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EXISTENCE AND MULTIPLICITY SOLUTIONS FOR A p(x) -LAPLACIAN-LIKE EQUATION VIA THE GENUS THEORY

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Abstract. We study the existence and multiplicity of nontrivial weak solutions for the following $p(x)$-Laplacian-like equation:

$$\begin{cases} -\triangle^{l}_{p(x)} u(x) = h(x,u(x)) + a(x)|u(x)|^{q(x)-2}u(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^N$. By using the variational method and Krasnoselskii's genus theory, we would show the existence and multiplicity of the solutions. Moreover, we would show the closedness for the set of eigenvalues in the case that $p(x) \equiv p$ is constant.

1. Introduction

In this paper, we consider the following problem:

$$\begin{cases} -\triangle^{l}_{p(x)} u(x) = h(x,u(x)) + a(x)|u(x)|^{p(x)-2}u(x), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^N$ with smooth enough boundary. Set

$$C_+(\bar{\Omega}) := \left\{ s \in C(\bar{\Omega}); s(x) > 1, \quad \text{for all} \quad x \in \bar{\Omega} \right\}.$$

Let λ be a positive real parameter and let $p \in C_+(\bar{\Omega})$ such that $1 < p(x) \le p^*(x)$, where $p^*(x) = \dfrac{Np(x)}{N-p(x)}$. Moreover,

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$$-\triangle^l_{p(x)}u := div\left(\left(1+\frac{|\nabla u|^{p(x)}}{\sqrt{1+|\nabla u|^{2p(x)}}}\right)|\nabla u|^{p(x)-2}\nabla u\right),$$

denotes $p(x)$-Laplacian-like operator. We would consider the following hypotheses:

($\mathbf{H_1}$) $h : \overline{\Omega}\times\mathbb{R}\to\mathbb{R}$ is a continuous function such that

$$C_2t^{\beta(x)-1}\le|h(x,t)|\le C_1t^{\alpha(x)-1}$$

, for all $t\ge 0$ and for all $x\in\overline{\Omega}$, where C_1, C_2 are positive constants and $\alpha,\beta\in C_+(\overline{\Omega})$ such that $1<\alpha^+<p^-\le p^+<p^*(x)$ for all $x\in\overline{\Omega}$. Moreover, $\beta^+\le q^+<p^-$.

($\mathbf{H_2}$) The map h is an odd function with respect to t, that is, $h(x,t)=-h(x,-t)$ for all $t\in\mathbb{R}$ and for $x\in\overline{\Omega}$.

($\mathbf{H_3}$) $a\in L^{(r(x)}(\Omega)$, where $q(x)r'(x)<p^*(x)$ and $\alpha^+, \frac{r'^+q^+}{r'^-}<p^-$ for all $x\in\bar{\Omega}$, where $r(x), r'(x)\in C_+(\bar{\Omega})$ such that $\frac{1}{r(x)}+\frac{1}{r'(x)}=1$.

Subject of $p(x)$-Laplacian problems is an interesting topic in recent years (see [4, 3, 5, 7, 15, 16, 22, 26]). Existence and multiplicity solutions for problems involving the $p(x)$-Laplacian-like operator under different boundary conditions, were studied by some researchers (see [1, 9, 10, 8, 24, 25]). In [1], the authors investigated the following $p(x)$-Laplacian-like problem:

$$\begin{cases}-div\left(\left(1+\frac{|\nabla u|^{p(x)}}{\sqrt{1+|\nabla u|^{2p(x)}}}\right)|\nabla u|^{p(x)-2}\nabla u\right)=\lambda f(x,u), & in\ \Omega,\\ u=0, & on\ \partial\Omega.\end{cases}\tag{1.2}$$

Using variational methods, they established the existence of one, two, and three solutions for the problem (1.2). Using the theory of variable-exponent Sobolev spaces, the authors in [10] studied the existence of a weak solution for the following Neumann problem:

$$\begin{cases}-div\left(|\nabla u|^{p(x)-2}\nabla u+\frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(x)}}}\right)=\lambda f(x,u,\nabla u), & in\ \Omega,\\ \left(|\nabla u|^{p(x)-2}\nabla u+\frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(x)}}}\right)\frac{\partial u}{\partial\eta}=0, & on\ \partial\Omega.\end{cases}\tag{1.3}$$

Vetro [24], by using variational methods and the critical point theory, considered the existence of at least one nontrivial weak solution and three weak solutions for the following Dirichlet problem:

$$\begin{cases}-\triangle^l_{p(x)}u(x)+|u(x)|^{p(x)-2}u(x)=\lambda g(x,u(x)), & in\ \Omega,\\ u=0, & on\ \partial\Omega.\end{cases}\tag{1.4}$$

Here, we would show the existence at least $\gamma(T)$ pairs ($T\subset X-\{0\}$) of different solutions for (1.1) by using variational method and Krasnoselskii's genus theory.

