Khayyam J. Math. 11 (2025), no. 1, 164-173
DOI: 10.22034/kjm.2025.450905.3177

Khayyam Journal of Mathematics
emis.de/journals/KJM
kjm-math.org

EXISTENCE OF SOLUTION FOR A GENERAL CLASS OF STRONGLY NONLINEAR ELLIPTIC PROBLEMS WITHOUT SIGN CONDITION

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Communicated by C. Zhang

Abstract. We study existence and L^∞-regularity results for a general class of strongly nonlinear elliptic problems associated with the differential inclusion

$$\beta(u) + A(u) + H(x, u, Du) \ni f,$$

where A is a Leray–Lions operator from $W_0^{1,p}(\Omega)$ into its dual, β is a maximal monotone mapping such that $0 \in \beta(0)$, while $H(x, s, \xi)$ is a nonlinear term, which satisfies some growth condition but no sign-condition on s. The right-hand side, f, is assumed to belong to $L^\infty(\Omega)$.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N (N \geq 1)$ with sufficiently smooth boundary $\partial\Omega$. The objective of this paper is to study the existence and L^∞-regularity results for the following strongly nonlinear elliptic inclusion

$$(E, f) \quad \begin{cases} \beta(u) + A(u) + H(x, u, Du) \ni f \text{ in } D'(\Omega), \\ \\ u \in W_0^{1,p}(\Omega), \; H(x, u, Du) \in L^1(\Omega), \end{cases}$$

where A is a Leray–Lions operator from $W_0^{1,p}(\Omega)$ into its dual $W^{-1,p'}(\Omega)$ $(1 < p < \infty)$ defined as $A(u) = -div(a(x, u, Du))$, $f \in L^\infty(\Omega)$, β is a maximal monotone mapping such that $0 \in \beta(0)$, and H is a nonlinear lower order term having a

Date: Received: 03 April 2024; Revised: 15 April 2025; Accepted: 20 April 2025.
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2020 *Mathematics Subject Classification.* Primary: 35J15; Secondary: 35J20, 35J60, 35D30.
Key words and phrases. L^∞-estimate, Leray–Lions operator, maximal monotone graph, Sobolev spaces, weak solution.

growth condition of the form $|H(x,s,\xi)| \leq g(s)|\xi|^p$ with $g \in L^1(\mathbb{R})$, $g \geq 0$ and without assuming any sign-condition on s.

We point out that, in the case of sign-condition, the solvability of the problem (E,f) is very well understood with L^∞ or L^1-data (see [1] and [2]). In these previous papers, Akdim and Ouboufettal proved the existence of weak solutions in appropriate Sobolev spaces through utilizing the Yosida approximation of the graph and via the strong convergence of the truncations.

In the case where $\beta \equiv 0$, Porretta [15] has proved the existence result for the problem where the right-hand side is a bounded Radon measure on Ω; more precisely, the problem treated in [15] is of the form

$$\begin{cases} A(u) + g(u)|Du|^p = \mu & \text{in } \Omega, \\ \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Another result in this direction can be found in [10] where the problem is studied with $f \in L^m(\Omega)$. In this last work, the authors proved that there exists a bounded weak solution for $m > \frac{N}{2}$ and unbounded entropy solution for $\frac{N}{2} > m > \frac{2N}{N+2}$. A different approach (without sign condition) was used in [9], under the assumption $b(x,s,\xi) = \lambda s - |\xi|^2$ with $\lambda > 0$. We recall also that the authors in [8] used the methods of lower and upper-solutions.

For the case of sign-condition, many important works have appeared during these last decades, namely, [5, 6, 7].

For $H \equiv 0$, the work in the L^1-theory for p-Laplacian type equations is [3], where the problem

$$\begin{cases} \beta(u) - div\,(a(x,Du)) \ni f & \text{in } \Omega, \\ \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

is studied for any β maximal monotone graph such that $0 \in \beta(0)$. It is proved that, for any $f \in L^1(\Omega)$, there exists a unique, named entropy solution.

The present paper is organized into four sections. The next one is on preliminaries. We include the precise hypotheses on our problem; we define weak solution; and we also state the main result. Section 3 is devoted to the proof of theorem. In the last section, we present an example for illustrating our abstract result.

