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NUMERICAL RADIUS INEQUALITIES OF PRODUCT AND TENSOR PRODUCT OF OPERATORS

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Abstract. In this paper, we give new numerical radius inequalities for bounded linear operators, product of bounded linear operators, and tensor product of two operators acting on complex Hilbert spaces. These inequalities improve and refine earlier well-known results. Also, we extend our obtained results to semi-Hilbert spaces.

1. Introduction

Throughout this paper, H and K are two complex Hilbert spaces. We denote by $B(H)$ the C^*-algebra of all bounded linear operators on H. For $T \in B(H)$, the numerical range of T is defined by

$$W(T) = \{\langle Tx, x\rangle : x \in H,\ \|x\| = 1\}, \tag{1.1}$$

and the numerical radius of T is given by

$$w(T) = \sup\{|z| : z \in W(T)\} = \sup_{x\in H,\ \|x\|=1} |\langle Tx, x\rangle|.$$

The numerical range of $T \in B(H)$, like the spectrum, is a subset of the complex plane. It is a celebrated result due to Hausdorff–Toeplitz [16, 25] that $W(T)$ is a bounded convex set, and its closure contains the convex hull of the spectrum $\sigma(T)$ of T. It is closed if $dim(H) < \infty$, but it is not always closed if $dim(H) = \infty$. It is known that the numerical radius $w(\cdot)$ defines a norm on $B(H)$, which is

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equivalent to the usual operator norm $\|\cdot\|$. In fact, for any $T \in B(H)$,

$$\frac{1}{2}\|T\| \leq w(T) \leq \|T\|. \tag{1.2}$$

These two inequality are sharp. Indeed, they become equalities, respectively, if $T^2 = 0$ and T is normal. Also, the numerical radius satisfies the power inequality:

$$w(T^n) \leq w^n(T),$$

for any natural number n. For a complete account about facts above, see, for example, [14].

A variation of (1.1) is the so-called maximal numerical range $W_0(T)$. It was introduced in [23] and defined as follows:

$$W_0(T) = \{\lim_n \langle Tx_n, x_n\rangle : x_n \in H, \|x_n\| = 1, \lim_n \|Tx_n\| = \|T\|\}.$$

In the same paper, it was shown that $W_0(T)$ is nonempty, closed, convex, and contained in the closure of the numerical range. In the case of finite-dimensional spaces, the maximal numerical range is produced by maximal vectors for T (those vectors $x \in H$ such that $\|x\| = 1$ and $\|Tx\| = \|T\|$). Note that the notion of the maximal numerical range was introduced in [23] (especially) for the purpose of calculating the norm of the inner derivation δ_A associated with $A \in B(H)$, which is defined by

$$\delta_A : B(H) \longrightarrow B(H),\ X \longmapsto AX - XA.$$

In fact, let $T \in B(H)$, and define

$$d(T) := \inf_{\lambda\in\mathbb{C}} \|T - \lambda I\|,$$

the distance of T to the scalar operators. Here, I is the identity operator. According to [23]

$$d^2(T) = \sup_{x\in H,\ \|x\|=1} \{\|Tx\|^2 - |\langle Tx, x\rangle|^2\}, \tag{1.3}$$

and there exists a unique scalar c_T, called the center of mass of T, such that

$$d(T) = \|T - c_T I\|,$$

and for any $A \in B(H)$,

$$\|\delta_A\| = 2d(A).$$

Recall that the tensor product of H and K is defined as the completion of the inner product space consisting of all elements of the form $\sum_{k=1}^{n} x_k \otimes y_k$ for any $x_k \in H$, $y_k \in K$ and any nonzero natural number n, with the inner product $\langle x \otimes y, z \otimes t\rangle := \langle x, z\rangle\langle y, t\rangle$. Here the notation $x \otimes y$ is defined algebraically so as to be bilinear in the two arguments x and y. Let $H \otimes K$ denote the tensor product of the spaces H and K. For $T \in B(H)$ and $S \in B(K)$, the tensor product $T \otimes S$ is defined by $(T \otimes S)(x \otimes y) = Tx \otimes Sy$, for all $x \otimes y \in H \otimes K$. Recall that we can write $T \otimes S$ as the product of $T \otimes I_K$ and $I_H \otimes S$ (I_H and I_K are the identity operators on H and K, respectively). Hence $T \otimes S \in B(H \otimes K)$ and $\|T \otimes S\| = \|T\|\|S\|$. Then, from inequalities (1.2), we get

$$\frac{1}{2}\|T\|\|S\| \leq w(T \otimes S) \leq \|T\|\|S\|. \tag{1.4}$$

It is also known, see [10], that

$$w(T)w(S) \leq w(T \otimes S) \leq \min\{w(T)\|S\|, w(S)\|T\|\}. \tag{1.5}$$

In [10], the authors showed that

$$\begin{aligned} w(T \otimes S) &\leq \min\{w(T)\left(w(S) + R(S)\right), w(S)\left(w(T) + R(T)\right)\} \\ &\leq 2w(T)w(S). \end{aligned} \tag{1.6}$$

Here $R(T) = \inf\{w(T - \lambda I) : \lambda \in \mathbb{C}\}$.

In Section 2 of the present paper, we improve inequality (1.6). Indeed, we show that

$$\begin{aligned} w(T \otimes S) &\leq \min\{w(T)\left(K(S) + R(S)\right), w(S)\left(K(T) + R(T)\right)\} \\ &\leq \min\{w(T)\left(w(S) + R(S)\right), w(S)\left(w(T) + R(T)\right)\} \\ &\leq \min\{w(T)\left(w(S) + K(S)\right), w(S)\left(w(T) + K(T)\right)\} \\ &\leq 2w(T)w(S), \end{aligned}$$

where $K(T) = \inf\{w(T - i\lambda z_T I) : \lambda \in \mathbb{C}\}$. Here z_T is the center of the smallest disk containing $W(T)$. Note that for any $T \in B(H)$, we have (see, [24])

$$R(T) \leq K(T) \leq w(T).$$

We give a refinement for the upper bound of the Kittaneh inequality [20]:

$$w(T) \leq \frac{1}{2}(\|T\| + \|T^2\|^{\frac{1}{2}}). \tag{1.7}$$

Namely, we prove that

$$w^2(T) \leq \frac{1}{4}\||T|^2 + |T^*|^2\| + \frac{1}{2}\|\Re(|T||T^*|)\| \leq \frac{1}{4}(\|T\| + \|T^2\|^{\frac{1}{2}})^2.$$

Here $\Re(T)$ and T^* stand for the real part and the adjoint of T, respectively. We also obtain new refinements of the classical inequality $w(T) \geq \frac{1}{2}\|T\|$.

We establish that

$$w^2(T) \geq \frac{\|T^*T + TT^*\|}{4} + \frac{|\|T + T^*\|^2 - \|T - T^*\|^2|}{8},$$

and

$$w^2(T) \geq \frac{\|T^*T + TT^*\|}{4} + \frac{|\|T + iT^*\|^2 - \|T - iT^*\|^2|}{8}.$$

Then, we improve the Kittaneh inequality [21]:

$$w^2(T) \geq \frac{\|T^*T + TT^*\|}{4}. \tag{1.8}$$

Next, we extend our obtained results to the product of two operators T and S on H such that commute or doubly commute (i.e., $TS = ST$ and $TS^* = S^*T$). We also obtain new numerical radius inequalities for the tensor product of two operators acting on complex Hilbert spaces. Section 3 is devoted to the generalization of our results to semi-Hilbert spaces.

2. Main results

We start our work with the following lemmas, which will be useful.

Lemma 2.1. [2] *Let $T \in B(H)$. Then*

$$w^2(T) \le \frac{1}{2}\|T^*T + TT^*\| - \frac{1}{2}m^2\left(|T| - |T^*|\right),$$

where $m(T)$ is the Crawford number of T given by $m(T) := \inf\{|z| : z \in W(T)\}$.

Lemma 2.2. [15] *Let $T \in B(H)$ and $x \in H$. Then*

$$|\langle Tx, x\rangle| \le \langle |T|x, x\rangle^{\frac{1}{2}} \langle |T^*|x, x\rangle^{\frac{1}{2}}.$$

Lemma 2.3. [18] *Let $T \in B(H)$ be a positive operator. For any unit $x \in H$ and $r \ge 1$, we have*

$$\langle Tx, x\rangle^r \le \langle T^r x, x\rangle.$$

Lemma 2.4. [19] *Let $T, S \in B(H)$ be positive operators. We have*

$$\|T + S\| \le \max\{\|T\|, \|S\|\} + \|T^{\frac{1}{2}} S^{\frac{1}{2}}\|.$$

Lemma 2.5. [24] *Let $T, S \in B(H)$ such that $TS = ST$. Then*

$$\begin{aligned} w(TS) &\le w(T)\left(K(S) + R(S)\right) \\ &\le w(T)\left(w(S) + R(S)\right) \\ &\le 2w(T)w(S). \end{aligned}$$

The following theorem gives a better estimation than inequality (1.6).

Theorem 2.6. *Let $T \in B(H)$ and $S \in B(K)$. Then*

$$\begin{aligned} w(T \otimes S) &\le \min\{w(T)\left(K(S) + R(S)\right), w(S)\left(K(T) + R(T)\right)\} \\ &\le \min\{w(T)\left(w(S) + R(S)\right), w(S)\left(w(T) + R(T)\right)\} \\ &\le \min\{w(T)\left(w(S) + K(S)\right), w(S)\left(w(T) + K(T)\right)\} \\ &\le 2w(T)w(S). \end{aligned}$$

Proof. Since $W(I_H \otimes S) = W(S)$, then $K(I_H \otimes S) = K(S)$ and $R(I_H \otimes S) = R(S)$. The operators $T \otimes I_K$ and $I_H \otimes S$ commute. Then by Lemma 2.5, we have

$$\begin{aligned} w(T \otimes S) &= w((T \otimes I_K)(I_H \otimes S)) \\ &\le w(T \otimes I_K)\left(K(I_H \otimes S) + R(I_H \otimes S)\right) \\ &= w(T)\left(K(S) + R(S)\right) \\ &\le w(T)\left(w(S) + R(S)\right) \\ &\le w(T)\left(w(S) + K(S)\right) \\ &\le 2w(T)w(S). \end{aligned}$$

□

Proposition 2.7. *Let $T, S \in B(H)$. If T and S doubly commute, then*

$$w^2(TS) \le \frac{1}{2}\|T^*TS^*S + TT^*SS^*\| - \frac{1}{2}m^2\left(|TS| - |T^*S^*|\right).$$