2. Preliminaries

First, we recall some necessary definitions and propositions concerning the Lebesgue and Sobolev spaces (for example, [5]):

For any $s \in C_+(\bar{\Omega})$, we set

$$s^- := \inf_{x\in\Omega} s(x) \quad \text{and} \quad s^+ := \sup_{x\in\Omega} s(x)$$

and

$$L^{s(x)}(\Omega) := \left\{ u : \Omega \to \mathbb{R} \text{ is a measurable function} : \int_\Omega |u(x)|^{s(x)} dx < +\infty \right\},$$

which is endowed with the following norm:

$$\|u\|_{s(x)} := \inf \left\{ \mu > 0\colon \int_\Omega \left| \frac{u(x)}{\mu} \right|^{s(x)} dx \leq 1 \right\}.$$

Then $L^{s(x)}(\Omega)$ is a separable reflexive Banach space [6, 13, 17].

The modular function on $L^{s(x)}(\Omega)$ is defined by

$$\sigma_{s(x)}(u) := \int_\Omega |u(x)|^{s(x)} dx.$$

Proposition 2.1. [11, 14]. *With the above notations, $\left(L^{s(x)}(\Omega), \|u\|_{s(x)}\right)$ is separable, uniformly convex, and reflexive and its conjugate space is $\left(L^{s'(x)}(\Omega), \|u\|_{s'(x)}\right)$, where $s, s' \in C_+(\bar{\Omega})$ and*

$$\frac{1}{s(x)} + \frac{1}{s'(x)} = 1, \qquad \textit{for all } x \in \Omega.$$

For all $u \in L^{s(x)}(\Omega)$, $w \in L^{s'(x)}(\Omega)$, we have

$$\left| \int_\Omega u\, w\, dx \right| \leq \left(\frac{1}{s^-} + \frac{1}{s'^-} \right) \|u\|_{s(x)}\, \|w\|_{s'(x)} \leq 2\|u\|_{s(x)}\, \|w\|_{s'(x)}. \tag{2.1}$$

Proposition 2.2. [12, 17] *Suppose that $u, u_n \in L^{s(x)}(\Omega)$. Then*

$\|u\|_{s(x)} > 1 \Rightarrow \|u\|_{s(x)}^{s^-} \leq \sigma_{s(x)}(u) \leq \|u\|_{s(x)}^{s^+}$;

$\|u\|_{s(x)} < 1 \Rightarrow \|u\|_{s(x)}^{s^+} \leq \sigma_{s(x)}(u) \leq \|u\|_{s(x)}^{s^-}$;

$\|u\|_{s(x)} > 1 \,(\text{respectively}, = 1; < 1) \Leftrightarrow \sigma_{s(x)}(u) > 1 \;(\text{respectively}, \; = 1; < 1)\,$;

$\|u_n\|_{s(x)} \longrightarrow 0(\text{respectively}, \longrightarrow +\infty) \Leftrightarrow \sigma_{s(x)}(u_n) \longrightarrow 0(\text{respectively}, \longrightarrow +\infty)$;

$\lim\limits_{n\to\infty} \|u_n - u\|_{s(x)} = 0 \Longleftrightarrow \lim\limits_{n\to\infty} \sigma_{s(x)}(u_n - u) = 0.$

The Sobolev space $W^{1,s(x)}(\Omega)$ is defined by

$$W^{1,s(x)}(\Omega) := \left\{ u \in L^{s(x)}(\Omega)\colon |\nabla u| \in L^{s(x)}(\Omega) \right\},$$

which is endowed with

$$\|u\|_{1,s(x)} = \|u\|_{s(x)} + \|\nabla u\|_{s(x)}.$$

$Moreover, W_0^{1,s(x)}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,s(x)}(\Omega)$ with respect to the norm $\|u\|_{1,s(x)}$. On $u \in W_0^{1,s(x)}(\Omega)$, $\|u\| = \|\nabla u\|_{s(x)}$ is an equivalent norm, following the Poincaré inequality (for more details, we refer to [6, 12, 16]).

