m(t_0)]\Bigg\} +^C \Delta_+^\alpha m(t_1). \quad (2.16)$$

Applying (2.10) to (2.16), we have

$$ {}^C\Delta_+^\alpha h(t_1) = -\frac{(t_1-t_0)^{-\alpha}}{\Gamma(1-\alpha)}[h(t_0) - m(t_0)] +^C \Delta_+^\alpha m(t_1),$$

however, based on the Lemma's statement, we know that

$$\begin{aligned} h(t) < m(t), &\qquad \text{for } t_0 \le t < t_1, \\ \Longrightarrow\ h(t) - m(t) < 0, &\qquad \text{for } t_0 \le t < t_1.\end{aligned}$$

Then, we obtain

$$ -\frac{(t_1-t_0)^{-\alpha}}{\Gamma(1-\alpha)}[h(t_0) - m(t_0)] > 0,$$

implying

$$ {}^C\Delta_+^\alpha h(t_1) >^C \Delta_+^\alpha m(t_1).$$

□

Lemma 2.25. *Assume that*

1. *$x^*(t; t_0, x_0), x^* \in C_{rd}^\alpha([t_0, T]_\mathbb{T}, \mathbb{R}^N)$ is a solution of the dynamic equation (1.1).*

2. $V(t,x^*) \in C_{rd}[\mathbb{T}\times\mathbb{R}^N,\mathbb{R}^N_+]$ *and for any* $t\in[t_0,\infty)_{\mathbb{T}}$, $x^*\in\mathbb{R}^N$,
$$^{C}\Delta^{\alpha}_{+}V(t,x^*) \le -\phi(\|x^*(t)\|), \tag{2.17}$$
where $\phi\in\mathcal{K}$.

Then for $t\in[t_0,T]_{\mathbb{T}}$, *the inequality*
$$V(t,x^*(t)) \le V(t_0,x_0) - \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}\phi(\|x^*(t)\|)\Delta s$$
holds.

Proof. Let
$$\begin{aligned} h(t) &= V(t,c^*(t)),\\ \text{with}\quad h(t_0) &= V(t_0,x_0), \end{aligned} \tag{2.18}$$
and
$$W(t) = \phi(\|x^*(t)\|). \tag{2.19}$$
Then, from (2.17) we have
$$^{C}\Delta^{\alpha}_{+}h(t) = {}^{C}\Delta^{\alpha}_{+}V(t,x^*(t)) \le -\phi(\|x^*(t)\|) = -W(t), \quad \text{for } t\in[t_0,T]_{\mathbb{T}}. \tag{2.20}$$
Consider the system
$$^{C}\Delta^{\alpha}\upsilon(t) = -W(t), \quad \upsilon(t_0) = h_\omega(t_0), \quad \text{where } h_\omega(t_0) = h(t_0)+\omega. \tag{2.21}$$
A solution $\upsilon(t) = \upsilon(t,t_0,\upsilon_0)$ of (2.21) will also satisfy the Volterra delta integral equation
$$\upsilon(t) = h_\omega(t_0) - \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}W(s)\Delta s. \tag{2.22}$$
We claim that
$$h(t) < \upsilon(t), \quad t\in[t_0,T]_{\mathbb{T}}. \tag{2.23}$$
In the event that this claim were untrue, there would be a time $t_1\in(t_0,T]_{\mathbb{T}}$, such that
$$h(t_1) = \upsilon(t_1) \quad \text{and} \quad h(t) < \upsilon(t) \quad \text{for} \quad t\in[t_0,t_1)_{\mathbb{T}}. \tag{2.24}$$
Lemma 2.24 is applied to (2.24) to obtain
$$\begin{aligned} &^{C}\Delta^{\alpha}_{+}h(t_1) > {}^{C}\Delta^{\alpha}_{+}\upsilon(t_1) = {}^{C}\Delta^{\alpha}\upsilon(t_1) = -W(t_1),\\ &\implies {}^{C}\Delta^{\alpha}_{+}h(t_1) > -W(t_1). \end{aligned} \tag{2.25}$$
Therefore, from (2.25) and for $t\in[t_0,T]_{\mathbb{T}}$, we obtain
$$^{C}\Delta^{\alpha}_{+}h(t) > -W(t). \tag{2.26}$$
Clearly, (2.26) contradicts (2.20), so we conclude that the claim in (2.23) holds.

Combining (2.18),(2.19),(2.22), and (2.23) we get
$$V(t,x^*(t)) = h(t) < h_\omega(t_0) - \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}\phi(\|x^*(s)\|)\Delta s, \tag{2.27}$$

which implies

$$V(t,x^*(t)) \quad \le \quad V(t_0,x_0) - \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}\phi(\|x^*(s)\|)\Delta s. \qquad (2.28)$$

If $\mathbb{T}$ contains right scattered points, and $V(t,\chi)$ is continuous, then (2.28) becomes

$$V(t,x^*(t)) \le V(t_0,x_0) - \frac{1}{\Gamma(\alpha)}\int_{t_0}^{\sigma(t)}(t-s)^{\alpha-1}\phi(\|x^*(s)\|)ds.$$

□

Theorem 2.26. *Assume the following statements:*
(i) $\Theta \in C_{rd}[\mathbb{T}\times\mathbb{R}_+^n,\mathbb{R}_+^n]$ *and* $\Theta(t,\kappa)\mu$ *is nondecreasing in* u.
(ii) $V \in C_{rd}[\mathbb{T}\times\mathbb{R}^N,\mathbb{R}_+^N]$ *is locally Lipschitzian in the second variable such that*

$$^{C}\Delta_+^{\alpha}V(t,x) \le \Theta(t,V(t,x)),(t,x)\in\mathbb{T}\times\mathbb{R}^N. \qquad (2.29)$$

(iii) $z(t)=z(t;t_0,u_0)$ *is the maximal solution of* (2.14) *existing on* $\mathbb{T}$.
Then

$$V(t,x(t)) \le z(t), \quad t\ge t_0, \qquad (2.30)$$

provided that

$$V(t_0,x_0)\le\kappa_0, \qquad (2.31)$$

where $x(t)=x(t;t_0,x_0)$ *is any solution of* (1.1), $t\in\mathbb{T}, \quad t\ge t_0$.

Proof. Apply the principle of induction as stated in Lemma 2.16 to the statement

$$\mathbf{S(t)}: V(t,x(t))\le z(t), \quad t\in\mathbb{T},\ t\ge t_0.$$

Then

(i) $\mathbf{S(t_0)}$ is true since $V(t_0,x_0)\le u_0$.
(ii) Let t be right-scattered and $\mathbf{S(t)}$ be true. We need to show that $\mathbf{S}(\sigma(\mathbf{t}))$ is true; that is,

$$V(\sigma(t),x(\sigma(t)))\le z(\sigma(t)). \qquad (2.32)$$

Set $h(t)=V(t,x(t))$; then $h(\sigma(t))=V(\sigma(t),x(\sigma(t)))$, but from (2.7), we have

$$^{C}\Delta_+^{\alpha}h(t)=\limsup_{\mu\to0+}\frac{1}{\mu^{\alpha}}\left\{\sum_{r=0}^{[\frac{(t-t_0)}{\mu}]}(-1)^{r\alpha}C_r[h(\sigma(t)-r\mu)-h(t_0)]\right\}, \quad t\ge t_0.$$

Also,

$$^{C}\Delta_+^{\alpha}z(t)=\limsup_{\mu\to0+}\frac{1}{\mu^{\alpha}}\left\{\sum_{r=0}^{[\frac{(t-t_0)}{\mu}]}(-1)^{r\alpha}C_r[z(\sigma(t)-r\mu)-z(t_0)]\right\}, \quad t\ge t_0,$$

so that

$$^{C}\Delta_+^{\alpha}z(t)-^{C}\Delta_+^{\alpha}h(t)$$

$$
\begin{aligned}
&= \limsup_{\mu\to 0+} \frac{1}{\mu^\alpha}\left\{\sum_{r=0}^{[\frac{(t-t_0)}{\mu}]} (-1)^{r\,\alpha}C_r[z(\sigma(t)-r\mu)-z(t_0)]\right\}\\
&\quad - \limsup_{\mu\to 0+} \frac{1}{\mu^\alpha}\left\{\sum_{r=0}^{[\frac{(t-t_0)}{\mu}]} (-1)^{r\,\alpha}C_r[h(\sigma(t)-r\mu)-h(t_0)]\right\},\\
&\left({}^C\Delta_+^\alpha z(t) - {}^C\Delta_+^\alpha h(t)\right)\mu^\alpha\\
&= \limsup_{\mu\to 0+}\left\{\sum_{r=0}^{[\frac{(t-t_0)}{\mu}]} (-1)^{r\,\alpha}C_r\Big[[z(\sigma(t)-r\mu)-z(t_0)]\right.\\
&\quad \left. -[h(\sigma(t)-r\mu)-h(t_0)]\Big]\right\}\\
&\le [z(\sigma(t))-z(t_0)]-[h(\sigma(t))-h(t_0)]\\
&\le [z(\sigma(t))-h(\sigma(t))]-[z(t_0)-h(t_0)].
\end{aligned}
$$

Then

$$[z(\sigma(t))-h(\sigma(t))] \ge \left({}^C\Delta_+^\alpha z(t) - {}^C\Delta_+^\alpha h(t)\right)\mu^\alpha + [z(t_0)-h(t_0)],$$

so that

$$
\begin{aligned}
[h(\sigma(t))-z(\sigma(t))] &\le \left({}^C\Delta_+^\alpha z(t) - {}^C\Delta_+^\alpha h(t)\right)\mu^\alpha + [h(t_0)-z(t_0)]\\
&\le \Big(\Theta(t,h(t))-\Theta(t,z(t))\Big)\mu^\alpha + [h(t_0)-z(t_0)].
\end{aligned}
$$

Since $g(t,u)\mu$ is nondecreasing in u and $\mathbf{S(t)}$ is true, then $h(\sigma(t)) - z(\sigma(t)) \le 0$ so (2.32) holds.