Proof. Since T and S doubly commute, by Lemma 2.1, we have

$$\begin{aligned} w^2(TS) &\leq \frac{1}{2}\|(TS)^*TS + TS(TS)^*\| - \frac{1}{2}m^2\left(|TS| - |(TS)^*|\right) \\ &= \frac{1}{2}\|S^*T^*TS + TSS^*T^*\| - \frac{1}{2}m^2\left(|TS| - |T^*S^*|\right) \\ &= \frac{1}{2}\|T^*TS^*S + TT^*SS^*\| - \frac{1}{2}m^2\left(|TS| - |T^*S^*|\right). \end{aligned}$$

□

Corollary 2.8. *Let $T \in B(H)$ and $S \in B(K)$. Then*

$$\begin{aligned} w^2(T\otimes S) &\leq \frac{1}{2}\|T^*T\otimes S^*S + TT^*\otimes SS^*\| - \frac{1}{2}m^2\left(|T\otimes S| - |T^*\otimes S^*|\right) \\ &\leq \|T\|^2\|S\|^2. \end{aligned}$$

Proof. Since $T\otimes I_K$ and $I_H\otimes S$ doubly commute on $H\otimes K$, by applying Proposition 2.7, we get

$$\begin{aligned} w^2(T\otimes S) &= w^2((T\otimes I_K)(I_H\otimes S)) \\ &\leq \frac{1}{2}\|(T\otimes I_K)^*(T\otimes I_K)(I_H\otimes S)^*(I_H\otimes S) \\ &\quad + (T\otimes I_K)(T\otimes I_K)^*(I_H\otimes S)(I_H\otimes S)^*\| \\ &\quad - \frac{1}{2}m^2\left(|(T\otimes I_K)(I_H\otimes S)| - |(T\otimes I_K)^*(I_H\otimes S)^*|\right) \\ &= \frac{1}{2}\|T^*T\otimes S^*S + TT^*\otimes SS^*\| - \frac{1}{2}m^2\left(|T\otimes S| - |T^*\otimes S^*|\right) \\ &\leq \frac{1}{2}\|T^*T\otimes S^*S\| + \frac{1}{2}\|TT^*\otimes SS^*\| \\ &\leq \frac{1}{2}\|T^*T\|\|S^*S\| + \frac{1}{2}\|TT^*\|\|SS^*\| \\ &= \|T\|^2\|S\|^2. \end{aligned}$$

□

Theorem 2.9. *Let $T, S \in B(H)$. If T and S doubly commute, then*

$$\begin{aligned} w^2(TS) &\leq \frac{1}{4}\||T|^2|S|^2 + |T^*|^2|S^*|^2\| + \frac{1}{2}\|\Re(|T||T^*||S||S^*|)\| \\ &\leq \frac{1}{4}(\|T\|\|S\| + \|T^2\|^{\frac{1}{2}}\|S^2\|^{\frac{1}{2}})^2 \\ &\leq \|T\|^2\|S\|^2. \end{aligned}$$

Proof. Let x be a unit vector of H. Then by Lemma 2.2, we have

$$\begin{aligned} |\langle TSx, x\rangle| &\leq \langle |TS|x, x\rangle^{\frac{1}{2}}\langle |(TS)^*|x, x\rangle^{\frac{1}{2}} \\ &\leq \frac{1}{2}\langle (|TS| + |(TS)^*|)x, x\rangle \\ &\leq \frac{1}{2}\langle (|TS| + |T^*S^*|)x, x\rangle. \end{aligned}$$

Therefore

$$|\langle TSx, x\rangle|^2 \leq \frac{1}{4}\langle(|TS| + |T^*S^*|)x, x\rangle^2.$$

By Lemma 2.3, we get

$$\begin{aligned}
|\langle TSx, x\rangle|^2 &\leq \frac{1}{4}\langle(|TS| + |T^*S^*|)^2x, x\rangle \\
&\leq \frac{1}{4}\langle(|TS|^2 + |T^*S^*|^2)x, x\rangle + \frac{1}{2}\langle\Re(|TS||T^*S^*|)x, x\rangle \\
&\leq \frac{1}{4}\langle(|T|^2|S|^2 + |T^*|^2|S^*|^2)x, x\rangle + \frac{1}{2}\langle\Re(|T||S||T^*||S^*|)x, x\rangle \\
&\leq \frac{1}{4}\langle(|T|^2|S|^2 + |T^*|^2|S^*|^2)x, x\rangle + \frac{1}{2}\langle\Re(|T||T^*||S||S^*|)x, x\rangle \\
&\leq \frac{1}{4}\||T|^2|S|^2 + |T^*|^2|S^*|^2\| + \frac{1}{2}\|\Re(|T||T^*||S||S^*|)\|.
\end{aligned}$$

By Lemma 2.4 and using the fact that $\||T||T^*|\| = \|T^2\|$ for any $T \in B(H)$, we have

$$\begin{aligned}
\||T|^2|S|^2 + |T^*|^2|S^*|^2\| &\leq \max\{\||T|^2|S|^2\|, \||T^*|^2|S^*|^2\|\} + \||T||S||T^*||S^*|\| \\
&\leq \|T\|^2\|S\|^2 + \||T||T^*|\|\||S||S^*|\| \\
&= \|T\|^2\|S\|^2 + \|T^2\|\|S^2\|.
\end{aligned}$$

Since

$$\begin{aligned}
\|\Re(|T||T^*||S||S^*|)\| &\leq \||T||T^*||S||S^*|\| \\
&\leq \||T||T^*|\|\||S||S^*|\| \\
&= \|T^2\|\|S^2\| \\
&\leq \|T\|\|T^2\|^{\frac{1}{2}}\|S\|\|S^2\|^{\frac{1}{2}},
\end{aligned}$$

then

$$\begin{aligned}
|\langle TSx, x\rangle|^2 &\leq \frac{1}{4}\||T|^2|S|^2 + |T^*|^2|S^*|^2\| + \frac{1}{2}\|\Re(|T||T^*||S||S^*|)\| \\
&\leq \frac{1}{4}\||T|^2|S|^2 + |T^*|^2|S^*|^2\| + \frac{1}{2}\|\Re(|T||T^*||S||S^*|)\| \\
&= \frac{1}{4}(\|T\|\|S\| + \|T^2\|^{\frac{1}{2}}\|S^2\|^{\frac{1}{2}})^2 \\
&\leq \|T\|^2\|S\|^2.
\end{aligned}$$

Thus

$$\begin{aligned}
w^2(TS) &\leq \frac{1}{4}\||T|^2|S|^2 + |T^*|^2|S^*|^2\| + \frac{1}{2}\|\Re(|T||T^*||S||S^*|)\| \\
&\leq \frac{1}{4}(\|T\|\|S\| + \|T^2\|^{\frac{1}{2}}\|S^2\|^{\frac{1}{2}})^2 \\
&\leq \|T\|^2\|S\|^2.
\end{aligned}$$

□

Taking $S = I$ in Theorem 2.9, we obtain a refinement of inequality (1.7) as follows.

Corollary 2.10. *Let $T \in B(H)$. Then*

$$\begin{aligned} w^2(T) &\leq \frac{1}{4}\||T|^2 + |T^*|^2\| + \frac{1}{2}\|\Re(|T||T^*|)\| \\ &\leq \frac{1}{4}(\|T\| + \|T^2\|^{\frac{1}{2}})^2 \\ &\leq \|T\|^2. \end{aligned}$$

Proposition 2.11. *Let $T \in B(H)$ and let $S \in B(K)$. Then*

$$\begin{aligned} w^2(T \otimes S) &\leq \frac{1}{4}\|T^*T \otimes S^*S + TT^* \otimes SS^*\| + \frac{1}{2}\|\Re(|T||T^*| \otimes |S||S^*|)\| \\ &\leq \frac{1}{4}(\|T\|\|S\| + \|T^2\|^{\frac{1}{2}}\|S^2\|^{\frac{1}{2}})^2 \\ &\leq \|T\|^2\|S\|^2. \end{aligned}$$

Proof. Since $T \otimes S = (T \otimes I_K)(I_H \otimes S)$, it suffices to apply Theorem 2.9 to the doubly commuting operators $T \otimes I_K$ and $I_H \otimes S$. □

The following theorem gives a refinement of the inequality $w(T) \geq \frac{\|T\|}{2}$.

Theorem 2.12. *Let $T \in B(H)$. Then*

$$w(T) \geq \frac{\|T\|}{2} + \frac{1}{4}|\|T + T^*\| - \|T - T^*\||.$$

Proof. Let $x \in H$ be a unit vector. Then

$$w(T) \geq |\langle Tx, x\rangle| \geq |\Re\langle Tx, x\rangle|.$$

Thus

$$w(T) \geq |\langle \Re(T)x, x\rangle|.$$

Taking the supremum over all unit vectors x, we get

$$w(T) \geq \|\Re(T)\|.$$

Similarly, we get

$$w(T) \geq \|\Im(T)\|.$$

Here $\Im(T)$ denotes the imaginary part of T. Hence

$$w(T) \geq \max\{\|\Re(T)\|, \|\Im(T)\|\}. \tag{2.1}$$

Thus

$$\begin{aligned} w(T) &\geq \max\{\|\Re(T)\|, \|\Im(T)\|\} \\ &= \frac{\|\Re(T)\| + \|\Im(T)\|}{2} + \frac{|\|\Re(T)\| - \|\Im(T)\||}{2} \\ &\geq \frac{\|\Re(T) + i\Im(T)\|}{2} + \frac{1}{4}|\|T + T^*\| - \|\frac{T - T^*}{i}\|| \\ &\geq \frac{\|T\|}{2} + \frac{1}{4}|\|T + T^*\| - \|T - T^*\||. \end{aligned}$$

□

Corollary 2.13. *Let $T \in B(H)$. If $w(T) = \dfrac{\|T\|}{2}$, then*
$$\|T + T^*\| = \|T - T^*\| = \|T\|.$$

Proof. Since $w(T) = \dfrac{\|T\|}{2}$, then by Theorem 2.34, we get
$$\|T + T^*\| = \|T - T^*\|.$$
We have
$$\|T + T^*\| = w(T + T^*) \leq 2w(T) = \|T\|.$$
On the other hand,
$$\begin{aligned}\|T\| &= \|\Re(T) + i\Im(T)\| \leq \|\Re(T)\| + \|\Im(T)\| \\ &\leq \frac{1}{2}(\|T + T^*\| + \|T - T^*\|) \\ &= \|T + T^*\|.\end{aligned}$$
□

Next, we give a characterization for the equality $w(T) = \dfrac{\|T\|}{2}$.