Proposition 2.3 (Poincaré inequality (see [3, 21])). *For $s \in C_+(\bar{\Omega})$, there is a constant $c > 0$ such that*

$$\|u\|_{s(x)} \leq c\|\nabla u\|_{s(x)}, \tag{2.2}$$

for all $u \in W_0^{1,s(x)}(\Omega)$.

It is well known that

$$W_0^{1,s(x)}(\Omega) = \left\{u \in L^{s(x)}(\Omega)\colon |\nabla u| \in L^{s(x)}(\Omega), u|_{\partial\Omega} = 0\right\}.$$

For more details, we refer to [6, 12, 21].

Proposition 2.4 (see [3, 21]). *For $s \in C_+(\bar{\Omega})$, $L^{s(x)}(\Omega)), W^{1,s(x)}(\Omega)$, and $W_0^{1,s(x)}(\Omega)$ are separable and reflexive Banach spaces.*

Proposition 2.5. *(Sobolev Embedding[12, 20]) For $s_1, s_2 \in C_+(\bar{\Omega})$ and $1 < s_2(x) < s_1^*(x)$ for all $x \in \bar{\Omega}$, there is a compact embedding*

$$W_0^{1,s_1(x)}(\Omega) \hookrightarrow L_{s_2(x)}(\Omega).$$

Therefore, there is a constant $c_0 > 0$ such that

$$\|u\|_{s_2(x)} \leq c_0\|u\|.$$

Proposition 2.6. [13, 20, 18] *Assume that Ω is bounded domain in which its boundary possesses the cone property and $s_1, s_2 \in C_+(\Omega)$ such that $s_2(x) < s_1^*(x)$. Then the embedding $W^{1,s_1(x)}$ in $L^{s_2(x)}$ is compact and continuous.*

It is direct to see that $\Lambda : W_0^{1,s(x)}(\Omega) \to R$ is defined by

$$\Lambda(u) = \int_\Omega \frac{1}{s(x)}(|\nabla u|^{s(x)} + \sqrt{1 + |\nabla u|^{2s(x)}})dx.$$

Then $\Lambda \in C^1(W_0^{1,s(x)}, R)$ and its weak derivative (weak sense) $\Lambda' : W_0^{1,s(x)} \to (W^{1,s(x)})^*$ is

$$(\Lambda'(u), v) = \int_\Omega (|\nabla u|^{s(x)-2}\nabla u + \frac{|\nabla u|^{2s(x)-2}\nabla u}{\sqrt{1 + |\nabla u|^{2p(x)}}})\nabla v dx,$$

for any $u, v \in W_0^{1,s(x)}$ (see [23]).

Proposition 2.7. [11, 14, 23] *The functional $\Lambda' : W_0^{1,s(x)}(\Omega) \to \left(W_0^{1,s(x)}(\Omega)\right)^*$ is a strictly monotone, bounded homeomorphism and is of type (S_+), namely, $u_n \to u$ (weakly) as $n \to \infty$ and $\overline{\lim}_{n\to\infty} (\Lambda(u_n), u_n - u) \leq 0$ implies that $u_n \to u$ (strongly) in $W_0^{1,s(x)}(\Omega)$.*

Definition 2.8. Let X be a real Banach space. Set $R := \{B \subset X - \{0\};\ B$ is compact and symmetric$\}$. For $B \in R$ the genus of B is defined as follows:

$$\gamma(B) := \inf\{m \geq 1;\ \exists g \in C(B, \mathbb{R}^m \backslash \{0\});\ g \text{ is odd}\}$$

and $\gamma(B) = \infty$, if does not exist such a map g. Moreover, $\gamma(\emptyset) = 0$ by the definition. For more details, we refer to [2].

3. Main results

Let $H : R \times \Omega \to R$ be a function defined by $H(x;t) = \int_0^t h(x;z)dz$, for all $x \in \Omega$ and $z \in R$.

For any $u \in W_0^{1,s(x)}(\Omega)$, we define $I : W_0^{1;s(x)}(\Omega) \to R$ by $I(u) =: \int H(x;u(x))dx$. Then I is well defined, continuously Gateaux differentiable with compact derivative in which whose Gateaux derivative is given by $(I'(u);v) = \int h(x;u(x))v(x)dx$ for all $u;v \in W_0^{1,s(x)}(\Omega)$ (see [1, 3]).

