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ON GENERALIZED VOLTERRA TYPE INTEGRAL OPERATORS ACTING BETWEEN FOCK–SOBOLEV SPACES

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Abstract. Recently, Mengestie and Takele characterized the bounded and compact properties of the generalized Volterra-type integral operator on weighted Fock spaces with weight functions growing faster than the Gaussian function defining the classical Fock spaces. Their result shows that the operator exhibits a richer bounded and compact structure when it acts between these spaces than the classical Fock spaces counterpart. A next question to raise is: what will happen to these properties if the weight function grows slower than the Gaussian weight function, specifically in the Fock–Sobolev spaces? So, the aim of this paper is to study the bounded and compact properties of the generalized Volterra-type integral operator on Fock–Sobolev spaces, with the goal of investigating the effects of slower growth of the weight function on these properties. Unlike the fast-growing case, our result shows that the operator has a similar bounded and compact structure as in the classical Fock spaces.

1. Introduction and preliminaries

The study of integral operators is an important area of research, as these operators are widely used in various fields of mathematics and science. One such type of integral operator is the Volterra type integral operator, which arises in many areas, such as control theory and mathematical modeling.

Recall that, for a given space $\mathcal{H}(\mathbb{C})$ of entire functions on $\mathbb{C}$, the Volterra-type integral operator on $\mathcal{H}(\mathbb{C})$, induced by an entire symbol function g, is defined as

$$V_g f(z) = \int_0^z f(w)g'(w)dw.$$

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The operator was first introduced by Pommerenke [16], in 1977, and then studied by other authors with the aim of investigating the connection between its behaviors and the function-theoretic properties of the symbol g. Pommerenke investigated the boundedness of the operator on the Hardy space H^p for $p = 2$, and almost two decades later, it was extended to the entire H^p space for $0 < p < \infty$ by Aleman and Siskakis [1], who further studied its compactness property. In [2], they described analogous characterizations in the Bergman space as well. Constantin [4] and Mengestie [12] considered the problem, along with other properties, over a space defined over the whole complex plane $\mathbb{C}$, namely, the classical Fock spaces $\mathcal{F}_p$, $0 < p \leq \infty$, which consists of entire functions f for which

$$\|f\|_p = \begin{cases} \left(\frac{p}{2\pi}\int_{\mathbb{C}} |f(z)|^p e^{-p\frac{|z|^2}{2}} dA(z)\right)^{\frac{1}{p}} < \infty, & 0 < p < \infty \\ \sup_{z\in\mathbb{C}} |f(z)| e^{-\frac{|z|^2}{2}} < \infty, & p = \infty \end{cases}$$

where dA denotes the Lebesgue area measure.

Later, in [5, 14], the authors described bounded and compact properties of V_g on weighted Fock spaces $\mathcal{F}^p_\varphi$, consisting of entire functions f for which

$$\|f\|_{\varphi,p} = \begin{cases} \left(\int_{\mathbb{C}} |f(z)|^p e^{-p\varphi(z)} dA(z)\right)^{\frac{1}{p}} < \infty, & 0 < p < \infty \\ \sup_{z\in\mathbb{C}} |f(z)| e^{-p\varphi(z)} < \infty, & p = \infty, \end{cases}$$

with weight function φ growing faster than the Gaussian weight function $\frac{|z|^2}{2}$ in the classical Fock spaces, and satisfies the following smoothness conditions:

(1) $\varphi : [0,\infty) \to [0,\infty)$ is a twice continuously differentiable function, which can be extended to the whole complex plane by setting $\varphi(z) = \varphi(|z|)$, $z \in \mathbb{C}$.

(2) $\Delta\varphi > 0$ and

$$\tau(z) :\simeq \begin{cases} (\Delta\varphi(z))^{-1/2}, & |z| \geq 1 \\ 1, & 0 \leq |z| < 1 \end{cases}$$

is a radial positive differentiable function satisfying the admissibility conditions

$$\lim_{r\to\infty} \tau(r) = 0 \quad \text{and} \quad \lim_{r\to\infty} \tau'(r) = 0,$$

and there exists a constant $C > 0$ such that $\tau(r)r^C$ increases for large r or

$$\lim_{r\to\infty} \tau'(r) \log \frac{1}{\tau(r)} = 0.$$

The notation $U(z) \simeq V(z)$ means both $U(z) \lesssim V(z)$ and $V(z) \lesssim U(z)$, where $U(z) \lesssim V(z)$ (or equivalently $V(z) \gtrsim U(z)$) means that there is a constant C such that $U(z) \leq CV(z)$ holds for all z in the set of a question.

Li and Stević [7] extended the Volterra type integral operator V_g into a more general operator called generalized Volterra type integral operator,

$$V_{(g,\psi)}f(z) = \int_0^z f(\psi(w))g'(w)dw,$$

induced by a pair of entire symbol functions (g, ψ). In particular, if $\psi(z) = z$, then $V_{(g,\psi)}$ reduces to the Volterra type integral operator V_g. Li and Stević studied some operator theoretic properties of $V_{(g,\psi)}$ in terms of the inducing pair of symbols (g, ψ) on some spaces of analytic functions on the unit disk. Boundedness and compactness of the operator on the classical Fock spaces were studied in [12, 15]. Recently, Mengestie and Takele [13] considered the problem over weighted Fock spaces $\mathcal{F}^p_\varphi$ with weight functions satisfying the above smoothness conditions. Their result, and the result in [5, 14], shows that the operators $V_{(g,\psi)}$ and V_g experience richer bounded and compact property on weighted Fock spaces $\mathcal{F}^p_\varphi$ than the classical Fock spaces. A next question to raise is: what will happen to these properties if the weight function grows slower than the Gaussian weight function, specifically, on the Fock–Sobolev spaces with weight function growing much slower than Gaussian weight in the classical Fock spaces? The aim of this paper is to characterize boundedness and compactness of generalized Volterra type integral operator, $V_{(g,\psi)}$, on Fock–Sobolev spaces, and investigate whether the operator show similar structure or not, when compared with the classical Fock spaces case.

Recall that, for $0 < p \leq \infty$ and a nonnegative integer m, Fock–Sobolev space $\mathcal{F}_{(m,p)}$, is a space of all entire functions f for which

$$\|f\|_{(m,p)} = \begin{cases} \left(\int_{\mathbb{C}} |f(z)|^p e^{-p(\frac{1}{2}|z|^2 - m\log(1+|z|))} dA(z)\right)^{\frac{1}{p}} < \infty, & 0 < p < \infty, \\ \sup_{z\in\mathbb{C}} |f(z)| e^{-p(\frac{1}{2}|z|^2 - m\log(1+|z|))} < \infty, & p = \infty. \end{cases}$$

In particular, the choice of $m = 0$ reduces the space to the classical Fock space.