Proposition 2.14. *Let $T \in B(H)$. Then $w(T) = \dfrac{\|T\|}{2}$ if and only if*
$$\|e^{i\theta}T + e^{-i\theta}T^*\| = \|e^{i\theta}T - e^{-i\theta}T^*\| = \|T\|, \qquad \textit{for all } \theta \in \mathbb{R}.$$

Proof. Suppose that $w(T) = \dfrac{\|T\|}{2}$. Then for all $\theta \in \mathbb{R}$
$$w(e^{i\theta}T) = \frac{\|e^{i\theta}T\|}{2}.$$
Applying Corollary 2.13, we have
$$\|e^{i\theta}T + e^{-i\theta}T^*\| = \|e^{i\theta}T - e^{-i\theta}T^*\| = \|T\|, \quad \text{for all } \theta \in \mathbb{R}.$$
The other implication is clear. □

Corollary 2.15. *Let $T, S \in B(H)$ such that T and S commute and $\|TS\| = \|T\|\|S\|$. Then*
$$w(TS) \geq \frac{\|T\|\|S\|}{2} + \frac{1}{4}\left|\|TS + T^*S^*\| - \|TS - T^*S^*\|\right|.$$

Corollary 2.16. *Let $T \in B(H)$ and let $S \in B(K)$. Then*
$$\begin{aligned} w(T \otimes S) \quad \geq \quad & \frac{1}{4}\left|\|T \otimes S + T^* \otimes S^*\| - \|T \otimes S - T^* \otimes S^*\|\right| \\ & + \frac{\|T\|\|S\|}{2}. \end{aligned}$$

Proof. It suffices to apply Corollary 2.15 to the operators $T \otimes I_K$ and $I_H \otimes S$. □

The following result gives a refinement of the inequality $w(T) \geq \dfrac{\|T\|}{2}$.

Theorem 2.17. *Let $T \in B(H)$. Then*

$$w(T) \geq \frac{\|T\|}{2} + \frac{1}{4}|\|T + iT^*\| - \|T - iT^*\||.$$

Proof. Let $x \in H$ be a unit vector. Then

$$\begin{aligned} |\langle Tx, x\rangle|^2 &= \langle \Re(T)x, x\rangle^2 + \langle \Im(T)x, x\rangle^2 \\ &\geq \frac{1}{2}(|\langle \Re(T)x, x\rangle| + |\langle\ \Im(T)x, x\rangle|)^2 \\ &\geq \frac{1}{2}|\langle \Re(T)x, x\rangle \pm \langle \Im(T)x, x\rangle|^2. \end{aligned}$$

Hence

$$|\langle Tx, x\rangle| \geq \frac{1}{\sqrt{2}}|\langle \Re(T)x, x\rangle \pm \langle \Im(T)x, x\rangle|.$$

Taking the supremum over all unit vectors x, we get

$$w(T) \geq \frac{\|\Re(T) \pm \Im(T)\|}{\sqrt{2}}. \tag{2.2}$$

It follows that

$$\begin{aligned} w(T) &\geq \frac{1}{\sqrt{2}} \max\{\|\Re(T) + \Im(T)\|, \|\Re(T) - \Im(T)\|\} \\ &= \frac{\|\Re(T) + \Im(T)\| + \|\Re(T) - \Im(T)\|}{2\sqrt{2}} \\ &\quad + \frac{|\|\Re(T) + \Im(T)\| - \|\Re(T) - \Im(T)\||}{2\sqrt{2}} \\ &\geq \frac{\|\Re(T) + \Im(T) + i(\Re(T) - \Im(T))\|}{2\sqrt{2}} \\ &\quad + \frac{|\|\Re(T) + \Im(T)\| - \|\Re(T) - \Im(T)\||}{2\sqrt{2}} \\ &= \frac{\|(1+i)T^*\|}{2\sqrt{2}} + \frac{|\|\Re(T) + \Im(T)\| - \|\Re(T) - \Im(T)\||}{2\sqrt{2}} \\ &= \frac{\|T\|}{2} + \frac{|\|T + iT^*\| - \|T - iT^*\||}{4}. \end{aligned}$$

□

Corollary 2.18. *Let $T \in B(H)$. If $w(T) = \frac{\|T\|}{2}$, then*

$$\|T + iT^*\| = \|T - iT^*\| = \|T\|.$$

Proof. We proceed as in the proof of Corollary 2.13 by using Theorem 2.17 and the fact that $T + iT^*$ is a normal operator. □

Corollary 2.19. *Let $T, S \in B(H)$. If T and S commute and $\|TS\| = \|T\|\|S\|$, then*

$$w(TS) \geq \frac{\|T\|\|S\|}{2} + \frac{1}{4}|\|TS + iT^*S^*\| - \|TS - iT^*S^*\||.$$

Corollary 2.20. *Let $T \in B(H)$ and $S \in B(K)$. Then*

$$\begin{aligned} w(T \otimes S) &\geq \frac{1}{4}\big|\|T \otimes S + iT^* \otimes S^*\| - \|T \otimes S - iT^* \otimes S^*\|\big| \\ &\quad + \frac{\|T\|\|S\|}{2}. \end{aligned}$$

Proof. It suffices to apply Theorem 2.17 to the operators $T \otimes I_K$ and $I_H \otimes S$. □

Now, we give a refinement of inequality (1.8).

Theorem 2.21. *Let $T \in B(H)$. Then*

$$w^2(T) \geq \frac{\|T^*T + TT^*\|}{4} + \frac{\big|\|T + T^*\|^2 - \|T - T^*\|^2\big|}{8}. \tag{2.3}$$

Proof. By inequality (2.1), we have

$$\begin{aligned} w^2(T) &\geq \max\{\|\Re(T)\|^2, \|\Im(T)\|^2\} \\ &= \frac{\|\Re(T)\|^2 + \|\Im(T)\|^2}{2} + \frac{\big|\|\Re(T)\|^2 - \|\Im(T)\|^2\big|}{2} \\ &\geq \frac{\|\Re(T)^2 + \Im(T)^2\|}{2} + \frac{\big|\|\Re(T)\|^2 - \|\Im(T)\|^2\big|}{2} \\ &= \frac{\|T^*T + TT^*\|}{4} + \frac{\big|\|T + T^*\|^2 - \|T - T^*\|^2\big|}{8}. \end{aligned}$$

□

The following theorem gives another refinement of inequality (1.8).

Theorem 2.22. *Let $T \in B(H)$. Then*

$$w^2(T) \geq \frac{\|T^*T + TT^*\|}{4} + \frac{\big|\|T + iT^*\|^2 - \|T - iT^*\|^2\big|}{8}.$$

Proof. By inequality (2.2), we have

$$\begin{aligned} w^2(T) &\geq \frac{1}{2}\max\{\|\Re(T) + \Im(T)\|^2, \|\Re(T) - \Im(T)\|^2\} \\ &= \frac{\|\Re(T) + \Im(T)\|^2 + \|\Re(T) - \Im(T)\|^2}{4} \\ &\quad + \frac{\big|\|\Re(T) + \Im(T)\|^2 - \|\Re(T) - \Im(T)\|^2\big|}{4} \\ &\geq \frac{\|(\Re(T) + \Im(T))^2 + (\Re(T) - \Im(T))^2\|}{4} \\ &\quad + \frac{\big|\|\Re(T) + \Im(T)\|^2 - \|\Re(T) - \Im(T)\|^2\big|}{4} \\ &= \frac{\|\Re(T)^2 + \Im(T)^2\|}{2} + \frac{\big|\|\Re(T) + \Im(T)\|^2 - \|\Re(T) - \Im(T)\|^2\big|}{4} \\ &= \frac{\|T^*T + TT^*\|}{4} + \frac{\big|\|T + iT^*\|^2 - \|T - iT^*\|^2\big|}{8}. \end{aligned}$$

□

We give a necessary and sufficient condition for the equality $w^2(T) = \frac{\|T^*T+TT^*\|}{4}$.

Proposition 2.23. *Let $T \in B(H)$. Then $w^2(T) = \dfrac{\|T^*T + TT^*\|}{4}$ if and only if $\|e^{i\theta}T + e^{-i\theta}T^*\| = \|e^{i\theta}T - e^{-i\theta}T^*\| = \|T^*T + TT^*\|$, for all $\theta \in \mathbb{R}$.*

Proof. Suppose $w^2(T) = \dfrac{\|T^*T + TT^*\|}{4}$. Then for all $\theta \in \mathbb{R}$

$$w^2(e^{i\theta}T) = \frac{\|(e^{i\theta}T)^*e^{i\theta}T + e^{i\theta}T(e^{i\theta}T)^*\|}{4}.$$

Hence from inequality (2.3), we have

$$\|e^{i\theta}T + e^{-i\theta}T^*\|^2 = \|e^{i\theta}T - e^{-i\theta}T^*\|^2, \qquad \text{for all } \theta \in \mathbb{R}.$$

Let $\theta \in \mathbb{R}$. Then we get

$$\begin{aligned}
\|T^*T + TT^*\| &= \frac{1}{2}\|(e^{i\theta}T + e^{-i\theta}T^*)^2 - (e^{i\theta}T - e^{-i\theta}T^*)^2\| \\
&\leq \frac{1}{2}(\|e^{i\theta}T + e^{-i\theta}T^*\|^2 + \|e^{i\theta}T - e^{-i\theta}T^*\|^2) \\
&= \|e^{i\theta}T + e^{-i\theta}T^*\|^2 \\
&= w^2(e^{i\theta}T + e^{-i\theta}T^*) \\
&\leq 4w^2(T) = \|T^*T + TT^*\|.
\end{aligned}$$

The other implication holds trivially. □

Corollary 2.24. *Let $T, S \in B(H)$. If T and S doubly commute, then*

$$w^2(TS) \geq \frac{|\|TS + T^*S^*\|^2 - \|TS - T^*S^*\|^2|}{8} + \frac{\||T|^2|S|^2 + |T^*|^2|S^*|^2\|}{4}.$$

Corollary 2.25. *Let $T \in B(H)$ and $S \in B(K)$. Then*

$$\begin{aligned}
w^2(T \otimes S) \quad \geq \quad & \frac{|\|T \otimes S + T^* \otimes S^*\|^2 - \|T \otimes S - T^* \otimes S^*\|^2|}{8} \\
& + \frac{\|T^*T \otimes S^*S + TT^* \otimes SS^*\|}{4}.
\end{aligned}$$

Proof. It suffices to apply Corollary 2.24 to the operators $T \otimes I_K$ and $I_H \otimes S$. □

