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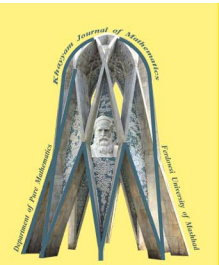

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ON LYAPUNOV STABILITY OF CAPUTO FRACTIONAL DYNAMIC EQUATIONS ON TIME SCALE USING VECTOR LYAPUNOV FUNCTIONS

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Abstract. This work extends the stability analysis of nonlinear fractional dynamic systems on time scales by introducing a new definition of the Caputo fractional delta derivative and Caputo delta Dini derivative for vector Lyapunov functions. The proposed framework effectively captures continuous and discrete dynamics, making it well-suited for hybrid systems. Based on this new formulation for the Caputo fractional delta derivative of vector Lyapunov function, comparison results, stability and asymptotic stability criteria for the investigated Caputo fractional dynamic equations are presented with practical examples demonstrating the applicability and effectiveness of our method.

1. Introduction and preliminaries

Dynamic systems often exhibit behaviors characterized by smooth and abrupt changes occurring almost simultaneously. These systems, which involve multiple time scales or frequencies, are effectively modeled using time scale calculus, as introduced in [7]. Time scale calculus unifies the study of differential and difference equations (see [10]), allowing for the analysis of systems that involve both continuous and discrete components.

In the qualitative theory of dynamic equations, stability of the solutions is one of the primary focus. We can compare how different starting points for solutions behave using stability theory. The study of stability in dynamic systems is a fundamental aspect of mathematical modeling with wide-ranging applications

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across various scientific and engineering disciplines. Classical differential equations have been extensively used to describe the behavior of dynamic systems. However, as the need to model complex systems with memory-dependent and non-local properties arises, the incorporation of fractional calculus has become increasingly prevalent.

Consider the Caputo fractional dynamic system on a time scale $\mathbb{T}$,

$$\begin{aligned} {}^{C}\Delta^{\alpha}x &= f(t,x), \quad t \in \mathbb{T}, \\ x(t_0) &= x_0, \quad t_0 \geq 0, \end{aligned} \tag{1.1}$$

where $f \in C_{rd}[\mathbb{T}\times\mathbb{R}^N, \mathbb{R}^N]$, $f(t,0) \equiv 0$, and ${}^{C}\Delta^{\alpha}x$ is the Caputo fractional delta derivative of $x \in \mathbb{R}^N$ of order α with respect to $t \in \mathbb{T}$. Let $x(t) = x(t,t_0,x_0) \in C^{\alpha}_{rd}[\mathbb{T},\mathbb{R}^N]$ be a solution to(1.1), and assume the existence and uniqueness of the solution as established in [3].

In [1, 2, 11], the stability properties of (1.1) was examined for $\mathbb{T} = \mathbb{R}$, while in [5, 14, 16, 17] the system (1.1) was studied with $\mathbb{T} = \mathbb{N}_0$. However, these studies did not address systems that undergo both continuous and discrete changes effectively.

To address these limitations and to better capture the nuances of fractional dynamic systems, we propose a new definition of the fractional derivative for scalar Lyapunov functions, defined along the solution path of (1.1) in [13, 12]. This new definition, called the Caputo fractional delta derivative and Caputo fractional delta Dini derivative on a time scale, preserves the properties of fractional derivatives while requiring only right dense continuity of the function. Moreover, it depends on the initial data $V(t_0,x_0)$ and is defined for discrete, continuous, or combined domains. Specifically, we define the new generalized Caputo fractional delta Dini derivative for a scalar Lyapunov function $V(t,x)$ as

$$\begin{aligned} {}^{C}\Delta^{\alpha}_{+}V(t,\mathfrak{x}) &= \limsup_{\mu\to 0^+}\frac{1}{\mu^{\alpha}}\Bigg[\sum_{r=0}^{[\frac{t-t_0}{\mu}]}(-1)^r({}^{\alpha}C_r)(V(\sigma(t)-r\mu,\mathfrak{x}(\sigma(t))-\mu^{\alpha}\mathfrak{f}(t,\mathfrak{x}(t))) \\ &\quad -V(t_0,\mathfrak{x}_0))\Bigg], \end{aligned} \tag{1.2}$$

where $t \in \mathbb{T}$, $\mathfrak{x},\mathfrak{x}_0 \in \mathbb{R}^n$, $\mu = \sigma(t) - t$, and $\mathfrak{x}(\sigma(t)) - \mu^{\alpha}\mathfrak{f}(t,\mathfrak{x}) \in \mathbb{R}^n$.

Still, in certain cases, a system may appear unstable or pose complexity when analyzed using scalar Lyapunov functions. This challenge can however be tackled by introduction of vector Lyapunov functions which provides a more refined perspective, enabling potential stability to be identified within specific subsystems or components. By employing vector Lyapunov functions, we can examine the stability properties of complex systems, which would otherwise present significant analytical challenges when using scalar Lyapunov functions alone.

In this work, we present the Lyapunov stability analysis for the nonlinear fractional dynamic system (1.1) using a new generalized derivative for vector Lyapunov functions, adopting the comparison principle approach. The comparison principle is a powerful tool in analyzing differential equations and dynamic systems, involving the comparison of the stability properties of a given system with

those of a simpler, well-understood system. By establishing relationships between the solutions of both systems, the stability of the simpler comparison system can be used to infer the stability of the original system.

For the purpose of this work, we shall use the Caputo fractional dynamic system of the form

$$ {}^{C}\Delta^{\alpha}\kappa = \Theta(t,\kappa),\ \kappa(t_0) = \kappa_0 \geq 0, \tag{1.3}$$

where $\kappa \in \mathbb{R}^n_+$, $\Theta : \mathbb{T} \times \mathbb{R}^n_+ \to \mathbb{R}^n_+$ and $\Theta(t,0) \equiv 0$. System (1.3) is called the comparison system. For this work, we will assume that the function $\Theta \in [\mathbb{T} \times \mathbb{R}^n_+, \mathbb{R}^n_+]$, is such that for any initial data $(t_0, \kappa_0) \in \mathbb{T} \times \mathbb{R}^n_+$, the system (1.3) with $\kappa(t_0) = \kappa_0$ has a unique solution $\kappa(t) = \kappa(t; t_0, \kappa_0) \in C^{\alpha}_{rd}[\mathbb{T}, \mathbb{R}^n_+]$. See [3, 6, 9].

The remainder of this paper is structured as follows: In Section 2, we provide a review of time scales and fractional derivatives. In Section 3, we introduce the new definition of the Caputo fractional delta derivative and Caputo delta Dini derivative for vector Lyapunov functions. Section 4 presents new stability results for nonlinear fractional dynamic systems on arbitrary time scales. Finally, Section 5 demonstrates the applicability of our results through an illustration.

Definition 1.1 ([4])**.** For $t \in \mathbb{T}$, the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined as

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\},$$

while the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined as

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\}.$$

(i) If $\sigma(t) > t$, then t is right scattered.
(ii) If $\rho(t) < t$, then t is left scattered.
(iii) If $t < \max \mathbb{T}$ and $\sigma(t) = t$, then t is called right dense.
(iv) If $t > \min \mathbb{T}$ and $\rho(t) = t$, then t is called left dense.

Definition 1.2 ([4])**.** The graininess function $\mu : \mathbb{T} \to [0, \infty)$ for $t \in \mathbb{T}$ is defined as

$$\mu(t) = \sigma(t) - t.$$

The derivative uses the set $\mathbb{T}^k$, which is derived from the time scale $\mathbb{T}$ as follows. If $\mathbb{T}$ has a left scattered maximum M, then $\mathbb{T}^k = \mathbb{T} \setminus \{M\}$. Otherwise, $\mathbb{T}^k = \mathbb{T}$.

Definition 1.3 ([4])**.** Let $h : \mathbb{T} \to \mathbb{R}$ and let $t \in \mathbb{T}^k$. We define the delta derivative h^{Δ} also known as the Hilger derivative as

$$h^{\Delta}(t) = \lim_{s \to t} \frac{h(\sigma(t)) - h(s)}{\sigma(t) - s}, \quad s \neq \sigma(t).$$

provided the limit exists.

The function $h^{\Delta} : \mathbb{T} \to \mathbb{R}$ is called the (Delta) derivative of h on $\mathbb{T}^k$.

If t is right dense, then the delta derivative of $h : \mathbb{T} \to \mathbb{R}$, becomes

$$h^{\Delta}(t) = \lim_{s \to t} \frac{h(t) - h(s)}{t - s},$$

and if t is right scattered, then the Delta derivative becomes

$$h^{\Delta}(t) = \frac{h^{\sigma}(t) - h(t)}{\mu(t)}.$$

For a function $h : \mathbb{T} \to \mathbb{R}$, h^{σ} denotes $h(\sigma(t))$.

Definition 1.4 ([8])**.** A function $h : \mathbb{T} \to \mathbb{R}$ is right dense continuous if it is continuous at all right dense points of $\mathbb{T}$ and its left sided limits exist and is finite at left dense points of $\mathbb{T}$. The set of all right dense continuous functions are denoted by

$$C_{rd} = C_{rd}(\mathbb{T}).$$

Definition 1.5 ([8])**.** Assume that $[a, b]$ is a closed and bounded interval in $\mathbb{T}$. Then a function $H : [a, b] \to \mathbb{R}$ is called a delta antiderivative of $h : [a, b] \to \mathbb{R}$ provided H is continuous on $[a, b]$, delta differentiable on $[a, b)$, and $H^{\Delta}(t) = h(t)$ for all $t \in [a, b)$. Then, we define the Delta integral by

$$\int_a^b h(t) = H(b) - H(a), \qquad \text{for all } a, b \in \mathbb{T}.$$

Remark 1.6 ([8]). All right dense continuous functions are delta integrable.

Definition 1.7 ([8])**.** A function $\phi : [0, r] \to [0, \infty)$ is of class $\mathcal{K}$ if it is continuous, and strictly increasing on $[0, r]$ with $\phi(0) = 0$.

Definition 1.8 ([8])**.** A continuous function $\mathcal{V} : \mathbb{R}^n \to \mathbb{R}$ with $\mathcal{V}(0) = 0$ is called positive definite (negative definite) on the domain D if there exists a function $\phi \in \mathcal{K}$ such that $\phi(|x|) \leq \mathcal{V}(x)$ $(\phi(|x|) \leq -\mathcal{V}(x))$ for $x \in D$.

Definition 1.9 ([8])**.** A continuous function $\mathcal{V} : \mathbb{R}^n \to \mathbb{R}$ with $\mathcal{V}(0) = 0$ is called positive semidefinite (negative semi-definite) on D if $\mathcal{V}(x) \geq 0$ $(\mathcal{V}(x) \leq 0)$ for all $x \in D$, and it can also vanish for some $x \neq 0$.

Definition 1.10 ([4])**.** Let $a, b \in \mathbb{T}$ and $h \in C_{rd}$. Then we define the integration on a time scale $\mathbb{T}$ as follows:

(i) If $\mathbb{T} = \mathbb{R}$, then

$$\int_a^b h(t)\Delta t = \int_a^b h(t)dt,$$

where $\int_a^b h(t)dt$ is the usual Riemann integral from calculus.