1.1. **Carleson measures.** Characterization of Carleson measures in the working space is one the main tool to describe bounded and compact properties of (generalized) Volterra-type integral operators. These type of measures was initially introduced in the Hardy space setting [6]. The characterization for Fock–Sobolev space $\mathcal{F}_{(m,p)}$ for $m \neq 0$, was obtained by Mengestie [9], generalizing the characterization obtained for the case $m = 0$, obtained by Isralowitz and Zhu [8], and Mengestie [12].

Recall that, for $0 < p \leq \infty$ and $0 < q < \infty$, a nonnegative measure μ on $\mathbb{C}$ is called a (p, q) Fock–Carleson measure for Fock–Sobolev space if

$$\int_{\mathbb{C}} |f(z)|^q e^{-\frac{q}{2}|z|^2} d\mu(z) \lesssim \|f\|^q_{(m,p)}$$

for all $f \in \mathcal{F}_{(m,p)}$, and a vanishing (p, q) Fock–Carleson measure for Fock–Sobolev space if

$$\lim_{j\to\infty} \int_{\mathbb{C}} |f_j(z)|^q e^{-\frac{q}{2}|z|^2} d\mu(z) = 0,$$

whenever f_j is a uniformly bounded sequence in $\mathcal{F}_{(m,p)}$ that converges to zero on a compact subset of $\mathbb{C}$. In other words, μ is a (p,q) Fock–Carleson measure for Fock–Sobolev space if and only if the embedding map $I_\mu : \mathcal{F}_{(m,p)} \to L^q(\lambda_q)$, where $d\lambda_q(z) = e^{-\frac{q}{2}|z|^2} d\mu(z)$, is bounded, and a vanishing (p,q) Fock–Carleson measure for Fock–Sobolev space if and only if I_μ is compact.

We record the following characterization of such a measure in Fock–Sobolev spaces from [9]. For this, we define (t,s)-Berezin type integral transform, $t,s>0$, of μ by

$$\tilde{\mu}_{(t,s)}(w) = \int_{\mathbb{C}} \frac{e^{-\frac{t}{2}|z-w|^2}}{(1+|z|)^s} d\mu(z).$$

Theorem 1.1. *Let μ be a nonnegative measure. Then*

(i) *for $0 < p \le q < \infty$,*

(a) *μ is a (p,q) Fock–Carleson measure for Fock–Sobolev space if and only if $\tilde{\mu}_{(t,mq)} \in L^\infty(\mathbb{C}, dA)$ for some (or any) $t > 0$. Moreover,*

$$\|\mu\|^q \simeq \|\tilde{\mu}_{(t,mq)}\|_{L^\infty(\mathbb{C},dA)}.$$

(b) *μ is a vanishing (p,q) Fock–Carleson measure for Fock–Sobolev space if and only if $\tilde{\mu}_{(t,mq)} \to 0$ as $|z| \to \infty$ for some (or any) $t > 0$.*

(ii) *for $0 < q < p \le \infty$, the following conditions are equivalent:*

(a) *μ is a (p,q) Fock–Carleson measure for Fock–Sobolev space.*

(b) *μ is a vanishing (p,q) Fock–Carleson measure for Fock–Sobolev space.*

(c) $\tilde{\mu}_{(t,mq)} \in \begin{cases} L^{\frac{p}{p-q}}(\mathbb{C}, dA), & p < \infty \\ L^1(\mathbb{C}, dA), & p = \infty. \end{cases}$ *for some (or any) $t > 0$.*

Moreover, $\|\mu\|^q \simeq \begin{cases} \|\tilde{\mu}_{(t,mq)}\|_{L^{\frac{p}{p-q}}(\mathbb{C},dA)}, & p < \infty \\ \|\tilde{\mu}_{(t,mq)}\|_{L^1(\mathbb{C},dA)}, & p = \infty. \end{cases}$

1.2. **Littlewood–Paley type formula.** In [10, 11], the spaces $\mathcal{F}_{(m,p)}$ have been characterized in terms of Littlewood–Paley type derivative formula, and the characterization for the case $m = 0$ was given by Constantin [4] and Mengestie [12]. Such a characterization plays important role in the study of integral operators with the role of eliminating the integral emanating from the definition of the operator. This will be stated by the following lemma.

Lemma 1.2. *Let $0 < p \le \infty$ and $f \in \mathcal{F}_{(m,p)}$. Then*

$$\|f\|_{(m,p)} \simeq \begin{cases} \left(|f(0)|^p + \int_{\mathbb{C}} \frac{|f'(z)|^p (1+|z|)^{mp+p}}{(1+|z|+||z|^2+|z|-m|)^p} e^{-p\frac{1}{2}|z|^2} dA(z) \right)^{\frac{1}{p}}, & p < \infty, \\ |f(0)| + \sup_{z\in\mathbb{C}} \frac{|f'(z)|(1+|z|)^{m+1}}{1+|z|+||z|^2+|z|-m|} e^{-\frac{1}{2}|z|^2}, & p = \infty. \end{cases}$$

1.3. **Kernel function.** The pointwise estimate from [3], given by

$$|f(z)| \le \frac{\|f\|_{(m,p)}}{(1+|z|)^m} e^{\frac{1}{2}|z|^2} \tag{1.1}$$

for $0 < p \le \infty$, in particular, shows that the point evaluation functionals are bounded on $\mathcal{F}_{(m,2)}$. By the Riesz representation theorem, there exists a unique function $K_{(m,w)}$, called reproducing kernel function, in $\mathcal{F}_{(m,2)}$ such that

$$f(w) = \langle f, K_{(m,w)} \rangle = \int_{\mathbb{C}} f(z)\overline{K_{(m,w)}(z)} e^{-|z|^2+2m\log(1+|z|)} dA(z)$$

for all f in $\mathcal{F}_{(m,2)}$. Thus, $\mathcal{F}_{(m,2)}$ is a reproducing kernel Hilbert space. However, an explicit expression for $K_{(m,w)}$ for $m \neq 0$ is not known, and an explicit expression for $K_{(0,w)}$, shortly K_w, is given by $K_w(z) = e^{z\overline{w}}$, and the normalized kernel is $k_w(z) = e^{z\overline{w}-\frac{|w|^2}{2}}$. Note that the function K_w belongs to $\mathcal{F}_{(m,p)}$ for all integer $m \geq 0$ and for all $0 < p \leq \infty$, with the norm estimate $\|K_w\|_{(m,p)} \simeq |w|^m e^{\frac{|w|^2}{2}}$. Thus, it is natural to define a sequence of test functions $\eta_{(m,w)}$, to play the role of normalized kernel function, by

$$\eta_{(m,w)}(z) := \frac{k_w(z)}{(1+|w|)^m},$$

which belongs to all $\mathcal{F}_{(m,p)}$ with $\|\eta_{(m,w)}\|_{(m,p)} \lesssim \frac{|w|^m}{(1+|w|)^m} \leq 1$.