Corollary 2.26. *Let $T \in B(H)$. Assume that $w^2(T) = \dfrac{\|T^*T + TT^*\|}{4}$; then*

$$\|T + iT^*\|^2 = \|T - iT^*\|^2 = \|T^*T + TT^*\|.$$

Proof. Since $w^2(T) = \dfrac{\|T^*T + TT^*\|}{4}$, then by Theorem 2.22, we get

$$\|T + iT^*\|^2 = \|T - iT^*\|^2.$$

Since the operator $T + iT^*$ is normal, then

$$\|T + iT^*\|^2 = w^2(T + iT^*) \leq 4w^2(T) = \|T^*T + TT^*\|.$$

On the other hand,

$$\begin{aligned}\|T^*T+TT^*\| &= \frac{1}{2}\|(T+iT^*)(T+iT^*)^*+(T-iT^*)(T-iT^*)^*\|\\ &\leq \frac{1}{2}(\|(T+iT^*)(T+iT^*)^*\|+\|(T-iT^*)(T-iT^*)^*\|)\\ &\leq \frac{1}{2}(\|T+iT^*\|^2+\|T-iT^*\|^2)=\|T+iT^*\|^2.\end{aligned}$$

Then

$$\|T+iT^*\|^2=\|T-iT^*\|^2=\|T^*T+TT^*\|.$$

□

Corollary 2.27. *Let $T,S\in B(H)$. If T and S doubly commute, then*

$$\begin{aligned}w^2(TS) &\geq \frac{|\|TS+iT^*S^*\|^2-\|TS-iT^*S^*\|^2|}{8}\\ &\quad+\frac{\||T|^2|S|^2+|T^*|^2|S^*|^2\|}{4}.\end{aligned}$$

Corollary 2.28. *Let $T\in B(H)$ and let $S\in B(K)$. Then*

$$\begin{aligned}w^2(T\otimes S) &\geq \frac{|\|T\otimes S+iT^*\otimes S^*\|^2-\|T\otimes S-iT^*\otimes S^*\|^2|}{8}\\ &\quad+\frac{\|T^*T\otimes S^*S+TT^*\otimes SS^*\|}{4}.\end{aligned}$$

Proposition 2.29. *Let $T\in B(H)$. Then*

$$d^2(T)=\inf_{\lambda\in\mathbb{C}}\{\|T-\lambda I\|^2-m^2(T-\lambda I)\}.$$

Proof. Let $\lambda\in\mathbb{C}$. Then

$$\begin{aligned}d^2(T) &= d^2(T-\lambda I)\\ &= \sup_{x\in H,\ \|x\|=1}\{\|(T-\lambda I)x\|^2-|\langle (T-\lambda I)x,x\rangle|^2\}\ (\text{by } (1.3))\\ &\leq \|T-\lambda I\|^2-m^2(T-\lambda I).\end{aligned}$$

Thus

$$\begin{aligned}d^2(T) &\leq \inf_{\lambda\in\mathbb{C}}\{\|T-\lambda I\|^2-m^2(T-\lambda I)\}\\ &\leq \|T-c_TI\|^2-m^2(T-c_TI)\\ &\leq \|T-c_TI\|^2\\ &= d^2(T).\end{aligned}$$

It results that

$$d^2(T)=\inf_{\lambda\in\mathbb{C}}\{\|T-\lambda I\|^2-m^2(T-\lambda I)\}.$$

□

Remark 2.30. In [8], the authors proved that for any $T \in B(H)$

$$\|T\|^2 - w_0'^2(T) \leq d^2(T) - (w_0'^2(T) - |c_T|)^2, \tag{2.4}$$

where $w_0'(T) := \inf\{|z| : z \in W_0(T)\}$. Then by Proposition 2.29, inequality (2.4), and the trivial inequality $w_0'^2(T) \leq w^2(T)$, we obtain a refinement of the inequality

$$\|T \otimes S\|^2 - w^2(T \otimes S) \leq \inf_{\lambda \in \mathbb{C}} \{\|T \otimes S - \lambda I_H \otimes I_K\|^2 - m^2(T \otimes S - \lambda I_H \otimes I_K)\},$$

established in [10].

Proposition 2.31. *Let $T, S \in B(H)$ such that T and S commute and $\|TS\| = \|T\|\|S\|$. Then*

$$\begin{aligned} w(TS) \geq\ & \frac{\|T\|\|S\|}{2} + \frac{|\|TS + T^*S^*\| - \|T\|\|S\||}{8} \\ & + \frac{|\|TS - T^*S^*\| - \|T\|\|S\||}{8}. \end{aligned}$$

Proof. According to [17], we have

$$\frac{\|T\|}{2} + \frac{|\|\Re(T)\| - \frac{\|T\|}{2}|}{4} + \frac{|\|\Im(T)\| - \frac{\|T\|}{2}|}{4} \leq w(T).$$

Then

$$\frac{\|TS\|}{2} + \frac{|\|\Re(TS)\| - \frac{\|TS\|}{2}|}{4} + \frac{|\|\Im(TS)\| - \frac{\|TS\|}{2}|}{4} \leq w(TS).$$

Since T and S commute and $\|TS\| = \|T\|\|S\|$, then

$$\begin{aligned} w(TS) \geq\ & \frac{\|T\|\|S\|}{2} + \frac{|\|TS + T^*S^*\| - \|T\|\|S\||}{8} \\ & + \frac{|\|TS - T^*S^*\| - \|T\|\|S\||}{8}. \end{aligned}$$

□

The following proposition gives a refinement of the inequality

$$\frac{\|T\|\|S\|}{2} \leq w(T \otimes S);$$

the left-hand side of inequality (1.4).

Proposition 2.32. *Let $T \in B(H)$ and let $S \in B(K)$. Then*

$$\begin{aligned} w(T \otimes S) \geq\ & \frac{\|T\|\|S\|}{2} + \frac{|\|T \otimes S + T^* \otimes S^*\| - \|T\|\|S\||}{8} \\ & + \frac{|\|T \otimes S - T^* \otimes S^*\| - \|T\|\|S\||}{8}. \end{aligned}$$

Proof. It suffices to apply Proposition 2.31 to $T \otimes I_K$ and $I_H \otimes S$. □

Proposition 2.33. [2] *Let $T \in B(H)$. Then for all $r \geq 1$*

$$w^{2r}(T) \leq \frac{1}{2}\||T|^{2r} + |T^*|^2|\|^2 - \frac{1}{2}m^2\left(|T|^r - |T^*|^r\right).$$

Proposition 2.34. *Let $T, S \in B(H)$. If T and S doubly commute, then for all $r \geq 1$,*

$$w^{2r}(TS) \leq \frac{1}{2}\||T|^r|S|^r + |T^*|^r|S^*|^r\|^2 - \frac{1}{2}m^2\left(|T|^{\frac{r}{2}}|S|^{\frac{r}{2}} - |T^*|^{\frac{r}{2}}|S^*|^{\frac{r}{2}}\right).$$

Proof. By Proposition 2.33, we have for all $r \geq 1$,

$$\begin{aligned} w^{2r}(TS) &\leq \frac{1}{2}\||TS|^{2r} + |(TS)^*|^{2r}\|^2 - \frac{1}{2}m^2\left(|TS|^r - |(TS)^*|^r\right) \\ &= \frac{1}{2}\|(S^*T^*TS)^r + (TSS^*T^*)^r\|^2 - \frac{1}{2}m^2\left((S^*T^*TS)^{\frac{r}{2}} - (TSS^*T^*)^{\frac{r}{2}}\right) \\ &= \frac{1}{2}\||T|^r|S|^r + |T^*|^r|S^*|^r\|^2 - \frac{1}{2}m^2\left(|T|^{\frac{r}{2}}|S|^{\frac{r}{2}} - |T^*|^{\frac{r}{2}}|S^*|^{\frac{r}{2}}\right). \end{aligned}$$

□

Proposition 2.35. *Let $T \in B(H)$ and let $S \in B(K)$. Then for all $r \geq 1$*

$$\begin{aligned} w^{2r}(T \otimes S) \quad \leq \quad & \frac{1}{2}\|(T^*T \otimes S^*S)^r + (TT^* \otimes SS^*)^r\|^2 \\ & -\frac{1}{2}m^2\left((T^*T \otimes S^*S)^{\frac{r}{2}} - (TT^* \otimes SS^*)^{\frac{r}{2}}\right). \end{aligned}$$

Proof. It suffices to apply Proposition 2.34 to $T \otimes I_K$ and $I_H \otimes S$. □

The following proposition gives a refinement of the inequality

$$w(T \otimes S) \leq \|T\|\|S\|.$$

Proposition 2.36. *Let $T \in B(H)$ and let $S \in B(K)$. Then*

$$\begin{aligned} w^2(T \otimes S) &\leq \frac{1}{2}\left(w(T^2 \otimes S^2) + \|T\|^2\|S\|^2\right) \\ &\leq \frac{1}{2}\left(\min\{w(T^2)\|S^2\|, w(S^2)\|T^2\|\} + \|T\|^2\|S\|^2\right) \\ &\leq \frac{1}{2}\left(\min\{w^2(T)\|S^2\|, w(S^2)\|T^2\|\} + \|T\|^2\|S\|^2\right) \\ &\leq \frac{1}{2}\left(\min\{w^2(T)\|S^2\|, w^2(S)\|T^2\|\} + \|T\|^2\|S\|^2\right) \\ &\leq \|T\|^2\|S\|^2. \end{aligned}$$

Proof. From [12], we have

$$\begin{aligned} w^2(T \otimes S) &\leq \frac{1}{2}\left(w((T \otimes S)^2) + \|T \otimes S\|^2\right) \\ &\leq \frac{1}{2}\left(w((T \otimes S)(T \otimes S)) + \|T\|^2\|S\|^2\right) \\ &\leq \frac{1}{2}\left(w(T^2 \otimes S^2) + \|T\|^2\|S\|^2\right) \\ &\leq \frac{1}{2}\left(\min\{w(T^2)\|S^2\|, w(S^2)\|T^2\|\} + \|T\|^2\|S\|^2\right) \\ &\leq \frac{1}{2}\left(\min\{w^2(T)\|S^2\|, w(S^2)\|T^2\|\} + \|T\|^2\|S\|^2\right) \end{aligned}$$

$$\leq \frac{1}{2}\left(\min\{w^2(T)\|S^2\|, w^2(S)\|T^2\|\} + \|T\|^2\|S\|^2\right)$$
$$\leq \|T\|^2\|S\|^2.$$

□

Proposition 2.37. *Let $T, S \in B(H)$ such that*

$$w(T+S) = w(T) + w(S).$$

Then

$$w(T)w(S) \leq \frac{1}{2}(w(ST) + \|T\|\|S\|).$$

Proof. Since $w(T+S) = w(T) + w(S)$ then, from the proof of [1, Proposition 3.6], there exists (x_n) of unit vectors in H such that $\lim_n |\langle Tx_n, x_n\rangle| = w(T)$ and $\lim_n |\langle Sx_n, x_n\rangle| = w(S)$. By the Dragomir inequality, for all $a, b, e \in H$ with $\|e\| = 1$, we have

$$2|\langle a, e\rangle\langle e, b\rangle| \leq |\langle a, b\rangle| + \|a\|\|b\|.$$

Taking $e = x_n$, $a = Tx_n$ and $b = S^*x_n$, we get

$$\begin{aligned} 2|\langle Tx_n, x_n\rangle\langle x_n, S^*x_n\rangle| &\leq |\langle Tx_n, S^*x_n\rangle + \|Tx_n\|\|S^*x_n\|| \\ &\leq w(ST) + \|T\|\|S\|. \end{aligned}$$

So, we get

$$2w(T)w(S) \leq w(ST) + \|T\|\|S\|,$$

as desired. □

Remark 2.38. Note that we can deduce the inequality

$$w^2(T) \leq \frac{1}{2}\left(w(T^2) + \|T\|^2\right)$$

established by Dragomir in [12] by letting $T = S$ in Proposition 2.37.