(ii) If $[a, b]$ consists of only isolated points, then

$$\int_a^b h(t)\Delta t = \begin{cases} \sum_{t\in[a,b)} \mu(t)h(t) & \text{if} \quad a < b, \\ 0 & \text{if} \quad a = b, \\ -\sum_{t\in[b,a)} \mu(t)h(t) & \text{if} \quad a > b. \end{cases}$$

(iii) If there exists a point $\sigma(t) > t$, then

$$\int_t^{\sigma(t)} h(s)\Delta s = \mu(t)f(t).$$

Definition 1.11 ([3]). **(Fractional integral on time scales)** Let $\alpha \in (0,1)$, let $[a,b]$ be an interval on $\mathbb{T}$, and let h be an integrable function on $[a,b]$. Then the fractional integral of order α of h is defined by

$$ {}^{\mathbb{T}}_{a}I^{\alpha}_{t}h^{\Delta}(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\Delta s. $$

Definition 1.12 ([3]). **(Caputo Derivative on Time Scale)**(Definition 2.8 in [3]) Let $\mathbb{T}$ be a time scale, let $t \in \mathbb{T}$, let $0 < \alpha < 1$, and let $h : \mathbb{T} \to \mathbb{R}$. The Caputo fractional derivative of order α of h is defined by

$$ {}_{a}\Delta^{\alpha}_{t}h(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-s)^{-\alpha} h^{\Delta^n}(s)\Delta s. $$

Definition 1.13. Let $\mathbb{T}$ be a time scale. A point $t_0 \in \mathbb{T}$ is said to be a minimal element of $\mathbb{T}$ if, for any $t \in \mathbb{T}$, $t > t_0$ whenever $t \neq t_0$.

Remark 1.14. The concept of minimal element is essential in studying dynamic equations because it establishes a starting point, a reference time from which the dynamics of the system evolve. In the study of difference equations (a discrete-time setting), t_0 represents the initial time step. Similarly, in differential equations (a continuous-time setting), t_0 represents the initial time instant.

Lemma 1.15 ([15]). *Let $\mathbb{T}$ be a time scale with the minimal element $t_0 \geq 0$. Suppose that for each $t \in \mathbb{T}$, there exists a statement $\mathbf{S}(t)$ such that the following conditions hold:*

(i) *$\mathbf{S}(t_0)$ is true;*
(ii) *If t is right-scattered and $\mathbf{S}(t)$ is true, then $\mathbf{S}(\sigma(t))$ is also true;*
(iii) *For every right-dense t, there exists a neighborhood $\mathcal{U}$ such that if $\mathbf{S}(t)$ is true, then $\mathbf{S}(t^*)$ is also true for all $t^* \in \mathcal{U}$ with $t^* \geq t$;*
(iv) *For left-dense t, if $\mathbf{S}(t^*)$ is true for all t^* in $[t_0, t)$, then $\mathbf{S}(t)$ is true.*

Then the statement $\mathbf{S}(t)$ is true for all $t \in \mathbb{T}$.

Remark 1.16. When $\mathbb{T} = \mathbb{N}$, then Lemma 1.15 reduces to the well-known principle of mathematical induction. That is,

(1) $\mathbf{S}(t_0)$ is true is equivalent to the statement is true for $n = 1$;
(2) If $\mathbf{S}(t)$ is true, then $\mathbf{S}(\sigma(t))$ is true is equivalent to if the statement is true for $n = k$, then the statement is true for $n = k + 1$.

Definition 1.17. Let $h \in C^{\alpha}_{rd}[\mathbb{T}, \mathbb{R}^n]$. Then the Grunwald–Letnikov fractional delta derivative is given by

$$ {}^{GL}\Delta^{\alpha}_{0}h(t) = \lim_{\mu \to 0+} \frac{1}{\mu^{\alpha}} \sum_{r=0}^{[\frac{(t-t_0)}{\mu}]} (-1)^{r\alpha} C_r[h(\sigma(t) - r\mu)], \quad t \geq t_0, \tag{1.4} $$

and the Grunwald–Letnikov fractional delta dini derivative is given by

$$ {}^{GL}\Delta^{\alpha}_{0+}h(t) = \limsup_{\mu \to 0+} \frac{1}{\mu^{\alpha}} \sum_{r=0}^{[\frac{(t-t_0)}{\mu}]} (-1)^{r\alpha} C_r[h(\sigma(t) - r\mu)], \quad t \geq t_0, \tag{1.5} $$

where $0 < \alpha < 1$, ${}^{\alpha}C_r = \frac{\alpha(\alpha-1)\cdots(\alpha-r+1)}{r!}$, and $[\frac{(t-t_0)}{\mu}]$ denotes the integer part of the fraction $\frac{(t-t_0)}{\mu}$.

Observe that if the domain is $\mathbb{R}$, then (1.5) becomes

$$ {}^{GL}\Delta_{0+}^{\alpha}h(t) = \limsup_{d\to 0+}\frac{1}{d^{\alpha}}\sum_{r=0}^{[\frac{(t-t_0)}{d}]}(-1)^{r\alpha}C_r[h(t-rd)], \quad t \geq t_0. $$

Remark 1.18. It is necessary to note that the relationship between the Caputo fractional delta derivative and the Grunwald–Letnikov fractional delta derivative is given by

$$ {}^{C}\Delta_0^{\alpha}h(t) = {}^{GL}\Delta_0^{\alpha}[h(t) - h(t_0)]. \tag{1.6} $$

Substituting (1.4) into (1.28), we have that the Caputo fractional delta derivative becomes

$$ {}^{C}\Delta_0^{\alpha}h(t) = \lim_{\mu\to 0+}\frac{1}{\mu^{\alpha}}\sum_{r=0}^{[\frac{(t-t_0)}{\mu}]}(-1)^{r\alpha}C_r[h(\sigma(t)-r\mu) - h(t_0)], \quad t \geq t_0, $$

$$ \begin{aligned} {}^{C}\Delta_0^{\alpha}h(t) &= \lim_{\mu\to 0+}\frac{1}{\mu^{\alpha}}\Bigg\{h(\sigma(t)) - h(t_0) \\ &\quad + \sum_{r=1}^{[\frac{(t-t_0)}{\mu}]}(-1)^{r\alpha}C_r[h(\sigma(t)-r\mu) - h(t_0)]\Bigg\}, \end{aligned} \tag{1.7} $$

and the Caputo fractional delta Dini derivative becomes

$$ {}^{C}\Delta_{0+}^{\alpha}h(t) = \limsup_{\mu\to 0+}\frac{1}{\mu^{\alpha}}\sum_{r=0}^{[\frac{(t-t_0)}{\mu}]}(-1)^{r\alpha}C_r[h(\sigma(t)-r\mu) - h(t_0)], \quad t \geq t_0, \tag{1.8} $$

which is equivalent to

$$ \begin{aligned} {}^{C}\Delta_{0+}^{\alpha}h(t) &= \limsup_{\mu\to 0+}\frac{1}{\mu^{\alpha}}\Bigg\{h(\sigma(t)) - h(t_0) \\ &\quad + \sum_{r=1}^{[\frac{(t-t_0)}{\mu}]}(-1)^{r\alpha}C_r[h(\sigma(t)-r\mu) - h(t_0)]\Bigg\}. \end{aligned} \tag{1.9} $$

For notation simplicity, we shall represent the Caputo fractional delta derivative of order α as ${}^{C}\Delta^{\alpha}$ and the Caputo fractional delta dini derivative of order α as ${}^{C}\Delta_{+}^{\alpha}$.

Given that $\lim_{N\to\infty}\sum_{r=0}^{N}(-1)^{r\alpha}C_r = 0$ where $\alpha \in (0,1)$, and $\lim_{\mu\to 0^+}[\frac{(t-t_0)}{\mu}] = \infty$, then it is easy to see that

$$ \lim_{\mu\to 0^+}\sum_{r=1}^{[\frac{(t-t_0)}{\mu}]}(-1)^{r\alpha}C_r = -1. \tag{1.10} $$

Also, from (1.8) and since the Caputo and Riemann–Liouville formulations coincide when $h(t_0) = 0$, then we have

$$\limsup_{\mu\to 0^+} \frac{1}{\mu^\alpha} \sum_{r=0}^{[\frac{(t-t_0)}{\mu}]} (-1)^{r\alpha} C_r =^{RL} \Delta^\alpha(1) = \frac{(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}, \quad t \geq t_0. \tag{1.11}$$

Now, we introduce the derivative of the Lyapunov function using the Caputo fractional delta Dini derivative of $h(t)$ given in (1.8).

Definition 1.19. We define the Caputo fractional delta Dini derivative of the Lyapunov function $V(t,x) \in C_{rd}[\mathbb{T}\times\mathbb{R}^N, \mathbb{R}^N_+]$ (which is locally Lipschitzian with respect to its second argument and $V(t,0) \equiv 0$) along the trajectories of solutions of the system (1.1) as follows:

$$\begin{aligned} &{}^C\Delta^\alpha_+ V(t,x) \\ &= \limsup_{\mu\to 0^+} \frac{1}{\mu^\alpha}\Bigg[\sum_{r=0}^{[\frac{t-t_0}{\mu}]} (-1)^r ({}^\alpha C_r)[V(\sigma(t) - r\mu, x(\sigma(t)) - \mu^\alpha f(t,x(t))) \\ &\quad - V(t_0,x_0)]\Bigg], \end{aligned} \tag{1.12}$$

which can further be expanded to

$$\begin{aligned} {}^C\Delta^\alpha_+ V(t,x) = \limsup_{\mu\to 0^+} \frac{1}{\mu^\alpha}\Bigg\{ & V(\sigma(t), x(\sigma(t))) - V(t_0,x_0) \\ & - \sum_{r=1}^{[\frac{t-t_0}{\mu}]} (-1)^{r+1} ({}^\alpha C_r)[V(\sigma(t) - r\mu, x(\sigma(t)) - \mu^\alpha f(t,x(t))) - V(t_0,x_0)]\Bigg\}, \end{aligned} \tag{1.13}$$

where $t \in \mathbb{T}, x, x_0 \in \mathbb{R}^N$, $\mu = \sigma(t) - t$ and $x(\sigma(t)) - \mu^\alpha f(t,x) \in \mathbb{R}^N$.

If $\mathbb{T}$ is continuous, that is $\mathbb{T} = \mathbb{R}$, and $V(t,x(t))$ is continuous at t, then we have

$$\begin{aligned} {}^C\Delta^\alpha_+ V(t,x) = \limsup_{d\to 0^+} \frac{1}{d^\alpha}\Bigg\{ & V(t,x(t)) - V(t_0,x_0) \\ & - \sum_{r=1}^{[\frac{t-t_0}{d}]} (-1)^{r+1} ({}^\alpha C_r)[V(t - rd, x(t) - d^\alpha f(t,x(t))) - V(t_0,x_0)]\Bigg\}, \end{aligned} \tag{1.14}$$

where $d > 0$.