2. Main results

We define the function $\mathcal{B}^{\infty}_{(m,\psi)}(|g|)$ to be

$$\mathcal{B}^{\infty}_{(m,\psi)}(|g|)(w) := \frac{|g'(w)|(1+|w|)^{m+1}}{(1+|\psi(w)|)^m(1+|w|+||w|^2+|w|-m|)} e^{\frac{1}{2}(|\psi(w)|^2-|w|^2}$$

and the Berezin type integral transform $\mathcal{B}_{(m,\psi)}(|g|^p)$ to be

$$\mathcal{B}_{(m,\psi)}(|g|^p)(w) := \int_{\mathbb{C}} \frac{|k_w(\psi(z))|^p|g'(z)|^p(1+|z|)^{mp+p}e^{-\frac{p}{2}|z|^2}}{(1+|\psi(z)|)^{mp}(1+|z|+||z|^2+|z|-m|)^p} dA(z),$$

for simplicity, and begin the section with the following useful lemma in our consideration.

Lemma 2.1. *Let $0 < p < \infty$ and let (g,ψ) be a pair of nonconstant entire functions. Then if $\mathcal{B}^{\infty}_{(m,\psi)}(|g|)$ or $\mathcal{B}_{(m,\psi)}(|g|^p)$ belongs to $L^{\infty}(\mathbb{C}, dA)$, then $\psi(z) = az+b$ for some $a, b \in \mathbb{C}$ with $|a| \leq 1$, and $b = 0$ when $|a| = 1$.*

Proof. First, suppose that $\mathcal{B}^{\infty}_{(m,\psi)}(|g|)$ belongs to $L^{\infty}(\mathbb{C}, dA)$. Then,

$$|g'(z)| \lesssim \frac{(1+|\psi(z)|)^m(1+|z|+||z|^2+|z|-m|)}{(1+|z|)^{(m+1)}e^{\frac{1}{2}(|\psi(z)|^2-|z|^2)}}.$$

Since g is nonconstant, we must have $|\psi(z)|^2 - |z|^2 \leq 0$. Thus, $|\psi(z)| \leq |z|$ and by Liouville's theorem ψ has the form $\psi(z) = az+b$ with $|a| \leq 1$, and $b = 0$ when $|a| = 1$.

On the other hand, if $\mathcal{B}_{(m,\psi)}(|g|^p)$ belongs to $L^{\infty}(\mathbb{C}, dA)$, then

$$\begin{aligned}\mathcal{B}_{(m,\psi)}(|g|^p)(w) &= \int_{\mathbb{C}} \frac{|k_w(\psi(z))|^p|g'(z)|^p(1+|z|)^{mp+p}e^{-\frac{p}{2}|z|^2}}{(1+|\psi(z)|)^{mp}(1+|z|+||z|^2+|z|-m|)^p} dA(z) \\ &\geq \int_{D(\zeta,1)} \frac{|k_w(\psi(z))|^p|g'(z)|^p(1+|z|)^{mp+p}e^{-\frac{p}{2}|z|^2}}{(1+|\psi(z)|)^{mp}(1+|z|+||z|^2+|z|-m|)^p} dA(z)\end{aligned}$$

for all $w, \zeta \in \mathbb{C}$, where $D(\zeta, 1)$ is a disc of center ζ and radius 1.

For $z \in D(\zeta, 1)$, using the following estimates:

$$1 + |z| \simeq 1 + |\zeta|,$$

$$1 + |\psi(z)| \simeq 1 + |\psi(\zeta)|,$$

$$1 + |z| + ||z|^2 + |z| - m| \simeq 1 + |\zeta| + ||\zeta|^2 + |\zeta| - m|,$$

and subharmonicity of $|k_w(\psi)g'|$, we further estimate the above integral as

$$\begin{aligned}\int_{D(\zeta,1)} & \frac{|k_w(\psi(z))|^p |g'(z)|^p (1+|z|)^{mp+p} e^{-\frac{p}{2}|z|^2}}{(1+|\psi(z)|)^{mp}(1+|z|+||z|^2+|z|-m|)^p} dA(z) \\ & \simeq \left(\frac{(1+|\zeta|)^{mp+p}}{(1+|\psi(\zeta)|)^{mp}(1+|\zeta|+||\zeta|^2+|\zeta|-m|)^p} \right) \\ & \quad \times \int_{D(\zeta,1)} |k_w(\psi(z))|^p |g'(z)|^p e^{-\frac{p}{2}|z|^2} dA(z) \\ & \gtrsim \frac{(1+|\zeta|)^{mp+p} |k_w(\psi(\zeta))|^p |g'(\zeta)|^p e^{-\frac{p}{2}|\zeta|^2}}{(1+|\psi(\zeta)|)^{mp}(1+|\zeta|+||\zeta|^2+|\zeta|-m|)^p}.\end{aligned}$$

Setting $w = \psi(\zeta)$, we obtain

$$\mathcal{B}_{(m,\psi)}(|g|^p)(\psi(\zeta)) \gtrsim \left(\mathcal{B}^{\infty}_{(m,\psi)}(|g|)\right)^p(\zeta) \tag{2.1}$$

and the conclusion follows from boundedness of $\mathcal{B}^{\infty}_{(m,\psi)}(|g|)$. □

We may now state our first main result, which characterizes boundedness and compactness in terms of the Berezin type integral transform $\mathcal{B}_{(m,\psi)}(|g|^q)$, and the simplified result in terms of $\mathcal{B}^{\infty}_{(m,\psi)}(|g|)$ will be given later.

Theorem 2.2. *Let $0 < p, q \leq \infty$ and let (g, ψ) be a pair of entire functions on $\mathbb{C}$.*

(i) *If $0 < p \leq q < \infty$, then $V_{(g,\psi)} : \mathcal{F}_{(m,p)} \to \mathcal{F}_{(m,q)}$ is*
  (a) *bounded if and only if $\mathcal{B}_{(m,\psi)}(|g|^q) \in L^{\infty}(\mathbb{C}, dA)$. Moreover,*

$$\|V_{(g,\psi)}\| \simeq \|\mathcal{B}_{(m,\psi)}(|g|^q)\|^{\frac{1}{q}}_{L^{\infty}(\mathbb{C},dA)}.$$

  (b) *compact if and only if $\mathcal{B}_{(m,\psi)}(|g|^q)(z)$ goes to zero as $|z| \to \infty$.*

(ii) *If $0 < q < p \leq \infty$, then*
  (a) *for $p < \infty$, $V_{(g,\psi)} : \mathcal{F}_{(m,p)} \to \mathcal{F}_{(m,q)}$ is bounded if and only if it is compact if and only if $\mathcal{B}_{(m,\psi)}(|g|^q) \in L^{\frac{p}{p-q}}(\mathbb{C}, dA)$. Moreover,*

$$\|V_{(g,\psi)}\| \simeq \|\mathcal{B}_{(m,\psi)}(|g|^q)\|^{\frac{1}{q}}_{L^{\frac{p}{p-q}}(\mathbb{C},dA)}.$$