2. Assumptions and main result

Let Ω be a bounded domain in $\mathbb{R}^N (N \geq 1)$ with sufficiently smooth boundary $\partial\Omega$. Our aim is to show existence and L^∞-regularity results to the strongly nonlinear elliptic inclusion problem with Dirichlet boundary conditions

$$(E,f) \quad \begin{cases} \beta(u) + A(u) + H(x,u,Du) \ni f \text{ in } D'(\Omega), \\ \\ u \in W_0^{1,p}(\Omega), \ \ H(x,u,Du) \in L^1(\Omega), \end{cases}$$

in which A is a nonlinear operator from $W_0^{1,p}(\Omega)$ into its dual $W^{-1,p'}(\Omega)$ ($\frac{1}{p}+\frac{1}{p'}=1$) defined by $A(u)=-div(a(x,u,Du))$, where $a:\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}^N$ is a Carathéodory function satisfying the following assumptions:
Assumption (H_1):

$$a(x,s,\xi)\cdot\xi\geq\lambda|\xi|^p, \tag{2.1}$$

$$|a(x,s,\xi)|\leq\Lambda(a_0(x)+|s|^{p-1}+|\xi|^{p-1}), \tag{2.2}$$

$$(a(x,s,\xi)-a(x,s,\eta))\cdot(\xi-\eta)>0 \tag{2.3}$$

for all $\xi\neq\eta\in\mathbb{R}^N$, where $a_0(x)\in L^{p'}(\Omega)$, $a_0\geq 0$, $\Lambda>0$, and $\lambda>0$.

Moreover, $H:\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ is a Carathéodory function such that
Assumption (H_2):

$$|H(x,s,\xi)|\leq g(s)|\xi|^p, \tag{2.4}$$

where $g\in L^1(\mathbb{R})$ and $g\geq 0$.

As to the nonlinearity β in the problem (E,f), we assume that
Assumption (H_3):
$\beta:\mathbb{R}\to 2^{\mathbb{R}}$ a set valued, maximal monotone mapping such that $0\in\beta(0)$.

We consider that
Assumption (H_4):
$f\in L^\infty(\Omega)$.

Definition 2.1. A weak solution to (E,f) is a pair of functions $(u,b)\in W_0^{1,p}(\Omega)\times L^1(\Omega)$ such that $u(x)\in D(\beta(x))$, $b(x)\in\beta(u(x))$ a.e. in Ω, $H(x,u,Du)\in L^1(\Omega)$ and

$$\int_\Omega b\varphi+\int_\Omega a(x,u,Du)\cdot D\varphi+\int_\Omega H(x,u,Du)\varphi=\int_\Omega f\varphi \tag{2.5}$$

holds for all $\varphi\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$.

Our objective is to prove the following result.

Theorem 2.2. *Under the assumptions $(H_1)-(H_4)$, there exists at least one weak solution $(u,b)\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)\times L^\infty(\Omega)$ of the problem (E,f).*

3. Proof of Theorem 2.2

This section is devoted to proving Theorem 2.2. The main idea in our proofs is to consider test functions of exponential type, these resemble, in some sense, those used in the proofs of [10].

From now on, we will use the standard truncation function T_k, $k>0$, defined for all $s\in\mathbb{R}$ by $T_k(s)=\max\{-k,\min\{s,k\}\}$. We also consider $G_k(s)=s-T_k(s)$ and $\gamma(s)=\displaystyle\int_0^s\frac{g(t)}{\lambda}dt$, where g is the function appears in (2.4).

`Step 1: The approximation problem`

First, we introduce the approximate problem

$$(E_\varepsilon, f) \quad \begin{cases} \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) - div(a(x, u_\varepsilon, Du_\varepsilon)) + H_\varepsilon(x, u_\varepsilon, Du_\varepsilon) = f & \text{in } \Omega, \\ \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

where

$$H_\varepsilon(x, s, \xi) = \frac{H(x, s, \xi)}{1 + \varepsilon |H(x, s, \xi)|}$$

satisfies

$$|H_\varepsilon(x, s, \xi)| \leq |H(x, s, \xi)|, \ |H_\varepsilon(x, s, \xi)| \leq \frac{1}{\varepsilon}$$

and $\beta_\varepsilon(\cdot) : \mathbb{R} \to \mathbb{R}$ is the Yosida approximation of $\beta(\cdot)$. Note that, for any $u \in W_0^{1,p}(\Omega)$, we have

$$\langle \beta_\varepsilon(u), u \rangle \geq 0, \ |\beta_\varepsilon(u)| \leq \frac{1}{\varepsilon}|u| \ \text{ and } \ \lim_{\varepsilon \to 0} \beta_\varepsilon(u) = \beta(u).$$

We recommend that the reader consult [11], for additional information on the maximal monotone mapping.