Given that $\lim_{N\to\infty} \sum_{r=0}^N (-1)^{r\alpha} C_r = 0$, where $\alpha \in (0,1)$, and $\lim_{\mu\to 0^+} [\frac{(t-t_0)}{\mu}] = \infty$, then it is easy to see that

$$\lim_{\mu\to 0^+} \sum_{r=1}^{[\frac{(t-t_0)}{\mu}]} (-1)^{r\alpha} C_r = -1. \tag{1.15}$$

Also, from (1.8) and since the Caputo and Riemann–Liouville formulations coincide when $h(t_0) = 0$, then we have

$$\begin{aligned} {}^{C}\Delta_{+}^{\alpha}h(t) &= {}^{RL}\Delta_{+}^{\alpha}h(t) \\ &= \limsup_{\mu\to 0+}\frac{1}{\mu^{\alpha}}\left\{\sum_{r=0}^{[\frac{(t-t_0)}{\mu}]}(-1)^{r\alpha}C_r[h(\sigma(t)-r\mu)]\right\}, \quad t\geq t_0, \end{aligned}$$

which can further be simplified to obtain

$$\begin{aligned} {}^{C}\Delta_{+}^{\alpha}h(t) &= \limsup_{\mu\to 0^+}\frac{1}{\mu^{\alpha}}\left\{\sum_{r=0}^{[\frac{(t-t_0)}{\mu}]}(-1)^{r\alpha}C_r\right\} = {}^{RL\mathbb{T}}D^{\alpha}(1) \\ &= \frac{(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}, \quad t\geq t_0. \end{aligned} \tag{1.16}$$

Applying (1.16) to (1.13), we obtain

$$\begin{aligned} {}^{C}\Delta_{+}^{\alpha}V(t,x) &= \limsup_{\mu\to 0^+}\frac{1}{\mu^{\alpha}}\Big\{V(\sigma(t),x(\sigma(t))) \\ &\quad + \sum_{r=1}^{[\frac{t-t_0}{\mu}]}(-1)^{r}({}^{\alpha}C_r)[V(\sigma(t)-r\mu,x(\sigma(t))-\mu^{\alpha}f(t,x(t))]\Big\} \\ &\quad - \frac{V(t_0,x_0)(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}. \end{aligned} \tag{1.17}$$

Definition 1.20. The trivial solution $x = 0$ of (1.1) is called

- S_1 stable if given $\epsilon > 0$ and $t_0 \in \mathbb{T}$, there exists a $\delta = \delta(\epsilon, t_0) > 0$ such that for any $x_0 \in \mathbb{R}^n$ the inequality $\|x_0\| < \delta$ implies $\|x(t; t_0, x_0)\| < \epsilon$, for $t \geq t_0$.
- S_2 asymptotically stable if it is stable and there exist numbers $\delta_0 = \delta_0(\epsilon)$ and $T = T(\epsilon)$ such that for $t \geq t_0 + T$, the inequality $\|x_0\| \leq \delta$ implies $x(t; t_0, x_0) < \epsilon$.

Lemma 1.21. *Assume that h and $m \in C_{rd}[\mathbb{T}, \mathbb{R}^n]$. Suppose there exists $t_1 > t_0$, where $t_1 \in \mathbb{T}$, such that $h(t_1) = m(t_1)$ and $h(t) < m(t)$ for $t_0 \leq t < t_1$. Then, if the Caputo fractional delta Dini derivatives of h and m exist at t_1, then the inequality ${}^{C}\Delta_{+}^{\alpha}h(t_1) >^{C} \Delta_{+}^{\alpha}m(t_1)$ holds.*

Proof. Applying (1.8), we have

$$\begin{aligned} {}^{C}\Delta_{+}^{\alpha}(h(t)-m(t)) &= \limsup_{\mu\to 0^+}\frac{1}{\mu^{\alpha}}\Big\{\sum_{r=0}^{[\frac{t-t_0}{\mu}]}(-1)^{r\alpha}C_r[h(\sigma(t)-r\mu)-m(\sigma(t)-r\mu)] \\ &\quad -[h(t_0)-m(t_0)]\Big\} \end{aligned}$$

$$
\begin{aligned}
{}^{C}\Delta_{+}^{\alpha}h(t) -{}^{C}\Delta_{+}^{\alpha}m(t) &= \limsup_{\mu\to 0^{+}}\frac{1}{\mu^{\alpha}}\Bigg\{\sum_{r=0}^{[\frac{t-t_0}{\mu}]}(-1)^{r\alpha}C_r[h(\sigma(t)-r\mu)\\
&\quad -m(\sigma(t)-r\mu)]-[h(t_0)-m(t_0)]\Bigg\}.
\end{aligned}
$$

At t_1, we have

$$
{}^{C}\Delta_{+}^{\alpha}h(t_1) = -\limsup_{\mu\to 0^{+}}\frac{1}{\mu^{\alpha}}\Bigg\{\sum_{r=0}^{[\frac{t_1-t_0}{\mu}]}(-1)^{r\alpha}C_r[h(t_0)-m(t_0)]\Bigg\}+{}^{C}\Delta_{+}^{\alpha}m(t_1). \quad (1.18)
$$

Applying (1.16) to (1.18), we have

$$
{}^{C}\Delta_{+}^{\alpha}h(t_1) = -\frac{(t_1-t_0)^{-\alpha}}{\Gamma(1-\alpha)}[h(t_0)-m(t_0)]+{}^{C}\Delta_{+}^{\alpha}m(t_1).
$$

However, based on the Lemma's statement, we know that

$$
\begin{aligned}
&h(t)<m(t),\ \text{for } t_0\le t<t_1,\\
\Longrightarrow\ &h(t)-m(t)<0,\ \text{for } t_0\le t<t_1,
\end{aligned}
$$

where 0 is taken as the zero vector; then, we obtain

$$
-\frac{(t_1-t_0)^{-\alpha}}{\Gamma(1-\alpha)}[h(t_0)-m(t_0)]>0,
$$

implying

$$
{}^{C}\Delta_{+}^{\alpha}h(t_1) >{}^{C}\Delta_{+}^{\alpha}m(t_1).
$$

□

Lemma 1.22. *Assume that*

1. $x^{*}(t;t_0,x_0), x^{*}\in C_{rd}^{\alpha}([t_0,T]_{\mathbb{T}},\mathbb{R}^N)$ *is a solution of the dynamic equation* (1.1).
2. $V(t,x^{*})\in C_{rd}[\mathbb{T}\times\mathbb{R}^N,\mathbb{R}_{+}^N]$ *and for any* $t\in[t_0,\infty)_{\mathbb{T}}$, $x^{*}\in\mathbb{R}^N$,

$$
{}^{C}\Delta_{+}^{\alpha}V(t,x^{*})\le-\phi(\|x^{*}(t)\|), \quad (1.19)
$$

where $\phi\in\mathcal{K}$.

Then, for $t\in[t_0,T]_{\mathbb{T}}$, *the inequality*

$$
V(t,x^{*}(t))\le V(t_0,x_0)-\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}\phi(\|x^{*}(t)\|)\Delta s,
$$

holds.

Proof. Let

$$
\begin{aligned}
h(t)&=V(t,x^{*}(t)),\\
\text{with}\quad h(t_0)&=V(t_0,x_0),
\end{aligned} \quad (1.20)
$$

and

$$
W(t)=\phi(\|x^{*}(t)\|). \quad (1.21)
$$

Then, from (1.19) we have

$$ {}^{C}\Delta_{+}^{\alpha}h(t) = {}^{C}\Delta_{+}^{\alpha}V(t,x^{*}(t)) \leq -\phi(\|x^{*}(t)\|) = -W(t), \quad \text{for } t \in [t_0,T]_{\mathbb{T}}. \tag{1.22} $$

Consider the system

$$ {}^{C}\Delta^{\alpha}\upsilon(t) = -W(t), \quad \upsilon(t_0) = h_{\omega}(t_0), \quad \text{where } h_{\omega}(t_0) = h(t_0) + \omega. \tag{1.23} $$

A solution $\upsilon(t) = \upsilon(t,t_0,\upsilon_0)$ of (1.23) will also satisfy the Volterra delta integral equation

$$ \upsilon(t) = h_{\omega}(t_0) - \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}W(s)\Delta s. \tag{1.24} $$

We claim that

$$ h(t) < \upsilon(t), \quad t \in [t_0,T]_{\mathbb{T}}. \tag{1.25} $$

In the event that this claim were untrue, there would be a time $t_1 \in (t_0,T]_{\mathbb{T}}$, such that

$$ h(t_1) = \upsilon(t_1) \quad \text{and} \quad h(t) < \upsilon(t) \quad \text{for} \quad t \in [t_0,t_1)_{\mathbb{T}}. \tag{1.26} $$

Lemma 1.21 is applied to (1.26) to obtain

$$ \begin{aligned} {}^{C}\Delta_{+}^{\alpha}h(t_1) &> {}^{C}\Delta_{+}^{\alpha}\upsilon(t_1) = {}^{C}\Delta^{\alpha}\upsilon(t_1) = -W(t_1), \\ &\Longrightarrow {}^{C}\Delta_{+}^{\alpha}h(t_1) > -W(t_1). \end{aligned} \tag{1.27} $$

Therefore, from (1.27) and for $t \in [t_0,T]_{\mathbb{T}}$, we obtain

$$ {}^{C}\Delta_{+}^{\alpha}h(t) > -W(t). \tag{1.28} $$

Clearly, (1.28) contradicts (1.22), so we conclude that the claim in (1.25) holds. Combining (1.20),(1.21),(1.24), and (1.25), we get

$$ V(t,x^{*}(t)) \quad = \quad h(t) < h_{\omega}(t_0) - \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}\phi(\|x^{*}(s)\|)\Delta s, \tag{1.29} $$

which implies

$$ V(t,x^{*}(t)) \quad \leq \quad V(t_0,x_0) - \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}\phi(\|x^{*}(s)\|)\Delta s. \tag{1.30} $$

If $\mathbb{T}$ contains right scattered points, and $V(t,\chi)$ is continuous, then (1.30) becomes

$$ V(t,x^{*}(t)) \leq V(t_0,x_0) - \frac{1}{\Gamma(\alpha)}\int_{t_0}^{\sigma(t)}(t-s)^{\alpha-1}\phi(\|x^{*}(s)\|)ds. $$

□

Theorem 1.23. *Assume that*

(*i*) $\Theta \in C_{rd}[\mathbb{T}\times\mathbb{R}_{+}^{n},\mathbb{R}_{+}^{n}]$ *and* $\Theta(t,\kappa)\mu$ *is non-decreasing in* κ.
(*ii*) $V \in C_{rd}[\mathbb{T}\times\mathbb{R}^{N},\mathbb{R}_{+}^{N}]$ *is locally Lipschitzian in the second variable such that*

$$ {}^{C}\Delta_{+}^{\alpha}V(t,x) \leq \Theta(t,V(t,x)), (t,x) \in \mathbb{T}\times\mathbb{R}^{N}. \tag{1.31} $$

(iii) $z(t) = z(t; t_0, u_0)$ *is the maximal solution of (1.3) existing on* $\mathbb{T}$.

Then

$$V(t, x(t)) \leq z(t), \quad t \geq t_0 \tag{1.32}$$

provided that

$$V(t_0, x_0) \leq \kappa_0, \tag{1.33}$$

where $x(t) = x(t; t_0, x_0)$ *is any solution of* (1.1), $t \in \mathbb{T}, \quad t \geq t_0$.

