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NONLINEAR MIXED ζ-JORDAN TRIPLE $*$-DERIVATIONS ON $*$-ALGEBRAS

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Abstract. Let $\Re$ be a unital $*$-algebra containing a nontrivial projection. Let ζ be a nonzero scalar. In this paper, under some mild conditions on $\Re$, we prove that a mapping $\Psi : \Re \longrightarrow \Re$ satisfies

$$\Psi(E \circledcirc F \circledast_\zeta G) = \Psi(E) \circledcirc F \circledast_\zeta G + E \circledcirc \Psi(F) \circledast_\zeta G + E \circledcirc F \circledast_\zeta \Psi(G),$$

for all $E, F, G \in \Re$ if and only if Ψ is an additive $*$-derivation and $\Psi(\zeta E) = \zeta\Psi(E)$ for all $E \in \Re$.

1. Introduction and preliminaries

Let $\Re$ be a $*$-algebra over the complex field $\mathbb{C}$. For $E, F \in \Re$, $E \circledcirc F = EF + FE$ and $E \circledast_\zeta F = EF + \zeta FE^*$ denote Jordan and ζ-Jordan $*$-product of E and F, respectively. If $\zeta = -1$, then ζ-Jordan $*$-product becomes the skew Lie product. They were extensively studied because these naturally arise in the problem of representing quadratic functionals (for details see [8, 9, 10, 11]) with sesquilinear functionals and in the problem of characterizing ideals (for details see [2] and [6]). A mapping Ψ between $*$-algebras $\Re$ and $\mathcal{F}$ is said to preserve the ζ-Jordan $*$-product if $\Psi(E \circledast_\zeta F) = \Psi(E) \circledast_\zeta \Psi(F)$ for all $E, F \in \Re$. Recently, many authors have paid more attention to the mappings preserving the ζ-Jordan $*$-product between $*$-algebra (for details see [1, 3, 4, 5]).

Recall that an additive map $\Psi : \Re \longrightarrow \Re$ is said to be an additive derivation if $\Psi(EF) = \Psi(E)F + E\Psi(F)$ for all $E, F \in \Re$. Furthermore, Ψ is said to be an

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additive $*$-derivation if it is an additive derivation and satisfies $\Psi(E^*) = \Psi(E)^*$ for all $E \in \Re$. Let $\Psi : \Re \longrightarrow \Re$ be a mapping (without the additivity assumption). We say Ψ is a nonlinear Jordan derivation if

$$\Psi(E \circledcirc F) = \Psi(E) \circledcirc F + E \circledcirc \Psi(F),$$

and in the similar lines, Ψ is said to be a ζ-Jordan $*$-derivation if

$$\Psi(E \circledast_\zeta F) = \Psi(E) \circledast_\zeta F + E \circledast_\zeta \Psi(F)$$

holds for all $E, F \in \Re$.

Taking into account of nonlinear Jordan derivation or ζ-Jordan $*$-derivation, we can enhance their development in a natural manner. A map $\Psi : \Re \longrightarrow \Re$ is said to be a nonlinear Jordan triple derivation if

$$\Psi(E \circledcirc F \circledcirc G) = \Psi(E) \circledcirc F \circledcirc G + E \circledcirc \Psi(F) \circledcirc G + E \circledcirc F \circledcirc \Psi(G),$$

and in the similar way, Ψ is said to be a ζ-Jordan $*$-triple derivation if

$$\Psi(E \circledast_\zeta F \circledast_\zeta G) = \Psi(E) \circledast_\zeta F \circledast_\zeta G + E \circledast_\zeta \Psi(F) \circledast_\zeta G + E \circledast_\zeta F \circledast_\zeta \Psi(G)$$

holds for all $E, F, G \in \Re$. Here $E \circledast_\zeta F \circledast_\zeta G$ stands for $(E \circledast_\zeta F) \circledast_\zeta G$.

In the past few years, there has been a notable surge in research dedicated to the investigation of mixed Lie or Jordan products, capturing the interest of numerous researchers in different algebras. Yaoxian and Jianhua [13] proved that every nonlinear mixed Lie triple derivation on factor von Neumann algebras is an additive $*$-derivation. Taghavi , Rohi and Darvish [12] proved that every nonlinear $*$-Jordan derivations on von Neumann algebras is an additive $*$-derivation. In, Zhou, Yang, and Zhang [14] proved that every nonlinear mixed Lie triple derivation on prime $*$-algebras is an additive $*$-derivation. Moreover, Rehman et al. in [7], proved that every nonlinear mixed Jordan triple derivation on unital $*$-algebras is an additive $*$-derivation.

Motivated by the above results, in this paper, we prove that a nonlinear mixed ζ-Jordan triple $*$-derivation on $*$-algebras is an additive $*$-derivation. A map $\Psi : \Re \longrightarrow \Re$ is said to be a nonlinear mixed ζ-Jordan triple $*$-derivation if

$$\Psi(E \circledcirc F \circledast_\zeta G) = \Psi(E) \circledcirc F \circledast_\zeta G + E \circledcirc \Psi(F) \circledast_\zeta G + E \circledcirc F \circledast_\zeta \Psi(G)$$

for all $E, F, G \in \Re$.

2. Main results

The following is the main result presented in this paper.

Theorem 2.1. *Let $\Re$ be a unital $*$-algebra with unit I containing a nontrivial projection $\mathcal{P}$ such that*

$$X\Re\mathcal{P} = 0 \Longrightarrow X = 0, \tag{2.1}$$

$$X\Re(I - \mathcal{P}) = 0 \Longrightarrow X = 0, \tag{2.2}$$

and let ζ be a nonzero scalar. Then a mapping $\Psi : \Re \longrightarrow \Re$ satisfies

$$\Psi(E \circledcirc F \circledast_\zeta G) = \Psi(E) \circledcirc F \circledast_\zeta G + E \circledcirc \Psi(F) \circledast_\zeta G + E \circledcirc F \circledast_\zeta \Psi(G) \tag{2.3}$$

for all $E, F, G \in \Re$ if and only if Ψ is an additive $*$-derivation and $\Psi(\zeta E) = \zeta\Psi(E)$ for all $E \in \Re$.