(iii) Let t be right dense and let $\mathcal{N}$ be a right neighborhood of $t\in\mathbb{T}$. We need to show that $\mathbf{S(t^*)}$ is true for $t^*\in\mathcal{N}$.

Let ω be an arbitrary small vector such that $\|\omega\| \le B_{\mathbb{T}}$ (where $B_{\mathbb{T}}$ is a small enough number on the time scale $\mathbb{T}$), and consider the initial value problem

$$ {}^C\Delta^\alpha\kappa = \Theta(t^*,\kappa)+\omega, \quad \kappa(t_0)=\kappa_0+\omega, \tag{2.33}$$

for $t^*\in\mathcal{N}$.

The function $\kappa_\omega(t^*) = \kappa(t^*)+\omega$ is a solution of (2.33) if and only if it satisfies the delta integral equation

$$\kappa_\omega(t^*) = \kappa_0+\omega+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t^*}(t^*-s)^{\alpha-1}(\Theta(s,\kappa_\omega(s))+\omega)\Delta s, \tag{2.34}$$

where $t^*, s\in\mathcal{N}$.

Let $h(t^*) \in C_{rd}(\mathbb{T}, \mathbb{R}^n_+)$ be such that $h(t^*) = V(t^*, x^*(t^*))$, where $x^*(t^*)$ is any other solution of (1.1). We show that

$$h(t^*) < \kappa_\omega(t^*), \quad \text{for } t^* \in \mathcal{N}. \tag{2.35}$$

The inequality (2.35) holds for $t^* = t_0$ since

$$h(t_0) = V(t_0, x_0) \le \kappa_0 < \kappa_\alpha(t_0).$$

Assume that the inequality (2.35) is not true, then there exists a point $t_1^* > t_0$ such that

$$h(t_1^*) = \kappa_\omega(t_1^*) \quad \text{and} \quad h(t^*) < \kappa_\omega(t^*) \quad \text{for } t_0 \le t^* < t_1^*,\ t^*, t_1^* \in \mathcal{N}.$$

From Lemma (2.24) it follows that

$${}^C\Delta_+^\alpha h(t_1^*) >^C \Delta_+^\alpha \kappa_\omega(t_1^*),$$

which implies

$${}^C\Delta_+^\alpha V(t_1^*, x^*(t_1^*)) >^C \Delta_+^\alpha \kappa_\omega(t_1^*).$$

Then, from (2.33), we arrive at

$${}^C\Delta_+^\alpha V(t_1^*, x^*(t_1^*)) > \Theta(t_1^*, \kappa_\omega(t_1^*)) + \omega) > \Theta(t_1^*, h(t_1^*)).$$

Therefore,

$${}^C\Delta_+^\alpha h(t_1^*) > \Theta(t_1^*, h(t_1^*)). \tag{2.36}$$

Now, for $t^* \in \mathcal{N}$,

$$\begin{aligned}
&{}^C\Delta_+^\alpha h(t^*) \\
&= \limsup_{\mu \to 0^+} \frac{1}{\mu^\alpha}\left\{h(t^*) - h(t_0) - \sum_{r=1}^{[\frac{t^*-t_0}{\mu}]} (-1)^{r+1}({}^\alpha C_r)[h(t^*-r\mu) - h(t_0)]\right\}, \\
&= \limsup_{\mu \to 0^+} \frac{1}{\mu^\alpha}\{V(t^*, x^*(t^*)) - V(t_0, x_0) \\
&\quad - \sum_{r=1}^{[\frac{t^*-t_0}{\mu}]} (-1)^{r+1}({}^\alpha C_r)[V(t^*-r\mu, x^*(t^*-r\mu) - V(t_0, x_0)]\} \\
&= \limsup_{\mu \to 0^+} \frac{1}{\mu^\alpha}\Bigg\{V(t^*, x^*(t)) - V(t_0, x_0) \\
&\quad - \sum_{r=1}^{[\frac{t^*-t_0}{\mu}]} (-1)^{r+1}({}^\alpha C_r)\Bigg[[V(t^*-r\mu, x^*(t^*) - \mu^\alpha f(t^*, x^*(t^*))) - V(t_0, x_0)] \\
&\quad -[V(t^*-r\mu, x^*(t^*) - \mu^\alpha f(t^*, x^*(t^*))) - V(t_0, x_0)] \\
&\quad\quad +[V(t^*-r\mu, x^*(t^*-r\mu)) - V(t_0, x_0)]\Bigg]\Bigg\} \\
&= \limsup_{\mu \to 0^+} \frac{1}{\mu^\alpha}\Bigg\{V(t^*, x^*(t^*)) - V(t_0, x_0)
\end{aligned}$$

$$- \sum_{r=1}^{[\frac{t^*-t_0}{\mu}]} (-1)^{r+1}({}^{\alpha}C_r)\Bigg[[V(t^* - r\mu, x^*(t^*) - \mu^{\alpha} f(t^*, x^*(t^*))) - V(t_0, x_0)]$$

$$-V[(t^* - r\mu, x^*(t^*) - \mu^{\alpha} f(t^*, x^*(t^*)))] + V(t^* - r\mu, x^*(t^* - r\mu))\Bigg]\Bigg\}$$

$$= \limsup_{\mu \to 0^+} \frac{1}{\mu^{\alpha}} \Bigg\{ V(t^*, x^*(t^*)) - V(t_0, x_0)$$

$$- \sum_{r=1}^{[\frac{t^*-t_0}{\mu}]} (-1)^{r+1}({}^{\alpha}C_r)\Bigg[[V(t^* - r\mu, x^*(t^*) - \mu^{\alpha} f(t^*, x^*(t^*))) - V(t_0, x_0)]$$

$$+[V(t^* - r\mu, x^*(t^* - r\mu)) - [V(t^* - r\mu, x^*(t^*) - \mu^{\alpha} f(t^*, x^*(t^*)))]\Bigg]\Bigg\}$$

$$= \limsup_{\mu \to 0^+} \frac{1}{\mu^{\alpha}} \Bigg\{ V(t^*, x^*(t^*)) - V(t_0, x_0)$$

$$- \sum_{r=1}^{[\frac{t^*-t_0}{\mu}]} (-1)^{r+1}({}^{\alpha}C_r[V(t^* - r\mu, x^*(t^* - \mu^{\alpha} f(t^*, x^*(t^*)) - V(t_0, x_0)]\Bigg\}$$

$$- \limsup_{\mu \to 0^+} \frac{1}{\mu^{\alpha}} \Bigg\{ \sum_{r=1}^{[\frac{t^*-t_0}{\mu}]} (-1)^{r+1}({}^{\alpha}C_r)[V(t^* - r\mu, x^*(t^* - r\mu))$$

$$-V(t^* - r\mu, x^*(t^*) - \mu^{\alpha} f(t^*, x^*(t^*)))]\Bigg\}.$$

Since $V(t^*, x)$ is locally Lipschtzian in the second variable, we have

$$\begin{aligned} {}^{C}\Delta_{+}^{\alpha} h(t^*) \quad &\leq \quad {}^{C}\Delta_{+}^{\alpha} V(t^*, x^*(t^*)) + L \limsup_{\mu \to 0^+} \frac{1}{\mu^{\alpha}} \sum_{r=1}^{[\frac{t^*-t_0}{\mu}]} (-1)^{r}({}^{\alpha}C_r)\|x^*(t^* - r\mu) \\ &\quad -(x^*(t^*) - \mu^{\alpha} f(t^*, x^*(t^*)))\|, \end{aligned}$$

where $L > 0$ is a Lipschitz constant.