Proof. Apply the principle of induction as stated in Lemma 1.15 to the statement

$$\mathbf{S(t)} : V(t, x(t)) \leq z(t), \quad t \in \mathbb{T}, \qquad t \geq t_0.$$

Then

(i) $\mathbf{S(t_0)}$ is true since $V(t_0, x_0) \leq u_0$,

(ii) Let t be right-scattered and let $\mathbf{S(t)}$ be true. We need to show that $\mathbf{S}(\sigma(\mathbf{t}))$ is true; that is,

$$V(\sigma(t), x(\sigma(t))) \leq z(\sigma(t)). \tag{1.34}$$

Set $h(t) = V(t, x(t))$; then $h(\sigma(t)) = V(\sigma(t), x(\sigma(t)))$ but from (1.8), we have

$${}^{C}\Delta_{+}^{\alpha} h(t) = \limsup_{\mu \to 0+} \frac{1}{\mu^{\alpha}} \left\{ \sum_{r=0}^{[\frac{(t-t_0)}{\mu}]} (-1)^{r\alpha} C_r [h(\sigma(t) - r\mu) - h(t_0)] \right\}, \quad t \geq t_0.$$

Also,

$${}^{C}\Delta_{+}^{\alpha} z(t) = \limsup_{\mu \to 0+} \frac{1}{\mu^{\alpha}} \left\{ \sum_{r=0}^{[\frac{(t-t_0)}{\mu}]} (-1)^{r\alpha} C_r [z(\sigma(t) - r\mu) - z(t_0)] \right\}, \quad t \geq t_0,$$

so that

$$\begin{aligned}
& {}^{C}\Delta_{+}^{\alpha} z(t) - {}^{C}\Delta_{+}^{\alpha} h(t) \\
= \; & \limsup_{\mu \to 0+} \frac{1}{\mu^{\alpha}} \left\{ \sum_{r=0}^{[\frac{(t-t_0)}{\mu}]} (-1)^{r\alpha} C_r [z(\sigma(t) - r\mu) - z(t_0)] \right\} \\
& - \limsup_{\mu \to 0+} \frac{1}{\mu^{\alpha}} \left\{ \sum_{r=0}^{[\frac{(t-t_0)}{\mu}]} (-1)^{r\alpha} C_r [h(\sigma(t) - r\mu) - h(t_0)] \right\} \\
& \left({}^{C}\Delta_{+}^{\alpha} z(t) - {}^{C}\Delta_{+}^{\alpha} h(t) \right) \mu^{\alpha} \\
= \; & \limsup_{\mu \to 0+} \left\{ \sum_{r=0}^{[\frac{(t-t_0)}{\mu}]} (-1)^{r\,\alpha} C_r \Big[[z(\sigma(t) - r\mu) - z(t_0)] \right. \\
& \left. - [h(\sigma(t) - r\mu) - h(t_0)] \Big] \right\} \\
\leq \; & [z(\sigma(t)) - z(t_0)] - [h(\sigma(t)) - h(t_0)]
\end{aligned}$$

$$\leq \quad [z(\sigma(t)) - h(\sigma(t))] - [z(t_0) - h(t_0)].$$

Then,

$$[z(\sigma(t)) - h(\sigma(t))] \geq \Big({}^{C}\Delta_{+}^{\alpha} z(t) - {}^{C}\Delta_{+}^{\alpha} h(t)\Big)\mu^{\alpha} + [z(t_0) - h(t_0)],$$

so that,

$$\begin{aligned}[h(\sigma(t)) - z(\sigma(t))] &\leq \Big({}^{C}\Delta_{+}^{\alpha} z(t) - {}^{C}\Delta_{+}^{\alpha} h(t)\Big)\mu^{\alpha} + [h(t_0) - z(t_0)] \\ &\leq \Big(\Theta(t, h(t)) - \Theta(t, z(t))\Big)\mu^{\alpha} + [h(t_0) - z(t_0)].\end{aligned}$$

Since $\Theta(t,\kappa)\mu$ is nondecreasing in κ and $\mathbf{S}(\mathbf{t})$ is true, then $h(\sigma(t)) - z(\sigma(t)) \leq 0$ so (1.34) holds.

(iii) Let t be right dense and let $\mathcal{N}$ be a right neighborhood of $t \in \mathbb{T}$. We need to show that $\mathbf{S}(\mathbf{t}^*)$ is true for $t^* \in \mathcal{N}$.

Let ω be an arbitrary small vector such that $\|\omega\| \leq B_{\mathbb{T}}$ (where $B_{\mathbb{T}}$ is a small enough number on the time scale $\mathbb{T}$) and consider the initial value problem

$${}^{C}\Delta^{\alpha}\kappa = \Theta(t^*, \kappa) + \omega, \quad \kappa(t_0) = \kappa_0 + \omega, \tag{1.35}$$

for $t^* \in \mathcal{N}$.

The function $\kappa_\omega(t^*) = \kappa(t^*) + \omega$ is a solution of (1.35) if and only if it satisfies the delta integral equation

$$\kappa_\omega(t^*) = \kappa_0 + \omega + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t^*} (t^* - s)^{\alpha-1} (\Theta(s, \kappa_\omega(s)) + \omega)\Delta s, \tag{1.36}$$

for $t^*, s \in \mathcal{N}$.

Let $h(t^*) \in C_{rd}[\mathbb{T}, \mathbb{R}_+^n]$ be such that $h(t^*) = V(t^*, x^*(t^*))$, where $x^*(t^*)$ is any other solution of (1.1). We show that

$$h(t^*) < \kappa_\omega(t^*), \quad \text{for } t^* \in \mathcal{N}. \tag{1.37}$$

The inequality (1.37) holds for $t^* = t_0$ since

$$h(t_0) = V(t_0, x_0) \leq \kappa_0 < \kappa_\alpha(t_0).$$

Assume that the inequality (1.37) is not true. Then there exist a point $t_1^* > t_0$ such that

$$h(t_1^*) = \kappa_\omega(t_1^*) \quad \text{and} \quad h(t^*) < \kappa_\omega(t^*) \quad \text{for} \quad t_0 \leq t^* < t_1^*,\ t^*, t_1^* \in \mathcal{N}.$$

From Lemma (1.21) it follows that

$${}^{C}\Delta_{+}^{\alpha} h(t_1^*) > {}^{C}\Delta_{+}^{\alpha} \kappa_\omega(t_1^*),$$

which implies

$${}^{C}\Delta_{+}^{\alpha} V(t_1^*, x^*(t_1^*)) > {}^{C}\Delta_{+}^{\alpha} \kappa_\omega(t_1^*).$$

Then, from (1.35), we arrive at

$${}^{C}\Delta_{+}^{\alpha} V(t_1^*, x^*(t_1^*)) > \Theta(t_1^*, \kappa_\omega(t_1^*)) + \omega) > \Theta(t_1^*, h(t_1^*)).$$

Therefore,

$$ {}^{C}\Delta_{+}^{\alpha} h(t_1^*) > \Theta(t_1^*, h(t_1^*)). \tag{1.38}$$

Now, for $t^* \in \mathcal{N}$,

$$
\begin{aligned}
&{}^{C}\Delta_{+}^{\alpha} h(t^*) \\
&= \limsup_{\mu\to 0^+} \frac{1}{\mu^\alpha}\Bigg\{ h(t^*) - h(t_0) - \sum_{r=1}^{[\frac{t^*-t_0}{\mu}]} (-1)^{r+1} ({}^{\alpha}C_r)[h(t^* - r\mu) - h(t_0)]\Bigg\}, \\
&= \limsup_{\mu\to 0^+} \frac{1}{\mu^\alpha}\{V(t^*, x^*(t^*)) - V(t_0, x_0) \\
&\quad - \sum_{r=1}^{[\frac{t^*-t_0}{\mu}]} (-1)^{r+1} ({}^{\alpha}C_r)[V(t^* - r\mu, x^*(t^* - r\mu) - V(t_0, x_0)]\} \\
&= \limsup_{\mu\to 0^+} \frac{1}{\mu^\alpha}\Bigg\{ V(t^*, x^*(t)) - V(t_0, x_0) \\
&\quad - \sum_{r=1}^{[\frac{t^*-t_0}{\mu}]} (-1)^{r+1} ({}^{\alpha}C_r)\Bigg[[V(t^* - r\mu, x^*(t^*) - \mu^\alpha f(t^*, x^*(t^*))) - V(t_0, x_0)] \\
&\quad -[V(t^* - r\mu, x^*(t^*) - \mu^\alpha f(t^*, x^*(t^*))) - V(t_0, x_0)] \\
&\quad +[V(t^* - r\mu, x^*(t^* - r\mu)) - V(t_0, x_0)]\Bigg]\Bigg\} \\
&= \limsup_{\mu\to 0^+} \frac{1}{\mu^\alpha}\Bigg\{ V(t^*, x^*(t^*)) - V(t_0, x_0) \\
&\quad - \sum_{r=1}^{[\frac{t^*-t_0}{\mu}]} (-1)^{r+1} ({}^{\alpha}C_r)\Bigg[[V(t^* - r\mu, x^*(t^*) - \mu^\alpha f(t^*, x^*(t^*))) - V(t_0, x_0)] \\
&\quad -V[(t^* - r\mu, x^*(t^*) - \mu^\alpha f(t^*, x^*(t^*)))] + V(t^* - r\mu, x^*(t^* - r\mu))\Bigg]\Bigg\} \\
&= \limsup_{\mu\to 0^+} \frac{1}{\mu^\alpha}\Bigg\{ V(t^*, x^*(t^*)) - V(t_0, x_0) \\
&\quad - \sum_{r=1}^{[\frac{t^*-t_0}{\mu}]} (-1)^{r+1} ({}^{\alpha}C_r)\Bigg[[V(t^* - r\mu, x^*(t^*) - \mu^\alpha f(t^*, x^*(t^*))) - V(t_0, x_0)] \\
&\quad +[V(t^* - r\mu, x^*(t^* - r\mu)) - [V(t^* - r\mu, x^*(t^*) - \mu^\alpha f(t^*, x^*(t^*)))]\Bigg]\Bigg\} \\
&= \limsup_{\mu\to 0^+} \frac{1}{\mu^\alpha}\Bigg\{ V(t^*, x^*(t^*)) - V(t_0, x_0)
\end{aligned}
$$

$$- \sum_{r=1}^{[\frac{t^*-t_0}{\mu}]} (-1)^{r+1} ({}^{\alpha}C_r[V(t^*-r\mu, x^*(t^*-\mu^{\alpha}f(t^*,x^*(t^*))-V(t_0,x_0)]\Big\}$$

$$-\limsup_{\mu\to 0^+}\frac{1}{\mu^{\alpha}}\Big\{\sum_{r=1}^{[\frac{t^*-t_0}{\mu}]}(-1)^{r+1}({}^{\alpha}C_r)[V(t^*-r\mu,x^*(t^*-r\mu))$$

$$-V(t^*-r\mu,x^*(t^*)-\mu^{\alpha}f(t^*,x^*(t^*)))]\Big\}.$$

Since $V(t^*,x)$ is locally Lipschtzian in the second variable, we have

$$\begin{aligned} {}^{C}\Delta_{+}^{\alpha}h(t^*) \;\leq\; & {}^{C}\Delta_{+}^{\alpha}V(t^*,x^*(t^*)) \\ & +L\limsup_{\mu\to 0^+}\frac{1}{\mu^{\alpha}}\sum_{r=1}^{[\frac{t^*-t_0}{\mu}]}(-1)^{r}({}^{\alpha}C_r)\|x^*(t^*-r\mu) \\ & -(x^*(t^*)-\mu^{\alpha}f(t^*,x^*(t^*)))\|, \end{aligned}$$

where $L>0$ is a Lipschitz constant.

As $\mu\to 0$, $\|x^*(t^*-r\mu)-(x^*(t^*)-\mu^{\alpha}f(t^*,x^*(t^*)))\|\to 0$, so that from (1.31) we arrive at

$${}^{C}\Delta_{+}^{\alpha}h(t^*) = {}^{C}\Delta_{+}^{\alpha}V(t^*,x^*(t^*)) \leq \Theta(t^*,V(t^*,x^*(t^*))) = \Theta(t^*,h(t^*)) \quad (1.39)$$

Now, (1.39) with $t^*=t_1^*$ contradicts (1.38); hence (1.37) is true.