Let $\mathcal{P}_1 = \mathcal{P}$ and let $\mathcal{P}_2 = I - \mathcal{P}$. Denote $\Re_{i,j} = \mathcal{P}_i \Re \mathcal{P}_j$ for $i, j = 1, 2$. By the Peirce decomposition of $\Re$, we have $\Re = \sum_{i,j=1}^{2} \Re_{i,j}$. In all that follows, when we write $E_{i,j}$, it indicates that $E_{i,j} \in \Re_{i,j}$. It is evident that our focus should be on proving the necessity. To accomplish this, we will present a series of lemmas to complete the proof of the theorem.

Lemma 2.2. *With above notation,* $\Psi(0) = 0$.

Proof. it follows from $\Psi(0) = \Psi(0 \circledcirc 0 \circledast_\zeta 0) = \Psi(0) \circledcirc 0 \circledast_\zeta 0 + 0 \circledcirc \Psi(0) \circledast_\zeta 0 + 0 \circledcirc 0 \circledast_\zeta \Psi(0) = 0$. □

Lemma 2.3. *For* $i, j, k \in \{1, 2\}, i \neq j, E_{kk} \in \Re_{kk}, F_{ij} \in \Re_{ij}$, *we have*

$$\Psi(E_{kk} + F_{ij}) = \Psi(E_{kk}) + \Psi(F_{ij}).$$

Proof. We only prove the case when $i = k = 1, j = 2$. The proof of the other case is similar. Let $W = \Psi(E_{11} + F_{12}) - \Psi(E_{11}) - \Psi(F_{12})$. We only need to prove that $W = 0$. For any $\lambda \in \mathbb{C}$, since $\frac{I}{2} \circledcirc (\lambda \mathcal{P}_2) \circledast_\zeta E_{11} = 0$ and $\frac{I}{2} \circledcirc (\lambda \mathcal{P}_2) \circledast_\zeta (E_{11} + F_{12}) = \frac{I}{2} \circledcirc (\lambda \mathcal{P}_2) \circledast_\zeta F_{12}$. It follows from Lemma 2.2, that

$$\begin{aligned}
&\Psi(\frac{I}{2}) \circledcirc (\lambda \mathcal{P}_2) \circledast_\zeta (E_{11} + F_{12}) + \frac{I}{2} \circledcirc \Psi(\lambda \mathcal{P}_2) \circledast_\zeta (E_{11} + F_{12}) \\
&\quad + \frac{I}{2} \circledcirc (\lambda \mathcal{P}_2) \circledast_\zeta \Psi(E_{11} + F_{12}) \\
=&\Psi(\frac{I}{2} \circledcirc (\lambda \mathcal{P}_2) \circledast_\zeta (E_{11} + F_{12}) \\
=&\Psi(\frac{I}{2} \circledcirc (\lambda \mathcal{P}_2) \circledast_\zeta E_{11}) + \Psi(\frac{I}{2} \circledcirc (\lambda \mathcal{P}_2) \circledast_\zeta F_{12}) \\
=&\Psi(\frac{I}{2}) \circledcirc (\lambda \mathcal{P}_2) \circledast_\zeta E_{11} + \frac{I}{2} \circledcirc \Psi(\lambda \mathcal{P}_2) \circledast_\zeta E_{11} + \frac{I}{2} \circledcirc (\lambda \mathcal{P}_2) \circledast_\zeta \Psi(E_{11}) \\
&\quad + \Psi(\frac{I}{2}) \circledcirc (\lambda \mathcal{P}_2) \circledast_\zeta F_{12} + \frac{I}{2} \circledcirc \Psi(\lambda \mathcal{P}_2) \circledast_\zeta F_{12} + \frac{I}{2} \circledcirc (\lambda \mathcal{P}_2) \circledast_\zeta \Psi(F_{12}).
\end{aligned}$$

Hence $\frac{I}{2} \circledcirc (\lambda \mathcal{P}_2) \circledast_\zeta W = 0$, that is $\lambda \mathcal{P}_2 W + \overline{\lambda} \zeta W \mathcal{P}_2 = 0$, for any $\lambda \in \mathbb{C}$. Let $\lambda + \overline{\lambda}\zeta \neq 0$. We get $W_{12} = 0, W_{21} = 0$ and $W_{22} = 0$.

Similarly, since $\frac{I}{2} \circledcirc (\lambda\zeta \mathcal{P}_1 - \overline{\lambda} \mathcal{P}_2) \circledast_\zeta F_{12} = 0$ and $\frac{I}{2} \circledcirc (\lambda\zeta \mathcal{P}_1 - \overline{\lambda} \mathcal{P}_2) \circledast_\zeta (E_{11} + F_{12}) = \frac{I}{2} \circledcirc (\lambda\zeta \mathcal{P}_1 - \overline{\lambda} \mathcal{P}_2) \circledast_\zeta E_{11}$. it follows that

$$\begin{aligned}
&\Psi(\frac{I}{2}) \circledcirc (\lambda\zeta \mathcal{P}_1 - \overline{\lambda} \mathcal{P}_2) \circledast_\zeta (E_{11} + F_{12}) + \frac{I}{2} \circledcirc \Psi(\lambda\zeta \mathcal{P}_1 - \overline{\lambda} \mathcal{P}_2) \circledast_\zeta (E_{11} + F_{12}) \\
&\quad + \frac{I}{2} \circledcirc (\lambda\zeta \mathcal{P}_1 - \overline{\lambda} \mathcal{P}_2) \circledast_\zeta \Psi(E_{11} + F_{12}) \\
=&\Psi(\frac{I}{2} \circledcirc (\lambda\zeta \mathcal{P}_1 - \overline{\lambda} \mathcal{P}_2) \circledast_\zeta (E_{11} + F_{12})) \\
=&\Psi(\frac{I}{2} \circledcirc (\lambda\zeta \mathcal{P}_1 - \overline{\lambda} \mathcal{P}_2) \circledast_\zeta E_{11}) + \Psi(\frac{I}{2} \circledcirc (\lambda\zeta \mathcal{P}_1 - \overline{\lambda} \mathcal{P}_2) \circledast_\zeta F_{12}) \\
=&\Psi(\frac{I}{2}) \circledcirc (\lambda\zeta \mathcal{P}_1 - \overline{\lambda} \mathcal{P}_2) \circledast_\zeta E_{11} + \frac{I}{2} \circledcirc \Psi(\lambda\zeta \mathcal{P}_1 - \overline{\lambda} \mathcal{P}_2) \circledast_\zeta E_{11} \\
&\quad + \frac{I}{2} \circledcirc (\lambda\zeta \mathcal{P}_1 - \overline{\lambda} \mathcal{P}_2) \circledast_\zeta \Psi(E_{11}) + \Psi(\frac{I}{2}) \circledcirc (\lambda\zeta \mathcal{P}_1 - \overline{\lambda} \mathcal{P}_2) \circledast_\zeta F_{12}
\end{aligned}$$