As $\mu \to 0$, $\|x^*(t^* - r\mu) - (x^*(t^*) - \mu^{\alpha} f(t^*, x^*(t^*)))\| \to 0$, so that from (2.29) we arrive at

$${}^{C}\Delta_{+}^{\alpha} h(t^*) ={}^{C} \Delta_{+}^{\alpha} V(t^*, x^*(t^*)) \leq \Theta(t^*, V(t^*, x^*(t^*))) = \Theta(t^*, h(t^*)). \tag{2.37}$$

Now, (2.37) with $t^* = t_1^*$ contradicts (2.36), hence (2.35) is true. For $t^* \in \mathcal{N}$, we now show that whenever $\omega_1 < \omega_2$, then

$$\kappa_{\omega_1}(t^*) < \kappa_{\omega_2}(t^*). \tag{2.38}$$

Note that (2.38) holds for $t^* = t_0$ since $\kappa(t_0) + \omega_1 < \kappa(t_0) + \omega_2 \implies \omega_1 < \omega_2$. Assume the inequality (2.38) is not true. Then there exists a point t_1^* such that $\kappa_{\omega_1}(t_1^*) = \kappa_{\omega_2}(t_1^*)$ and $\kappa_{\omega_1}(t^*) < \kappa_{\omega_2}(t^*)$ for $t_0 \leq t^* < t_1^*$ $t^* \in \mathcal{N}$.

By Lemma (2.24), we have

$${}^{C}\Delta_{+}^{\alpha} \kappa_{\omega_1}(t_1^*) >{}^{C} \Delta_{+}^{\alpha} \kappa_{\omega_2}(t_1^*).$$

However,

$$\begin{aligned}
&{}^C\Delta_+^\alpha \kappa_{\omega_1}(t_1^*) -^C \Delta_+^\alpha \kappa_{\omega_2}(t_1^*) \\
&= \Theta(t_1^*, \kappa_{\omega_1}(t_1^*)) + \omega_1 - [\Theta(t_1^*, \kappa_{\omega_2}(t_1^*)) + \omega_2] \\
&= \omega_1 - \omega_2 < 0,
\end{aligned}$$

which is a contradiction, and so (2.38) is true. Now from (2.38) and since $\omega \leq B_{\mathbb{T}}$, we deduce that

$$\kappa_{\omega_1}(t^*) < \kappa_{\omega_2}(t^*) < \cdots < \kappa(t^*) + \omega_i \leq |\kappa(t^*) + B_{\mathbb{T}}| \leq M,$$

and therefore we can say that the family of solutions $\{\kappa_{\omega_i}(t^*)\}$ is uniformly bounded with bound $M > 0$ on $\mathbb{T}$. This means that $\|\kappa_{\omega_i}(t^*)\| \leq M$ for $t^* \in \mathcal{N}$ and $\|\omega\| \leq B_{\mathbb{T}}$].

We now show that the family $\{\kappa_{\omega_i}(t^*)\}$ is equicontinuous on $\mathbb{T}$. Assume $\mathcal{S} = \sup\{|\Theta(t^*, x^*)| : (t^*, x^*) \in \mathcal{N} \times [-M, M]\}$. Now, let us take $\{\omega_i\}_{i=1}^\infty(t^*)$, as a decreasing sequence, such that $\lim\limits_{i\to\infty} \omega_i = 0$, and consider a sequence of functions $\kappa_{\omega_i}(t^*)$, and take $t_1^*, t_2^* \in \mathcal{N}$ with $t_1^* < t_2^*$. Then we have the following estimate

$$\begin{aligned}
&|\kappa_{\omega_i}(t_2^*) - \kappa_{\omega_i}(t_1^*)| \\
&= \Bigg|\kappa_0 + \omega_i + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_2^*} (t_2^* - s)^{\alpha-1} (\Theta(s, \kappa_{\omega_i}(s)) + \alpha_i) \Delta s \\
&\quad -(\kappa_0 + \omega_i + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_1^*} (t_1^* - s)^{\alpha-1} (\Theta(s, \kappa_{\omega_i}(s)) + \omega_i)) \Delta s\Bigg| \\
&= \frac{1}{\Gamma(\alpha)} \Bigg| \int_{t_0}^{t_2^*} (t_2^* - s)^{\alpha-1} (\Theta(s, \kappa_{\omega_i}(s))) \Delta s \\
&\quad - \int_{t_0}^{t_1^*} (t_1^* - s)^{\alpha-1} (\Theta(s, \kappa_{\omega_i}(s))) \Delta s\Bigg| \\
&\leq \frac{1}{\Gamma(\alpha)} \Bigg[\Bigg| \int_{t_0}^{t_2^*} (t_2^* - s)^{\alpha-1} \Bigg| \Delta s \Bigg| (\Theta(s, \kappa_{\omega_i}(s))) \Bigg| \\
&\quad + \Bigg| \int_{t_0}^{t_1^*} (t_1^* - s)^{\alpha-1} \Delta s \Bigg| \Bigg| (\Theta(s, \kappa_{\omega_i}(s))) \Bigg| \Bigg] \\
&\leq \frac{\mathcal{S}}{\Gamma(\alpha)} \Bigg[\Bigg| \int_{t_0}^{t_2^*} (t_2^* - s)^{\alpha-1} \Delta s \Bigg| + \Bigg| \int_{t_0}^{t_1^*} (t_1^* - s)^{\alpha-1} \Delta s \Bigg| \Bigg] \\
&= \frac{\mathcal{S}}{\Gamma(\alpha)} \Bigg[\Bigg| \int_{t_0}^{t_1^*} (t_2^* - s)^{\alpha-1} \Delta s + \int_{t_1^*}^{t_2^*} (t_2^* - s)^{\alpha-1} \Delta s \Bigg| \\
&\quad + \Bigg| \int_{t_0}^{t_1^*} (t_1^* - s)^{\alpha-1} \Delta s \Bigg| \Bigg] \\
&= \frac{\mathcal{S}}{\Gamma(\alpha)} \Bigg[\Bigg| - \Bigg[\frac{(t_2^* - t_1^*)^\alpha}{\alpha} - \frac{(t_2^* - t_0)^\alpha}{\alpha} \Bigg] + \frac{(t_2^* - t_1^*)^q}{\alpha} \Bigg| \\
&\quad + \Bigg| \frac{(t_1^* - t_0)^\alpha}{\alpha} \Bigg| \Bigg]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\mathcal{S}}{\Gamma(\alpha)}\left[\left|-\frac{(t_2^*-t_1^*)^\alpha}{\alpha}+\frac{(t_2^*-t_0)^\alpha}{\alpha}+\frac{(t_2^*-t_1^*)^\alpha}{\alpha}\right|+\left|\frac{(t_1^*-t_0)^\alpha}{\alpha}\right|\right]\\
&= \frac{\mathcal{S}}{\Gamma(\alpha)}\left[\left|\frac{(t_2^*-t_0)^\alpha}{\alpha}\right|+\left|\frac{(t_1^*-t_0)^\alpha}{\alpha}\right|\right]\\
&= \frac{\mathcal{S}}{\Gamma(\alpha+1)}\left[(t_2^*-t_0)^\alpha+(t_1^*-t_0)^\alpha\right]\\
&\leq \frac{2\mathcal{S}}{\Gamma(\alpha+1)}\left[(t_2^*-t_0)^\alpha\right].
\end{aligned}
$$

A family of solutions $\{\kappa_{\omega_i}(t^*)\}$ is said to be equicontinuous if given $\epsilon > 0$, we can find $\delta > 0$ such that $|\kappa_{\omega_i}(t_2^*) - \kappa_{\omega_i}(t_1^*)| < \epsilon$ whenever $|t_2^* - t_1^*| < \delta$, implying that $|\kappa_{\omega_i}(t_2^*) - \kappa_{\omega_i}(t_1^*)| \leq \frac{2\mathcal{S}}{\Gamma(\alpha+1)}\left[(t_2^*-t_0)^\alpha\right] < \epsilon$ provided $|t_2^* - t_1^*| < \delta$.