Definition 3.1. For $p(x) \in C_+(\bar{\Omega})$, $u \in W_0^{1,p(x)}(\Omega)$ is called a weak solution of (1.1) if

$$\begin{aligned}\int_\Omega |\nabla u|^{p(x)-2}\nabla u\,\nabla w\,dx + \int_\Omega \frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(x)}}}\nabla w\,dx &= \int_\Omega h(x,u)w\,dx \\ &\quad + \int_\Omega a(x)|u|^{p(x)-2}u\,w\,dx\end{aligned}$$

for all $w \in W_0^{1,p(x)}(\Omega)$.

The energy functional associated with problem (1.1) is

$$\begin{aligned}\tau(u) = &\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx + \int_\Omega \frac{1}{p(x)}\left[\sqrt{1+|\nabla u|^{2p(x)}}\right]\,dx \\ &- \int_\Omega H(x,u)\,dx - \int_\Omega \frac{a(x)}{q(x)}|u|^{q(x)}\,dx,\end{aligned}$$

for all $u \in W_0^{1,p(x)}(\Omega)$. It is well defined, C^1 functional and for all $u, w \in W_0^{1,p(x)}(\Omega)$

$$\begin{aligned}\langle \tau'(u), w\rangle &= \int_\Omega |\nabla u|^{p(x)-2}\nabla u\,\nabla w\,dx + \int_\Omega \frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(x)}}}\nabla w\,dx \\ &\quad - \int_\Omega h(x,u)w\,dx - \int_\Omega a(x)|u|^{q(x)-2}.\end{aligned}$$

Definition 3.2. We recall that τ satisfies in the Palais–Smale condition (PS) if for any sequence $\{u_n\} \subset W_0^{1,p(x)}(\Omega)$ in which $(\tau(u_n))_n$ is bounded and

$$\tau'(u_n) \to 0 \text{ } as \text{ } n \to \infty,$$

$(u_n)_n$ has a convergence subsequence.

Theorem 3.3. [2]. *Let $\tau \in C^1(W_0^{1,p(x)}(\Omega), \mathbb{R})$ and satisfy the PS condition. We assume the following conditions:*

i) *τ is even and bounded from below;*
ii) *There exists $T \in R$ such that $\gamma(T) = m$ and $\sup_{x\in T}\tau(x) < \tau(0)$.*

Then problem $\tau'(u) = 0$ has at least m pairs of distinct critical points and their corresponding critical values are less than $\tau(0)$.

Theorem 3.4. *If* $(\mathbf{H_1}), (\mathbf{H_2})$, *and* $(\mathbf{H_3})$ *hold, then there are at least* m *pairs of distinct critical point for* (1.1).

Lemma 3.5. *Under assumptions* $(\mathbf{H_1}), (\mathbf{H_2})$, *and* $(\mathbf{H_3})$, τ *is coercive on* $W_0^{1,p(x)}(\Omega)$ *and bounded from below.*

Proof. To prove coercivity, for any $u \in W_0^{1,p(x)}(\Omega)$, we may assume that $\|u\| > 1$. Then

$$\begin{aligned}\tau(u) =& \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx + \int_\Omega \frac{1}{p(x)}\left[\sqrt{1+|\nabla u|^{2p(x)}}\right]\,dx\\ &- \int_\Omega H(x,u)\,dx - \int_\Omega \frac{a(x)}{q(x)}|u|^{q(x)}\,dx\\ \geq& \frac{2}{p^+}\|u\|^{p^-} - \frac{C_1S_1}{\alpha^+}\|u\|^{\alpha^+} - S_2\|a(x)\|_{r(x)}\||u|^{q(x)}\|_{r'(x)}.\end{aligned}$$

By a direct computation, one can easily deduce that $\||u|^{q(x)}\|_{r'(x)} \leq \|u\|^{\frac{r'^+q^+}{r'^-}}$, so

$$\tau(u) \geq \frac{2}{p^+}\|u\|^{p^-} - \frac{C_1S_1}{\alpha^+}\|u\|^{\alpha^+} - S_4\|u\|^{\frac{r'^+q^+}{r'^-}}.$$

Therefore, τ is coercive and bounded from below on $W_0^{1,p(x)}(\Omega)$. □

Lemma 3.6. *If* $(\mathbf{H_1})$–$(\mathbf{H_3})$ *hold, then* τ *satisfies* PS *condition.*

Proof. Let $\{u_n\} \subset W_0^{1,p(x)}(\Omega)$ be a PS sequence. Lemma 3.5 shows that $(u_n)_n$ is a bounded sequence.