$$+\frac{I}{2}\circledcirc\Psi(\lambda\zeta\mathcal{P}_1-\overline{\lambda}\mathcal{P}_2)\circledast_\zeta F_{12}+\frac{I}{2}\circledcirc(\lambda\zeta\mathcal{P}_1-\overline{\lambda}\mathcal{P}_2)\circledast_\zeta\Psi(F_{12}).$$

Hence $\frac{I}{2}\circledcirc(\lambda\zeta\mathcal{P}_1-\overline{\lambda}\mathcal{P}_2)\circledast_\zeta W=0$, that is $(\lambda+\overline{\lambda\zeta})\zeta W_{11}=0$, for any $\lambda\in\mathbb{C}$ and then $W_{11}=0$. Now we get $W=0$, hence $\Psi(E_{11}+F_{12})=\Psi(E_{11})+\Psi(F_{12})$. $\square$

Lemma 2.4. *For $E_{11}\in\Re_{11},F_{22}\in\Re_{22}$, we have $\Psi(E_{11}+F_{22})=\Psi(E_{11})+\Psi(F_{22})$.*

Proof. We only need to show that $W=\Psi(E_{11}+F_{22})-\Psi(E_{11})-\Psi(F_{22})=0$. For any $\lambda\in\mathbb{C}$, $\frac{I}{2}\circledcirc(\lambda\mathcal{P}_1)\circledast_\zeta F_{22}=0$ and $\frac{I}{2}\circledcirc(\lambda\mathcal{P}_1)\circledast_\zeta(E_{11}+F_{22})=\frac{I}{2}\circledcirc(\lambda\mathcal{P}_1)\circledast_\zeta E_{11}$. It follows that

$$\begin{aligned}
&\Psi(\frac{I}{2})\circledcirc(\lambda\mathcal{P}_1)\circledast_\zeta(E_{11}+F_{22})+\frac{I}{2}\circledcirc\Psi(\lambda\mathcal{P}_1)\circledast_\zeta(E_{11}+F_{22})\\
&\quad+\frac{I}{2}\circledcirc(\lambda\mathcal{P}_1)\circledast_\zeta\Psi(E_{11}+F_{22})\\
=&\Psi(\frac{I}{2}\circledcirc(\lambda\mathcal{P}_1)\circledast_\zeta(E_{11}+F_{22}))\\
=&\Psi(\frac{I}{2}\circledcirc(\lambda\mathcal{P}_1)\circledast_\zeta E_{11})+\Psi(\frac{I}{2}\circledcirc(\lambda\mathcal{P}_1)\circledast_\zeta F_{22})\\
=&\Psi(\frac{I}{2})\circledcirc(\lambda\mathcal{P}_1)\circledast_\zeta E_{11}+\frac{I}{2}\circledcirc\Psi(\lambda\mathcal{P}_1)\circledast_\zeta E_{11}+\frac{I}{2}\circledcirc(\lambda\mathcal{P}_1)\circledast_\zeta\Psi(E_{11})\\
&\quad+\Psi(\frac{I}{2})\circledcirc(\lambda\mathcal{P}_1)\circledast_\zeta F_{22}+\frac{I}{2}\circledcirc\Psi(\lambda\mathcal{P}_1)\circledast_\zeta F_{22}+\frac{I}{2}\circledcirc(\lambda\mathcal{P}_1)\circledast_\zeta\Psi(F_{22}).
\end{aligned}$$

Hence $\frac{I}{2}\circledcirc(\lambda\mathcal{P}_1)\circledast_\zeta W=0$, that is $\lambda\mathcal{P}_1W+\overline{\lambda}\zeta W\mathcal{P}_1=0$, for any $\lambda\in\mathbb{C}$. Let $\lambda+\overline{\lambda}\zeta\neq0$. We get $W_{11}=0,W_{12}=0$ and $W_{21}=0$. Similarly, we can get $W_{22}=0$. Therefore, the Lemma has been proven. $\square$

Lemma 2.5. *For $E_{12}\in\Re_{12},F_{21}\in\Re_{21}$, we have $\Psi(E_{12}+F_{21})=\Psi(E_{12})+\Psi(F_{21})$.*