Since $|\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))| \leq \frac{1}{\varepsilon^2}$ and $|H_\varepsilon(x, u_\varepsilon, Du_\varepsilon)| \leq \frac{1}{\varepsilon}$, for any fixed $\varepsilon > 0$, there exists at least one solution u_ε of (E_ε, f) (see [13, 14]). That is, for each $0 < \varepsilon \leq 1$ and $f \in W^{-1,p'}(\Omega)$, there exists at least one solution $u_\varepsilon \in W_0^{1,p}(\Omega)$ such that

$$\int_\Omega \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))\varphi + \int_\Omega a(x, u_\varepsilon, Du_\varepsilon) \cdot D\varphi + \int_\Omega H_\varepsilon(x, u_\varepsilon, Du_\varepsilon)\varphi = \langle f, \varphi \rangle \tag{3.1}$$

holds for all $\varphi \in W_0^{1,p}(\Omega)$, where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $W_0^{1,p}(\Omega)$ and $W^{-1,p'}(\Omega)$.

`Step 2: A priori estimates`

Let $\phi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ with $\phi \geq 0$. Taking $\varphi = \exp(\gamma(u_\varepsilon))\phi$ as a test function in (3.1), we obtain

$$\begin{aligned} \int_\Omega \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) \exp(\gamma(u_\varepsilon))\phi + \int_\Omega a(x, u_\varepsilon, Du_\varepsilon) \cdot D(\exp(\gamma(u_\varepsilon))\phi) \\ + \int_\Omega H_\varepsilon(x, u_\varepsilon, Du_\varepsilon) \exp(\gamma(u_\varepsilon))\phi = \int_\Omega f \exp(\gamma(u_\varepsilon))\phi, \end{aligned}$$

then

$$\begin{aligned} &\int_\Omega \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) \exp(\gamma(u_\varepsilon))\phi + \int_\Omega \frac{g(u_\varepsilon)}{\lambda} \exp(\gamma(u_\varepsilon))\phi \ a(x, u_\varepsilon, Du_\varepsilon) \cdot Du_\varepsilon \\ &+ \int_\Omega a(x, u_\varepsilon, Du_\varepsilon) \exp(\gamma(u_\varepsilon)) \cdot D\phi + \int_\Omega H_\varepsilon(x, u_\varepsilon, Du_\varepsilon) \exp(\gamma(u_\varepsilon))\phi \\ &= \int_\Omega f \exp(\gamma(u_\varepsilon))\phi \end{aligned}$$

by (2.1) and (2.4) we get

$$\int_\Omega \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))\exp(\gamma(u_\varepsilon))\phi + \int_\Omega \exp(\gamma(u_\varepsilon))a(x,u_\varepsilon,Du_\varepsilon)\cdot D\phi$$
$$\leq \int_\Omega f\exp(\gamma(u_\varepsilon))\phi. \tag{3.2}$$

Similarly, for each $\phi \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$, $\phi \geq 0$. Taking $\varphi = \exp(-\gamma(u_\varepsilon))\phi$ as a test function in (3.1), we obtain

$$\int_\Omega \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))\exp(-\gamma(u_\varepsilon))\phi + \int_\Omega a(x,u_\varepsilon,Du_\varepsilon)\exp(-\gamma(u_\varepsilon))\cdot D\phi$$
$$\geq \int_\Omega f\exp(-\gamma(u_\varepsilon))\phi. \tag{3.3}$$

Let Φ be a locally Lipschitz continuous and increasing real function such that $\Phi(0)=0$. Thus, taking $\phi = \Phi(u_\varepsilon^+)$ in (3.2) and $\phi = \Phi(u_\varepsilon^-)$ in (3.3), it yields

$$\int_\Omega \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))\exp(\gamma(u_\varepsilon))\Phi(u_\varepsilon^+) + \int_\Omega \exp(\gamma(u_\varepsilon))\Phi'(u_\varepsilon)a(x,u_\varepsilon,Du_\varepsilon)\cdot Du_\varepsilon^+$$
$$\leq \int_\Omega f\exp(\gamma(u_\varepsilon))\Phi(u_\varepsilon^+)$$

and

$$-\int_\Omega \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))\exp(-\gamma(u_\varepsilon))\Phi(u_\varepsilon^-) - \int_\Omega \exp(-\gamma(u_\varepsilon))\Phi'(u_\varepsilon)a(x,u_\varepsilon,Du_\varepsilon)\cdot Du_\varepsilon^-$$
$$\leq -\int_\Omega f\exp(-\gamma(u_\varepsilon))\Phi(u_\varepsilon^-).$$