For $t^*\in\mathcal{N}$, we now show that whenever $\omega_1<\omega_2$, then

$$\kappa_{\omega_1}(t^*)<\kappa_{\omega_2}(t^*). \quad (1.40)$$

Note that (1.40) holds for $t^*=t_0$ since $\kappa(t_0)+\omega_1<\kappa(t_0)+\omega_2 \implies \omega_1<\omega_2$. Assume the inequality (1.40) is not true. Then there exists a point t_1^* such that $\kappa_{\omega_1}(t_1^*)=\kappa_{\omega_2}(t_1^*)$ and $\kappa_{\omega_1}(t^*)<\kappa_{\omega_2}(t^*)$ for $t_0\leq t^*<t_1^*$ $t^*\in\mathcal{N}$.

By Lemma (1.21), we have

$${}^{C}\Delta_{+}^{\alpha}\kappa_{\omega_1}(t_1^*) > {}^{C}\Delta_{+}^{\alpha}\kappa_{\omega_2}(t_1^*).$$

However,

$$\begin{aligned} {}^{C}\Delta_{+}^{\alpha}\kappa_{\omega_1}(t_1^*) - {}^{C}\Delta_{+}^{\alpha}\kappa_{\omega_2}(t_1^*) &= \Theta(t_1^*,\kappa_{\omega_1}(t_1^*))+\omega_1 \\ &\quad -[\Theta(t_1^*,\kappa_{\omega_2}(t_1^*))+\omega_2] \\ &= \omega_1-\omega_2<0, \end{aligned}$$

which is a contradiction, and so (1.40) is true. Now, from (1.40) and since $\omega\leq B_{\mathbb{T}}$, we deduce that

$$\kappa_{\omega_1}(t^*)<\kappa_{\omega_2}(t^*)<\cdots<\kappa(t^*)+\omega_i\leq|\kappa(t^*)+B_{\mathbb{T}}|\leq M,$$

Therefore we can say that the family of solutions $\{\kappa_{\omega_i}(t^*)\}$ is uniformly bounded with bound $M>0$ on $\mathbb{T}$. This means that $\|\kappa_{\omega_i}(t^*)\|\leq M$ for $t^*\in\mathcal{N}$ and $\|\omega\|\leq B_{\mathbb{T}}$]

We now show that the family $\{\kappa_{\omega_i}(t^*)\}$ is equicontinuous on $\mathbb{T}$. Assume $\mathcal{S}=\sup\{|\Theta(t^*,x^*)|:(t^*,x^*)\in\mathcal{N}\times[-M,M]\}$. Now let us take

$\{\omega_i\}_{i=1}^{\infty}(t^*)$, as a decreasing sequence, such that $\lim_{i\to\infty} \omega_i = 0$, and consider a sequence of functions $\kappa_{\omega_i}(t^*)$ and take $t_1^*, t_2^* \in \mathcal{N}$ with $t_1^* < t_2^*$. Then we have the following estimate:

$$
\begin{aligned}
& |\kappa_{\omega_i}(t_2^*) - \kappa_{\omega_i}(t_1^*)| \\
= & \left| \kappa_0 + \omega_i + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_2^*} (t_2^* - s)^{\alpha-1} (\Theta(s, \kappa_{\omega_i}(s)) + \alpha_i) \Delta s \right. \\
& \left. -(\kappa_0 + \omega_i + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_1^*} (t_1^* - s)^{\alpha-1} (\Theta(s, \kappa_{\omega_i}(s)) + \omega_i)) \Delta s \right| \\
= & \frac{1}{\Gamma(\alpha)} \left| \int_{t_0}^{t_2^*} (t_2^* - s)^{\alpha-1} (\Theta(s, \kappa_{\omega_i}(s))) \Delta s \right. \\
& \left. - \int_{t_0}^{t_1^*} (t_1^* - s)^{\alpha-1} (\Theta(s, \kappa_{\omega_i}(s))) \Delta s \right| \\
\leq & \frac{1}{\Gamma(\alpha)} \left[\left| \int_{t_0}^{t_2^*} (t_2^* - s)^{\alpha-1} \right| \Delta s \left| (\Theta(s, \kappa_{\omega_i}(s))) \right| \right. \\
& \left. + \left| \int_{t_0}^{t_1^*} (t_1^* - s)^{\alpha-1} \Delta s \right| \left| (\Theta(s, \kappa_{\omega_i}(s))) \right| \right] \\
\leq & \frac{\mathcal{S}}{\Gamma(\alpha)} \left[\left| \int_{t_0}^{t_2^*} (t_2^* - s)^{\alpha-1} \Delta s \right| + \left| \int_{t_0}^{t_1^*} (t_1^* - s)^{\alpha-1} \Delta s \right| \right] \\
= & \frac{\mathcal{S}}{\Gamma(\alpha)} \left[\left| \int_{t_0}^{t_1^*} (t_2^* - s)^{\alpha-1} \Delta s + \int_{t_1^*}^{t_2^*} (t_2^* - s)^{\alpha-1} \Delta s \right| \right. \\
& \left. + \left| \int_{t_0}^{t_1^*} (t_1^* - s)^{\alpha-1} \Delta s \right| \right] \\
= & \frac{\mathcal{S}}{\Gamma(\alpha)} \left[\left| - \left[\frac{(t_2^* - t_1^*)^{\alpha}}{\alpha} - \frac{(t_2^* - t_0)^{\alpha}}{\alpha} \right] + \frac{(t_2^* - t_1^*)^{q}}{\alpha} \right| + \left| \frac{(t_1^* - t_0)^{\alpha}}{\alpha} \right| \right] \\
= & \frac{\mathcal{S}}{\Gamma(\alpha)} \left[\left| - \frac{(t_2^* - t_1^*)^{\alpha}}{\alpha} + \frac{(t_2^* - t_0)^{\alpha}}{\alpha} + \frac{(t_2^* - t_1^*)^{\alpha}}{\alpha} \right| + \left| \frac{(t_1^* - t_0)^{\alpha}}{\alpha} \right| \right] \\
= & \frac{\mathcal{S}}{\Gamma(\alpha)} \left[\left| \frac{(t_2^* - t_0)^{\alpha}}{\alpha} \right| + \left| \frac{(t_1^* - t_0)^{\alpha}}{\alpha} \right| \right] \\
= & \frac{\mathcal{S}}{\Gamma(\alpha+1)} \left[(t_2^* - t_0)^{\alpha} + (t_1^* - t_0)^{\alpha} \right] \\
\leq & \frac{2\mathcal{S}}{\Gamma(\alpha+1)} \left[(t_2^* - t_0)^{\alpha} \right].
\end{aligned}
$$

A family of solutions $\{\kappa_{\omega_i}(t^*)\}$ is said to be equicontinuous if given $\epsilon > 0$, we can find $\delta > 0$ such that $|\kappa_{\omega_i}(t_2^*) - \kappa_{\omega_i}(t_1^*)| < \epsilon$ whenever $|t_2^* - t_1^*| < \delta$, implying that $|\kappa_{\omega_i}(t_2^*) - \kappa_{\omega_i}(t_1^*)| \leq \frac{2\mathcal{S}}{\Gamma(\alpha+1)} \left[(t_2^* - t_0)^{\alpha} \right] < \epsilon$ provided $|t_2^* - t_1^*| < \delta$.

Now, we choose $\delta = \left(\frac{\epsilon\Gamma(\alpha+1)}{2\mathcal{S}}\right)^{\frac{1}{\alpha}}$, $\left(\frac{\epsilon\Gamma(\alpha+1)}{2\mathcal{S}}\right)^{\frac{1}{\alpha}} > \left(\frac{2\mathcal{S}(t_2^*-t_0)^\alpha}{\Gamma(\alpha+1)} \times \frac{\Gamma(\alpha+1)}{2\mathcal{S}}\right)^{\frac{1}{\alpha}} = (t_2^* - t_0)$ but $(t_2^* - t_0) > |t_2^* - t_1^*|$ so since $(t_2^* - t_0) < \delta$, then $|t_2^* - t_1^*| < \delta$. Proving that the family of solutions $\{\kappa_\omega(t^*)\}$ is equicontinuous. By the Arzela–Ascoli theorem, $\{\kappa_{\omega_i}(t^*)\}$ has a subsequence $\{\kappa_{\omega_{i_j}}(t^*)\}$, which converges uniformly to a function $z(t^*)$ on $\mathbb{T}$. We then show that $z(t^*)$ is a solution of (1.3). Equation (1.36) becomes

$$\begin{aligned}\kappa_{\omega_{i_j}}(t^*) &= \kappa_0 + \omega_{i_j} \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t^*}(t^*-s)^{\alpha-1}(\Theta(s,\kappa_{\omega_{i_j}}(s)) + \omega_{i_j})\Delta s. \end{aligned} \tag{1.41}$$

Taking the limit as $i_j \to \infty$, then $\kappa_{\omega_{i_j}}(t^*) \to z(t^*)$ on $\mathbb{T}$. Now, (1.41) yields

$$z(t^*) = \kappa_0 + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t^*}(t^*-s)^{\alpha-1}(\Theta(s,z(t^*)))\Delta s. \tag{1.42}$$

Thus, $z(t^*)$ is a solution of (1.3) on $\mathbb{T}$. Since $\lim_{j\to\infty}\kappa_{\omega_{i_j}}(t^*) = z(t)$ exists, then for any κ_{ω_i} that satisfies the dynamic equation (1.3), $\kappa_\omega(t^*) \le z(t^*)$. So from (1.37), we have that $h(t^*) < \kappa_\omega(t^*) \le z(t^*)$ on $\mathbb{T}$.

(iv) Let t be left-dense and let $\mathbf{S}(s)$ be true for all $s > t$. We want to show that $\mathbf{S}(t)$ is true. This immediately follows right-dense continuity of $V(t,x)$, $x(t)$ and the maximal solution $z(t)$ Therefore by induction principle, the statement $\mathbf{S}(\mathbf{t})$ is true for all $t \in \mathbb{T}$, and this completes the proof.

□

2. Main result

In this section, we give the stability criteria for the fractional dynamic system (1.1).

Theorem 2.1. *Assume that the following conditions are satisfied:*

(1) *The function $V(t,x(t)) \in C_{rd}[\mathbb{T}\times\mathbb{R}^N, \mathbb{R}_+^N]$, $V(t,x(t))$ is locally Lipschitzian with respect to x, $V(t,0) \equiv 0$ and the inequality*

$$\phi(\|x\|) \le V_0(t,x(t)), \tag{2.1}$$

holds for all $(t,x) \in \mathbb{T}\times\mathbb{R}^N$, $V_0(t,x) = \sum_{i=1}^N V_i(t,x(t))$ and $\phi \in \mathcal{K}$;

(2) *$\Theta \in C_{rd}[\mathbb{T}\times\mathbb{R}_+^n, \mathbb{R}_+^n]$ is quasimonotone nondecreasing with respect to the second variable at all $t \in \mathbb{T}$, $g(t,0) \equiv 0$, and*

$$^C\Delta_+^\alpha V(t,x(t)) \le \Theta(t, V(t,x(t)));$$

(3) *The zero solution of the comparison (1.3) is stable.*

Then the zero solution of the system (1.1) is stable.