  (b) *for $p = \infty$, $V_{(g,\psi)} : \mathcal{F}_{(m,\infty)} \to \mathcal{F}_{(m,q)}$ is bounded if and only if it is compact if and only if $\mathcal{B}_{(m,\psi)}(|g|^q) \in L^1(\mathbb{C}, dA)$. Moreover,*

$$\|V_{(g,\psi)}\| \simeq \|\mathcal{B}_{(m,\psi)}(|g|^q)\|^{\frac{1}{q}}_{L^1(\mathbb{C},dA)}.$$

Proof. An application of Littlewood–Paley type estimate in Lemma 1.2 gives

$$\begin{aligned}\|V_{(g,\psi)}f\|^q_{(m,q)} &\simeq \int_{\mathbb{C}} \frac{|f(\psi(z))|^q|g'(z)|^q(1+|z|)^{mq+q}}{(1+|z|+||z|^2+|z|-m|)^q} e^{-\frac{q}{2}|z|^2}dA(z)\\ &= \int_{\mathbb{C}} |f(z)|^q d\beta_{(m,q)}(z),\end{aligned}$$

where $\beta_{(m,q)}$ is a pull back measure on $\mathbb{C}$, defined by

$$\beta_{(m,q)}(E) = \int_{\psi^{-1}(E)} \frac{|g'(z)|^q(1+|z|)^{mq+q}}{(1+|z|+||z|^2+|z|-m|)^q} e^{-\frac{q}{2}|z|^2}dA(z),$$

for every Borel subset E of $\mathbb{C}$. Let $d\lambda_{(m,q)}(z) = e^{\frac{q}{2}|z|^2}d\beta_{(m,q)}(z)$. Then

$$\|V_{(g,\psi)}f\|^q_{(m,q)} \simeq \int_{\mathbb{C}} |f(z)|^q e^{-\frac{q}{2}|z|^2} d\lambda_{(m,q)}(z). \tag{2.2}$$

(i) (a) From the above estimate we conclude that $V_{(g,\psi)} : \mathcal{F}_{(m,p)} \to \mathcal{F}_{(m,q)}$ is bounded if and only if $\lambda_{(m,q)}$ is a (p,q) Fock–Carleson measure for Fock–Sobolev space. By Theorem 1.1, this holds if and only if $\tilde{\lambda}_{(m,q)}$ belongs to $L^\infty(\mathbb{C}, dA)$. Indeed, substituting back $d\lambda_{(m,q)}$ and $d\beta_{(m,q)}$, we get

$$\begin{aligned}\tilde{\lambda}_{(m,q)}(w) &= \int_{\mathbb{C}} \frac{e^{-\frac{q}{2}|z-w|^2}}{(1+|z|)^{mq}} d\lambda_{(m,q)}(z)\\ &= \int_{\mathbb{C}} \frac{e^{\frac{q}{2}|z|^2}e^{-\frac{q}{2}|z-w|^2}}{(1+|z|)^{mq}} d\beta_{(m,q)}(z)\\ &= \int_{\mathbb{C}} \frac{|k_w(\psi(z))|^q|g'(z)|^q(1+|z|)^{mq+q}e^{-\frac{q}{2}|z|^2}}{(1+|\psi(z)|)^{mq}(1+|z|+||z|^2+|z|-m|)^q} dA(z)\\ &= \mathcal{B}_{(m,\psi)}(|g|^q)(w).\end{aligned} \tag{2.3}$$

From which the conclusion follows. Moreover, we have the norm estimate

$$\|V_{(g,\psi)}\| \simeq \|\tilde{\lambda}_{(m,q)}\|^{\frac{1}{q}}_{L^\infty(\mathbb{C},dA)} = \|\mathcal{B}_{(m,\psi)}(|g|^q)\|^{\frac{1}{q}}_{L^\infty(\mathbb{C},dA)}.$$

(b) Similarly, the operator $V_{(g,\psi)} : \mathcal{F}_{(m,p)} \to \mathcal{F}_{(m,q)}$ is compact if and only if $\lambda_{(m,q)}$ is a vanishing (p,q) Fock–Carleson measure for Fock–Sobolev space, and by Theorem 1.1, this holds if and only if $\tilde{\lambda}_{(m,q)}(z)$ goes to zero as $|z| \to \infty$. Hence, from estimate (2.3), we conclude that $V_{(g,\psi)}$ is compact if and only if $\mathcal{B}_{(m,\psi)}(|g|^q)(z)$ goes to zero as $|z| \to \infty$.

(ii) (a) From the estimate in (2.2) and by Theorem 1.1, $V_{(g,\psi)}$ is bounded if and only if it is compact if and only if $\tilde{\lambda}_{(m,q)}$ belongs to $L^{\frac{p}{p-q}}(\mathbb{C}, dA)$. Indeed, from the estimate in (2.3), we conclude that $V_{(g,\psi)}$ is bounded (or compact) if and only if $\mathcal{B}_{(m,\psi)}(|g|^q)$ belongs to $L^{\frac{p}{p-q}}(\mathbb{C}, dA)$. Moreover, we have the norm estimate

$$\|V_{(g,\psi)}\| \simeq \|\tilde{\lambda}_{(m,q)}\|^{\frac{1}{q}}_{L^{\frac{p}{p-q}}(\mathbb{C},dA)} \simeq \|\mathcal{B}_{(m,\psi)}(|g|^q)\|^{\frac{1}{q}}_{L^{\frac{p}{p-q}}(\mathbb{C},dA)}.$$

(b) By similar argument as above, $V_{(g,\psi)}$ is bounded (or compact) if and only if $\mathcal{B}_{(m,\psi)}(|g|^q)$ belongs to $L^1(\mathbb{C}, dA)$. Moreover,

$$\|V_{(g,\psi)}\| \simeq \|\mathcal{B}_{(m,\psi)}(|g|^q)\|_{L^1(\mathbb{C},dA)}^{\frac{1}{q}}.$$

□

We may now state the simplified form of the above theorem, including the case when the operator maps from $\mathcal{F}_{(m,p)}$ to $\mathcal{F}_{(m,\infty)}$, and $0 < p < \infty$.