Now, we choose $\delta = \left(\frac{\epsilon\Gamma(\alpha+1)}{2\mathcal{S}}\right)^{\frac{1}{\alpha}}$, $\left(\frac{\epsilon\Gamma(\alpha+1)}{2\mathcal{S}}\right)^{\frac{1}{\alpha}} > \left(\frac{2\mathcal{S}(t_2^*-t_0)^\alpha}{\Gamma(\alpha+1)} \times \frac{\Gamma(\alpha+1)}{2\mathcal{S}}\right)^{\frac{1}{\alpha}} = (t_2^* - t_0)$ but $(t_2^* - t_0) > |t_2^* - t_1^*|$ so since $(t_2^* - t_0) < \delta$. Then $|t_2^* - t_1^*| < \delta$. Proving that the family of solutions $\{\kappa_\omega(t^*)\}$ is equi-continuous. By the Arzela–Ascoli theorem, $\{\kappa_{\omega_i}(t^*)\}$ has a subsequence $\{\kappa_{\omega_{i_j}}(t^*)\}$, which converges uniformly to a function $z(t^*)$ on $\mathbb{T}$. We then show that $z(t^*)$ is a solution of (2.14). Equation (2.34) becomes

$$
\kappa_{\omega_{i_j}}(t^*) = \kappa_0 + \omega_{i_j} + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t^*}(t^*-s)^{\alpha-1}(\Theta(s,\kappa_{\omega_{i_j}}(s)) + \omega_{i_j})\Delta s. \tag{2.39}
$$

Taking the limit as $i_j \to \infty$, then $\kappa_{\omega_{i_j}}(t^*) \to z(t^*)$ on $\mathbb{T}$. Now, (2.39) yields

$$
z(t^*) = \kappa_0 + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t^*}(t^*-s)^{\alpha-1}(\Theta(s,z(t^*)))\Delta s. \tag{2.40}
$$

Thus, $z(t^*)$ is a solution of (2.14) on $\mathbb{T}$. Since $\lim\limits_{j\to\infty}\kappa_{\omega_{i_j}}(t^*) = z(t)$ exists, then for any κ_{ω_i} that satisfies the dynamic equation (2.14), $\kappa_\omega(t^*) \leq z(t^*)$. So from (2.35), we have that $h(t^*) < \kappa_\omega(t^*) \leq z(t^*)$ on $\mathbb{T}$.

(iv) Let t be left-dense, and let $\mathbf{S}(s)$ be true for all $s > t$. We want to show that $\mathbf{S}(t)$ is true. This immediately follows the right-dense continuity of $V(t, x^*)$, $x^*(t)$ and the maximal solution $z(t)$. Therefore by the induction principle, the statement $\mathbf{S}(\mathbf{t})$ is true for all $t \in \mathbb{T}$.

□

3. Main result

In this section, we give the stability criteria for the fractional dynamic system (1.1) in terms of two measures $m_0, m \in \mathfrak{M}$.

Theorem 3.1 ((m_0, m)-Practical Stability)**.** *Assume that*

($\mathfrak{m}_1$) $0 < \lambda < A$;

($\mathfrak{m}_2$) *$m_0, m \in \mathfrak{M}$, and m_0 is uniformly finer than m, implying $m(t,x) \leq \phi(m_0(t,x))$, $\phi \in \mathcal{K}$, whenever $m_0(t,x) < \lambda$;*

($\mathfrak{m}_3$) *there exit $\mathfrak{Q} \in C_{rd}[\mathbb{R}^g_+, \mathbb{R}^n_+]$ and $V \in C_{rd}[\mathbb{T} \times \mathbb{R}^N, \mathbb{R}^g_+]$ such that if $\mathfrak{V}(t,x) \equiv \mathfrak{Q}(V(t,x))$, then $\mathfrak{V}(t,\mathfrak{u})$ is locally Lipschitzian in $\mathfrak{u}$ and for $(t,x) \in \mathfrak{S}(m,A) = \{(t,x) \in \mathbb{T} \times \mathbb{R}^N : m(t,x) < A\}$,*

$$\begin{aligned} \mathfrak{b}(m(t,x)) \quad &\leq \quad \sum_{i=0}^{n} \mathfrak{V} \qquad\qquad \text{if} \quad m(t,x) < A, \\ \sum_{0}^{\infty} \mathfrak{V}_i(t,x) \quad &\leq \quad \mathfrak{a}(m_0(t,x)) \qquad \text{if} \quad m_0(t,x) < \lambda; \end{aligned} \tag{3.1}$$

($\mathfrak{m}_4$) *for $(t,x) \in \mathfrak{S}(m,A)$,*

$${}^{C}\Delta^{\alpha}_{+}(t,x) \leq \Theta(t, \mathfrak{V}(t,x)), \tag{3.2}$$

where $\Theta \in C_{rd}[\mathbb{T} \times \mathbb{R}^n_+, \mathbb{R}^n]$, $\Theta(t,\kappa)$ is quasimonotone nondecreasing in κ with $\Theta(t,\kappa) \equiv 0$, and for each i, $1 \leq i \leq n$, $\Theta_i(t,\kappa)\mu^*(t) + \kappa_i$ is nondecreasing in κ for all $t \in \mathbb{T}$;

($\mathfrak{m}_5$) *$\phi(\lambda) < A$ and $\mathfrak{a}(\lambda) < \mathfrak{b}(A)$ hold.*

Then the practical stability properties of (2.14) imply the corresponding (m_0, m)-practical stability properties of (1.1).

Proof. By the assumption of the practical stability of (2.14), we have for some $t_0 \in \mathbb{R}_+$ and $0 < \lambda < A$,

$$\sum_{i=0}^{n} \kappa_{0_i} < \mathfrak{a}(\lambda) \implies \sum_{i=1}^{n} \kappa_i(t; t_0, \mathfrak{u}_\mathfrak{o}) < \mathfrak{b}(A), \qquad t \geq t_0. \tag{3.3}$$

We claim that (1.1) is (m, m_0)-practically stable with respect to (λ, A).

If this where false, there would exists a solution $x(t) = x(t; t_0, x_0)$ of (1.1) with $m_0(t_0, x_0) < \lambda$ and a time $t_1 > t_0$ such that

$$m(t_1, x(t_1)) = A \ \text{ and } \ m(t, x(t)) \leq A \qquad \text{for } t \in [t_0, t_1). \tag{3.4}$$

Setting

$$\mathfrak{u}_{0_1} = \mathfrak{V}_i(t_0, x_0),$$

and from (2.30) of the comparison Theorem 2.26, we have

$$\mathfrak{V}(t, x(t)) \leq z(t, t_0, \mathfrak{u}), \tag{3.5}$$

where $z(t)$ is the maximal solution of (2.14).

Also, from assumptions (m_2) and (m_5), it is clear to see that $m(t_0, x_0) \leq \phi(m_0, (t_0, x_0)) < \phi(\lambda) < A$.

Obviously,

$$\sum_{i=1}^{n} < \sum_{0}^{\infty} \mathfrak{V}_i(t_0, x_0) \leq a(m_0(t_0, x_0)) < a(\lambda). \tag{3.6}$$

Combining these estimates, (3.3), (3.4), (3.5), and (3.6), we obtain

$$\mathfrak{b}(A) = b(m(t_1, x(t_1))) \leq \sum_{i=1}^{n} \mathfrak{V}_i(t_1, x(t_1)) \leq \sum_{i=1}^{n} z_i(t_1; t_0, \mathfrak{u}_\mathfrak{o}) < b(A). \tag{3.7}$$

Equation (3.7) is a contradiction, so the assumption of the (m_0, m)-practical stability of system (1.1) is true. □

Similarly, we shall proceed to make the proof other major concept such as (m_0, m)-practical asymptotic stability and (m_0, m)-strong practical stability.

Theorem 3.2 ((m_0, m)-Practical Asymptotic Stability)**.** *Assume that conditions $(m_1) - (m_4)$ of Theorem 3.1 are satisfied, and that (2.14) is practically quasi stable. Then (1.1), is (m_0, m)-practically asymptotically stable.*

Proof. By the assumption of the practical quasi stability of (2.14), given $\mathfrak{a}(\lambda) > 0$, $\epsilon > 0$ and $t_0 \in \mathbb{T}$, there exists $T = T(t_0, \epsilon, \mathfrak{a}(\lambda)) > 0$ such that

$$\sum_{i=0}^{n} \kappa_{0_i} < \mathfrak{a}(\lambda) \implies \sum_{i=1}^{n} \kappa_i(t; t_0, \mathfrak{u}_0) < \mathfrak{b}(\epsilon), \quad t \geq t_0 + T. \tag{3.8}$$

Set $\kappa_{0_i} = \mathfrak{V}_i(t_0, x_0)$, with the assumption of $m_0(t_0, x_0) < \lambda$, and from (2.30) of the comparison Theorem 2.26, we have

$$\mathfrak{V}(t, x(t)) \leq z(t, t_0, \mathfrak{u}), \tag{3.9}$$

where $z(t)$ is the maximal solution of (2.14).