Then, we prove that $\{u_n\}$ has a convergent subsequence in $W_0^{1,p(x)}(\Omega)$. It follows from Proposition 2.5 and the reflexivity of $W_0^{1,p(x)}(\Omega)$ that

$$u_n \rightharpoonup u \text{ in } W_0^{1,p(x)}(\Omega),\ u_n \to u \text{ in } L^{s(x)}(\Omega),\ u_n(x) \to u(x),\ a.e.\ in\ \Omega, \tag{3.1}$$

where $1 \leq s(x) < p^*(x)$.

By $(\mathbf{H_1})$, (2.1), and (3.1), we have

$$\begin{aligned}\left|\int_\Omega h(x,u_n)(u_n-u)\,dx\right| &\leq C_2\left|\int_\Omega |u_n|^{\alpha(x)-2}u_n(u_n-u)dx\right|\\ &\leq C_2\int_\Omega |u_n|^{\alpha(x)-1}|u_n-u|dx\\ &\leq C_4\left||u_n|^{\alpha(x)-1}\right|_{\frac{\alpha(x)}{\alpha(x)-1}}|u_n-u|_{\alpha(x)}.\end{aligned}$$

Because $\{u_n\}$ converges strongly to u in $L^{\alpha(x)}(\Omega)$, that is, $|u_n-u|_{\alpha(x)} \to 0$ as $n \to \infty$, so

$$\begin{aligned}\left|\int_\Omega h(x,u_n)(u_n-u)dx\right| &\leq C_3\int |u_n|^{\alpha(x)-1}|u_n-u|\\ &\leq C_3\,\||u_n|^{\alpha(x)-1}\|_{\frac{\alpha(x)}{\alpha(x)-1}}\|u_n-u\|_{\alpha(x)}\\ &\leq C_4\|u_n-u\|_{\alpha(x)} \to 0.\end{aligned}$$

Choosing $s(x) \in C_+(\bar{\Omega})$ in which $\frac{1}{r(x)} + \frac{1}{\frac{p(x)}{p(x)-1}} + \frac{1}{s(x)} = 1$, in fact $s(x) = \frac{r(x)p(x)}{r(x)-p(x)} < p^*(x)$, then

$$
\begin{aligned}
|\int_\Omega a(x)|u_n|^{p(x)-2}|u_n(u_n-u)| &\leq C_5\|a(x)\|_{r(x)}\||u_n|^{p(x)-1}\|_{\frac{p(x)}{p(x)-1}}\|u_n-u\|_{s(x)} \\
&\leq C_6\|u_n-u\|_{s(x)} \to 0.
\end{aligned}
$$

Indeed,

$$\langle \tau'(u_n), u_n - u\rangle \to 0,$$

and so

$$\langle \Lambda'(u_n), u_n - u\rangle \to 0,$$

where $\Lambda(u) = \int_\Omega \dfrac{1}{p(x)}(|\nabla u|^{p(x)} + \sqrt{1+|\nabla u|^{2p(x)}}dx$. From Proposition 2.7, the sequence $\{u_n\}$ converges strongly to u in $W_0^{1,p(x)}(\Omega)$. Therefore, τ satisfies the PS_c condition. □

Proof of Theorem 3.4. Set $R_m = \{B \subset R;\ \gamma(B) \geq m\}$ and
$d_m = \inf\limits_{B\in R_m} \sup\limits_{u\in B}\tau(u),\ m = 1, 2, \ldots.$
Then

$$-\infty < d_1 \leq d_2 \leq \cdots \leq d_m \leq d_{m+1} \leq \cdots .$$

We will show that $d_m < 0$ for every $m \in \mathbb{N}$. Because $W_0^{1,p(x)}$ is a separable Banach space, for any $m \in \mathbb{N}$, let X_m be an m-dimensional linear subspace of $W_0^{1,p(x)}$ such that $X_m \subset C_0^\infty(\Omega)$. We note that norms on X_m are equivalent. Assume that $0 < \rho_m < 1$ and $S^m_{\rho_m} = \{u \in X_m;\ \|u\| = \rho_m\}$. For $u \in S^m_{\rho_m}$, we have

$$\int_\Omega H(x,u)\,dx \geq C_7 \int \frac{1}{\beta(x)}|u|^{\beta(x)} \geq C_8\rho_m^{\beta^+}.$$