Theorem 2.3. *Let $0 < p, q \leq \infty$ and let (g, ψ) be a pair of nonconstant entire functions.*

- (i) *If $0 < p \leq q \leq \infty$, then $V_{(g,\psi)} : \mathcal{F}_{(m,p)} \to \mathcal{F}_{(m,q)}$ is*
  - (a) *bounded if and only if $\mathcal{B}^{\infty}_{(m,\psi)}(|g|)$ belongs to $L^{\infty}(\mathbb{C}, dA)$.*
  - (b) *compact if and only if $\mathcal{B}^{\infty}_{(m,\psi)}(|g|)(z)$ goes to zero as $|z| \to \infty$.*
- (ii) *If $0 < q < p \leq \infty$, then*
  - (a) *for $p < \infty$, $V_{(g,\psi)} : \mathcal{F}_{(m,p)} \to \mathcal{F}_{(m,q)}$ is bounded if and only if it is compact if and only if $\mathcal{B}^{\infty}_{(m,\psi)}(|g|) \in L^{\frac{pq}{p-q}}(\mathbb{C}, dA)$.*
  - (b) *for $p = \infty$, $V_{(g,\psi)} : \mathcal{F}_{(m,\infty)} \to \mathcal{F}_{(m,q)}$ is bounded if and only if it is compact if and only if $\mathcal{B}^{\infty}_{(m,\psi)}(|g|) \in L^{q}(\mathbb{C}, dA)$.*

Proof. (i) (a) **Case $q < \infty$:** If $V_{(g,\psi)}$ is bounded, then by Theorem 2.2, $\mathcal{B}_{(m,\psi)}(|g|^q)$ is uniformly bounded over $\mathbb{C}$, and from the estimate in (2.1), $\mathcal{B}^{\infty}_{(m,\psi)}(|g|)$ is also bounded..

On the other hand, if $\mathcal{B}^{\infty}_{(m,\psi)}(|g|)$ is uniformly bounded over $\mathbb{C}$, then by Lemma 2.1, $\psi(z) = az + b$ with $0 < |a| \leq 1$, and hence

$$\begin{aligned}
\mathcal{B}_{(m,\psi)}(|g|^q)(w) &= \int_{\mathbb{C}} \frac{|k_w(\psi(z))|^q |g'(z)|^q (1+|z|)^{mq+q} e^{-\frac{q}{2}|z|^2}}{(1+|\psi(z)|)^{mq}(1+|z|+||z|^2+|z|-m|)^q} dA(z) \\
&\leq \left(\sup_{z\in\mathbb{C}} \left(\mathcal{B}^{\infty}_{(m,\psi)}(|g|)\right)^q(z)\right) \int_{\mathbb{C}} |k_w(\psi(z))|^q e^{-\frac{q}{2}|\psi(z)|^2} dA(z) \\
&= \left(\sup_{z\in\mathbb{C}} \left(\mathcal{B}^{\infty}_{(m,\psi)}(|g|)\right)^q(z)\right) \frac{1}{|a|^2} \int_{\mathbb{C}} |k_w(z)|^q e^{-\frac{q}{2}|z|^2} dA(z) \\
&= \frac{1}{|a|^2} \sup_{z\in\mathbb{C}} \left(\mathcal{B}^{\infty}_{(m,\psi)}(|g|)\right)^q(z).
\end{aligned}$$

From Theorem 2.2, we conclude that the operator $V_{(g,\psi)}$ is bounded.
Case $q = \infty$: If $V_{(g,\psi)}$ is bounded, then applying $V_{(g,\psi)}$ to the sequence of test functions $\eta_{(m,w)}(z) = (1+|w|)^{-m} k_w(z)$,

$$\begin{aligned}
\infty > \|V_{(g,\psi)}\| &\geq \|V_{(g,\psi)}\eta_{(m,w)}\|_{(m,\infty)} \\
&\simeq \sup_{z\in\mathbb{C}} \frac{|\eta_{(m,w)}(\psi(z))||g'(z)|(1+|z|)^{m+1}}{1+|z|+||z|^2+|z|-m|} e^{-\frac{1}{2}|z|^2} \\
&\geq \frac{|g'(z)|(1+|z|)^{m+1}}{(1+|w|)^m(1+|z|+||z|^2+|z|-m|)} e^{\frac{1}{2}(|\psi(z)|^2-|\psi(z)-w|^2-|z|^2}
\end{aligned}$$

for all w and z in $\mathbb{C}$. Setting $w = \psi(z)$ in the above estimate gives

$$\infty > \|V_{(g,\psi)}\| \gtrsim \mathcal{B}^{\infty}_{(m,\psi)}(|g|)(z),$$

and therefore $\mathcal{B}^{\infty}_{(m,\psi)}(|g|)$ belongs to $L^{\infty}(\mathbb{C}, dA)$.

On the other hand, if $\mathcal{B}^{\infty}_{(m,\psi)}(|g|)$ belongs to $L^{\infty}(\mathbb{C}, dA)$, then using Lemma 1.2 and the pointwise estimate in (1.1),

$$\begin{aligned}
\|V_{(g,\psi)}f\|_{(m,\infty)} &\simeq \sup_{z\in\mathbb{C}} \frac{|f(\psi(z))||g'(z)|(1+|z|)^{m+1}}{1+|z|+||z|^2+|z|-m|} e^{-\frac{1}{2}|z|^2} \\
&\leq \sup_{z\in\mathbb{C}} \frac{\|f\|_{(m,p)}|g'(z)|(1+|z|)^{m+1}}{(1+|\psi(z)|)^m(1+|z|+||z|^2+|z|-m|)} e^{\frac{1}{2}(|\psi(z)|^2-|z|^2)} \\
&= \|f\|_{(m,p)} \sup_{z\in\mathbb{C}} \mathcal{B}^{\infty}_{(m,\psi)}(|g|)(z).
\end{aligned}$$

This implies that

$$\|V_{(g,\psi)}\| \lesssim \sup_{z\in\mathbb{C}} \mathcal{B}^{\infty}_{(m,\psi)}(|g|)(z) = \|\mathcal{B}^{\infty}_{(m,\psi)}(|g|)\|_{L^{\infty}(\mathbb{C},dA)} < \infty,$$

and hence $V_{(g,\psi)}$ is bounded.

(b) **Case** $q < \infty$**:** If the operator $V_{(g,\psi)}$ is compact, then from Theorem 2.2, and the estimate in (2.1), it follows that $\mathcal{B}^{\infty}_{(m,\psi)}(|g|)(z)$ goes to zero as $|z| \to \infty$.