3. Extension to semi-Hilbert spaces

In this section, we extend our obtained results to semi-Hilbert spaces. In order to state our results in detail, we first recall some notations and results from the literature. In all that follows, $A \in B(H)$ is a positive operator on H, that is, $\langle Ax, x\rangle \geq 0$, for all $x \in H$. The operator A induces a positive semi-definite sesquilinear form in the following manner:

$$\langle\cdot,\cdot\rangle_A : H \times H \longrightarrow \mathbb{C},\ (x, y) \longmapsto \langle x, y\rangle_A := \langle Ax, y\rangle = \langle A^{1/2}x, A^{1/2}y\rangle,$$

where $A^{1/2}$ represents the square root of A. We denote by $\|\cdot\|_A$ the seminorm induced by $\langle\cdot,\cdot\rangle_A$, which is given by $\|x\|_A = \sqrt{\langle x, x\rangle_A} = \sqrt{\|A^{1/2}x\|}$, for every $x \in H$. It is easy to verify that $\|.\|_A$ is a norm if and only if A is injective and the space $(H, \|\cdot\|_A)$ is complete if and only if the range of A is closed. Let $T \in B(H)$. If there exists a constant $c > 0$ such that $\|Tx\|_A \leq c\|x\|_A$ for all $x \in \overline{\mathcal{R}(A)}$; the closure of $\mathcal{R}(A)$, where $\mathcal{R}(A)$ is the range of A, then the A-operator semi-norm of T is given by

$$\|T\|_A = \sup_{x\in\overline{\mathcal{R}(A)},\ x\neq 0} \frac{\|Tx\|_A}{\|x\|_A} = \sup_{x\in\overline{\mathcal{R}(A)},\ \|x\|=1} \|Tx\|_A < +\infty.$$

Note that, if $A = I$, then we get the classical norm of an operator T, which will simply be denoted by $\|T\|$. Set

$$B^A(H) = \{T \in B(H) : \|T\|_A < +\infty\}.$$

It is worth noting that $B^A(H)$ is not generally a subalgebra of $B(H)$ (see, [13]). Furthermore, it is not difficult to check that $\|T\|_A = 0$ if and only if $ATA = 0$. Recently, there are many papers that study operators defined on a semi-Hilbert space $(H, \|\cdot\|_A)$. One may see [6, 7, 9, 22] and their references.

Let $T \in B(H)$. An operator $S \in B(H)$ is called an A-adjoint operator of T if $\langle Tx, y\rangle_A = \langle x, Sy\rangle_A$ for all $x, y \in H$ (see, [4]). So, S is an A-adjoint of T if and only if $AS = T^*A$, that is, S is a solution in $B(H)$ of the equation $AX = T^*A$. Note that this type of operator equations can be studied by using the following famous theorem due to Douglas.

Theorem 3.1. [11] *If $T, U \in B(H)$, then the following statements are equivalent:*

(1) $\mathcal{R}(U) \subseteq \mathcal{R}(T)$,
(2) $TS = U$ *for some* $S \in B(H)$,
(3) *There exists* $\lambda > 0$ *such that* $\|U^*x\| \leq \lambda|T^*x\|$ *for all* $x \in H$.

If one of these conditions holds, then there exists a unique solution of the operator equation $TX = U$, denoted by Q, such that $\mathcal{R}(Q) \subseteq \overline{\mathcal{R}(T^*)}$. Such Q is called the reduced solution of $TX = U$.

Let $B_A(H)$ denote the set of all operators that admit A-adjoints. An application of Theorem 3.1 shows that

$$B_{A^{1/2}}(\mathcal{H}) = \{T \in B(H)\,;\ \exists\,\lambda > 0 \ \text{such that}\ \|Tx\|_A \leq \lambda\|x\|_A \ \text{for all}\ x \in \mathcal{H}\}.$$

From [13], we have

$$B_A(H) \subset B_{A^{\frac{1}{2}}}(H) \subset B^A(H) \subset B(H).$$

Let $\mathcal{R}(A^{\frac{1}{2}})$ be the Hilbert space endowed with the inner product

$$(A^{\frac{1}{2}}x, A^{\frac{1}{2}}y) := \langle Px, Py\rangle,$$

for all $x, y \in H$, where P denotes the orthogonal projection of H onto $\overline{\mathcal{R}(A)}$. Starting now, we will use the shorthand notation $\mathbf{R}(A^{\frac{1}{2}})$ for the Hilbert space $(\mathcal{R}(A^{\frac{1}{2}}), (\cdot,\cdot))$. Moreover, the norm induced by $(\cdot,\cdot)$ on $\mathbf{R}(A^{1/2})$ will be denoted by $\|\cdot\|_{\mathbf{R}(A^{1/2})}$. It is important to highlight that $\mathcal{R}(A)$ is dense in $\mathbf{R}(A^{1/2})$ (see [13]). Moreover, since $\mathcal{R}(A) \subseteq \mathcal{R}(A^{1/2})$, we have

$$(Ax, Ay) = (A^{1/2}A^{1/2}x, A^{1/2}A^{1/2}y) = \langle PA^{1/2}x, PA^{1/2}y\rangle = \langle x, y\rangle_A,$$

for any $x, y \in H$ and so

$$\|Ax\|_{\mathbf{R}(A^{1/2})} = \|x\|_A,$$

for any $x \in H$. For more details about the Hilbert space $\mathbf{R}(A^{1/2})$, we refer the interested reader to [5, 13].

Before we move on, we need to introduce the following concepts and lemmas. Let $T \in B(H)$. The A-numerical range of T is given by

$$W_A(T) = \{\langle Tx, x\rangle_A : x \in H, \|x\|_A = 1\},$$

The A-numerical radius of T is defined as

$$w_A(T) = \sup\{|z| : z \in W_A(T)\},$$

and the A-Crawford number of T is the positive real number defined by

$$m_A(T) = \inf\{|z| : z \in W_A(T)\}.$$

For more details for all these concepts, see [5, 3, 4, 6] and the references therein.

Now, let us consider the operator Z_A defined by

$$Z_A : H \longrightarrow \mathbf{R}(A^{1/2}),\ x \longmapsto Z_A x = Ax.$$

We have the following result.

Lemma 3.2. [5] *Let $T \in B(H)$. Then $T \in B_{A^{\frac{1}{2}}}(H)$ if and only if there exists a unique $\tilde{T} \in B(\boldsymbol{R}(A^{\frac{1}{2}}))$ such that $Z_A T = \tilde{T} Z_A$. Moreover, $\|\tilde{T}\|_{B(\boldsymbol{R}(A^{\frac{1}{2}}))} = \|T\|_A$.*

Lemma 3.3. [13] *Let $T \in B_{A^{\frac{1}{2}}}(H)$. Then*

$$\overline{W(\tilde{T})} = \overline{W_A(T)}.$$

Lemma 3.4. [24] *Let $T \in B_{A^{\frac{1}{2}}}(H)$. Then*

(1) $R(\tilde{T}) = R_A(T)$,
(2) $K(\tilde{T}) = K_A(T)$,

where

$$R_A(T) = \inf\{w_A(T - \lambda I) : \lambda \in \mathbb{C}\}$$

and

$$K_A(T) = \inf\{w_A(T - i\lambda z_T I) : \lambda \in \mathbb{C}\}.$$

Now, we can state our first results. In the next, let A and B be two positive operators acting on H and K, respectively. Note that the operator $A \otimes B$ is positive on $H \otimes K$.

Lemma 3.5. *Let $T \in B_{A^{\frac{1}{2}}}(H)$ and let $S \in B_{A^{\frac{1}{2}}}(K)$. Then $T \otimes S \in B_{A^{\frac{1}{2}} \otimes B^{\frac{1}{2}}}(H \otimes K)$ and*

$$\widetilde{T \otimes S} = \tilde{T} \otimes \tilde{S}.$$

Proof. We have

$$\begin{aligned} Z_{A\otimes B}(T \otimes S) &= (Z_A \otimes Z_B)(T \otimes S) \\ &= Z_A T \otimes Z_B S \\ &= \tilde{T} A \otimes \tilde{S} B \\ &= (\tilde{T} \otimes \tilde{S})(A \otimes B). \end{aligned}$$

Then by Lemma 3.2, $T \otimes S \in B_{A^{\frac{1}{2}} \otimes B^{\frac{1}{2}}}(H \otimes K)$ and

$$\widetilde{T \otimes S} = \widetilde{T} \otimes \widetilde{S}.$$

□

Theorem 3.6. *Let $T \in B_{A^{\frac{1}{2}}}(H)$ and let $S \in B_{B^{\frac{1}{2}}}(K)$. Then*

$$\begin{aligned} w_{A\otimes B}(T \otimes S) &\leq \min\{w_A(T)\left(K_B(S) + R_B(S)\right), w_B(S)\left(K_A(T) + R_A(T)\right)\} \\ &\leq \min\{w_A(T)\left(w_B(S) + R_B(S)\right), w_B(S)\left(w_A(T) + R_A(T)\right)\} \\ &\leq \min\{w_A(T)\left(w_B(S) + K_B(S)\right), w_B(S)\left(w_A(T) + K_A(T)\right)\} \\ &\leq 2w_A(T)w_B(S). \end{aligned}$$

Proof. By Lemmas 3.3 and 3.5, we have

$$w_{A\otimes B}(T \otimes S) = w(\widetilde{T \otimes S}) = w(\widetilde{T} \otimes \widetilde{S}). \tag{3.1}$$