Proof. We only need to prove that $W=\Psi(E_{12}+F_{21})-\Psi(E_{12})-\Psi(F_{21})=0$. For any $\lambda\in\mathbb{C}$, $\frac{I}{2}\circledcirc(\lambda\zeta\mathcal{P}_1-\overline{\lambda}\mathcal{P}_2)\circledast_\zeta E_{12}=0$ and $\frac{I}{2}\circledcirc(\lambda\zeta\mathcal{P}_1-\overline{\lambda}\mathcal{P}_2)\circledast_\zeta(E_{12}+F_{21})=\frac{I}{2}\circledcirc(\lambda\zeta\mathcal{P}_1-\overline{\lambda}\mathcal{P}_2)\circledast_\zeta F_{21}$. It follows that

$$\begin{aligned}
&\Psi(\frac{I}{2})\circledcirc(\lambda\zeta\mathcal{P}_1-\overline{\lambda}\mathcal{P}_2)\circledast_\zeta(E_{12}+F_{21})+\frac{I}{2}\circledcirc\Psi(\lambda\zeta\mathcal{P}_1-\overline{\lambda}\mathcal{P}_2)\circledast_\zeta(E_{12}+F_{21})\\
&\quad+\frac{I}{2}\circledcirc(\lambda\zeta\mathcal{P}_1-\overline{\lambda}\mathcal{P}_2)\circledast_\zeta\Psi(E_{12}+F_{21})\\
=&\Psi(\frac{I}{2}\circledcirc(\lambda\zeta\mathcal{P}_1-\overline{\lambda}\mathcal{P}_2)\circledast_\zeta(E_{12}+F_{21}))\\
=&\Psi(\frac{I}{2}\circledcirc(\lambda\zeta\mathcal{P}_1-\overline{\lambda}\mathcal{P}_2)\circledast_\zeta E_{12})+\Psi(\frac{I}{2}\circledcirc(\lambda\zeta\mathcal{P}_1-\overline{\lambda}\mathcal{P}_2)\circledast_\zeta F_{21})\\
=&\Psi(\frac{I}{2})\circledcirc(\lambda\zeta\mathcal{P}_1-\overline{\lambda}\mathcal{P}_2)\circledast_\zeta E_{12}+\frac{I}{2}\circledcirc\Psi(\lambda\zeta\mathcal{P}_1-\overline{\lambda}\mathcal{P}_2)\circledast_\zeta E_{12}\\
&\quad+\frac{I}{2}\circledcirc(\lambda\zeta\mathcal{P}_1-\overline{\lambda}\mathcal{P}_2)\circledast_\zeta\Psi(E_{12})+\Psi(\frac{I}{2})\circledcirc(\lambda\zeta\mathcal{P}_1-\overline{\lambda}\mathcal{P}_2)\circledast_\zeta F_{21}\\
&\quad+\frac{I}{2}\circledcirc\Psi(\lambda\zeta\mathcal{P}_1-\overline{\lambda}\mathcal{P}_2)\circledast_\zeta F_{21}+\frac{I}{2}\circledcirc(\lambda\zeta\mathcal{P}_1-\overline{\lambda}\mathcal{P}_2)\circledast_\zeta\Psi(F_{21}).
\end{aligned}$$

Hence $\frac{I}{2} \circledcirc (\lambda\zeta\mathcal{P}_1 - \overline{\lambda}\mathcal{P}_2) \circledast_\zeta W = 0$, that is $(\lambda + \overline{\lambda\zeta})\zeta W_{11} = 0$, for any $\lambda \in \mathbb{C}$, and then $W_{11} = W_{22} = 0$. Now since $\frac{I}{2} \circledcirc E_{12} \circledast_\zeta \mathcal{P}_1 = 0$, it follows that

$$\begin{aligned}
&\Psi(\frac{I}{2}) \circledcirc (E_{12} + F_{21}) \circledast_\zeta \mathcal{P}_1 + \frac{I}{2} \circledcirc \Psi(E_{12} + F_{21}) \circledast_\zeta \mathcal{P}_1 + \frac{I}{2} \circledcirc (E_{12} + F_{21}) \circledast_\zeta \Psi(\mathcal{P}_1)\\
=&\Psi(\frac{I}{2} \circledcirc (E_{12} + F_{21}) \circledast_\zeta \mathcal{P}_1)\\
=&\Psi(\frac{I}{2} \circledcirc E_{12} \circledast_\zeta \mathcal{P}_1) + \Psi(\frac{I}{2} \circledcirc F_{21} \circledast_\zeta \mathcal{P}_1)\\
=&\Psi(\frac{I}{2}) \circledcirc E_{12} \circledast_\zeta \mathcal{P}_1 + \frac{I}{2} \circledcirc \Psi(E_{12}) \circledast_\zeta \mathcal{P}_1 + \frac{I}{2} \circledcirc E_{12} \circledast_\zeta \Psi(\mathcal{P}_1)\\
&+ \Psi(\frac{I}{2}) \circledcirc F_{21} \circledast_\zeta \mathcal{P}_1 + \frac{I}{2} \circledcirc \Psi(F_{21}) \circledast_\zeta \mathcal{P}_1 + \frac{I}{2} \circledcirc F_{21} \circledast_\zeta \Psi(\mathcal{P}_1).
\end{aligned}$$

Hence $\frac{I}{2} \circledcirc W \circledast_\zeta \mathcal{P}_1 = 0$, from which we get that $W_{21} = 0$. Similarly, we have $W_{12} = 0$. Proving the Lemma. □

Lemma 2.6. *For any $E_{11} \in \Re_{11}, F_{12} \in \Re_{12}, G_{21} \in \Re_{21}$, we have*

$$\begin{aligned}
\Psi(E_{11} + F_{12} + G_{21}) &= \Psi(E_{11}) + \Psi(F_{12}) + \Psi(G_{21}) \text{ and}\\
\Psi(E_{22} + F_{12} + G_{21}) &= \Psi(E_{22}) + \Psi(F_{12}) + \Psi(G_{21}).
\end{aligned}$$

Proof. We only prove, $W = \Psi(E_{11} + F_{12} + G_{21}) - \Psi(E_{11}) - \Psi(F_{12}) - \Psi(G_{21}) = 0$. Similarly, we can show that $\Psi(E_{22} + F_{12} + G_{21}) = \Psi(E_{22}) + \Psi(F_{12}) + \Psi(G_{21})$.