We will prove the other side. Assume $\lim_{|z|\to\infty} \mathcal{B}^{\infty}_{(m,\psi)}(|g|)(z) = 0$ and therefore $\mathcal{B}^{\infty}_{(m,\psi)}(|g|)$ is bounded over $\mathbb{C}$. By Lemma 2.1, $\psi(z) = az + b$ with $0 < |a| \leq 1$. Using this, we get

$$\begin{aligned}
\mathcal{B}_{(m,\psi)}(|g|^q)(w) &= \int_{\mathbb{C}} \frac{|k_w(\psi(z))|^q |g'(z)|^q (1+|z|)^{mq+q} e^{-\frac{q}{2}|z|^2}}{(1+|\psi(z)|)^{mq}(1+|z|+||z|^2+|z|-m|)^q} dA(z) \\
&= \int_{|z|\leq|w|} \frac{\left(\mathcal{B}^{\infty}_{(m,\psi)}(|g|)\right)^q(z)|k_w(\psi(z))|^q}{e^{\frac{q}{2}|\psi(z)|^2}} dA(z) \\
&\quad + \int_{|z|>|w|} \frac{\left(\mathcal{B}^{\infty}_{(m,\psi)}(|g|)\right)^q(z)|k_w(\psi(z))|^q}{e^{\frac{q}{2}|\psi(z)|^2}} dA(z) \\
&\lesssim \sup_{z:|z|\leq|w|} |k_w(\psi(z))|^q \int_{|z|\leq|w|} \frac{\left(\mathcal{B}^{\infty}_{(m,\psi)}(|g|)\right)^q(z)}{e^{-\frac{q}{2}|\psi(z)|^2}} dA(z) \\
&\quad + \left(\sup_{z:|z|>|w|} \left(\mathcal{B}^{\infty}_{(m,\psi)}(|g|)\right)^q(z)\right) \int_{|z|>|w|} \frac{|k_w(\psi(z))|^q}{e^{\frac{q}{2}|\psi(z)|^2}} dA(z) \\
&\leq \left(\sup_{z:|z|\leq|w|} |k_w(\psi(z))|^q\right)\left(\sup_{z\in\mathbb{C}} \left(\mathcal{B}^{\infty}_{(m,\psi)}(|g|)\right)^q(z)\right) \int_{|z|\leq|w|} e^{-\frac{q}{2}|\psi(z)|^2} dA(z) \\
&\quad + \frac{1}{|a|^2} \sup_{z:|z|>|w|} \left(\mathcal{B}^{\infty}_{(m,\psi)}(|g|)\right)^q(z) \\
&\lesssim \sup_{z:|z|\leq|w|} |k_w(\psi(z))|^q + \sup_{z:|z|>|w|} \left(\mathcal{B}^{\infty}_{(m,\psi)}(|g|)\right)^q(z).
\end{aligned}$$

Since k_w uniformly converges to zero on a compact subsets of $\mathbb{C}$ as $|w| \to \infty$, and $\sup_{z:|z|>|w|} \left(\mathcal{B}^{\infty}_{(m,\psi)}(|g|)\right)^q(z)$ goes to zero as $|w| \to \infty$, we have

$$\lim_{|w|\to\infty} \mathcal{B}_{(m,\psi)}(|g|^q)(w) = 0.$$

Thus, by Theorem 2.2, we conclude that the operator is compact.

Case $q = \infty$**:** First, suppose that $V_{(g,\psi)}$ is compact. Observe that the sequence $\eta_{(m,w)} \to 0$ as $|w| \to \infty$ uniformly on a compact subsets of $\mathbb{C}$, and hence

$$\limsup_{|w|\to\infty} \|V_{(g,\psi)}\eta_{(m,w)}\|_{(m,\infty)} = 0.$$

Now, using Lemma 1.2, and then putting $w = \psi(z)$, we have

$$\begin{aligned}
0 &= \limsup_{|w|\to\infty} \|V_{(g,\psi)}\eta_{(m,w)}\|_{(m,\infty)} \\
&\simeq \limsup_{|w|\to\infty} \left(\sup_{z\in\mathbb{C}} \frac{|\eta_{(m,w)}(\psi(z))||g'(z)|(1+|z|)^{m+1}}{1+|z|+||z|^2+|z|-m|} e^{-\frac{1}{2}|z|^2} \right) \\
&\gtrsim \limsup_{|\psi(z)|\to\infty} \left(\frac{|g'(z)|(1+|z|)^{m+1}}{1+|z|+||z|^2+|z|-m|} e^{\frac{1}{2}(|\psi(z)|^2-|z|^2)} \right) \\
&= \limsup_{|\psi(z)|\to\infty} \mathcal{B}^{\infty}_{(m,\psi)}(|g|)(z).
\end{aligned}$$

Therefore, $\lim_{|\psi(z)|\to\infty} \mathcal{B}^{\infty}_{(m,\psi)}(|g|)(z) = 0$.

On the other hand, if $\lim_{|\psi(z)|\to\infty} \mathcal{B}^{\infty}_{(m,\psi)}(|g|)(z) = 0$, then we have the following facts:

(1) By (i) above, $V_{(g,\psi)}$ is bounded, and applying $V_{(g,\psi)}$ to the constant function $f(z) = 1$, we have

$$\begin{aligned}
\infty > \|V_{(g,\psi)}\| &\gtrsim \|V_{(g,\psi)}f\|_{(m,\infty)} \\
&\simeq \sup_{z\in\mathbb{C}} \frac{|g'(z)|(1+|z|)^{m+1}}{1+|z|+||z|^2+|z|-m|} e^{-\frac{1}{2}|z|^2} \simeq \|g\|_{(m,\infty)},
\end{aligned}$$

which implies that $g \in \mathcal{F}_{(m,\infty)}$.

(2) For each $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\mathcal{B}^{\infty}_{(m,\psi)}(|g|)(z) < \epsilon$ for all $|\psi(z)| > N$.

Now, let f_j be a sequence of bounded functions in $\mathcal{F}_{(m,p)}$ converging to zero uniformly on a compact subsets of $\mathbb{C}$ as $j \to \infty$. Then applying Lemma 1.2 and the estimate in (1.1) together with the above two facts, we get

$$\begin{aligned}
&\|V_{(g,\psi)}f_j\|_{(m,\infty)} \\
&\simeq \sup_{z\in\mathbb{C}} \frac{|f_j(\psi(z))||g'(z)|(1+|z|)^{m+1}}{1+|z|+||z|^2+|z|-m|} e^{-\frac{1}{2}|z|^2} \\
&\leq \sup_{z:|\psi(z)|\leq N} \frac{|f_j(\psi(z))||g'(z)|(1+|z|)^{m+1}}{1+|z|+||z|^2+|z|-m|} e^{-\frac{1}{2}|z|^2} \\
&\quad + \sup_{z:|\psi(z)|>N} \frac{|f_j(\psi(z))||g'(z)|(1+|z|)^{m+1}}{1+|z|+||z|^2+|z|-m|} e^{-\frac{1}{2}|z|^2}
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \|g\|_{(m,\infty)} \sup_{z:|\psi(z)|\le N} |f_j(\psi(z))| \\
&\quad + \|f_j\|_{(m,p)} \sup_{z:|\psi(z)|>N} \frac{|g'(z)|(1+|z|)^{m+1}}{(1+|\psi(z)|)^m(1+|z|+||z|^2+|z|-m|)} e^{\frac{1}{2}(|\psi(z)|^2-|z|^2)} \\
&\lesssim \sup_{z:|\psi(z)|\le N} |f_j(\psi(z))| + \sup_{z:|\psi(z)|>N} \mathcal{B}^{\infty}_{(m,\psi)}(|g|)(z) \\
&< \sup_{z:|\psi(z)|\le N} |f_j(\psi(z))| + \epsilon.
\end{aligned}
$$

Letting $\epsilon \to 0$ and then $j \to \infty$, we get $\lim_{j\to\infty} \|V_{(g,\psi)} f_j\|_{(m,\infty)} = 0$, and therefore the operator $V_{(g,\psi)}$ is compact.