Applying Theorem 2.6 to $\widetilde{T}$ and $\widetilde{S}$ and using Lemma 3.4, we have the desired inequalities. □

The following theorem gives a refinement of the inequality $w_A(T) \geq \dfrac{\|T\|_A}{2}$ established in [6]. For this, we need the following auxiliary lemma. We will denote the A-adjoint of T by $T^{\sharp_A}$, for any $T \in B_A(H)$. Note that $T = \Re_A(T) + i\Im_A(T)$, where

$$\Re_A(T) = \frac{T + T^{\sharp_A}}{2} \quad \text{and} \quad \Im_A(T) = \frac{T - T^{\sharp_A}}{2i}.$$

Lemma 3.7. [22] *Let $T \in B_A(H)$. Then*

$$\widetilde{T^{\sharp_A}} = (\widetilde{T})^*, \ \widetilde{(T^{\sharp_A})^{\sharp_A}} = \widetilde{T}.$$

Theorem 3.8. *Let $T \in B_A(H)$. Then*

$$w_A(T) \geq \frac{\|T\|_A}{2} + \frac{1}{4}\left|\|T + T^{\sharp_A}\|_A - \|T - T^{\sharp_A}\|_A\right|.$$

Proof. By Lemma 3.3, Theorem 2.12 and Lemma 3.7, we have

$$\begin{aligned} w_A(T) &= w(\widetilde{T}) \\ &\geq \frac{\|\widetilde{T}\|_{B(\mathbf{R}(A^{\frac{1}{2}}))}}{2} + \frac{1}{4}\left|\|\widetilde{T} + (\widetilde{T})^*\|_{B(\mathbf{R}(A^{\frac{1}{2}}))} - \|\widetilde{T} - (\widetilde{T})^*\|_{B(\mathbf{R}(A^{\frac{1}{2}}))}\right| \\ &= \frac{\|\widetilde{T}\|_{B(\mathbf{R}(A^{\frac{1}{2}}))}}{2} + \frac{1}{4}\left|\|\widetilde{T} + \widetilde{T^{\sharp_A}}\|_{B(\mathbf{R}(A^{\frac{1}{2}}))} - \|\widetilde{T} - \widetilde{T^{\sharp_A}}\|_{B(\mathbf{R}(A^{\frac{1}{2}}))}\right| \\ &= \frac{\|\widetilde{T}\|_{B(\mathbf{R}(A^{\frac{1}{2}}))}}{2} + \frac{1}{4}\left|\|\widetilde{T + T^{\sharp_A}}\|_{B(\mathbf{R}(A^{\frac{1}{2}}))} - \|\widetilde{T - T^{\sharp_A}}\|_{B(\mathbf{R}(A^{\frac{1}{2}}))}\right| \\ &= \frac{\|T\|_A}{2} + \frac{1}{4}\left|\|T + T^{\sharp_A}\|_A - \|T - T^{\sharp_A}\|_A\right|. \end{aligned}$$

The last equality follows from Lemma 3.2. □

Corollary 3.9. *Let $T \in B_A(H)$. Suppose $w_A(T) = \dfrac{\|T\|_A}{2}$; then*

$$\|T + T^{\sharp_A}\|_A = \|T - T^{\sharp_A}\|_A = \|T\|_A.$$

Proof. Applying Corollary 2.13 to $\widetilde{T}$, we get the desired result. □

Proposition 3.10. *Let $T \in B_A(H)$. Then $w_A(T) = \dfrac{\|T\|_A}{2}$ if and only if*

$$\|e^{i\theta}T + e^{-i\theta}T^{\sharp_A}\|_A = \|e^{i\theta}T - e^{-i\theta}T^{\sharp_A}\|_A = \|T\|_A, \quad \text{for all } \theta \in \mathbb{R}.$$

Proof. Applying Proposition 2.14 to $\widetilde{T}$ and using the fact that $\widetilde{\alpha T + \beta S} = \alpha\widetilde{T} + \beta\widetilde{S}$ for all complex numbers α, β, we obtain the desired equalities. □

Corollary 3.11. *Let $T, S \in B_A(H)$. If T and S commute and $\|TS\|_A = \|T\|_A\|S\|_A$, then*

$$w_A(TS) \geq \frac{\|T\|_A\|S\|_A}{2} + \frac{1}{4}|\|TS + T^{\sharp_A}S^{\sharp_A}\|_A - \|TS - T^{\sharp_A}S^{\sharp_A}\|_A|.$$

Proof. Since T and S commute and $\widetilde{TS} = \widetilde{T}\widetilde{S}$, then $\widetilde{T}$ and $\widetilde{S}$ also commute. Hence we can apply Corollary 2.15 to $\widetilde{T}$ and $\widetilde{S}$. □

Corollary 3.12. *Let $T \in B_A(H)$ and let $S \in B_B(K)$. Then*

$$\begin{aligned} w_{A\otimes B}(T \otimes S) \;\; &\geq \;\; \frac{1}{4}|\|T \otimes S + T^{\sharp_A} \otimes S^{\sharp_B}\|_{A\otimes B} - \|T \otimes S - T^{\sharp_A} \otimes S^{\sharp_B}\|_{A\otimes B}| \\ &\quad + \frac{\|T\|_A\|S\|_B}{2}. \end{aligned}$$

Proof. Using Corollary 2.16 to $\widetilde{T \otimes S}$, the result follows. □

Proposition 3.13. *Let $T \in B_A(H)$. Then*

$$w_A(T) \geq \frac{\|T\|_A}{2} + \frac{1}{4}|\|T + iT^{\sharp_A}\|_A - \|T - iT^{\sharp_A}\|_A|.$$

Proof. It is a simple consequence of Theorem 2.34 applied to $\widetilde{T}$. □

Proposition 3.14. *Let $T, S \in B_{A^{\frac{1}{2}}}(H)$ such that $TS = ST$ and $\|TS\|_A = \|T\|_A\|S\|_A$. Then*

$$w_A(TS) \geq \frac{\|TS\|_A}{2} + \frac{1}{4}|\|TS + iT^{\sharp_A}S^{\sharp_A}\|_A - \|TS - iT^{\sharp_A}S^{\sharp_A}\|_A|.$$

Proof. Applying Corollary 2.19 to $\widetilde{T}$ and $\widetilde{S}$, we get the result. □

Corollary 3.15. *Let $T \in B_A(H)$ and let $S \in B_B(K)$. Then*

$$\begin{aligned} w_{A\otimes B}(T \otimes S) \;\; &\geq \;\; \frac{1}{4}|\|T \otimes S + iT^{\sharp_A} \otimes S^{\sharp_B}\|_{A\otimes B} - \|T \otimes S - iT^{\sharp_A} \otimes S^{\sharp_B}\|_{A\otimes B}| \\ &\quad + \frac{\|T\|_A\|S\|_B}{2}. \end{aligned}$$

Proof. It suffices to apply Proposition 3.14 to $\widetilde{T \otimes I_K}$ and $\widetilde{I_H \otimes S}$. □

Proposition 3.16. *Let $T, S \in B_A(H)$ such that $TS = ST$ and $T^{\sharp_A}S = ST^{\sharp_A}$. Then*

$$w_A^2(TS) \leq \frac{1}{2}\|T^{\sharp_A}TS^{\sharp_A}S + TT^{\sharp_A}SS^{\sharp_A}\|_A - \frac{1}{2}m^2\left(|\widetilde{TS}| - |\widetilde{T^{\sharp_A}S^{\sharp_A}}|\right).$$

Proof. Since $TS = ST$, then $\widetilde{TS} = \widetilde{ST}$; thus $\widetilde{T}\widetilde{S} = \widetilde{S}\widetilde{T}$. We have $\widetilde{T^{\sharp_A}S} = \widetilde{ST^{\sharp_A}}$. Then

$$\widetilde{T^{\sharp_A}}\widetilde{S} = \widetilde{S}\widetilde{T^{\sharp_A}}.$$

By Lemma 3.7, we get

$$\left(\widetilde{T}\right)^*\widetilde{S} = \widetilde{S}\left(\widetilde{T}\right)^*.$$

Hence $\widetilde{T}$ and $\widetilde{S}$ doubly commute, so the result is obtained by using Proposition 2.7. ◻

Corollary 3.17. *Let $T \in B_A(H)$ and let $S \in B_B(K)$. Then*

$$w_{A\otimes B}^2(T\otimes S) \leq \frac{1}{2}\|T^{\sharp_A}T\otimes S^{\sharp_B}S + TT^{\sharp_A}\otimes SS^{\sharp_B}\|_{A\otimes B},$$

$$-\frac{1}{2}m^2\left(|\widetilde{T\otimes S}| - |\widetilde{T^{\sharp_A}\otimes S^{\sharp_B}}|\right) \leq \|T\|_A^2\|S\|_B^2.$$

Proof. Using equality (3.1) and applying Corollary 2.8 to $\widetilde{T}$ and $\widetilde{S}$, we obtain the required inequalities. ◻

Theorem 3.18. *Let $T, S \in B_A(H)$ such that $TS = ST$ and $T^{\sharp_A}S = ST^{\sharp_A}$. Then*

$$\begin{aligned} w_A^2(TS) &\leq \frac{1}{4}\|T^{\sharp_A}TS^{\sharp_A}S + TT^{\sharp_A}SS^{\sharp_A}\|_A + \frac{1}{2}\|\Re(|\widetilde{T}||\widetilde{T^{\sharp_A}}||\widetilde{S}||\widetilde{S^{\sharp_A}}|)\|_{B(\boldsymbol{R}(A^{\frac{1}{2}}))} \\ &\leq \frac{1}{4}(\|T\|_A\|S\|_A + \|T^2\|_A^{\frac{1}{2}}\|S^2\|_A^{\frac{1}{2}})^2 \\ &\leq \|T\|_A^2\|S\|_A^2. \end{aligned}$$

Proof. We get the desired result by applying Theorem 2.9 to $\widetilde{T}$ and $\widetilde{S}$. ◻

Feki [13] proved that

$$w_A^2(T) \leq \frac{1}{2}(\|T\|_A + \|T^2\|_A^{\frac{1}{2}}). \tag{3.2}$$

The following corollary gives a refinement of inequality (3.2).