For any $\lambda \in \mathbb{C}$, since $\frac{I}{2} \circledcirc (\lambda\mathcal{P}_2) \circledast_\zeta E_{11} = 0$. It follows from Lemma 2.5, that

$$\begin{aligned}
&\Psi(\frac{I}{2}) \circledcirc (\lambda\mathcal{P}_2) \circledast_\zeta (E_{11} + F_{12} + G_{21}) + \frac{I}{2} \circledcirc \Psi(\lambda\mathcal{P}_2) \circledast_\zeta (E_{11} + F_{12} + G_{21})\\
&\quad + \frac{I}{2} \circledcirc (\lambda\mathcal{P}_2) \circledast_\zeta \Psi(E_{11} + F_{12} + G_{21})\\
=&\Psi(\frac{I}{2} \circledcirc (\lambda\mathcal{P}_2) \circledast_\zeta (E_{11} + F_{12} + G_{21}))\\
=&\Psi(\frac{I}{2} \circledcirc (\lambda\mathcal{P}_2) \circledast_\zeta E_{11}) + \Psi(\frac{I}{2} \circledcirc (\lambda\mathcal{P}_2) \circledast_\zeta (F_{12} + G_{21}))\\
=&\Psi(\frac{I}{2}) \circledcirc (\lambda\mathcal{P}_2) \circledast_\zeta E_{11} + \frac{I}{2} \circledcirc \Psi(\lambda\mathcal{P}_2) \circledast_\zeta E_{11} + \frac{I}{2} \circledcirc (\lambda\mathcal{P}_2) \circledast_\zeta \Psi(E_{11})\\
&\quad + \Psi(\frac{I}{2}) \circledcirc (\lambda\mathcal{P}_2) \circledast_\zeta (F_{12} + G_{21}) + \frac{I}{2} \circledcirc \Psi(\lambda\mathcal{P}_2) \circledast_\zeta (F_{12} + G_{21})\\
&\quad + \frac{I}{2} \circledcirc (\lambda\mathcal{P}_2) \circledast_\zeta (\Psi(F_{12}) + \Psi(G_{21})).
\end{aligned}$$

Hence $\frac{I}{2} \circledcirc (\lambda\mathcal{P}_2) \circledast_\zeta W = 0$, for any $\lambda \in \mathbb{C}$. Let $\lambda + \overline{\lambda}\zeta \neq 0$. We get $W_{12} = 0, W_{21} = 0$ and $W_{22} = 0$. Since $\frac{I}{2} \circledcirc (\overline{\lambda}\mathcal{P}_1 - \lambda\zeta\mathcal{P}_2) \circledast_\zeta G_{21} = 0$, it follows from Lemma 2.2 that

$$\begin{aligned}
&\Psi(\frac{I}{2}) \circledcirc (\overline{\lambda}\mathcal{P}_1 - \lambda\zeta\mathcal{P}_2) \circledast_\zeta (E_{11} + F_{12} + G_{21})\\
&\quad + \frac{I}{2} \circledcirc \Psi(\overline{\lambda}\mathcal{P}_1 - \lambda\zeta\mathcal{P}_2) \circledast_\zeta (E_{11} + F_{12} + G_{21})
\end{aligned}$$

$$
\begin{aligned}
&+\frac{I}{2}\circledcirc(\overline{\lambda}\mathcal{P}_1-\lambda\zeta\mathcal{P}_2)\circledast_\zeta\Psi(E_{11}+F_{12}+G_{21})\\
=&\Psi(\frac{I}{2}\circledcirc(\overline{\lambda}\mathcal{P}_1-\lambda\zeta\mathcal{P}_2)\circledast_\zeta(E_{11}+F_{12}+G_{21}))\\
=&\Psi(\frac{I}{2}\circledcirc(\overline{\lambda}\mathcal{P}_1-\lambda\zeta\mathcal{P}_2)\circledast_\zeta(E_{11}+F_{12}))+\Psi(\frac{I}{2}\circledcirc(\overline{\lambda}\mathcal{P}_1-\lambda\zeta\mathcal{P}_2)\circledast_\zeta G_{21})\\
=&\Psi(\frac{I}{2}\circledcirc(\overline{\lambda}\mathcal{P}_1-\lambda\zeta\mathcal{P}_2)\circledast_\zeta E_{11})+\Psi(\frac{I}{2}\circledcirc(\overline{\lambda}\mathcal{P}_1-\lambda\zeta\mathcal{P}_2)\circledast_\zeta F_{12})\\
=&\Psi(\frac{I}{2})\circledcirc(\overline{\lambda}\mathcal{P}_1-\lambda\zeta\mathcal{P}_2)\circledast_\zeta(E_{11}+F_{12}+G_{21})\\
&+\frac{I}{2}\circledcirc\Psi(\overline{\lambda}\mathcal{P}_1-\lambda\zeta\mathcal{P}_2)\circledast_\zeta(E_{11}+F_{12}+G_{21})\\
&+\frac{I}{2}\circledcirc(\overline{\lambda}\mathcal{P}_1-\lambda\zeta\mathcal{P}_2)\circledast_\zeta(\Psi(E_{11})+\Psi(F_{12})+\Psi(G_{21})).
\end{aligned}
$$

Hence $\frac{I}{2}\circledcirc(\overline{\lambda}\mathcal{P}_1-\lambda\zeta\mathcal{P}_2)\circledast_\zeta W=0$, for any $\lambda\in\mathbb{C}$. From which we get that $W_{11}=0$, proving the Lemma. □

Lemma 2.7. *For $E_{ij}\in\Re_{ij}, F_{ij}\in\Re_{ij}, 1\leq i\neq j\leq 2,$ we have* $\Psi(E_{ij}+F_{ij})=\Psi(E_{ij})+\Psi(F_{ij})$.