(ii) (a) By Theorem 2.2, it is enough to show that $\mathcal{B}_{(m,\psi)}(|g|^q) \in L^{\frac{p}{p-q}}(\mathbb{C}, dA)$ if and only if $\mathcal{B}^{\infty}_{(m,\psi)}(|g|) \in L^{\frac{pq}{p-q}}(\mathbb{C}, dA)$. First we suppose that $\mathcal{B}_{(m,\psi)}(|g|^q) \in L^{\frac{p}{p-q}}(\mathbb{C}, dA)$ and proceed to show $\mathcal{B}^{\infty}_{(m,\psi)}(|g|) \in L^{\frac{pq}{p-q}}(\mathbb{C}, dA)$. Using the estimate in (2.1) and Lemma 2.1, that is, $\psi(z) = az+b$ for some $a, b \in \mathbb{C}$ with $0 < |a| \le 1$, we get

$$
\begin{aligned}
\int_{\mathbb{C}} \left(\mathcal{B}^{\infty}_{(m,\psi)}(|g|)\right)^{\frac{pq}{p-q}}(z) dA(z) &\lesssim \int_{\mathbb{C}} \mathcal{B}^{\frac{p}{p-q}}_{(m,\psi)}(|g|^q)(\psi(z)) dA(z) \\
&= \frac{1}{|a|^2} \int_{\mathbb{C}} \mathcal{B}^{\frac{p}{p-q}}_{(m,\psi)}(|g|^q)(z) dA(z) < \infty.
\end{aligned}
$$

On the other hand, if $\mathcal{B}^{\infty}_{(m,\psi)}(|g|) \in L^{\frac{pq}{p-q}}(\mathbb{C}, dA)$, then applying Hölder's inequality, we obtain

$$
\begin{aligned}
&\mathcal{B}^{\frac{p}{p-q}}_{(m,\psi)}(|g|^q)(w) \\
&= \left(\int_{\mathbb{C}} \frac{|k_w(\psi(z))|^q |g'(z)|^q (1+|z|)^{mq+q} e^{-\frac{q|z|^2}{2}}}{(1+|\psi(z)|)^{mq}(1+|z|+||z|^2+|z|-m|)^q} dA(z) \right)^{\frac{p}{p-q}} \\
&\le \int_{\mathbb{C}} \frac{|k_w(\psi(z))|^q |g'(z)|^{\frac{pq}{p-q}} (1+|z|)^{(m+1)\frac{pq}{p-q}}}{(1+|\psi(z)|)^{\frac{mpq}{p-q}}(1+|z|+||z|^2+|z|-m|)^{\frac{pq}{p-q}}} e^{-\frac{pq|z|^2}{2(p-q)}} e^{\frac{q}{2}(\frac{p}{p-q}-1)|\psi(z)|^2} dA(z) \\
&\quad \times \left(\int_{\mathbb{C}} |k_w(\psi(z))|^q e^{-\frac{q|\psi(z)|^2}{2}} dA(z) \right)^{\frac{q}{p-q}} \\
&\lesssim \int_{\mathbb{C}} \frac{|k_w(\psi(z))|^q |g'(z)|^{\frac{pq}{p-q}} (1+|z|)^{(m+1)\frac{pq}{p-q}}}{(1+|\psi(z)|)^{\frac{mpq}{p-q}}(1+|z|+||z|^2+|z|-m|)^{\frac{pq}{p-q}}} e^{-\frac{pq|z|^2}{2(p-q)}} e^{\frac{q}{2}(\frac{p}{p-q}-1)|\psi(z)|^2} dA(z).
\end{aligned}
$$

Applying Fubini's theorem and using the fact that

$$
\int_{\mathbb{C}} |K_w(z)|^q e^{-\frac{q}{2}|z|^2} dA(z) = e^{\frac{q}{2}|w|^2},
$$

we further get

$$
\int_{\mathbb{C}} \mathcal{B}^{\frac{p}{p-q}}_{(m,\psi)}(|g|^q)(w) dA(w)
$$

$$\lesssim \int_{\mathbb{C}}\int_{\mathbb{C}} \frac{|k_w(\psi(z))|^q |g'(z)|^{\frac{pq}{p-q}} (1+|z|)^{(m+1)\frac{pq}{p-q}}}{(1+|\psi(z)|)^{\frac{mpq}{p-q}} (1+|z|+||z|^2+|z|-m|)^{\frac{pq}{p-q}}}$$

$$\times e^{-\frac{pq|z|^2}{2(p-q)}} e^{\frac{q}{2}(\frac{p}{p-q}-1)|\psi(z)|^2} dA(z)dA(w)$$

$$= \int_{\mathbb{C}} \frac{|g'(z)|^{\frac{pq}{p-q}} (1+|z|)^{(m+1)\frac{pq}{p-q}} e^{-\frac{pq|z|^2}{2(p-q)}+\frac{q}{2}(\frac{p}{p-q}-1)|\psi(z)|^2}}{(1+|\psi(z)|)^{\frac{mpq}{p-q}} (1+|z|+||z|^2+|z|-m|)^{\frac{pq}{p-q}}} \int_{\mathbb{C}} \frac{|K_{\psi(z)}(w)|^q}{e^{\frac{q|w|^2}{2}}} dA(w)dA(z)$$

$$= \int_{\mathbb{C}} \frac{|g'(z)|^{\frac{pq}{p-q}} (1+|z|)^{(m+1)\frac{pq}{p-q}} e^{-\frac{pq}{2(p-q)}(|\psi(z)|^2-|z|^2)}}{(1+|\psi(z)|)^{\frac{mpq}{p-q}} (1+|z|+||z|^2+|z|-m|)^{\frac{pq}{p-q}}} dA(z)$$

$$= \int_{\mathbb{C}} \left(\mathcal{B}^{\infty}_{(m,\psi)}(|g|)\right)^{\frac{pq}{p-q}}(z)dA(z) < \infty.$$

From which we conclude that $\mathcal{B}_{(m,\psi)}(|g|^q) \in L^{\frac{p}{p-q}}(\mathbb{C}, dA)$.