Proof. Let $\epsilon > 0$ be an arbitrary small number, the stability of the zero solution. Then $\kappa = 0$ of (1.3) implies that for $\phi(\epsilon) > 0$, and $t_0 \in \mathbb{T}$. We can find a $\lambda = \lambda(t_0, \epsilon) > 0$ such that

$$\sum_{i=1}^{n} \kappa_i(t; t_0, \kappa_0) < \phi(\epsilon), \quad \text{at all } t \geq t_0, \tag{2.2}$$

whenever $\sum_{i=1}^{n} \kappa_0 < \lambda$, where $\kappa(t) = \kappa(t, t_0, \kappa_0)$ is any solution of the comparison system (1.3). Consequent of (2.2), we have

$$\sum_{i=1}^{n} z_i(t; t_0, \kappa_0) < \phi(\epsilon), \quad \text{at all } t \geq t_0, \tag{2.3}$$

where $z(t)$ is the maximal solution of (1.3). Now, $V(t, 0) = 0$ and $V \in C_{rd}$. This implies that V is continuous at the origin. Then given $\lambda > 0$, we can find $\delta = \delta(t_0, \lambda) > 0$ such that for $x_0 \in \mathbb{R}^n$. We have $\|x_0\| < \delta$ implies $V_0(t_0, x_0) < \lambda$.

Claim that $\|x_0\| < \delta$ implies $\|x(t)\| < \epsilon$ at all $t \geq t_0$ where $x(t) = x(t, t_0, x_0)$ is any solution of the system (1.1). Assuming this claim is false, then there would exists a time $t_1 \in \mathbb{T}$, $t_1 > t_0$ such that the solution $x(t)$ of the dynamic system (1.1) at the instant time t_1 leaves the $\epsilon-$neighborhood of the zero solution. That is, $\|x(t)\| < \epsilon$ at $t_0 \leq t < t_1$ and

$$\|x(t_1)\| \geq \epsilon, \tag{2.4}$$

but from Theorem 1.23, we have

$$V(t, x(t)) \leq z(t), \quad t_0 \leq t \leq t_1, \tag{2.5}$$

provided $V(t_0, x_0) \leq \kappa_0$, where $z(t)$ is maximal solution of the comparison system (1.3). Since the inequalities on vectors is component-wise, then from (2.5), we deduce that

$$V_0(t, x(t)) \leq \sum_{i=1}^{n} z_i(t; t_0, \kappa_0). \tag{2.6}$$

Combining (2.1),(2.3), (2.6), and (2.4) for $t = t_1$, we obtain

$$\phi(\|x(t_1)\|) \leq V_0(t_1, x(t_1)) \leq \sum_{i=1}^{n} z_i(t_1; t_0, \kappa_0) < \phi(\epsilon) \leq \phi(\|x(t_1))\|$$

$$\implies \phi(\|x(t_1)\|) < \phi(\|x(t_1))\|. \tag{2.7}$$

The contradiction (2.7) shows that $t_1 \notin \mathbb{T}$ and therefore $\|x(t)\| < \epsilon$ at all $t \in \mathbb{T}$ whenever $\|x_0\| < \delta$ and such the zero solution (1.1) is stable. □

Theorem 2.2. *Assume the following conditions:*

1. $V(t, x) \in C_{rd}[\mathbb{T} \times \mathbb{R}^N, \mathbb{R}^N_+]$ *and Lipschitzian in* x.
2. $\Theta \in C_{rd}[\mathbb{T} \times \mathbb{R}^n_+, \mathbb{R}^n_+]$ *and* $g(t, u)$ *is nondecreasing in* u *with* $\Theta(t, \kappa) \equiv 0$,
3. *For* $(t, x) \in \mathbb{T} \times \mathbb{R}^N$,

$$^{C}\Delta^{\alpha}_{+} V_0(t, x) \leq -c(\|x\|),$$

where $c \in \mathcal{K}$.

4. $b(\|x\|) \leq V_0(t,x) \leq a(\|x\|)$, *where* $V_0(t,x) = \sum_{i=1}^{N} V_i(t,x(t))$ *and* $a, b \in \mathcal{K}$. *Then the trivial solution* $x = 0$ *of the fractional dynamic equation* (1.1) *is asymptotically stable.*

Proof. Let $x^*(t)$ be any solution of (1.1). Then $x^* = 0$ is asymptotically stable, if it is stable and there exist numbers $\delta_0 = \delta_0(\epsilon)$ and $T = T(\epsilon)$ such that for $t \geq t_0 + T$, the inequality $\|x_0\| \leq \delta$ implies $x(t; t_0, x_0) < \epsilon$. By Theorem 2.1, $x^* = 0$ is stable .

We now show that for any $\eta > 0$, there exist $T = T(\eta) > 0$ such that for any $t_0 \in \mathbb{T}, x_0 \in \mathbb{R}^N$ with $\|x_0\| \leq \delta$, the inequality $\|x^*(t; t_0, x_0)\| < \eta$ holds for $t \geq t_0 + T$.

Let $\lambda \in (0, \delta)$ be a constant such that

$$a(\lambda) < b(\delta), \quad \text{with} \quad \|x_0\| < \lambda. \tag{2.8}$$

Combining condition (4) and (2.8) at (t_0, x_0), we have

$$b(\|x_0\|) \leq V_0(t_0, x_0) \leq a(\|x_0\|) < a(\lambda) < b(\delta). \tag{2.9}$$

We can see clearly from (2.9) that $\|x_0\| < \delta$.

Let $\eta = \eta(\epsilon), \eta \in (0, \epsilon)$.

We claim that

$$\|x_0\| < \delta \quad \Longrightarrow \quad \|x^*(t)\| < \eta, \quad \text{for } t \geq t_0 + T. \tag{2.10}$$

Assume that (2.10) is not true. Then there exist at least one $t \geq t_0 + T$ such that

$$\|x_0\| < \delta \quad \Longrightarrow \quad \|x^*(t)\| \geq \eta. \tag{2.11}$$

Since $c \in \mathcal{K}$, condition (2) of the theorem can be written as

$$^{C}\Delta_{+}^{\alpha} V_0(t, x^*) \leq -c(\eta), \tag{2.12}$$

which from Lemma 1.22, is equivalent to the delta integral equation

$$V_0(t, x^*(t)) \leq V_0(t_0, x_0) - \frac{c(\eta)}{\Gamma(\alpha)} \int_{t_0}^{t} (t - s)^{\alpha - 1} \Delta s \quad \text{for} \quad t \geq t_0 + T. \tag{2.13}$$

If $\mathbb{T} = \mathbb{R}$, then (2.13) becomes

$$V_0(t, x^*(t)) \leq V_0(t_0, x_0) - \frac{c(\eta)(t - t_0)}{\alpha\Gamma(\alpha)}, \tag{2.14}$$

and for sufficiently large t, (2.14) contradicts the condition (4) of the theorem and in-fact definition of $V(t, x(t))$, and so (2.10) holds when $\mathbb{T} = \mathbb{R}$.

If $\mathbb{T}$ consists of only isolated points, then (2.13) becomes

$$V_0(t, x^*(t)) \leq V_0(t_0, x_0) - \frac{c(\eta)}{\Gamma(\alpha)} \sum_{j=1}^{i} \mu(t_j) r(t_j), \tag{2.15}$$

for all $t_i, t_j \in \mathbb{T}$ and $r(t) = (t - s)^{\alpha - 1}$.

As $i \to \infty$, the right-hand side of (2.15) tends to $-\infty$, which is also a contradiction.

If $\mathbb{T}$ consists of t_i such that $\sigma(t) > t$, then (2.13) becomes

$$V_0(t, x^*(t)) \leq V_0(t_0, x_0) - \frac{c(\eta)}{\Gamma(\alpha)}\mu(t_i)r(t_i), \tag{2.16}$$

where $r(t) = (t-s)^{\alpha-1}$. As $i \to \infty$, the right-hand side of (2.16) approaches $-\infty$, which is another contradiction, implying that (2.10) holds across different time scales. Therefore the zero solution of (1.1) is asymptotically stable. □

3. Application

Example 3.1. Consider the Caputo fractional dynamic system

$$\begin{aligned} {}^C\Delta^\alpha x(t) &= -10x - x^2\exp(x) + 2y - y^2\exp(x) \\ {}^C\Delta^\alpha y(t) &= 2x - x^2\exp(y) - 4y - y^2\exp(y) \end{aligned} \tag{3.1}$$

for $t \geq t_0$, with initial conditions

$$x(t_0) = x_0 \quad \text{and} \quad y(t_0) = y_0,$$

where $(x, y) \in \mathbb{R}^2$, $f = (f_1, f_2)$.

Consider a vector $V = (V_1, V_2)^T$, where $V_1(t, x, y) = |x|$ and $V_2(t, x, y) = |y|$, for $t \in \mathbb{T}$ and $(x, y) \in \mathbb{R}^2$. Then condition 1 of Theorem 2.1 is satisfied, for $\phi = \frac{1}{2}r$, where $\phi \in \mathcal{K}$, so that the associated norm $\|x\| = \sqrt{x^2 + y^2}$.

Moreover,

$$V_0(t, x, y) = \sum_{i=1}^{2} V_i(t, x, y) = |x| + |y|;$$

then $\phi(\|x\|) \leq V(t, x, y)$. From (1.17), we compute the Caputo fractional Dini derivative for $V_1(t, x, y) = |x|$ as follows:

$$\begin{aligned} {}^C\Delta_+^\alpha V_1(t, x) &= \limsup_{\mu \to 0^+} \frac{1}{\mu^\alpha}\Bigg\{ V_1(\sigma(t), x(\sigma(t)) \\ &\quad + \sum_{r=1}^{[\frac{t-t_0}{\mu}]} (-1)^r ({}^\alpha C_r)[V_1(\sigma(t) - r\mu, x(\sigma(t)) - \mu^\alpha f_1(t, x(t))]\Bigg\} \\ &\quad - \frac{V_1(t_0, x_0)(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)} \\ &= \limsup_{\mu \to 0^+} \frac{1}{\mu^\alpha}\left\{ |x(\sigma(t))| + \sum_{r=1}^{[\frac{t-t_0}{\mu}]} (-1)^r ({}^\alpha C_r)[|x(\sigma(t)) - \mu^\alpha f_1(t, x)|]\right\} \\ &\quad - \frac{|x_0|(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)} \\ &\leq \limsup_{\mu \to 0^+} \frac{1}{\mu^\alpha}\left\{ |x(\sigma(t))| + \sum_{r=1}^{[\frac{t-t_0}{\mu}]} (-1)^r ({}^\alpha C_r)[|x(\sigma(t))| + |\mu^\alpha f_1(t, x)|]\right\} \end{aligned}$$

$$
\begin{aligned}
&-\frac{|x_0|(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}\\
\leq\ & \limsup_{\mu\to 0^+}\frac{1}{\mu^\alpha}\Bigg\{|x(\sigma(t))|+\sum_{r=1}^{[\frac{t-t_0}{\mu}]}(-1)^r({}^\alpha C_r)|x(\sigma(t))|\\
&+\sum_{r=1}^{[\frac{t-t_0}{\mu}]}(-1)^r({}^\alpha C_r)|\mu^\alpha f_1(t,x)|\Bigg\}-\frac{|x_0|(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}\\
\leq\ & |x(\sigma(t))|\limsup_{\mu\to 0^+}\frac{1}{\mu^\alpha}\sum_{r=0}^{[\frac{t-t_0}{\mu}]}(-1)^r({}^\alpha C_r)\\
&+|f_1(t,x)|\limsup_{\mu\to 0^+}\sum_{r=1}^{[\frac{t-t_0}{\mu}]}(-1)^r({}^\alpha C_r)-\frac{|x_0|(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}.
\end{aligned}
$$

Applying (1.15) and (1.16), we have

$$
{}^C\Delta_+^\alpha V_1(t,x)=\frac{|x(\sigma(t))|(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}-|f_1(t;x)|-\frac{|x_0|(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}
$$

$$
{}^C\Delta_+^\alpha V_1\leq\frac{|x(\sigma(t))|(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}-|f_1(t;x)|.
$$