It is clear that system (1.1) is (m_0, m)-practically stable from Theorem 3.1. Suppose that there is a possibility that although (m_0, m)-practically stable, system (1.1) is not (m_0, m)-practically asymptotically stable. Then there would exit a sequence $\{t_j\}_{j\geq 1}$, $t_j \geq t_0 + T$, $t_j \to \infty$ such that

$$\mathfrak{b}(\epsilon) \leq \sum_{i=1}^{n} x_i(t_j, t_0, x_0), \tag{3.10}$$

whenever $\sum_{i=1}^{n} x_{0_i} < \mathfrak{a}(\lambda)$, where $x(t, t_0, x_0)$ is any solution of (1.1). Then, from (3.10), (3.9), and (3.8), we obtain a contradiction

$$\mathfrak{b}(\epsilon) \leq \sum_{i=1}^{n} \mathfrak{V}_i(t_j, x(t_j)) \leq \sum_{i=1}^{n} z_i(t_j, t_0, \mathfrak{u}_0) < \mathfrak{b}(\epsilon). \tag{3.11}$$

The contradiction (3.11), concertizes that (1.1) is not only (m_0, m)-practically stable but also (m_0, m)-practically asymptotically stable. □

Theorem 3.3 ((m, m_0)-Practical quasi-stability)**.** *Assume that conditions $(m_1) - (m_4)$ of Theorem 3.1 are satisfied, and that (2.14) is practically quasi-stable. Then, (1.1) is (m, m_0)-practically quasi-stable.*

Proof. By the assumption, (2.14) is (m_0, m)-practically quasi-stable for $(\mathfrak{a}(\lambda), \mathfrak{b}(\lambda), \mathfrak{b}(B), T) > 0$; so that for some $t_0 \in \mathbb{T}$,

$$\sum_{i=1}^{n} \kappa_{0_i} < \mathfrak{a}(\lambda) \implies \sum_{i=1}^{n} \kappa_i(t, t_0, \mathfrak{u}_0) < \mathfrak{b}(B), \quad t \geq t_0 + T, \tag{3.12}$$

where $\kappa_i(t, t_0, \mathfrak{u}_0)$ is a any solution of (2.14). By the practical quasi-stability of (2.14), we make the assumption that $m_0(t_0, x_0) < \lambda$ so that $m(t, x(t)) < A$ for $t \geq t_0$. Setting $\kappa_{0_i} = \mathfrak{V}_i(t_0, x_0)$ and by Theorem 2.26, we have the estimate

$$\mathfrak{V}(t, x(t)) \leq z(t, t_0, \mathfrak{u}), \tag{3.13}$$

where $z(t)$ is the maximal solution of (2.14). Following this argument closely, and from (3.12) and (3.13), we obtain

$$\mathfrak{b}(m(t,x(t))) \leq \sum_{i=1}^{n} \mathfrak{V}_i(t,x(t)) \leq \sum_{i=1}^{n} z_i(t,t_0,\mathfrak{u}_\circ) < \mathfrak{b}(B), \quad t \geq t_0 + T. \tag{3.14}$$

So that whenever $m_0(t_0,x_0) < \lambda$, $m(t,x(t)) < B$ for $t \geq t_0 + T$. This leads to the conclusion that (1.1) is (m_0,m)-practically quasi-stable. □

Corollary 3.4 ((m,m_0)-Strongly practical stability)**.** *Assume that conditions $(m_1) - (m_4)$ of Theorem 3.1 are satisfied, and that (2.14) is strongly practically stable. Then, (1.1) is (m,m_0)-Strongly practically stable.*

Proof. The proof follows directly from Theorems 3.1 and 3.3 since by Definition 2.22, (1.1) is (m_0,m)-strongly practically stable if it is (m_0,m)-practically stable and (m_0,m)-practically quasi-stable together. □

We shall consider a special case of Theorem 3.1, where $\Theta(t,u)$ will be in the form $\gamma(t)$ with $\gamma \in C_{rd}[\mathbb{T},\mathbb{R}^n_+]$.

Theorem 3.5. *Assume that conditions $\mathfrak{m}_1 - \mathfrak{m}_4$ of Theorem 3.1 holds for $\Theta(t,u) = \gamma(t)$, with $\gamma \in C_{rd}[\mathbb{T},\mathbb{R}^n_+]$, $\gamma_i \in \mathbf{L}^1$, $1 \leq i \leq n$. Then the practical stability of the comparison system (2.14), implies the (m_0,m)-practical stability of (1.1).*

Proof. From the assumption that $\Theta(t,u) = \gamma(t)$, consider the system

$$^{C}\Delta^{\alpha}\kappa = \gamma(t) \quad \kappa(t_0) = \kappa_0, \tag{3.15}$$

for $t \in \mathbb{T}$.

The function $\kappa(t)$ is a solution of (3.15) if and only if it satisfies the delta integral equation

$$\kappa(t) = \kappa(t_0) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1}(\gamma(s))\Delta s, \quad t,s \in \mathbb{T}. \tag{3.16}$$

By the properties of γ_i, and for $1 \leq i \leq n$, we obtain

$$\kappa_i(t) \leq \kappa_i(t_0) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{\infty} (t-s)^{\alpha-1}(\gamma_i(s))\Delta s \leq \kappa_{i0} + \mathfrak{R}_i(t_0), \quad t \geq t_0, \tag{3.17}$$

where $\mathfrak{R}_i(t_0) = \int_{t_0}^{\infty}(t-s)^{\alpha-1}(\gamma_i(s))\Delta s$. Let $0 < \lambda < A$ be given, and assume $m_0(t_0,x_0) < \lambda$, setting $\kappa_{i0} = \mathfrak{V}_i(t_0,x_0)$. Then

$$\sum_{i=0}^{n} \mathfrak{V}_i(t,x(t)) \leq \sum_{i=0}^{n} \mathfrak{V}_i(t_0,x_0) + \sum_{i=1}^{n} \mathfrak{R}_i(t_0), \tag{3.18}$$

which in view of (3.1), we obtain

$$\sum_{i=0}^{n} \mathfrak{V}_i(t,x(t)) \leq \mathfrak{a}(m_0(t_0,x_0)) + \sum_{i=1}^{n} \mathfrak{R}_i(t_0) < \mathfrak{a}(\lambda) + \sum_{i=1}^{n} \mathfrak{R}_i(t_0). \tag{3.19}$$

To conclude the practical stability of the comparison system, we have to show that there exists at least a time $t_0 \in \mathbb{T}$ when

$$\mathfrak{a}(\lambda) + \sum_{i=1}^{n} \mathfrak{R}_i(t_0) < \mathfrak{b}(A).$$

However, from the assumptions in condition $(\mathfrak{m}_4)$ of Theorem 3.1, and those on γ_i, it is obvious to see that such time t_0 exists. Hence, we can conclude that system (1.1) is (m_0, m)-practically stable. □

In the same way, we shall give a special case of Theorem 3.2, which cannot be inferred by Theorem 3.1.

Theorem 3.6. *Assume that conditions $\mathfrak{m}_1 - \mathfrak{m}_4$ of Theorem 3.1 hold except for (3.2), which will be modified as follows:*

$$\begin{aligned} {}^{C}\Delta_{+}\mathfrak{V}_k(t,x) &\leq -\xi(m(t,x)) + \gamma_k(t), \quad 1 \leq k \leq n, \\ {}^{C}\Delta_{+}\mathfrak{V}_i(t,x) &\leq \gamma_i(t), \qquad\qquad\qquad 1 \leq i \leq n, \quad i \neq k, \end{aligned} \tag{3.20}$$

where $\xi \in \mathcal{K}$, $\gamma \in C_{rd}[\mathbb{T}, \mathbb{R}^n_+]$, $\gamma_i \in \mathbb{L}^1$, $1 \leq i \leq n$. Let $m(t,x)$ be locally Lipschitzian in x and let $C^{\Delta}_{+}m(t,x)$ be bounded on $\mathbb{T} \times \mathbb{R}^n$. Then system (1.1) is (m_0, m)-practically asymptotically stable.

Proof. To show that (1.1) is (m_0, m)-practically asymptotically stable, we have to show that (1.1) is (m_0, m)-practically stable and that any solution $x(t)$ of (1.1) with $m_0(t_0, x_0) < \lambda$. Then $\lim_{t\to\infty} m(t, x(t)) = 0$. Theorem 3.6 guarantees the (m_0, m)-practical stability of (1.1). So we are left to show that $\lim_{t\to\infty} m(t, x(t)) = 0$ whenever $m_0(t_0, x_0) < \lambda$.

We shall make this proof by contradiction and in two phases.

Phase 1

Claim: $\liminf_{t\to\infty} m(t, x(t)) = 0$ whenever $m_0(t_0, x_0) < \lambda$.

If this claim were to be false, then from (3.20), we would obtain that as $t \to \infty$, $\mathfrak{V}(t, x(t)) \to -\infty$ which contradicts the positive definiteness of $\mathfrak{V}(t, x(t))$, implying our claim is correct.

Phase 2

Claim: $\limsup_{t\to\infty} m(t, x(t)) = 0$ whenever $m(t, x(t)) < \lambda$.

Similarly, if this claim were false, that is $\limsup_{t\to\infty} m(t, x(t)) \neq 0$, then for any $\epsilon > 0$, there would exist a divergent sequence of time $\{t_i\}$ where $t_i \in \mathbb{T}$ and $t_i \to \infty$ as $i \to \infty$, such that

$$m(t_i, x(t_i)) \geq \epsilon. \tag{3.21}$$

With this, t_i can be one of the following cases:

Case 1: left-scattered, right-scattered,
Case 2: left-dense, right-dense,
Case 3: left-dense, right-scattered,
Case 4: left-scattered, right-dense.