Disregarding first the nonnegative terms and taking into account (2.1) along with the fact that $\exp(\pm\gamma(u_\varepsilon))$ are bounded, we add up both formulas obtaining $C>0$ such that

$$\int_\Omega \Phi'(u)|Du|^p \leq C\int_\Omega |f\Phi(u)|. \tag{3.4}$$

Taking $\Phi(u_\varepsilon) = G_k(u_\varepsilon)$ in (3.4), it yields

$$\int_\Omega |DG_k(u_\varepsilon)|^p \leq C\int_\Omega |f||G_k(u_\varepsilon)|. \tag{3.5}$$

By applying Stampacchia's L^∞-regularity procedure (see [16]), it follows that there exists $C_\infty > 0$ satisfying

$$||u_\varepsilon||_\infty \leq C_\infty. \tag{3.6}$$

Let $\Phi(u_\varepsilon) = u_\varepsilon$ in (3.4). We obtain

$$\int_\Omega |Du_\varepsilon|^p \leq C\int_\Omega |f||u_\varepsilon|. \tag{3.7}$$

By (3.6), it yields

$$||u_\varepsilon||_{W_0^{1,p}(\Omega)} \leq C. \tag{3.8}$$

`Step 3: Basic convergence results`

As a consequence of (3.6) and (3.8), we can extract a subsequence, still denoted by u_ε, such that

$$u_\varepsilon \rightharpoonup u \text{ weakly in } W_0^{1,p}(\Omega) \tag{3.9}$$

and

$$u_\varepsilon \to u \text{ strongly in } L^q(\Omega) \text{ and a.e. in } \Omega, \text{ for any } q \geq 1, \tag{3.10}$$

with

$$-C_\infty \leq u \leq C_\infty. \tag{3.11}$$

Taking into account (3.11), we get that $|\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))|$ is uniformly bounded. Consequently, there exists $b \in L^\infty(\Omega)$ such that

$$\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) \overset{*}{\rightharpoonup} b \text{ in } L^\infty(\Omega). \tag{3.12}$$

Moreover,

$$-\beta^0(-C_\infty) \leq b \leq \beta^0(C_\infty), \tag{3.13}$$

where β^0 denotes the minimal selection of the graph of β. Namely, $\beta^0(l)$ is the minimal in the norm element of $\beta(l)$, that is,

$$\beta^0(l) = inf\{|r| : r \in \mathbb{R} \text{ and } r \in \beta(l)\}.$$

Since $u_\varepsilon \to u$ strongly in $L^1(\Omega)$, applying [4, Lemma G], it follows that $b(x) \in \beta(u(x))$ a.e. on Ω.

Let us see now that

$$u_\varepsilon \to u \text{ strongly in } W_0^{1,p}(\Omega). \tag{3.14}$$

To prove this, we will prove that

$$\lim_{\varepsilon \to 0} \int_\Omega (a(x, u_\varepsilon, Du_\varepsilon) - a(x, u_\varepsilon, Du)) \cdot D(u_\varepsilon - u) = 0.$$

To begin with the proof of it, let $\delta = \exp(-sup|\gamma|)$. Then

$$\begin{aligned}
&\delta \int_\Omega (a(x, u_\varepsilon, Du_\varepsilon) - a(x, u, Du)) \cdot D(u_\varepsilon - u) \\
&\leq \int_{\{u_\varepsilon - u \geq 0\}} \exp(\gamma(u_\varepsilon)) \, (a(x, u_\varepsilon, Du_\varepsilon) - a(x, u_\varepsilon, Du)) \cdot D(u_\varepsilon - u) \\
&+ \int_{\{u_\varepsilon - u \leq 0\}} \exp(-\gamma(u_\varepsilon)) \, ((x, u_\varepsilon, Du_\varepsilon) - a(x, u_\varepsilon, Du)) \cdot D(u_\varepsilon - u) \\
&= I_\varepsilon^1 - I_\varepsilon^2 + I_\varepsilon^3 - I_\varepsilon^4,
\end{aligned}$$