As $t\to\infty$, $\frac{|x(\sigma(t))|(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}\to 0$, then

$$
\begin{aligned}
{}^C\Delta_+^\alpha V_1 &\leq -|f_1(t;x)|\\
&= -\left[|-10x-x^2\exp(x)+2y-y^2\exp(x)|\right]\\
&= -\left[|-10x+2y-(x^2+y^2)\exp(x)|\right]\\
&\leq -\left[|-10x+2y|\right]\\
&\leq -\left[10|x|+2|y|\right]\\
&= -10|x|-2|y|\\
&= -10V_1-2V_2,
\end{aligned}
$$

so

$$
{}^C\Delta_+^\alpha V_1\leq -8V_1+V_2. \tag{3.2}
$$

Similarly, compute the Caputo fractional Dini derivative for $V_2(t,x,y)=|y|$ as follows:

$$
\begin{aligned}
{}^C\Delta_+^\alpha V_2(t,y) &= \limsup_{\mu\to 0^+}\frac{1}{\mu^\alpha}\Bigg\{V_2(\sigma(t),y(\sigma(t))\\
&\quad+\sum_{r=1}^{[\frac{t-t_0}{\mu}]}(-1)^r({}^\alpha C_r)[V_2(\sigma(t)-r\mu,y(\sigma(t))-\mu^\alpha f_2(t,y(t))]\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
& -\frac{V_2(t_0,y_0)(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)} \\
= & \limsup_{\mu\to 0^+}\frac{1}{\mu^\alpha}\left\{|y(\sigma(t))|+\sum_{r=1}^{[\frac{t-t_0}{\mu}]}(-1)^r({}^\alpha C_r)[|y(\sigma(t))-\mu^\alpha f_2(t,y)|]\right\} \\
& -\frac{|y_0|(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)} \\
\leq & \limsup_{\mu\to 0^+}\frac{1}{\mu^\alpha}\left\{|y(\sigma(t))|+\sum_{r=1}^{[\frac{t-t_0}{\mu}]}(-1)^r({}^\alpha C_r)[|y(\sigma(t))|+|\mu^\alpha f_2(t,y)|]\right\} \\
& -\frac{|y_0|(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)} \\
\leq & \limsup_{\mu\to 0^+}\frac{1}{\mu^\alpha}\Bigg\{|y(\sigma(t))|+\sum_{r=1}^{[\frac{t-t_0}{\mu}]}(-1)^r({}^\alpha C_r)|y(\sigma(t))| \\
& +\sum_{r=1}^{[\frac{t-t_0}{\mu}]}(-1)^r({}^\alpha C_r)|\mu^\alpha f_2(t,y)|\Bigg\}-\frac{|y_0|(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)} \\
\leq & |y(\sigma(t))|\limsup_{\mu\to 0^+}\frac{1}{\mu^\alpha}\sum_{r=0}^{[\frac{t-t_0}{\mu}]}(-1)^r({}^\alpha C_r) \\
& +|f_2(t,y)|\limsup_{\mu\to 0^+}\sum_{r=1}^{[\frac{t-t_0}{\mu}]}(-1)^r({}^\alpha C_r)-\frac{|y_0|(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}.
\end{aligned}
$$

Applying (1.15) and (1.16) we have

$$
{}^C\Delta_+^\alpha V_2=\frac{|y(\sigma(t))|(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}-|f_2(t;y)|-\frac{|y_0|(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}
$$

$$
{}^C\Delta_+^\alpha V_2\leq\frac{|y(\sigma(t))|(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}-|f_2(t;y)|.
$$

As $t\to\infty$, $\frac{|y(\sigma(t))|(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}\to 0$, then

$$
\begin{aligned}
{}^C\Delta_+^\alpha V_2 &\leq -|f_2(t;y)| \\
&= -\left[|2x-x^2\exp(y)-4y-y^2\exp(y)|\right] \\
&= -\left[|2x-4y-(x^2-y^2)\exp(y)|\right] \\
&\leq -\left[|2x-4y|\right] \\
&\leq -\left[2|x|+4|y|\right] \\
&= -2|x|-4|y| \\
&\leq 2|x|-4|y|.
\end{aligned}
$$

Therefore,

$$ {}^{C}\Delta_{+}^{\alpha} V_2 \leq 2V_1 - 4V_2. \tag{3.3}$$

Combining (3.2) and (3.3), we have that

$$ {}^{C}\Delta_{+}^{\alpha} V \leq \begin{pmatrix} -8 & 1 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \Theta(t, V). \tag{3.4}$$

Now, consider the comparison system

$$ {}^{C\mathbb{T}}D_{+}^{\alpha} \kappa^{\Delta} = \Theta(t, \kappa) = A\kappa, \tag{3.5}$$

where $A = \begin{pmatrix} -8 & 1 \\ 2 & -4 \end{pmatrix}$.

The vectorial inequality (3.4) and all other conditions of Theorem 2.1 are satisfied if A has eigen values with negative real parts, since the eigen values of A are $-6 - \sqrt{6}$ and $\sqrt{6} - 6$. Then (3.1) is stable. Therefore, we conclude that the trivial solution $x_0 = 0$ of the system (3.1) is stable.

Example 3.2. Consider the Caputo fractional dynamic system

$$\begin{aligned} {}^{C}\Delta^{\alpha}\psi_1(t) &= -4\psi_1 + \psi_2 + 3\psi_3 \cos t \\ {}^{C}\Delta^{\alpha}\psi_2(t) &= 3\psi_1 - 4\psi_1 \sin t + \psi_3 \\ {}^{C}\Delta^{\alpha}\psi_3(t) &= \psi_1 \sin t - 3\psi_3 \cos t, \end{aligned} \tag{3.6}$$

for $t \geq t_0$, with initial conditions

$$\psi_1(t_0) = \psi_{10}, \quad \psi_2(t_0) = \psi_{20}, \quad \text{and} \quad \psi_3(t_0) = \psi_{30}$$

where $x = (\psi_1, \psi_2, \psi_3)$, $f = (f_1, f_2, f_3)$.

Consider a vector $V = (V_1, V_2, V_3)^T$, where $V_1(t, \psi_1, \psi_2, \psi_3) = |\psi_1| \exp^{|\psi_1|}$, $V_2(t, \psi_1, \psi_2, \psi_3) = |\psi_2| \exp^{|\psi_2|}$ and $V_3(t, \psi_1, \psi_2, \psi_3) = |\psi_3| \exp^{|\psi_3|}$, for $t \in \mathbb{T}$ and $(\psi_1, \psi_2, \psi_3) \in \mathbb{R}^3$. Then condition 1 of Theorem 2.1 is satisfied, for $\phi = \frac{1}{2}r$, where $\phi \in \mathcal{K}$, so that the associated norm $\|\psi\| = \sqrt{\psi_1^2 + \psi_2^2 + \psi_3^2}$.

Moreover,

$$V_0(\psi_1, \psi_2, \psi_3) = \sum_{i=1}^{2} V_i(\psi_1, \psi_2, \psi_3) = |\psi_1| \exp^{|\psi_1|} + |\psi_2| \exp^{|\psi_2|} + |\psi_3| \exp^{|\psi_3|},$$

then $\phi(\|\psi\|) \leq V(\psi_1, \psi_2, \psi_3)$. From (1.17), we compute the Caputo fractional Dini derivative for $V_1(\psi_1, \psi_2, \psi_3) = |\psi_1| \exp^{|\psi_1|}$ as follows:

$$\begin{aligned} {}^{C}\Delta_{+}^{\alpha} V_1(\psi) &= \limsup_{\mu \to 0^+} \frac{1}{\mu^{\alpha}} \left\{ V_1(\psi(\sigma(t)) + \sum_{r=1}^{[\frac{t-t_0}{\mu}]} (-1)^r ({}^{\alpha}C_r) \left[V_1(\psi(\sigma(t)) - \mu^{\alpha} f_1(\psi) \right] \right\} \\ &\quad - \frac{V_1(\psi_0)(t - t_0)^{-\alpha}}{\Gamma(1 - \alpha)} \\ &= \limsup_{\mu \to 0^+} \frac{1}{\mu^{\alpha}} \Big\{ |\psi_1(\sigma(t))| \exp^{|\psi_1(\sigma(t))|} \end{aligned}$$

$$
\begin{aligned}
&+\sum_{r=1}^{[\frac{t-t_0}{\mu}]}(-1)^r({}^{\alpha}C_r)\left[|\psi_1(\sigma(t))-\mu^{\alpha}f_1(\psi)|\exp^{|\psi_1(\sigma(t))-\mu^{\alpha}f_1(t,\psi_1)|}\right]\Bigg\}\\
&-\frac{|\psi_{10}|\exp^{|\psi_{10}|}(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}\\
\leq\ &\limsup_{\mu\to 0^+}\frac{1}{\mu^{\alpha}}\Bigg\{|\psi_1(\sigma(t))|\exp^{|\psi_1(\sigma(t))|}\\
&+\sum_{r=1}^{[\frac{t-t_0}{\mu}]}(-1)^r({}^{\alpha}C_r)[|\psi_1(\sigma(t))|+|\mu^{\alpha}f_1(\psi)|]\exp^{|\psi_1(\sigma(t))|+|\mu^{\alpha}f_1(t,\psi_1)|}]\Bigg\}\\
&-\frac{|\psi_{10}|(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}\\
\leq\ &\limsup_{\mu\to 0^+}\frac{1}{\mu^{\alpha}}\Bigg\{|\psi_1(\sigma(t))|\exp^{|\psi_1(\sigma(t))|}\\
&+\sum_{r=1}^{[\frac{t-t_0}{\mu}]}(-1)^r({}^{\alpha}C_r)|\psi_1(\sigma(t))|\exp^{|\psi_1(\sigma(t))|+|\mu^{\alpha}f_1(t,\psi_1)|}\\
&+\sum_{r=1}^{[\frac{t-t_0}{\mu}]}(-1)^r({}^{\alpha}C_r)|\mu^{\alpha}f_1(t,\psi)|\exp^{|\psi_1(\sigma(t))|+|\mu^{\alpha}f_1(t,\psi_1)|}\Bigg\}\\
&-\frac{|\psi_{10}|(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}\\
\leq\ &|\psi_1(\sigma(t))|\exp^{|\psi_1(\sigma(t))|}\limsup_{\mu\to 0^+}\frac{1}{\mu^{\alpha}}\sum_{r=0}^{[\frac{t-t_0}{\mu}]}(-1)^r({}^{\alpha}C_r)\\
&+|f_1(t,x)|\exp^{|\psi_1(\sigma(t))|}\limsup_{\mu\to 0^+}\sum_{r=1}^{[\frac{t-t_0}{\mu}]}(-1)^r({}^{\alpha}C_r)\\
&-\frac{|\psi_{10}|(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}.
\end{aligned}
$$

Applying (1.15) and (1.16), we have

$${}^{C}\Delta_{+}^{\alpha}V_1\leq\frac{|\psi_1(\sigma(t))|\exp^{|\psi_1(\sigma(t))|}|(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}-|f_1(t;x)|\exp^{|\psi_1(\sigma(t))|}.$$