Proof. It can be easily shown that $\frac{I}{2}\circledcirc(\mathcal{P}_i+F_{ij})\circledast_\zeta(\mathcal{P}_j+F_{ij})=E_{ij}+F_{ij}+\zeta E_{ij}^*+\zeta F_{ij}E_{ij}^*$. It follows from Lemmas 2.3 and 2.6, that

$$
\begin{aligned}
&\Psi(E_{ij}+F_{ij})+\Psi(\zeta E_{ij}^*)+\Psi(\zeta F_{ij}E_{ij}^*)\\
=&\Psi(\frac{I}{2}\circledcirc(\mathcal{P}_i+F_{ij})\circledast_\zeta(\mathcal{P}_j+F_{ij}))\\
=&\Psi(\frac{I}{2})\circledcirc(\mathcal{P}_i+F_{ij})\circledast_\zeta(\mathcal{P}_j+F_{ij})+\frac{I}{2}\circledcirc\Psi(\mathcal{P}_i+F_{ij})\circledast_\zeta(\mathcal{P}_j+F_{ij})\\
&+\frac{I}{2}\circledcirc(\mathcal{P}_i+F_{ij})\circledast_\zeta\Psi(\mathcal{P}_j+F_{ij})\\
=&\Psi(\frac{I}{2})\circledcirc(\mathcal{P}_i+F_{ij})\circledast_\zeta(\mathcal{P}_j+F_{ij})+\frac{I}{2}\circledcirc(\Psi(\mathcal{P}_i)+\Psi(F_{ij}))\circledast_\zeta(\mathcal{P}_j+F_{ij})\\
&+\frac{I}{2}\circledcirc(\mathcal{P}_i+F_{ij})\circledast_\zeta(\Psi(\mathcal{P}_j)+\Psi(F_{ij}))\\
=&\Psi(\frac{I}{2}\circledcirc\mathcal{P}_i\circledast_\zeta\mathcal{P}_j)+\Psi(\frac{I}{2}\circledcirc\mathcal{P}_i\circledast_\zeta F_{ij})+\Psi(\frac{I}{2}\circledcirc E_{ij}\circledast_\zeta\mathcal{P}_j)+\Psi(\frac{I}{2}\circledcirc E_{ij}\circledast_\zeta F_{ij})\\
=&\Psi(F_{ij})+\Psi(E_{ij}+\zeta E_{ij}^*)+\Psi(\zeta F_{ij}E_{ij}^*)\\
=&\Psi(E_{ij})+\Psi(F_{ij})+\Psi(\zeta E_{ij}^*)+\Psi(\zeta F_{ij}E_{ij}^*).
\end{aligned}
$$

Consequently, $\Psi(E_{ij}+F_{ij})=\Psi(E_{ij})+\Psi(F_{ij})$. □

Lemma 2.8. *For $E_{ii}\in\Re_{ii}, F_{ii}\in\Re_{ii}, i=1,2,$ we have* $\Psi(E_{ii}+F_{ii})=\Psi(E_{ii})+\Psi(F_{ii})$.

Proof. Let $W=\Psi(E_{ii}+F_{ii})-\Psi(E_{ii})-\Psi(F_{ii})$. We only need to prove $W=0$. For any $\lambda\in\mathbb{C}$, since $\frac{I}{2}\circledcirc(\lambda\mathcal{P}_j)\circledast_\zeta E_{ii}=\frac{I}{2}\circledcirc(\lambda\mathcal{P}_j)\circledast_\zeta F_{ii}=\frac{I}{2}\circledcirc(\lambda\mathcal{P}_j)\circledast_\zeta(E_{ii}+F_{ii})=$

0 $(i \neq j)$, it follows that

$$\Psi(\frac{I}{2} \circledcirc (\lambda\mathcal{P}_j) \circledast_\zeta (E_{ii} + F_{ii})) = \Psi(\frac{I}{2} \circledcirc (\lambda\mathcal{P}_j) \circledast_\zeta E_{ii}) + \Psi(\frac{I}{2} \circledcirc (\lambda\mathcal{P}_j) \circledast_\zeta F_{ii}).$$

Hence $\frac{I}{2} \circledcirc (\lambda\mathcal{P}_j) \circledast_\zeta W = 0$. From which we get that $W_{ij} = W_{ji} = W_{jj} = 0$.

For any, $G_{ij} \in \Re_{ij}(i \neq j)$, it follows from Lemma 2.7, that

$$\begin{aligned}
&\Psi(\frac{I}{2}) \circledcirc (E_{ii} + F_{ii}) \circledast_\zeta G_{ij} + \frac{I}{2} \circledcirc \Psi(E_{ii} + F_{ii}) \circledast_\zeta G_{ij} + \frac{I}{2} \circledcirc (E_{ii} + F_{ii}) \circledast_\zeta \Psi(G_{ij})\\
=&\Psi(\frac{I}{2} \circledcirc (E_{ii} + F_{ii}) \circledast_\zeta G_{ij})\\
=&\Psi(\frac{I}{2} \circledcirc (E_{ii} \circledast_\zeta G_{ij})) + \Psi(\frac{I}{2} \circledcirc (F_{ii} \circledast_\zeta G_{ij}))\\
=&\Psi(\frac{I}{2}) \circledcirc (E_{ii} + F_{ii}) \circledast_\zeta G_{ij} + \frac{I}{2} \circledcirc (\Psi(E_{ii}) + \Psi(F_{ii})) \circledast_\zeta G_{ij}\\
&+ \frac{I}{2} \circledcirc (E_{ii} + F_{ii}) \circledast_\zeta \Psi(G_{ij}).
\end{aligned}$$

Hence $\frac{I}{2} \circledcirc W_{ii} \circledast_\zeta G_{ij} = 0$, that is $W_{ii}G_{ij} = 0$, for all $G_{ij} \in \Re_{ij}$, $W_{ii}G\mathcal{P}_j = 0$ for all $G \in \Re$. It follows from (2.1) and (2.2) that $W_{ii} = 0$. Proving the Lemma. □

Lemma 2.9. *For any $E_{11} \in \Re_{11}, F_{12} \in \Re_{12}, G_{21} \in \Re_{21}, H_{22} \in \Re_{22}$, we have*

$$\Psi(E_{11} + F_{12} + G_{21} + H_{22}) = \Psi(E_{11}) + \Psi(F_{12}) + \Psi(G_{21}) + \Psi(H_{22}).$$