(b) By Theorem 2.2, we plan to show, $\mathcal{B}_{(m,\psi)}(|g|^q) \in L^1(\mathbb{C}, dA)$ implies $\mathcal{B}^{\infty}_{(m,\psi)}(|g|)$ belongs to $L^q(\mathbb{C}, dA)$, and $\mathcal{B}^{\infty}_{(m,\psi)}(|g|) \in L^q(\mathbb{C}, dA)$ intern implies boundedness of the operator. If $\mathcal{B}_{(m,\psi)}(|g|^q) \in L^1(\mathbb{C}, dA)$, then a similar procedure as in the above proof shows that $\mathcal{B}^{\infty}_{(m,\psi)}(|g|)$ belongs to $L^q(\mathbb{C}, dA)$.

On the other hand, if $\mathcal{B}^{\infty}_{(m,\psi)}(|g|)$ belongs to $L^q(\mathbb{C}, dA)$, then applying Lemmas 1.2 and 2.1,

$$\|V_{(g,\psi)}f\|_{(m,q)}$$

$$\simeq \int_{\mathbb{C}} \frac{|f(\psi(z))|^q |g'(z)|^q (1+|z|)^{mq+q}}{(1+|z|+||z|^2+|z|-m|)^q} e^{-\frac{q}{2}|z|^2} dA(z)$$

$$\le \sup_{z\in\mathbb{C}} \left(|f(\psi(z))|^q e^{-\frac{q}{2}|\psi(z)|^2}\right) \int_{\mathbb{C}} \frac{|g'(z)|^q (1+|z|)^{mq+q}}{(1+|z|+||z|^2+|z|-m|)^q} e^{\frac{q}{2}(|\psi(z)|^2-|z|^2)} dA(z)$$

$$= \|f\|_{(m,\infty)} \int_{\mathbb{C}} \left(\mathcal{B}^{\infty}_{(m,\psi)}(|g|)\right)^q(z)dA(z),$$

which implies that

$$\|V_{(g,\psi)}\| \lesssim \int_{\mathbb{C}} \left(\mathcal{B}^{\infty}_{(m,\psi)}(|g|)\right)^q(z)dA(z) < \infty.$$

Therefore, $V_{(g,\psi)}$ is bounded. □

In particular, for $\psi(z) = z$, that is, for the Volterra type integral operator V_g, we have the following corollary of the above theorem. We remark that, this particular results were obtained by Mengestie [10, 11].

Corollary 2.4. *Let $0 < p, q \le \infty$ and let g be an entire function on $\mathbb{C}$.*

(i) *If $0 < p \le q \le \infty$, then $V_g : \mathcal{F}_{(m,p)} \to \mathcal{F}_{(m,q)}$ is*
   (a) *bounded if and only if $g(z) = az^2 + bz + c$, for some $a, b, c \in \mathbb{C}$.*
   (b) *compact if and only if $g(z) = az + b$, for some $a, b \in \mathbb{C}$.*

(ii) *If $0 < q < p \le \infty$, then*

(a) *for $p < \infty$, $V_g : \mathcal{F}_{(m,p)} \to \mathcal{F}_{(m,q)}$ is bounded if and only if it is compact if and only if $g(z) = az + b$ whenever $\frac{q}{2} > \frac{p-q}{p}$, and g is constant otherwise.*

(b) *for $p = \infty$, $V_g : \mathcal{F}_{(m,\infty)} \to \mathcal{F}_{(m,q)}$ is bounded if and only if it is compact if and only if $g(z) = az + b$ whenever $q > 2$, and g is constant otherwise.*

For $m = 0$, Theorem 2.3 gives a characterization of bounded and compact generalized Volterra type integral operators on the classical Fock spaces, obtained in [15]. We state it by the following theorem. We shortly write $M_{(g,\psi)}$ for the function $\mathcal{B}^{\infty}_{(0,\psi)}(|g|)$.

Theorem 2.5. *Let $0 < p, q \le \infty$ and let (g, ψ) be a pair of nonconstant entire functions.*

(i) *If $0 < p \le q \le \infty$, then $V_{(g,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$ is*
  (a) *bounded if and only if $M_{(g,\psi)}$ belongs to $L^{\infty}(\mathbb{C}, dA)$.*
  (b) *compact if and only if $M_{(g,\psi)}(z)$ goes to zero as $|z| \to \infty$.*

(ii) *If $0 < q < p \le \infty$, then*
  (a) *for $p < \infty$, $V_{(g,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$ is bounded if and only if it is compact if and only if $M_{(g,\psi)} \in L^{\frac{pq}{p-q}}(\mathbb{C}, dA)$.*
  (b) *for $p = \infty$, $V_{(g,\psi)} : \mathcal{F}_\infty \to \mathcal{F}_q$ is bounded if and only if it is compact if and only if $M_{(g,\psi)} \in L^{q}(\mathbb{C}, dA)$.*

Comparing Theorem 2.3 for $m \neq 0$, with Theorem 2.5, we conclude that the operator has similar bounded and compact structure as in the classical Fock spaces case. In fact we have the following proposition.

Proposition 2.6. *Let $0 < p, q \le \infty$, let m be a positive integer, and let (g, ψ) be a pair of nonconstant entire functions. Then $V_{(g,\psi)} : \mathcal{F}_{(m,p)} \to \mathcal{F}_{(m,q)}$ is bounded (respectively, compact) if and only if $V_{(g,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$ is bounded (respectively, compact).*

Proof. Suppose $V_{(g,\psi)} : \mathcal{F}_{(m,p)} \to \mathcal{F}_{(m,q)}$ is bounded. Then $\mathcal{B}^{\infty}_{(m,\psi)}(|g|)$ is bounded, and hence $1 + |\psi(z)| \le 1 + |z|$. Using this and the inequality,

$$1 + |z| + ||z|^2 + |z| - m| \le (1 + |z|)^2,$$

we get

$$M_{(g,\psi)}(z) \le \mathcal{B}^{\infty}_{(m,\psi)}(|g|)(z).$$

From this and Theorem 2.3, and Theorem 2.5, we conclude that $V_{(g,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$ is also bounded.

On the other hand, if $V_{(g,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$ is also bounded, then $M_{(g,\psi)}$ is bounded, and by Lemma 2.1, $\psi(z) = az + b$ for some $a, b \in \mathbb{C}$. Thus, the function,

$$\frac{(1 + |z|)^{m+2}}{(1 + |\psi(z)|)^m (1 + |z| + ||z|^2 + |z| - m|)}$$

is bounded. From which, we obtain the following estimate,

$$\mathcal{B}^{\infty}_{(m,\psi)}(|g|)(z) \lesssim M_{(g,\psi)}(z).$$

This together with Theorems 2.3 and 2.5 imply that $V_{(g,\psi)} : \mathcal{F}_{(m,p)} \to \mathcal{F}_{(m,q)}$ is bounded.

The compactness case follows from similar procedures as above. $\square$

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