Corollary 3.19. *Let $T \in B_A(H)$. Then*

$$\begin{aligned} w_A^2(T) &\leq \frac{1}{4}\|T^{\sharp_A}T + TT^{\sharp_A}\|_A + \frac{1}{2}\|\Re(|\widetilde{T}||\widetilde{T^{\sharp_A}}|)\|_{B(\boldsymbol{R}(A^{\frac{1}{2}}))} \\ &\leq \frac{1}{4}(\|T\|_A + \|T^2\|_A^{\frac{1}{2}})^2 \\ &\leq \|T\|_A^2. \end{aligned}$$

Proof. Letting $S = I_H$ in Theorem 3.18, the results follow. ◻

Proposition 3.20. *Let $T \in B_A(H)$ and let $S \in B_B(K)$. Then*

$$\begin{aligned} w_{A\otimes B}(T\otimes S) &\leq \frac{1}{4}\|T^{\sharp_A}T\otimes S^{\sharp_A}S+TT^{\sharp_A}\otimes SS^{\sharp_A}\|_A \\ &\quad+\frac{1}{2}\|\Re(|\widetilde{T}\otimes I_K||\widetilde{T^{\sharp_A}}\otimes I_K||I_H\otimes\widetilde{S}||I_H\otimes\widetilde{S^{\sharp_A}}|)\|_{B(\boldsymbol{R}(A^{\frac{1}{2}}))} \\ &\leq \frac{1}{4}(\|T\|_A\|S\|_A+\|T^2\|_A^{\frac{1}{2}}\|S^2\|_A^{\frac{1}{2}})^2 \\ &\leq \|T\|_A^2\|S\|_A^2. \end{aligned}$$

Proof. Applying Theorem 2.9 to the doubly commuting operators $\widetilde{T\otimes I_K}$ and $\widetilde{I_H\otimes S}$, we have the required result. □

Theorem 3.21. *Let $T \in B_A(H)$. Then*

$$w_A^2(T) \geq \frac{\|T^{\sharp_A}T+TT^{\sharp_A}\|_A}{4}+\frac{|\|T+T^{\sharp_A}\|_A^2-\|T-T^{\sharp_A}\|_A^2|}{8}.$$

Proof. It is a simple consequence of Theorem 2.21 applying to $\widetilde{T}$. □

We give a sufficient and necessary condition for $w_A^2(T)=\dfrac{\|T^{\sharp_A}T+TT^{\sharp_A}\|_A}{4}$.

Proposition 3.22. *Let $T \in B_A(H)$. Then*

$$w_A^2(T)=\frac{\|T^{\sharp_A}T+TT^{\sharp_A}\|_A}{4}$$

if and only if

$$\|e^{i\theta}T+e^{-i\theta}T^{\sharp_A}\|_A^2=\|e^{i\theta}T-e^{-i\theta}T^{\sharp_A}\|_A^2=\|T^{\sharp_A}T+TT^{\sharp_A}\|_A, \qquad \text{for all } \theta\in\mathbb{R}.$$

Proof. Applying Proposition 2.23 to $\widetilde{T}$, we get the suitable equivalence. □

Corollary 3.23. *Let $T, S \in B_A(H)$ such that $TS = ST$ and $T^{\sharp_A}S = ST^{\sharp_A}$. Then*

$$\begin{aligned} w^2(TS) \;\;&\geq\;\; \frac{|\|TS+T^{\sharp_A}S^{\sharp_A}\|_A^2-\|TS-T^{\sharp_A}S^{\sharp_A}\|_A^2|}{8} \\ &\quad+\frac{\|T^{\sharp_A}TS^{\sharp_A}S+TT^{\sharp_A}SS^{\sharp_A}\|_A}{4}. \end{aligned}$$

Proof. It suffices to apply Corollary 2.24 to the doubly commute operators $\widetilde{T}$ and $\widetilde{S}$. □

Corollary 3.24. *Let $T, S \in B_A(H)$. Then*

$$\begin{aligned} w_{A\otimes B}^2(T\otimes S) &\geq \frac{|\|T\otimes S+T^{\sharp_A}\otimes S^{\sharp_B}\|_A^2-\|T\otimes S-T^{\sharp_A}\otimes S^{\sharp_B}\|_A^2|}{8} \\ &\quad+\frac{\|T^{\sharp_A}T\otimes S^{\sharp_B}S+TT^{\sharp_A}\otimes SS^{\sharp_B}\|_A}{4}. \end{aligned}$$

Proof. The result follows easily from Corollary 2.25 applied to $\widetilde{T}$ and $\widetilde{S}$. □

Theorem 3.25. *Let $T \in B_A(H)$. Then*

$$w_A^2(T) \geq \frac{|\|T + iT^{\sharp_A}\|_A^2 - \|T - iT^{\sharp_A}\|_A^2|}{8} + \frac{\|T^{\sharp_A}T + TT^{\sharp_A}\|_A}{4}.$$

Proof. It is a consequence of Theorem 2.22 applied to $\widetilde{T}$. □

Corollary 3.26. *Let $T \in B_A(H)$. Suppose $w_A^2(T) = \frac{\|T^{\sharp_A}T+TT^{\sharp_A}\|_A}{4}$; then*

$$\|T + iT^{\sharp_A}\|_A^2 = \|T - iT^{\sharp_A}\|_A^2 = \|T^{\sharp_A}T + TT^{\sharp_A}\|_A.$$

Proof. Applying Corollary 2.26 to $\widetilde{T}$, we find the desired result. □

Corollary 3.27. *Let $T, S \in B_A(H)$ such that $TS = ST$ and $T^{\sharp_A}S = ST^{\sharp_A}$. Then*

$$\begin{aligned} w^2(TS) \;\; \geq \;\; & \frac{|\|TS + iT^{\sharp_A}S^{\sharp_A}\|_A^2 - \|TS - iT^{\sharp_A}S^{\sharp_A}\|_A^2|}{8} \\ & + \frac{\|T^{\sharp_A}TS^{\sharp_A}S + TT^{\sharp_A}SS^{\sharp_A}\|_A}{4}. \end{aligned}$$

Proof. The proof follows from Corollary 2.27 applying to $\widetilde{T}$ and $\widetilde{S}$. □

Corollary 3.28. *Let $T \in B_A(H)$ and let $S \in B_B(K)$. Then*

$$\begin{aligned} w_{A\otimes B}^2(T \otimes S) \geq & \frac{|\|T \otimes S + iT^{\sharp_A} \otimes S^{\sharp_B}\|_A^2 - \|T \otimes S - iT^{\sharp_A} \otimes S^{\sharp_B}\|_A^2|}{8} \\ & + \frac{\|T^{\sharp_A}T \otimes S^{\sharp_B}S + TT^{\sharp_A} \otimes SS^{\sharp_B}\|_A}{4}. \end{aligned}$$

Proof. Using Corollary 2.28 to $\widetilde{T}$ and $\widetilde{S}$, we get the required inequalities. □

Proposition 3.29. *Let $T \in B_{A^{\frac{1}{2}}}(H)$. Then*

$$d_A^2(T) = \inf_{\lambda \in \mathbb{C}} \{\|T - \lambda I\|_A^2 - m_A^2(T - \lambda I)\},$$

where

$$d_A(T) = \inf_{\lambda \in \mathbb{C}} \|T - \lambda I\|_A.$$

Proof. By Proposition 2.29 and Lemma 3.2, we get

$$\begin{aligned} d_A^2(T) &= \inf_{\lambda \in \mathbb{C}} \|T - \lambda I\|_A^2 = \inf_{\lambda \in \mathbb{C}} \|\widetilde{T - \lambda I}\|_{B(\mathbf{R}(A^{\frac{1}{2}}))}^2 = \inf_{\lambda \in \mathbb{C}} \|\widetilde{T} - \lambda I\|_{B(\mathbf{R}(A^{\frac{1}{2}}))}^2 \\ &= d_A^2(\widetilde{T}) = \inf_{\lambda \in \mathbb{C}} \{\|\widetilde{T} - \lambda I\|_{B(\mathbf{R}(A^{\frac{1}{2}}))}^2 - m_A^2(\widetilde{T} - \lambda I)\} \\ &= \inf_{\lambda \in \mathbb{C}} \{\|\widetilde{T - \lambda I}\|_{B(\mathbf{R}(A^{\frac{1}{2}}))}^2 - m_A^2(\widetilde{T - \lambda I})\} \\ &= \inf_{\lambda \in \mathbb{C}} \{\|T - \lambda I\|_A^2 - m_A^2(T - \lambda I)\}. \end{aligned}$$

□

Proposition 3.30. *Let $T \in B_A(H)$. Then*

$$\frac{\|T\|_A}{2} + \frac{|\|\Re_A(T)\|_A - \frac{\|T\|_A}{2}|}{4} + \frac{|\|\Im_A(T)\|_A - \frac{\|T\|_A}{2}|}{4} \leq w_A(T).$$

Proof. We have

$$\begin{aligned}\|\Re(\widetilde{T})\|_{B(\mathbf{R}(A^{\frac{1}{2}}))} &= \frac{1}{2}\|\widetilde{T}+(\widetilde{T})^*\|_{B(\mathbf{R}(A^{\frac{1}{2}}))} = \frac{1}{2}\|\widetilde{T}+\widetilde{T}^{\sharp_A}\|_{B(\mathbf{R}(A^{\frac{1}{2}}))}\\ &= \frac{1}{2}\|\widetilde{T+T^{\sharp_A}}\|_{B(\mathbf{R}(A^{\frac{1}{2}}))} = \frac{1}{2}\|T+T^{\sharp_A}\|_A\\ &= \|\frac{1}{2}\left(T+T^{\sharp_A}\right)\|_A = \|\Re_A(T)\|_A.\end{aligned}$$

Hence

$$\|\Re(\widetilde{T})\|_{B(\mathbf{R}(A^{\frac{1}{2}}))} = \|\Re_A(T)\|_A. \tag{3.3}$$

By the same manner, we get

$$\|\Im(\widetilde{T})\|_{B(\mathbf{R}(A^{\frac{1}{2}}))} = \|\Im_A(T)\|_A. \tag{3.4}$$

Using Proposition 3.30 and equalities (3.3) and (3.4), we obtain the desired inequality. ☐

Proposition 3.31. *Let $T, S \in B_A(H)$. If T and S commute A and $\|TS\|_A = \|T\|_A\|S\|_A$, then*

$$\begin{aligned}w_A(TS) \geq \frac{\|T\|_A\|S\|_A}{2} &+ \frac{|\|TS+T^{\sharp_A}S^{\sharp_A}\|_A - \|T\|_A\|S\|_A|}{8}\\ &+ \frac{|\|TS-T^{\sharp_A}S^{\sharp_A}\|_A - \|T\|_A\|S\|_A|}{8}.\end{aligned}$$

Proof. Applying Proposition 2.31 to $\widetilde{T}$, we find the required formula. ☐