Assume without loss of generality that there exists a sub-sequence $\{t_j\}$ of $\{t_i\}$, satisfying $t_j \to \infty$ as $j \to \infty$ and $m(t_j, x(t_j)) \geq \epsilon$. Also, if t_j belongs to one of the above four cases, then we can argue as follows:

Case 1:

It is clear that case 1 occurs at discrete times so that from (3.20) and with respect to (2.12), we have

$$\begin{aligned}
{}^{C}\Delta_{+}\mathfrak{V}_k(t,x) &= \frac{1}{\mu^{\alpha}(t_j)}\Bigg[\sum_{r=0}^{[\frac{t-t_0}{\mu}]}(-1)^r({}^{\alpha}C_r)(\mathfrak{V}_k(\sigma(t_j),x(\sigma(t_j)))-\mathfrak{V}_k(t_0,x_0))\Bigg]\\
&\leq -\xi(m(t_j,x(t_j)))+\gamma_{k,j}\\
&= \sum_{r=0}^{[\frac{t-t_0}{\mu}]}(-1)^r({}^{\alpha}C_r)(\mathfrak{V}_k(\sigma(t_j),x(\sigma(t_j))))\\
&\leq \sum_{r=0}^{[\frac{t-t_0}{\mu}]}(-1)^r({}^{\alpha}C_r)(\mathfrak{V}_k(t_0,x_0))+(-\xi(m(t_j,x(t_j)))+\gamma_{k,j})\,\mu^{\alpha}(t_j).
\end{aligned}$$

From the assumption of (3.21), and with $\tau = \max_i \mu(t_i)$, we have

$$\begin{aligned}
0 &< \sum_{r=0}^{[\frac{t-t_0}{\mu}]}(-1)^r({}^{\alpha}C_r)(\mathfrak{V}_k(\sigma(t_j),x(\sigma(t_j))))\\
&\leq \sum_{r=0}^{[\frac{t-t_0}{\mu}]}(-1)^r({}^{\alpha}C_r)(\mathfrak{V}_k(t_0,x_0))-(t_{j+1}-t_j)\xi(\epsilon)+\tau\sum_{i=0}^{\infty}\gamma_{k,i}. \qquad (3.22)
\end{aligned}$$

Since time is discrete in this case, $\gamma_k \in \mathbf{L}^1$ implies that $\sum_{i=0}^{\infty}\gamma_{k,i} < \infty$, so that the right-hand side of (3.22) will be negative for sufficiently large j, contradicting the positive definiteness of $\mathfrak{V}(t)$, and therefore proving that the claim $\limsup_{t\to\infty} m(t,x(t)) = 0$ whenever $m(t,x(t)) < \lambda$ is indeed correct for case 1.

Case 2

Case 2 occurs in continuous times, so that from (3.20) and in view of (2.13), we have

$$\begin{aligned}
&{}^{C}\Delta_{+}^{\alpha}\mathfrak{V}_k(t_j,x) = {}^{C}D_{+}^{\alpha}\mathfrak{V}_k(t_j,x)\\
&= \limsup_{\kappa\to 0^+}\frac{1}{\kappa^{\alpha}}\Bigg\{\mathfrak{V}_k(t_j,x(t_j))-\mathfrak{V}_k(t_0,x_0)\\
&\quad -\sum_{r=1}^{[\frac{t-t_0}{\kappa}]}(-1)^{r+1}({}^{\alpha}C_r)[\mathfrak{V}_k(t_j-r\kappa,x(t_j))-\kappa^{\alpha}f(t_j,x(t_j))-\mathfrak{V}_k(t_0,x_0)]\Bigg\}\\
&\leq -\xi(m(t_j,x(t_j)))+\gamma_{k,j}\\
&= \mathfrak{V}_k(t_j,x(t_j)) - \limsup_{\kappa\to 0^+}\frac{1}{\kappa^{\alpha}}\sum_{r=0}^{[\frac{t-t_0}{\kappa}]}(-1)^{r+1}({}^{\alpha}C_r)\mathfrak{V}_k(t_0,x_0)]
\end{aligned}$$

$$- \limsup_{\kappa\to 0^+} \frac{1}{\kappa^\alpha} \sum_{r=1}^{[\frac{t-t_0}{\kappa}]} (-1)^{r+1} ({}^\alpha C_r) \mathfrak{V}_k(t_j, x(t_j))$$
$$\leq -\xi(m(t_j, x(t_j))) + \gamma_{k,j}.$$

Applying (2.10), we obtain

$$\mathfrak{V}_k(t_j, x(t_j)) + \limsup_{\kappa\to 0^+} \frac{1}{\kappa^\alpha} \sum_{r=1}^{[\frac{t-t_0}{\kappa}]} (-1)^{r} ({}^\alpha C_r) \mathfrak{V}_k(t_j, x(t_j))$$
$$\leq -\xi(m(t_j, x(t_j))) + \gamma_{k,j} - \frac{\mathfrak{V}_k(t_0, x_0)(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}$$
$$\mathfrak{V}_k(t_j, x(t_j)) + \limsup_{\kappa\to 0^+} \sum_{r=1}^{[\frac{t-t_0}{\kappa}]} (-1)^{r} ({}^\alpha C_r) \mathfrak{V}_k(t_j, x(t_j))$$
$$\leq \left(-\xi(m(t_j, x(t_j))) + \gamma_{k,j} - \frac{\mathfrak{V}_k(t_0, x_0)(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)} \right) \kappa^\alpha.$$

Applying (2.9) and the assumption of (3.21) the right-hand side becomes negative, which is a contradiction further validating our claim for case 2.

In other cases, we assume a subsequence $\{t_{1j}\}$ of $\{t_i\}$ such that $t_{1j} \to \infty$ as $j \to \infty$, and we argue that either For **Cases (3) and (4)**, applying Lemma 2.25 to (3.20), and in view of Definition 2.10, we obtain

$$\mathfrak{V}_k(t_{1j}, x(t_{1j})) \leq \mathfrak{V}_k(t_0, x_0) - \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_{1j}} (t-s)^{\alpha-1} - \xi(m(t_{1j}, x(t_{1j}))) + \gamma_{k,j} \Delta s. \tag{3.23}$$

From the assumption of (3.21), we obtain

$$\mathfrak{V}_k(t_{1j}, x(t_{1j})) \leq \mathfrak{V}_k(t_0, x_0) - \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_{1j}} (t-s)^{\alpha-1} - \xi(\epsilon) + \gamma_{k,j} \Delta s,$$

$$0 \leq \mathfrak{V}_k(t_{1j}, x(t_{1j})) \leq \mathfrak{V}_k(t_0, x_0) - \frac{\xi(\epsilon) + \gamma_{k,j}(t_{1j} - t_0)^\alpha}{\alpha\Gamma(\alpha)}. \tag{3.24}$$

Since $\gamma_k \in \mathbf{L}^1$ implies that $\gamma_{k,j} < \infty$ and $t_{1j} \to \infty$ as $j \to \infty$, the right-hand side of (3.24) approaches $-\infty$. This also is a contradiction and validates our claim that $\limsup_{t\to\infty} m(t, x(t)) = 0$ whenever $m(t, x(t)) < \lambda$.

Since $\liminf_{t\to\infty} m(t, x(t)) = \limsup_{t\to\infty} m(t, x(t)) = 0$, then, $\lim_{t\to\infty} m(t, x(t)) = 0$ whenever $m_0(t_0, x_0) < \lambda$ guaranteeing that (1.1) is (m_0, m)-practically asymptotically stable. ∎

4. Application

Consider the Caputo fractional dynamic system:

$$
\begin{aligned}
{}^{C}\Delta^{\alpha}\upsilon_1(t) &= 6\upsilon_1-\frac{3\upsilon_2^2+3\upsilon_3^2}{\upsilon_1},\\
{}^{C}\Delta^{\alpha}\upsilon_2(t) &= -\frac{3\upsilon_1^2}{\upsilon_2}+4\upsilon_2,\\
{}^{C}\Delta^{\alpha}\upsilon_3(t) &= -\frac{\upsilon_1^2}{\upsilon_3}+3\upsilon_3,
\end{aligned}
\tag{4.1}
$$

for $t \geq t_0$, with initial conditions

$$\upsilon_1(t_0)=\upsilon_{10},\quad \upsilon_2(t_0)=\upsilon_{20},\quad \text{and}\quad \upsilon_3(t_0)=\upsilon_{30},$$

where $\upsilon=(\upsilon_1,\upsilon_2,\upsilon_3)$, and $f=(f_1,f_2,f_3)$.