where

$$\begin{aligned}
I_\varepsilon^1 &= \int_{\{u_\varepsilon - u \geq 0\}} \exp(\gamma(u_\varepsilon)) a(x, u_\varepsilon, Du_\varepsilon) \cdot D(u_\varepsilon - u) \\
I_\varepsilon^2 &= \int_{\{u_\varepsilon - u \geq 0\}} \exp(\gamma(u_\varepsilon)) a(x, u_\varepsilon, Du) \cdot D(u_\varepsilon - u) \\
I_\varepsilon^3 &= \int_{\{u_\varepsilon - u \leq 0\}} \exp(-\gamma(u_\varepsilon)) a(x, u_\varepsilon, Du_\varepsilon) \cdot D(u_\varepsilon - u) \\
I_\varepsilon^4 &= \int_{\{u_\varepsilon - u \leq 0\}} \exp(-\gamma(u_\varepsilon)) - a(x, u_\varepsilon, Du) \cdot D(u_\varepsilon - u).
\end{aligned}$$

Thus, we only have to prove that
$$\limsup_{\varepsilon\to 0} I_\varepsilon^1 - I_\varepsilon^2 + I_\varepsilon^3 - I_\varepsilon^4 \le 0.$$
Having in mind (2.2), (3.10) and that $\exp(\gamma(u_\varepsilon))$ is bounded, it follows that
$$\exp(\gamma(u_\varepsilon))a(x,u_\varepsilon,Du) \to \exp(\gamma(u))a(x,u_\varepsilon,Du) \text{ in } L^{p'}(\Omega),$$
and then (3.9) implies
$$\lim_{\varepsilon\to 0} I_\varepsilon^2 = 0.$$
In the same way,
$$\lim_{\varepsilon\to 0} I_\varepsilon^4 = 0.$$
Take $\phi = (u_\varepsilon - u)^+$ in (3.2), it yields
$$\int_\Omega \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) \exp(\gamma(u_\varepsilon))(u_\varepsilon - u)^+ + \int_\Omega \exp(\gamma(u_\varepsilon))a(x,u_\varepsilon,Du_\varepsilon)\cdot D(u_\varepsilon - u)^+$$
$$\le \int_\Omega f \exp(\gamma(u_\varepsilon))(u_\varepsilon - u)^+. \tag{3.15}$$
In view of (3.10), we have $u_\varepsilon \to u$ in $L^1(\Omega)$, and thanks to $\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) \overset{*}{\rightharpoonup} b$ in $L^\infty(\Omega)$, then
$$\lim_{\varepsilon\to 0} \int_\Omega \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) \exp(\gamma(u_\varepsilon))(u_\varepsilon - u)^+ = 0. \tag{3.16}$$
Similarly, we have f belongs to $L^\infty(\Omega)$. Then
$$\lim_{\varepsilon\to 0} \int_\Omega f \exp(\gamma(u_\varepsilon))(u_\varepsilon - u)^+ = 0. \tag{3.17}$$
Thus, from (3.15), (3.16) and (3.17), we conclude that
$$\limsup_{\varepsilon\to 0} I_\varepsilon^1 \le 0.$$
By taking $\phi = (u_\varepsilon - u)^-$ in (3.3), reasoning as above, one deduces that
$$\limsup_{\varepsilon\to 0} I_\varepsilon^3 \le 0.$$
Finally, a result in [8] (see also [12]) implies (3.14).

`Step 4: Passing to the limit`

As a consequence of (3.14), we have for a subsequence
$$Du_\varepsilon \to Du \text{ a.e. in } \Omega$$
which with
$$u_\varepsilon \to u \text{ a.e. in } \Omega$$
yields, since $a(x,u_\varepsilon,Du_\varepsilon)$ is bounded in $(L^{p'}(\Omega))^N$,
$$a(x,u_\varepsilon,Du_\varepsilon) \longrightarrow a(x,u,Du) \text{ strongly in } (L^{p'}(\Omega))^N. \tag{3.18}$$
Next, we need to prove that
$$H_\varepsilon(x,u_\varepsilon,Du_\varepsilon) \to H(x,u,Du) \text{ strongly in } L^1(\Omega). \tag{3.19}$$

First, we note that

$$H_\varepsilon(x,u_\varepsilon,Du_\varepsilon)\to H(x,u,Du) \text{ a.e in } \Omega. \tag{3.20}$$

So we need only the equi-integrability of the sequence $H_\varepsilon(x,u_\varepsilon,Du_\varepsilon)$ and Vitali's convergence theorem.