Proposition 3.32. *Let $T \in B_A(H)$ and let $S \in B_B(K)$. Then*

$$\begin{aligned}\frac{\|T\|_A\|S\|_B}{2} &+ \frac{|\|T\otimes S+T^{\sharp_A}\otimes S^{\sharp_B}\|_{A\otimes B} - \|T\|_A\|S\|_B|}{8}\\ &+ \frac{|\|T\otimes S-T^{\sharp_A}\otimes S^{\sharp_B}\|_{A\otimes B} - \|T\|_A\|S\|_B|}{8} \leq w_{A\otimes B}(T\otimes S).\end{aligned}$$

Proof. It suffices to apply Proposition 3.31 to $T\otimes I_K$ and $I_H\otimes S$. ☐

Proposition 3.33. *Let $T \in B_A(H)$. Then for all $r \geq 1$*

$$w_A^{2r}(T) \leq \frac{1}{2}\|(T^{\sharp_A}T)^r + (TT^{\sharp_A})^r\|_A^2 - \frac{1}{2}m_A\left(|\widetilde{T}|^r - |\widetilde{T^{\sharp_A}}|^r\right)^2.$$

Proof. It is a simple deduction from Proposition 2.33 applied to $\widetilde{T}$. ☐

Proposition 3.34. *Let $T \in B_A(H)$ and let $S \in B_B(K)$. Then for all $r \geq 1$*

$$\begin{aligned}w_{A\otimes B}^{2r}(T\otimes S) \leq \frac{1}{2}\|(T^{\sharp_A}T\otimes S^{\sharp_B}S)^r &+ (TT^{\sharp_A}\otimes SS^{\sharp_B})^r\|^2\\ &- \frac{1}{2}m_A\left(|\widetilde{T\otimes S}|^r - |\widetilde{T^{\sharp_A}\otimes S^{\sharp_B}}|^r\right)^2.\end{aligned}$$

Proof. Combining Proposition 3.33 applied to $T\otimes S$, and the equality $(T\otimes S)^{\sharp_{A\otimes B}} = T^{\sharp_A}\otimes S^{\sharp_B}$ yields the desired inequality. ☐

Proposition 3.35. *Let $T \in B_{A^{\frac{1}{2}}}(H)$ and let $S \in B_{B^{\frac{1}{2}}}(K)$. Then*

$$\begin{aligned}
w_{A\otimes B}^2(T\otimes S) &\le \frac{1}{2}\left(w_{A\otimes B}(T^2\otimes S^2) + \|T\|_A^2\|S\|_B^2\right)\\
&\le \frac{1}{2}\min\{w_A(T^2)\|S^2\|_B, w_B(S^2)\|T^2\|_A\} + \frac{1}{2}\|T\|_A^2\|S\|_B^2\\
&\le \frac{1}{2}\min\{w_A^2(T)\|S^2\|_B, w_B(S^2)\|T^2\|_A\} + \frac{1}{2}\|T\|_A^2\|S\|_B^2\\
&\le \frac{1}{2}\min\{w_A^2(T)\|S^2\|_B, w_B^2(S)\|T^2\|_A\} + \frac{1}{2}\|T\|_A^2\|S\|_B^2\\
&\le \|T\|_A^2\|S\|_B^2.
\end{aligned}$$

Proof. By using Lemmas 3.2, 3.3 and Proposition 2.36, whenever it is necessary, we have

$$\begin{aligned}
w_{A\otimes B}^2(T\otimes S) &= w^2(\widetilde{T\otimes S})\\
&= w^2(\widetilde{T}\otimes\widetilde{S})\\
&\le \frac{1}{2}\left(w((\widetilde{T})^2\otimes(\widetilde{S})^2) + \|\widetilde{T}\|^2_{B(\mathbf{R}(A^{\frac{1}{2}}))}\|\widetilde{S}\|^2_{B(\mathbf{R}(B^{\frac{1}{2}}))}\right)\\
&= \frac{1}{2}\left(w(\widetilde{T^2}\otimes\widetilde{S^2}) + \|T\|_A^2\|S\|_B^2\right)\\
&= \frac{1}{2}\left(w(\widetilde{T^2\otimes S^2}) + \|T\|_A^2\|S\|_B^2\right)\\
&= \frac{1}{2}\left(w_{A\otimes B}(T^2\otimes S^2) + \|T\|_A^2\|S\|_B^2\right)\\
&= \frac{1}{2}\left(w(\widetilde{T}^2\otimes\widetilde{S}^2) + \|T\|_A^2\|S\|_B^2\right)\\
&\le \frac{1}{2}\min\{w(\widetilde{T^2})\|\widetilde{S^2}\|_{B(\mathbf{R}(B^{\frac{1}{2}}))}, w(\widetilde{S^2})\|\widetilde{T^2}\|_{B(\mathbf{R}(A^{\frac{1}{2}}))}\} + \frac{1}{2}\|T\|_A^2\|S\|_B^2\\
&= \frac{1}{2}\min\{w_A(T^2)\|S^2\|_B, w_B(S^2)\|T^2\|_A\} + \frac{1}{2}\|T\|_A^2\|S\|_B^2\\
&\le \frac{1}{2}\min\{w_A^2(T)\|S^2\|_B, w_B(S^2)\|T^2\|_A\} + \frac{1}{2}\|T\|_A^2\|S\|_B^2\\
&\le \frac{1}{2}\min\{w_A^2(T)\|S^2\|_B, w_B^2(S)\|T^2\|_A\} + \frac{1}{2}\|T\|_A^2\|S\|_B^2\\
&\le \|T\|_A^2\|S\|_B^2.
\end{aligned}$$

□

Proposition 3.36. *Let $T, S \in B_A(H)$. If $w_A(T+S) = w_A(T) + w_A(S)$, then*

$$w_A(T)w_A(S) \le \frac{1}{2}(w_A(ST) + \|T\|_A\|S\|_A).$$

Proof. Applying Proposition 2.37 to $\widetilde{T}$ and $\widetilde{S}$, we get the required inequality. □

Remark 3.37. Note that we obtain the inequality

$$w_A^2(T) \le \frac{1}{2}\left(w_A(T^2) + \|T\|_A^2\right),$$

for any $T \in B_{A^{\frac{1}{2}}}(H)$, which is established in [13], by letting $T = S$ in Proposition 3.36.

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References

[1] A. Abu-Omar and F. Kittaneh, *Notes on some spectral radius and numerical radius inequalities*, Studia Math. **227** (2015), no. 2, 97–109.

[2] M.W. Alomari, *Improvements of Some Numerical Radius Inequalities*, Azerbaijan J. Math. **12** (2022), no. 1, 129–141.

[3] M.L. Arias, G. Corach and M.C. Gonzalez,*Metric properties of projections in semi-Hilbertian spaces*, Integr. Equ. Oper. Theory, **62** (2008), 11–28.

[4] M.L. Arias, G. Corach and M.C. Gonzalez, *Partial isometries in semi-Hilbertian spaces*, Linear Algebra Appl. **428** (2008), no. 7, 1460–1475.

[5] M.L. Arias, G. Corach and M.C. Gonzalez, *Lifting properties in operator ranges*, Acta Sci. Math. (Szeged), **75** (2009), no. 3–4, 635–653.

[6] H. Baklouti, K. Feki and O.A.M. Sid Ahmed, *Joint numerical ranges of operators in semi-Hilbertian spaces*, Linear Algebra Appl. **555**, (2018), 266–284.

[7] H. Baklouti and S. Namouri, *Closed operators in semi-Hilbertian spaces*, Linear Multilinear Algebra, **70** (2021), no. 20, 5847–5858.

[8] E.H. Benabdi, A. Baghdad and M.C. Kaadoud, *Birkhoff-James orthogonality and algebraic maximal numerical range in C*-algebras*, Linear Multilinear Algebra, **70** (2022), no. 21, 6523–6534.

[9] P. Bhunia, S. S. Dragomir, M.S. Moslehian and K. Paul, *Lectures on numerical radius inequalities*, Infosys Science Foundation Series in Mathematical Sciences, Springer, 2022.

[10] P. Bhunia, K. Paul and A. Sen, *Numerical radius inequalities for tensor product of operators*, Sen Proc. Indian Acad. Sci. (Math. Sci.), **133** (2023), no. 3, 1–12.

[11] R.G. Douglas, *On majorization, factorization and range inclusion of operators in Hilbert space*, Proc. Amer. Math. Soc. **17** (1966), 413–416.

[12] S.S. Dragomir, *Inequalities for the norm and the numerical radius of linear operators in Hilbert spaces*, Demonstratio Math. **40** (2007), no. 2, 411–417.

[13] K. Feki, *Spectral radius of semi-Hilbertian space operators and its applications*, Ann. Funct. Anal. **11** (2020), 929–946.

[14] K.E. Gustafson and D.K.M. Rao, *Numerical range: The field of values of linear operators and matrices*, Springer, New York, NY, USA, 1997.

[15] P.R. Halmos, *A Hilbert Space Problems Book*, Springer, New York, 1982.

[16] F. Hausdorff, *Der Wertvorrat einer Bilinearform,* Math. Z. **3** (1919), 314–316.

[17] O. Hirzallah, F. Kittaneh and K. Shebrawi, *Numerical radius inequalities for certain 2 × 2 operator matrices*, Integr. Equ. Oper. Theory, **71** (2011), 129–147.

[18] F. Kittaneh, *Notes on some inequalities for Hilbert space operators*, Publ. Res. Inst. Math. Sci. **24** (1988), 283–293.

[19] F. Kittaneh, *Norm inequalities for certain operator sums*, J. Funct. Anal. **143** (1997), 337–348.

[20] F. Kittaneh, *A numerical radius inequality and an estimate for the numerical radius of the Frobenius companion matrix*, Studia Math. **158** (2003), no. 1, 11–17.

[21] F. Kittaneh, *Numerical radius inequalities for Hilbert space operators*, Studia Math., **168** (2005), no. 1, 73–80.

[22] W. Majdak, N.A. Secelean and L. Suciu, *Ergodic properties of operators in some semi-Hilbertian spaces*, Linear Multilinear Algebra, **61** (2013), no. 2, 139–159.

[23] J.G. Stampfli, *The norm of derivation*, Pacifc J. Math. **33** (1970), 737–747.

[24] Z. Taki and M.C. Kaadoud, *Some new refnements of numerical radius inequalities for Hilbert and semi-Hilbert space operators*, Filomat, **37** (2023), no. 20, 6925–6947.

[25] O. Toeplitz, *Das algebraische Analogon zu einem Satze von Fej´er*, Math. Z. **2** (1918), no.1-2, 187–197.

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