Since, $\sqrt{1+|t\nabla u|^{2p(x)}} \leq (1+|t\nabla u|)^{p(x)}$, so from the hypothesis $(\mathbf{H_3})$ for $u \in S^m_{r_m}$ and $t \in (0,1)$,

$$
\begin{aligned}
\tau(tu) = &\int_\Omega \frac{1}{p(x)}|t\nabla u|^{p(x)}\,dx + \int_\Omega \frac{1}{p(x)}\left[\sqrt{1+|t\nabla u|^{2p(x)}}\right]dx - \int_\Omega \frac{a(x)}{q(x)}|tu|^{q(x)}\,dx \\
&- \int_\Omega H(x,tu)\,dx \\
&\leq \frac{t^{p^-}}{p^-}\int_\Omega |\nabla u|^{p(x)}\,dx + \frac{1}{p^-}\int_\Omega \left[\sqrt{1+|t\nabla u|^{2p(x)}}\right]dx - C_8t^{q^-}\|u\|^c - C_9t^{\beta^+}\|u\|^{\beta^+} \\
&\leq C_9t^{p^-} + C_{10} - C_{11}t^{q^+} - C_{12}t^{\beta^+}
\end{aligned}
$$

for some suitable positive constants.

Nevertheless, without loss of generality, we can assume that $\beta^+ \leq q^+$. Then

$$
\begin{aligned}
\tau(tu) &= C_9t^{p^-} + C_{10} - C_{11}t^{q^+} - C_{12}t^{\beta^+} \\
&\leq t^{\beta^+}[C_9t^{p^- - \beta^+} - C_{11}t^{q^+ - \beta^+} - C_{12}] + C_{10}.
\end{aligned}
$$

Hence, there is small enough $t_m \in (0,1)$ such that $\tau(v) \leq -C_{12} + C_{10}$ for any $v = t_m u \in S^m_{t_m\rho_m}$. Then $\sup_{v\in S^m_{t_m\rho_m}} \tau(v) < C_{10} = \tau(0)$. Moreover $(\mathbf{H_2})$ implies that τ is even. On the other hand, it is well known that $\gamma(S^m_{t_m r_m}) = k$, $d_m \leq -\frac{1}{2}\lambda\rho_m < 0$, so by Theorem 3.3, τ has least m pairs of different critical points and since m is arbitrary so it has infinitely many critical points. $\square$

Corollary 3.7. *If* $(\mathbf{H_1})$–$(\mathbf{H_3})$ *hold. Then there are infinitely many solutions for problem* (1.1).

Proof. Since m is arbitrary, so there are infinitely many critical points of τ. $\square$

4. Closedness of the set of eigenfunctions

We study closedness of the set of eigenfunctions of the problem (1.1) in typical conditions. We consider the following problem:

$$\begin{cases} -div\left(\left(1+\dfrac{|\nabla u|^p}{\sqrt{1+|\nabla u|^{2p}}}\right)|\nabla u|^{p-2}\nabla u\right) = \lambda |u|^{q-2}u, & in\ \Omega,\\ u = 0, & on\ \partial\Omega, \end{cases} \tag{4.1}$$

where

$$1 < q \leq p < p^*. \tag{4.2}$$

The pair $(u,\lambda) \in W^{1,p}_0 \times \mathbb{R}^+$ is an eigenpair of (4.1) if

$$\int_\Omega |\nabla u|^{p-2}\nabla u\,\nabla w\,dx + \int_\Omega \frac{|\nabla u|^{2p-2}\nabla u}{\sqrt{1+|\nabla u|^{2p}}}\nabla w\,dx = \lambda\int_\Omega |u|^{q-2}uw\,dx, \tag{4.3}$$

for all $w \in W^{1,p}_0(\Omega)$. Let

$$\langle Au, w\rangle = \int_\Omega |u|^{q-2}u\,w\,dx, \tag{4.4}$$

and let

$$\langle Bu, w\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u\,\nabla w\,dx + \int_\Omega \frac{|\nabla u|^{2p-2}\nabla u}{\sqrt{1+|\nabla u|^{2p}}}\nabla w\,dx. \tag{4.5}$$

Then (4.3) becomes $Bu = \lambda Au$.