Consider a vector $V=(V_1,V_2,V_3)^T$, where $V_1=\upsilon_1^2$, $V_2=\upsilon_2^2$ and $V_3=\upsilon_3^2$, for $t\in\mathbb{T}$ and $(\upsilon_1,\upsilon_2,\upsilon_3)\in\mathbb{R}^3$. Then condition 1 of Theorem 3.1 is satisfied, for $\phi=\frac{1}{2}r$, where $\phi\in\mathcal{K}$, so that the associated norm $\|\upsilon\|=\sqrt{\upsilon_1^2+\upsilon_2^2+\upsilon_3^2}$, and

$$V_0(\upsilon_1,\upsilon_2,\upsilon_3)=\sum_{i=1}^{2}V_i(\upsilon_1,\upsilon_2,\upsilon_3)=\upsilon_1^2+\upsilon_2^2+\upsilon_3^2.$$

Then $\phi(\|\psi\|)\leq V(\psi_1,\psi_2,\psi_3)$. From (**??**), we compute the Caputo fractional Dini derivative for $V_1=\upsilon_1^2$ as follows:

$$
\begin{aligned}
&{}^{C}\Delta_{+}^{\alpha}V_1\\
&=\limsup_{\mu\to0^+}\frac{1}{\mu^{\alpha}}\Bigg\{\left[(\upsilon_1(\sigma(t)))^2\right]-\left[(\upsilon_{10})^2\right]\\
&\quad+\sum_{r=1}^{\left[\frac{t-t_0}{\mu}\right]}(-1)^r\left({}^{\alpha}C_r\right)\left[(\upsilon_1(\sigma(t))-\mu^{\alpha}f_1(t,\upsilon))^2\right]-\left[((\upsilon_{10})^2\right]\Bigg\}\\
&=\limsup_{\mu\to0^+}\frac{1}{\mu^{\alpha}}\Bigg\{\left[(\upsilon_1(\sigma(t)))^2\right]-\left[(\upsilon_{10})^2\right]\\
&\quad+\sum_{r=1}^{\left[\frac{t-t_0}{\mu}\right]}(-1)^r\left({}^{\alpha}C_r\right)\left[(\upsilon_1(\sigma(t)))^2-2\upsilon_1(\sigma(t))\mu^{\alpha}f_1(t,\upsilon_1,\upsilon_2,\upsilon_3)\right.\\
&\quad\left.+\mu^{2\alpha}(f_1(t,\upsilon_1,\upsilon_2,\upsilon_3))^2\right]-\left[(\upsilon_{10})^2\right]\Bigg\}\\
&=-\limsup_{\mu\to0^+}\frac{1}{\mu^{\alpha}}\left\{\sum_{r=0}^{\left[\frac{t-t_0}{\mu}\right]}(-1)^r\left({}^{\alpha}C_r\right)\left[(\upsilon_{10})^2\right]\right\}\\
&\quad+\limsup_{\mu\to0^+}\frac{1}{\mu^{\alpha}}\left\{\sum_{r=0}^{\left[\frac{t-t_0}{\mu}\right]}(-1)^r\left({}^{\alpha}C_r\right)\left[(\upsilon_1(\sigma(t)))^2\right]\right\}
\end{aligned}
$$

$$-\limsup_{\mu\to 0^+}\left\{\sum_{r=1}^{\left[\frac{t-t_0}{\mu}\right]}(-1)^r\left({}^{\alpha}C_r\right)\left[2\upsilon_1(\sigma(t))\mu^{\alpha}f_1(t,\upsilon_1,\upsilon_2,\upsilon_3)\right]\right\}.$$

Applying (2.9) and (2.10), we obtain

$${}^{C}\Delta_{+}^{\alpha}V_1 \quad \le \frac{(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}\left[(\upsilon_1(\sigma(t)))^2\right]-\left[2\upsilon_1(\sigma(t))f_1(t,\upsilon_1,\upsilon_2,\upsilon_3)\right].$$

As $t\to\infty$, $\frac{(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}\left[(\upsilon_1(\sigma(t)))^2\right]\to 0$, which is

$${}^{C}\Delta_{+}^{\alpha}V_1 \;\le\; -2[\upsilon_1(\sigma(t))f_1(t,\upsilon_1,\upsilon_2,\upsilon_3)].$$

Applying $\upsilon(\sigma(t))\le \mu^{C}\Delta^{\alpha}\upsilon(t)+\upsilon(t)$, we have

$$\begin{aligned}
{}^{C}\Delta_{+}^{\alpha}V_1 &= -2\Big[\mu(t)f_1^2(t,\upsilon_1,\upsilon_2,\upsilon_3)+\upsilon_1(t)f_1(t,\upsilon_1,\upsilon_2,\upsilon_3)\Big]\\
&= -2\Bigg[\mu(t)\left(6\upsilon_1-\frac{3\upsilon_2^2+3\upsilon_3^2}{\upsilon_1}\right)^2+\upsilon_1\left(6\upsilon_1-\frac{3\upsilon_2^2+3\upsilon_3^2}{\upsilon_1}\right)\Bigg]\\
&= -2\mu(t)\left[\left(6\upsilon_1-\frac{3\upsilon_2^2+3\upsilon_3^2}{\upsilon_1}\right)^2\right]-2\upsilon_1\left[6\upsilon_1-\frac{3\upsilon_2^2+3\upsilon_3^2}{\upsilon_1}\right]. \qquad (4.2)
\end{aligned}$$

If $\mathbb{T}=\mathbb{R}$, then $\mu=0$, so that (4.2) becomes

$$\begin{aligned}
{}^{C}\Delta_{+}^{\alpha}V_1(\upsilon_1,\upsilon_2,\upsilon_3) &= -2\upsilon_1\left[6\upsilon_1-\frac{3\upsilon_2^2+3\upsilon_3^2}{\upsilon_1}\right]\\
&= -12\upsilon_1^2+6\upsilon_2^2+6\upsilon_3^2\\
&= (-12 \quad 6 \quad 6)\cdot(V_1 \quad V_2 \quad V_3)^T. \qquad (4.3)
\end{aligned}$$

Similarly, compute the Caputo fractional Dini derivative for $V_2(\upsilon)=\upsilon_2^2$ as follows:

$$\begin{aligned}
&{}^{C}\Delta_{+}^{\alpha}V_2(\upsilon)\\
&=\limsup_{\mu\to 0^+}\frac{1}{\mu^{\alpha}}\Bigg\{\left[(\upsilon_2(\sigma(t)))^2\right]-\left[(\upsilon_{20})^2\right]\\
&\quad+\sum_{r=1}^{\left[\frac{t-t_0}{\mu}\right]}(-1)^r\left({}^{\alpha}C_r\right)\left[(\upsilon_2(\sigma(t))-\mu^{\alpha}f_2(t,\upsilon))^2\right]-\left[((\upsilon_{20})^2\right]\Bigg\}\\
&=\limsup_{\mu\to 0^+}\frac{1}{\mu^{\alpha}}\Bigg\{\left[(\upsilon_2(\sigma(t)))^2\right]-\left[(\upsilon_{20})^2\right]\\
&\quad+\sum_{r=1}^{\left[\frac{t-t_0}{\mu}\right]}(-1)^r\left({}^{\alpha}C_r\right)\left[(\upsilon_2(\sigma(t)))^2-2\upsilon_2(\sigma(t))\mu^{\alpha}f_2(t,\upsilon_1,\upsilon_2,\upsilon_3)\right.\\
&\quad\left.+\mu^{2\alpha}(f_2(t,\upsilon_1,\upsilon_2,\upsilon_3))^2\right]-\left[(\upsilon_{20})^2\right]\Bigg\}
\end{aligned}$$

$$= -\limsup_{\mu\to 0^+} \frac{1}{\mu^\alpha}\left\{\sum_{r=0}^{\left[\frac{t-t_0}{\mu}\right]} (-1)^r \left({}^\alpha C_r\right)\left[(\upsilon_{20})^2\right]\right\}$$
$$+\limsup_{\mu\to 0^+} \frac{1}{\mu^\alpha}\left\{\sum_{r=0}^{\left[\frac{t-t_0}{\mu}\right]} (-1)^r \left({}^\alpha C_r\right)\left[(\upsilon_2(\sigma(t)))^2\right]\right\}$$
$$-\limsup_{\mu\to 0^+}\left\{\sum_{r=1}^{\left[\frac{t-t_0}{\mu}\right]} (-1)^r \left({}^\alpha C_r\right)\left[2\upsilon_2(\sigma(t))\mu^\alpha f_2(t,\upsilon_1,\upsilon_2,\upsilon_3)\right]\right\}.$$

Applying (2.9) and (2.10), we obtain

$$ {}^C\Delta_+^\alpha V_2(\upsilon) \leq \frac{(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}\left[(\upsilon_2(\sigma(t)))^2\right] - [2\upsilon_2(\sigma(t))f_2(t,\upsilon_1,\upsilon_2,\upsilon_3)].$$