As $t\to\infty$, $\frac{|\psi_1(\sigma(t))|\exp^{|\psi_1(\sigma(t))|}|(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}\to 0$, then

$$
\begin{aligned}
{}^{C}\Delta_{+}^{\alpha}V_1 &\leq -|f_1(t;\psi)|\exp^{|\psi_1(\sigma(t))|}\\
&\leq -\left[|-4\psi_1+\psi_2+3\psi_3\cos t|\right],
\end{aligned}
$$

so

$${}^{C}\Delta_{+}^{\alpha}V_1\leq -4V_1+V_2+3V_3. \tag{3.7}$$

Similarly, compute the Caputo fractional Dini derivative for $V_2(t,x,y) = |y|$ as follows:

$$
\begin{aligned}
{}^{C}\Delta_{+}^{\alpha}V_2(\psi) &= \limsup_{\mu\to 0^+}\frac{1}{\mu^{\alpha}}\left\{V_2(\psi(\sigma(t)) + \sum_{r=1}^{[\frac{t-t_0}{\mu}]}(-1)^r({}^{\alpha}C_r)\left[V_2(\psi(\sigma(t)) - \mu^{\alpha}f_2(\psi)\right]\right\} \\
&\quad -\frac{V_2(\psi_0)(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)} \\
&= \limsup_{\mu\to 0^+}\frac{1}{\mu^{\alpha}}\Bigg\{|\psi_2(\sigma(t))|\exp^{|\psi_2(\sigma(t))|} \\
&\quad + \sum_{r=1}^{[\frac{t-t_0}{\mu}]}(-1)^r({}^{\alpha}C_r)\left[|\psi_2(\sigma(t)) - \mu^{\alpha}f_2(\psi)|\exp^{|\psi_1(\sigma(t))-\mu^{\alpha}f_2(t,\psi_2)|}\right]\Bigg\} \\
&\quad -\frac{|\psi_{20}|\exp^{|\psi_{20}|}(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)} \\
&\leq \limsup_{\mu\to 0^+}\frac{1}{\mu^{\alpha}}\Bigg\{|\psi_2(\sigma(t))|\exp^{|\psi_2(\sigma(t))|} \\
&\quad + \sum_{r=1}^{[\frac{t-t_0}{\mu}]}(-1)^r({}^{\alpha}C_r)[|\psi_2(\sigma(t))| + |\mu^{\alpha}f_2(\psi)|]\exp^{|\psi_2(\sigma(t))|+|\mu^{\alpha}f_2(t,\psi_2)|}]\Bigg\} \\
&\quad -\frac{|\psi_{20}|(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)} \\
&\leq \limsup_{\mu\to 0^+}\frac{1}{\mu^{\alpha}}\Bigg\{|\psi_2(\sigma(t))|\exp^{|\psi_2(\sigma(t))|} \\
&\quad + \sum_{r=1}^{[\frac{t-t_0}{\mu}]}(-1)^r({}^{\alpha}C_r)|\psi_2(\sigma(t))|\exp^{|\psi_2(\sigma(t))|+|\mu^{\alpha}f_2(t,\psi_2)|} \\
&\quad + \sum_{r=1}^{[\frac{t-t_0}{\mu}]}(-1)^r({}^{\alpha}C_r)|\mu^{\alpha}f_2(t,\psi)|\exp^{|\psi_2(\sigma(t))|+|\mu^{\alpha}f_2(t,\psi_2)|}\Bigg\} \\
&\quad -\frac{|\psi_{20}|(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)} \\
&\leq |\psi_2(\sigma(t))|\exp^{|\psi_2(\sigma(t))|}\limsup_{\mu\to 0^+}\frac{1}{\mu^{\alpha}}\sum_{r=0}^{[\frac{t-t_0}{\mu}]}(-1)^r({}^{\alpha}C_r) \\
&\quad +|f_2(t,x)|\exp^{|\psi_2(\sigma(t))|}\limsup_{\mu\to 0^+}\sum_{r=1}^{[\frac{t-t_0}{\mu}]}(-1)^r({}^{\alpha}C_r) - \frac{|\psi_{20}|(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}.
\end{aligned}
$$

Applying (1.15) and (1.16) we have

$$ {}^{C}\Delta_{+}^{\alpha} V_2 \leq \frac{|\psi_2(\sigma(t))| \exp^{|\psi_2(\sigma(t))|} |(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)} - |f_2(t;x)| \exp^{|\psi_2(\sigma(t))|}. $$

As $t \to \infty$, $\frac{|\psi_2(\sigma(t))| \exp^{|\psi_2(\sigma(t))|} |(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)} \to 0$, then

$$ \begin{aligned} {}^{C}\Delta_{+}^{\alpha} V_2 &\leq -|f_2(t;x)| \exp^{|\psi_2(\sigma(t))|} \\ &\leq -\left[|3\psi_1 - 4\psi_1 \sin t + \psi_3|\right]. \end{aligned} $$

Therefore,

$$ {}^{C}\Delta_{+}^{\alpha} V_2 \leq 3V_1 - 4V_2 + V_3. \tag{3.8} $$

Similarly, compute the Caputo fractional Dini derivative for $V_3(t,x,y,z) = |z|$ as follows:

$$ \begin{aligned} {}^{C}\Delta_{+}^{\alpha} V_3(\psi) &= \limsup_{\mu \to 0^+} \frac{1}{\mu^{\alpha}} \left\{ V_3(\psi(\sigma(t)) + \sum_{r=1}^{[\frac{t-t_0}{\mu}]} (-1)^r ({}^{\alpha}C_r) \left[V_3(\psi(\sigma(t)) - \mu^{\alpha} f_3(\psi)\right] \right\} \\ &\quad - \frac{V_3(\psi_0)(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)} \\ &= \limsup_{\mu \to 0^+} \frac{1}{\mu^{\alpha}} \Bigg\{ |\psi_3(\sigma(t))| \exp^{|\psi_3(\sigma(t))|} \\ &\quad + \sum_{r=1}^{[\frac{t-t_0}{\mu}]} (-1)^r ({}^{\alpha}C_r) \left[|\psi_3(\sigma(t)) - \mu^{\alpha} f_3(\psi)| \exp^{|\psi_3(\sigma(t)) - \mu^{\alpha} f_3(t,\psi_3)|}\right] \Bigg\} \\ &\quad - \frac{|\psi_{30}| \exp^{|\psi_{30}|} (t-t_0)^{-\alpha}}{\Gamma(1-\alpha)} \\ &\leq \limsup_{\mu \to 0^+} \frac{1}{\mu^{\alpha}} \Bigg\{ |\psi_3(\sigma(t))| \exp^{|\psi_3(\sigma(t))|} \\ &\quad + \sum_{r=1}^{[\frac{t-t_0}{\mu}]} (-1)^r ({}^{\alpha}C_r) [|\psi_1(\sigma(t))| + |\mu^{\alpha} f_3(\psi)|] \exp^{|\psi_3(\sigma(t))| + |\mu^{\alpha} f_3(t,\psi_3)|}] \Bigg\} \\ &\quad - \frac{|\psi_{30}| (t-t_0)^{-\alpha}}{\Gamma(1-\alpha)} \\ &\leq \limsup_{\mu \to 0^+} \frac{1}{\mu^{\alpha}} \Bigg\{ |\psi_1(\sigma(t))| \exp^{|\psi_3(\sigma(t))|} \\ &\quad + \sum_{r=1}^{[\frac{t-t_0}{\mu}]} (-1)^r ({}^{\alpha}C_r) |\psi_3(\sigma(t))| \exp^{|\psi_3(\sigma(t))| + |\mu^{\alpha} f_1(t,\psi_3)|} \\ &\quad + \sum_{r=1}^{[\frac{t-t_0}{\mu}]} (-1)^r ({}^{\alpha}C_r) |\mu^{\alpha} f_3(t,\psi)| \exp^{|\psi_3(\sigma(t))| + |\mu^{\alpha} f_3(t,\psi_3)|} \Bigg\} \end{aligned} $$

$$
\begin{aligned}
&-\frac{|\psi_{30}|(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)} \\
\leq\ & |\psi_3(\sigma(t))|\exp^{|\psi_3(\sigma(t))|}\limsup_{\mu\to 0^+}\frac{1}{\mu^\alpha}\sum_{r=0}^{[\frac{t-t_0}{\mu}]}(-1)^r({}^\alpha C_r) \\
&+|f_3(t,x)|\exp^{|\psi_3(\sigma(t))|}\limsup_{\mu\to 0^+}\sum_{r=1}^{[\frac{t-t_0}{\mu}]}(-1)^r({}^\alpha C_r)-\frac{|\psi_{30}|(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}.
\end{aligned}
$$

Applying (1.15) and (1.16), we have

$$
{}^C\Delta_+^\alpha V_3 \leq \frac{|\psi_3(\sigma(t))|\exp^{|\psi_3(\sigma(t))|}|(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}-|f_3(t;x)|\exp^{|\psi_3(\sigma(t))|}.
$$

As $t\to\infty$, $\frac{|\psi_3(\sigma(t))|\exp^{|\psi_3(\sigma(t))|}|(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}\to 0$, then

$$
\begin{aligned}
{}^C\Delta_+^\alpha V_3 &\leq -|f_3(t;x)|\exp^{|\psi_3(\sigma(t))|} \\
&\leq -\left[|\psi_1\sin t-3\psi_3\cos t|\right],
\end{aligned}
$$

so

$$
{}^C\Delta_+^\alpha V_3 \leq V_1-0V_2-3V_3. \tag{3.9}
$$

Combining (3.7), (3.8), and (3.9), we have

$$
{}^C\Delta_+^\alpha V \leq \begin{pmatrix}-4 & 1 & 3\\ 3 & -4 & 1\\ 1 & 0 & -3\end{pmatrix}\begin{pmatrix}V_1\\V_2\\V_3\end{pmatrix}=\Theta(t,V). \tag{3.10}
$$

Now, consider the comparison system

$$
{}^C\Delta_+^\alpha \kappa^\Delta=\Theta(t,\kappa)=A\kappa, \tag{3.11}
$$

where $A=\begin{pmatrix}-4 & 1 & 3\\ 3 & -4 & 1\\ 1 & 0 & -3\end{pmatrix}$.

The vectorial inequality (3.10) and all other conditions of Theorem 2.1 are satisfied if A has eigen values with negative real parts, since the eigen values of A are $\lambda_1=-6.17741,\quad \lambda_2=-3.67836,\quad \lambda_3=-1.14423$, then (3.6) is stable. Therefore, we conclude that the trivial solution $x_0=0$ of the system (3.6) is not just stable but asymptotically stable.

4. Conclusion

In this work, we have extended the stability analysis of nonlinear Caputo fractional dynamic equations on time scales by developing a new generalized definition of the Caputo fractional derivative for vector Lyapunov functions. This new definition, termed the Caputo fractional delta Dini derivative, maintains the essential properties of fractional derivatives and allows for analysis across continuous, discrete, and hybrid domains. By extending the Lyapunov function framework from scalar to vector form, we provided a more nuanced understanding of system stability, particularly for complex, large-scale systems. The use of

vector Lyapunov functions facilitated a microscopic stability analysis, allowing us to decompose intricate systems into constituent components. This approach revealed stability properties that might be obscured when using traditional scalar Lyapunov functions, thereby offering a comprehensive framework for studying the stability of nonlinear systems. We also employed the comparison principle approach, leveraging simpler comparison systems to derive stability results for the original complex system. The applicability of the comparison principle made stability analysis much easier. Instead of dealing with complicated systems directly, we showed that stability can be determined by checking simpler matrix properties such as the quasimonotonicity (nonnegativity of the off diagonal element) and negativity of the real parts of the eigen value of the matrix. This new approach simplified the process of checking stability for both fractional differential and fractional difference equations. The results presented in this paper contribute significantly to the field of fractional dynamic systems by unifying stability theory for differential and difference equations and providing a robust method for stability analysis on arbitrary time scales.

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