Proof. Let $W = \Psi(E_{11} + F_{12} + G_{21} + H_{22}) - \Psi(E_{11}) - \Psi(F_{12}) - \Psi(G_{21}) - \Psi(H_{22})$. We only need to prove that $T = 0$. For any $\lambda \in \mathbb{C}$, since $\frac{I}{2} \circledcirc (\lambda\mathcal{P}_1) \circledast_\zeta H_{22} = 0$, it follows from Lemma 2.6, that

$$\begin{aligned}
&\Psi(\frac{I}{2}) \circledcirc (\bar{\lambda}\mathcal{P}_1) \circledast_\zeta (E_{11} + F_{12} + G_{21} + H_{22})\\
&\quad + \frac{I}{2} \circledcirc \Psi(\bar{\lambda}\mathcal{P}_1) \circledast_\zeta (E_{11} + F_{12} + G_{21} + H_{22})\\
&\quad + \frac{I}{2} \circledcirc (\bar{\lambda}\mathcal{P}_1) \circledast_\zeta \Psi(E_{11} + F_{12} + G_{21} + H_{22})\\
=&\Psi(\frac{I}{2} \circledcirc (\bar{\lambda}\mathcal{P}_1) \circledast_\zeta (E_{11} + F_{12} + G_{21} + H_{22}))\\
=&\Psi(\frac{I}{2} \circledcirc (\bar{\lambda}\mathcal{P}_1) \circledast_\zeta (E_{11} + F_{12} + G_{21})) + \Psi(\frac{I}{2} \circledcirc (\bar{\lambda}\mathcal{P}_1) \circledast_\zeta (H_{22}))\\
=&\Psi(\frac{I}{2}) \circledcirc (\bar{\lambda}\mathcal{P}_1) \circledast_\zeta (E_{11} + F_{12} + G_{21} + H_{22})\\
&\quad + \frac{I}{2} \circledcirc \Psi(\bar{\lambda}\mathcal{P}_1) \circledast_\zeta (E_{11} + F_{12} + G_{21} + H_{22})\\
&\quad + \frac{I}{2} \circledcirc (\bar{\lambda}\mathcal{P}_1) \circledast_\zeta (\Psi(E_{11}) + \Psi(F_{12}) + \Psi(G_{21}) + \Psi(H_{22})).
\end{aligned}$$

Hence $\frac{I}{2} \circledcirc (\lambda\mathcal{P}_1) \circledast_\zeta W = 0$, for any $\lambda \in \mathbb{C}$. From which we have $W_{11} = 0, W_{12} = 0, W_{21} = 0$.

Similarly, we get $W_{22} = 0$. Hence the Lemma stands proved. □

Lemma 2.10. Ψ *is additive.*

Proof. One can easily observe this by combining consequences of Lemmas 2.7–2.9. $\square$

Lemma 2.11. (*i*) $\mathcal{P}_1\Psi(\mathcal{P}_1)\mathcal{P}_2 = -\mathcal{P}_1\Psi(\mathcal{P}_2)\mathcal{P}_2$,
(*ii*) $\mathcal{P}_2\Psi(\mathcal{P}_1)\mathcal{P}_1 = -\mathcal{P}_2\Psi(\mathcal{P}_2)\mathcal{P}_1$,
(*iii*) $\mathcal{P}_1\Psi(\mathcal{P}_2)\mathcal{P}_1 = \mathcal{P}_2\Psi(\mathcal{P}_1)\mathcal{P}_2 = 0$.

Proof. (*i*) It follows from $\mathcal{P}_1 \circledcirc \mathcal{P}_1 \circledast_\zeta \mathcal{P}_2 = 0$ and by Lemma 2.2,

$$\begin{aligned}0 =&\Psi(\mathcal{P}_1 \circledcirc \mathcal{P}_1 \circledast_\zeta \mathcal{P}_2)\\ =&\Psi(\mathcal{P}_1) \circledcirc \mathcal{P}_1 \circledast_\zeta \mathcal{P}_2 + \mathcal{P}_1 \circledcirc \Psi(\mathcal{P}_1) \circledast_\zeta \mathcal{P}_2 + \mathcal{P}_1 \circledcirc \mathcal{P}_1 \circledast_\zeta \Psi(\mathcal{P}_2)\\ =&2\mathcal{P}_1\Psi(\mathcal{P}_1)\mathcal{P}_2 + 2\zeta\mathcal{P}_2\Psi(\mathcal{P}_1)^*\mathcal{P}_1 + 2\mathcal{P}_1\Psi(\mathcal{P}_2) + 2\zeta\Psi(\mathcal{P}_2)\mathcal{P}_1.\end{aligned}$$

Multiplying both sides by $\mathcal{P}_1$ from left and by $\mathcal{P}_2$ from right, we get

$$\mathcal{P}_1\Psi(\mathcal{P}_1)\mathcal{P}_2 = -\mathcal{P}_1\Psi(\mathcal{P}_2)\mathcal{P}_2$$

(*ii*) It follows from $\mathcal{P}_2 \circledcirc \mathcal{P}_2 \circledast_\zeta \mathcal{P}_1 = 0$ and by Lemma 2.2,

$$\begin{aligned}0 =&\Psi(\mathcal{P}_2 \circledcirc \mathcal{P}_2 \circledast_\zeta \mathcal{P}_1)\\ =&\Psi(\mathcal{P}_2) \circledcirc \mathcal{P}_2 \circledast_\zeta \mathcal{P}_1 + \mathcal{P}_2 \circledcirc \Psi(\mathcal{P}_2) \circledast_\zeta \mathcal{P}_1 + \mathcal{P}_2 \circledcirc \mathcal{P}_2 \circledast_\zeta \Psi(\mathcal{P}_1)\\ =&2\mathcal{P}_2\Psi(\mathcal{P}_2)\mathcal{P}_1 + 2\zeta\mathcal{P}_1\Psi(\mathcal{P}_2)^*\mathcal{P}_2 + 2\mathcal{P}_2\Psi(\mathcal{P}_1) + 2\zeta\Psi(\mathcal{P}_1)\mathcal{P}_2.\end{aligned}$$

Multiplying both sides by $\mathcal{P}_2$ from left and by $\mathcal{P}_1$ from right, we get

$$\mathcal{P}_2\Psi(\mathcal{P}_2)\mathcal{P}_1 = -\mathcal{P}_2\Psi(\mathcal{P}_1)\mathcal{P}_1.$$