As $t\to\infty$, $\frac{(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}\left[(\upsilon_2(\sigma(t)))^2\right]\to 0$, which is

$$ {}^C\Delta_+^\alpha V_2(\upsilon) \leq -2[\upsilon_2(\sigma(t))f_2(t,\upsilon_1,\upsilon_2,\upsilon_3)].$$

Applying $\upsilon(\sigma(t)) \leq \mu\, {}^C\Delta^\alpha \upsilon(t) + \upsilon(t)$, we have

$$\begin{aligned}
{}^C\Delta_+^\alpha V_2(\upsilon) &= -2\Big[\mu(t) f_2^2(t,\upsilon_1,\upsilon_2,\upsilon_3) + \upsilon_2(t) f_2(t,\upsilon_1,\upsilon_2,\upsilon_3)\Big]\\
&= -2\left[\mu(t)\left(-\frac{3\upsilon_1^2}{\upsilon_2}+4\upsilon_2\right)^2 + \upsilon_2\left(-\frac{3\upsilon_1^2}{\upsilon_2}+4\upsilon_2\right)\right]\\
&= -2\mu(t)\left[\left(-\frac{3\upsilon_1^2}{\upsilon_2}+4\upsilon_2\right)^2\right] - 2\upsilon_1\left[-\frac{3\upsilon_1^2}{\upsilon_2}+4\upsilon_2\right]. \qquad (4.4)
\end{aligned}$$

If $\mathbb{T}=\mathbb{R}$, then $\mu=0$, so that (4.4) becomes

$$\begin{aligned}
{}^C\Delta_+^\alpha V_2(\upsilon_1,\upsilon_2) &= -2\upsilon_2\left[-\frac{3\upsilon_1^2}{\upsilon_2}+4\upsilon_2\right]\\
&= 6\upsilon_1^2 - 8\upsilon_2^2\\
&= (6 \quad -8 \quad 0)\cdot(V_1 \quad V_2 \quad V_3)^T. \qquad (4.5)
\end{aligned}$$

Similarly, compute the Caputo fractional Dini derivative for $V_3(\upsilon_1,\upsilon_2,\upsilon_3)=\upsilon_3^2$ as follows:

$$\begin{aligned}
&{}^C\Delta_+^\alpha V_3\\
&= \limsup_{\mu\to 0^+}\frac{1}{\mu^\alpha}\Bigg\{\left[(\upsilon_3(\sigma(t)))^2\right] - \left[(\upsilon_{30})^2\right]\\
&\quad + \sum_{r=1}^{\left[\frac{t-t_0}{\mu}\right]} (-1)^r\left({}^\alpha C_r\right)\left[(\upsilon_3(\sigma(t)) - \mu^\alpha f_3(t,\upsilon))^2\right] - \left[((\upsilon_{30})^2\right]\Bigg\}\\
&= \limsup_{\mu\to 0^+}\frac{1}{\mu^\alpha}\Bigg\{\left[(\upsilon_3(\sigma(t)))^2\right] - \left[(\upsilon_{30})^2\right]
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{r=1}^{\left[\frac{t-t_0}{\mu}\right]}(-1)^r\left({}^{\alpha}C_r\right)\left[(\upsilon_3(\sigma(t)))^2-2\upsilon_3(\sigma(t))\mu^{\alpha}f_3(t,\upsilon_1,\upsilon_2,\upsilon_3)\right.\\
&\left.+\mu^{2\alpha}(f_3(t,\upsilon_1,\upsilon_2,\upsilon_3))^2\right]-\left[(\upsilon_{30})^2\right]\Bigg\}\\
=&-\limsup_{\mu\to0^+}\frac{1}{\mu^{\alpha}}\left\{\sum_{r=0}^{\left[\frac{t-t_0}{\mu}\right]}(-1)^r\left({}^{\alpha}C_r\right)\left[(\upsilon_{30})^2\right]\right\}\\
&+\limsup_{\mu\to0^+}\frac{1}{\mu^{\alpha}}\left\{\sum_{r=0}^{\left[\frac{t-t_0}{\mu}\right]}(-1)^r\left({}^{\alpha}C_r\right)\left[(\upsilon_3(\sigma(t)))^2\right]\right\}\\
&-\limsup_{\mu\to0^+}\left\{\sum_{r=1}^{\left[\frac{t-t_0}{\mu}\right]}(-1)^r\left({}^{\alpha}C_r\right)\left[2\upsilon_3(\sigma(t))\mu^{\alpha}f_3(t,\upsilon_1,\upsilon_2,\upsilon_3)\right]\right\}.
\end{aligned}
$$

Applying (2.9) and (2.10), we obtain

$$
{}^{C}\Delta_{+}^{\alpha}V_3 \quad\leq \frac{(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}\left[(\upsilon_3(\sigma(t)))^2\right]-\left[2\upsilon_1(\sigma(t))f_3(t,\upsilon_1,\upsilon_2,\upsilon_3)\right].
$$

As $t\to\infty$, $\frac{(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}\left[(\upsilon_3(\sigma(t)))^2\right]\to 0$, then

$$
{}^{C}\Delta_{+}^{\alpha}V_3 \;\leq\; -2[\upsilon_3(\sigma(t))f_3(t,\upsilon_1,\upsilon_2,\upsilon_3)].
$$

Applying $\upsilon(\sigma(t))\leq \mu^{C}\Delta^{\alpha}\upsilon(t)+\upsilon(t)$,

$$
\begin{aligned}
{}^{C}\Delta_{+}^{\alpha}V_3 &= -2\Big[\mu(t)f_3^2(t,\upsilon_3,\upsilon_2,\upsilon_3)+\upsilon_3(t)f_3(t,\upsilon_1,\upsilon_2,\upsilon_3)\Big]\\
&= -2\left[\mu(t)\left(-\frac{\upsilon_1^2}{\upsilon_3}+3\upsilon_3\right)^2+\upsilon_3\left(-\frac{\upsilon_1^2}{\upsilon_3}+3\upsilon_3\right)\right]\\
&= -2\mu(t)\left[\left(-\frac{\upsilon_1^2}{\upsilon_3}+3\upsilon_3\right)^2\right]-2\upsilon_3\left[-\frac{\upsilon_1^2}{\upsilon_3}+3\upsilon_3\right]. \qquad (4.6)
\end{aligned}
$$

If $\mathbb{T}=\mathbb{R}$, then $\mu=0$, so that (4.6) becomes

$$
\begin{aligned}
{}^{C}\Delta_{+}^{\alpha}V_3(\upsilon_1,\upsilon_2,\upsilon_3) &= -2\upsilon_3\left[-\frac{\upsilon_1^2}{\upsilon_3}+3\upsilon_3\right]\\
&= 2\upsilon_1^2-6\upsilon_3^2\\
&= (2 \quad 0 \quad -6)\cdot(V_1 \quad V_2 \quad V_3)^T. \qquad (4.7)
\end{aligned}
$$

Combining (4.3), (4.5), and (4.7), we have

$$
{}^{C}\Delta_{+}^{\alpha}V \leq \begin{pmatrix}-12 & 6 & 6\\ 6 & -8 & 0\\ 2 & 0 & -6\end{pmatrix}\begin{pmatrix}V_1\\ V_2\\ V_3\end{pmatrix}=\Theta(t,V). \qquad (4.8)
$$

Now consider the comparison system

$$^{C}\Delta_{+}^{\alpha}\kappa = \Theta(t,\kappa) = A\kappa, \tag{4.9}$$

where $A = \begin{pmatrix} -12 & 6 & 6 \\ 6 & -8 & 0 \\ 2 & 0 & -6 \end{pmatrix}$.

The vectorial inequality (4.8) and all other conditions of Theorems (3.1) and (3.2) are satisfied if A has eigenvalues with negative real parts, since the eigenvalues of A are $\lambda_1 = -17.059$, $\lambda_2 = -6.59407$, $\lambda_3 = -2.34691$, then (4.1) is stable. Therefore, we conclude that the trivial solution of the system (4.1) is stable.

5. Conclusion

In conclusion, we introduced a novel framework for assessing the practical stability of Caputo fractional dynamic equations on time scales, utilizing two measures, m and m_0, and employing the new Caputo fractional delta derivative and Caputo fractional delta Dini derivative of order $\alpha \in (0,1)$. This approach provides a unified method for stability analysis across both continuous and discrete time domains, enhancing the applicability of stability theory to hybrid systems that display both gradual and abrupt changes. By unifying several known stability concepts within a single setup, our framework offers a more comprehensive stability analysis that accommodates the complex dynamics of real-world systems. The practical stability results established here have been validated through an illustrative example. This work advances the theoretical understanding of stability for fractional dynamic systems on time scales and offers a versatile tool for researchers and practitioners analyzing complex hybrid systems.

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