Indeed, let $\phi=\displaystyle\int_0^{u_\varepsilon} g(s)\chi_{\{s>h\}}ds$ in (3.2). We obtain

$$\begin{aligned}&\int_\Omega \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))\exp(G(u_\varepsilon))\int_0^{u_\varepsilon} g(s)\chi_{\{s>h\}}ds\\ &+\int_\Omega \exp(G(u_\varepsilon))g(u_\varepsilon)\chi_{\{u_\varepsilon>h\}}a(x,u_\varepsilon,Du_\varepsilon)\cdot Du_\varepsilon\\ &\le \int_\Omega f\exp(G(u_\varepsilon))\int_0^{u_\varepsilon} g(s)\chi_{\{s>h\}}ds,\end{aligned}$$

as the first term on the left-hand side is nonnegative. Since

$$\int_0^{u_\varepsilon} g(s)\chi_{\{s>h\}}ds\le\int_h^{+\infty} g(s)ds$$

by (2.1), we get

$$\int_{\{u_\varepsilon>h\}} g(u_\varepsilon)|Du_\varepsilon|^p\le C\int_h^{+\infty} g(s)ds$$

and then since $g\in L^1(\mathbb{R})$, we deduce that

$$\lim_{h\to\infty}\sup_{\varepsilon\in]0;1]}\int_{\{u_\varepsilon>h\}} g(u_\varepsilon)|Du_\varepsilon|^p=0.$$

Similarly, taking $\phi=\displaystyle\int_{u_\varepsilon}^0 g(s)\chi_{\{s<-h\}}ds$ in (3.3), we obtain the corresponding result on the set $\{u_\varepsilon<-h\}$. Hence

$$\lim_{h\to\infty}\sup_{\varepsilon\in]0;1]}\int_{\{|u_\varepsilon|>h\}} g(u_\varepsilon)|Du_\varepsilon|^p=0. \tag{3.21}$$

A standard argument allows to conclude from (3.14) and (3.21) that $g(u_\varepsilon)|Du_\varepsilon|^p$ strongly converges in $L^1(\Omega)$ to $g(u)|Du|^p$. Then we deduce from (2.4) that the sequence $H_\varepsilon(x,u_\varepsilon,Du_\varepsilon)$ is equi-integrable and, by Vitali's theorem, that (3.19) holds true.

Finally, from (3.12), (3.18), and (3.19), we can pass to the limit in (3.1). We obtain

$$\int_\Omega b\varphi+\int_\Omega a(x,u,Du)\cdot D\varphi+\int_\Omega H(x,u,Du)\varphi=\int_\Omega f\varphi$$

for any $\varphi\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$.

Thus, with this last step, the proof of Theorem 2.2 is concluded.

4. Example

Let Ω be a bounded domain of $\mathbb{R}^N$. Let us consider the Carathéodory functions

$$a(x,s,\xi) = |\xi|^{p-2}\xi, \quad H(x,s,\xi) = \frac{1}{1+s^2}|\xi|^p$$

and β the maximal monotone graph defined by

$$\beta(s) = (s-1)^+ - (s+1)^-.$$

It is easy to show that the Carathéodory function $a(x,s,\xi)$ satisfies the growth condition (2.2), the coercivity (2.1), and the strict monotonie condition (2.3). Also, the Carathéodory function $H(x,s,\xi)$ satisfies the condition (2.4), where $g(s) = \frac{1}{1+s^2}$ is a function belongs to $L^1(\mathbb{R})$.

Finally, the hypotheses of Theorem 2.2 are satisfied. Therefore for all $f \in L^\infty(\Omega)$, the following problem

$$(E,f) \quad \begin{cases} \beta(u) - \Delta_p(u) + H(x,u,Du) \ni f \text{ in } D'(\Omega), \\ \\ u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), \ H(x,u,Du) \in L^1(\Omega), \end{cases}$$

has at least one solution.

Acknowledgement. The authors would like to thank the anonymous reviewer for offering valuable suggestions to improve this paper.

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