Theorem 4.1. *The sets of eigenvalues of the problem* (4.1) *are closed.*

Proof. Let $\{(u_n,\mu_n)\}$ be a sequence of eigenpairs of (4.3) such that $\mu_n \to \mu$ for some $\mu \geq 0$. We show that there is u such that $u_n \to u$ and (u,μ) is a eigenpair of (4.1). First, we prove that $\{u_n\}$ is bounded in $W^{1,p}_0(\Omega)$. Assume by contradiction that for large enough n, by (4.3), we have

$$\int_\Omega |\nabla u_n|^p\,dx + \int_\Omega \frac{|\nabla u_n|^{2p}}{\sqrt{1+|\nabla u_n|^{2p}}}\,dx = \mu_n\int_\Omega |u_n|^q\,dx.$$

So

$$\|u_n\|^p \leq \mu_n\|u_n\|_q.$$

By the Sobolev embedding, we have

$$\|u_n\|^p \leq \mu_n c_0 \|u_n\|^q.$$

It follows from (4.2), when we divide the last inequality by $\|u_n\|^q$ and pass to the limit as $n \to \infty$, we obtain a contradiction. Thus $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. We can assume $\|u_n\| = 1$, and thus $\{u_n\}$ has a weakly convergent subsequence. We may assume that $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$. By (4.3), (4.4), and (4.5), we have

$$\langle B(u_n), u_n - u\rangle = \mu_n \langle A(u_n), u_n - u\rangle. \tag{4.6}$$

By (2.1), we have

$$\langle A(u_n), u_n - u\rangle = |\int_\Omega |u_n|^{q-2} u_n (u_n - u)\, dx| \leq ||u_n|^{q-1}|_{\frac{q}{q-1}} |u_n - u|_q. \tag{4.7}$$

Because $\{u_n\}$ converges strongly to u in $L^q(\Omega)$, that is, $|u_n - u|_q \to 0$, as $n \to \infty$, we get $\langle A(u_n), u_n - u\rangle \to 0$, as $n \to \infty$. From (4.6), $\langle B(u_n), u_n - u\rangle \to 0$, as $n \to \infty$. Then by Proposition (2.7), the sequence $\{u_n\}$ converges strongly to u in $W_0^{1,p}(\Omega)$.

To show that μ is an eigenvalue of (4.3) and that u is an associated eigenfunction, we need to show for any $w \in W_0^{1,p}(\Omega)$ as $n \to \infty$,

$$\int_\Omega |\nabla u_n|^{p-2} \nabla u_n \nabla w \, dx \to \int_\Omega |\nabla u|^{p-2} \nabla u \nabla w \, dx, \tag{4.8}$$

$$\int_\Omega \frac{|\nabla u_n|^{2p-2} \nabla u_n}{\sqrt{1 + |\nabla u_n|^{2p}}} \nabla w \, dx \to \int_\Omega \frac{|\nabla u|^{2p-2} \nabla u}{\sqrt{1 + |\nabla u|^{2p}}} \nabla w \, dx, \tag{4.9}$$

and

$$\int_\Omega |u_n|^{q-2} u_n \, w \, dx \to \int_\Omega |u|^{q-2} u \, w \, dx. \tag{4.10}$$

Let $t_n = |\nabla u_n|^{p-2} \nabla u_n$ and let $t = |\nabla u|^{p-2} \nabla u$. Then as $u_n \to u$ in $W_0^{1,p}(\Omega)$, $t_n \to t$, a.e. in Ω and

$$\int_\Omega |t_n|^{\frac{p}{p-1}} \, dx \to \int_\Omega |t|^{\frac{p}{p-1}} \, dx.$$

It follows from [19, Lemma A.1] that $t_n \to t$ in $L^{\frac{p}{p-1}}(\Omega)$. Thus, by (2.1) we obtain (4.8). Similarly, we have (4.9) and (4.10). $\square$

5. Conclusion

Here, we proved multiplicity and infinitely of solutions for the problem (1.1) by using variational method. Our main tool in this work was genus theory. Moreover, we proved the closedness of the set of eigenfunctions for problem (4.1) such that $p(x) \equiv p$.

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