(*iii*) We assume that $\zeta\neq-1$. From (*i*) we have $2\mathcal{P}_1\Psi(\mathcal{P}_1)\mathcal{P}_2 + 2\zeta\mathcal{P}_2\Psi(\mathcal{P}_1)^*\mathcal{P}_1 + 2\mathcal{P}_1\Psi(\mathcal{P}_2) + 2\zeta\Psi(\mathcal{P}_2)\mathcal{P}_1 = 0$. Multiplying both sides of this equation by $\mathcal{P}_1$ from left and right respectively, we get $\mathcal{P}_1\Psi(\mathcal{P}_2)\mathcal{P}_1 = 0$. From (*ii*) we have $2\mathcal{P}_2\Psi(\mathcal{P}_2)\mathcal{P}_1 + 2\zeta\mathcal{P}_1\Psi(\mathcal{P}_2)^*\mathcal{P}_2 + 2\mathcal{P}_2\Psi(\mathcal{P}_1) + 2\zeta\Psi(\mathcal{P}_1)\mathcal{P}_2 = 0$. Multiplying both sides of this equation by $\mathcal{P}_2$ from left and right respectively, we get $\mathcal{P}_2\Psi(\mathcal{P}_1)\mathcal{P}_2 = 0$. $\square$

Lemma 2.12. $\mathcal{P}_1\Psi(\mathcal{P}_1)\mathcal{P}_1 = \mathcal{P}_2\Psi(\mathcal{P}_2)\mathcal{P}_2 = 0$.

Proof. For every $E_{12} \in \Re_{12}$, we have $\Psi(\mathcal{P}_1 \circledcirc \mathcal{P}_1 \circledast_\zeta E_{12}) = 2\Psi(E_{12})$.

On the other hand, we have

$$\begin{aligned}\Psi(\mathcal{P}_1 \circledcirc \mathcal{P}_1 \circledast_\zeta E_{12}) =&\Psi(\mathcal{P}_1) \circledcirc \mathcal{P}_1 \circledast_\zeta E_{12} + \mathcal{P}_1 \circledcirc \Psi(\mathcal{P}_1) \circledast_\zeta E_{12}\\ &+ \mathcal{P}_1 \circledcirc \mathcal{P}_1 \circledast_\zeta \Psi(E_{12})\\ =&2\Psi(\mathcal{P}_1)\mathcal{P}_1E_{12} + 2\mathcal{P}_1\Psi(\mathcal{P}_1)E_{12} + 2\zeta E_{12}\Psi(\mathcal{P}_1)^*\mathcal{P}_1\\ &+ 2\mathcal{P}_1\Psi(E_{12}) + 2\zeta\Psi(E_{12})\mathcal{P}_1.\end{aligned}$$

On comparing the above two equations, we get $\Psi(\mathcal{P}_1)\mathcal{P}_1E_{12} + \mathcal{P}_1\Psi(\mathcal{P}_1)E_{12} + \zeta E_{12}\Psi(\mathcal{P}_1)^*\mathcal{P}_1 + \mathcal{P}_1\Psi(E_{12}) + \zeta\Psi(E_{12})\mathcal{P}_1 - \Psi(E_{12}) = 0$. Multiplying last equation by $\mathcal{P}_1$ from the left and by $\mathcal{P}_2$ from the right, we get $\mathcal{P}_1\Psi(\mathcal{P}_1)E_{12} = 0$, that is, $\mathcal{P}_1\Psi(\mathcal{P}_1)\mathcal{P}_1E\mathcal{P}_2 = 0$, for all $E \in \Re$. It follows from (2.1) and (2.2) that $\mathcal{P}_1\Psi(\mathcal{P}_1)\mathcal{P}_1 = 0$. Similarly, we can prove that $\mathcal{P}_2\Psi(\mathcal{P}_2)\mathcal{P}_2 = 0$. $\square$

Lemma 2.13. (*i*) $\Psi(\mathcal{P}_1) = \mathcal{P}_1\Psi(\mathcal{P}_1)\mathcal{P}_2 + \mathcal{P}_2\Psi(\mathcal{P}_1)\mathcal{P}_1$, $\Psi(\mathcal{P}_2) = \mathcal{P}_1\Psi(\mathcal{P}_2)\mathcal{P}_2 + \mathcal{P}_2\Psi(\mathcal{P}_2)\mathcal{P}_1$,
(*ii*) $\Psi(I) = 0$.

Proof. (*i*) By using Peirce decomposition, we can write

$$\Psi(\mathcal{P}_1) = \mathcal{P}_1\Psi(\mathcal{P}_1)\mathcal{P}_1 + \mathcal{P}_1\Psi(\mathcal{P}_1)\mathcal{P}_2 + \mathcal{P}_2\Psi(\mathcal{P}_1)\mathcal{P}_1 + \mathcal{P}_2\Psi(\mathcal{P}_1)\mathcal{P}_2.$$

Using Lemmas 2.11 and 2.12 we get $\Psi(\mathcal{P}_1) = \mathcal{P}_1\Psi(\mathcal{P}_1)\mathcal{P}_2 + \mathcal{P}_2\Psi(\mathcal{P}_1)\mathcal{P}_1$. Similarly, we can show that $\Psi(\mathcal{P}_2) = \mathcal{P}_1\Psi(\mathcal{P}_2)\mathcal{P}_2 + \mathcal{P}_2\Psi(\mathcal{P}_2)\mathcal{P}_1$.

(*ii*) It follows from Lemmas 2.9-2.11, that $\Psi(I) = \Psi(\mathcal{P}_1) + \Psi(\mathcal{P}_2) = \mathcal{P}_1\Psi(\mathcal{P}_1)\mathcal{P}_2 + \mathcal{P}_2\Psi(\mathcal{P}_1)\mathcal{P}_1 + \mathcal{P}_1\Psi(\mathcal{P}_2)\mathcal{P}_2 + \mathcal{P}_2\Psi(\mathcal{P}_2)\mathcal{P}_1 = 0$. □

Lemma 2.14. $\Psi(E^*) = \Psi(E)^*$ *for all* $E \in \Re$.

Proof. From Lemma 2.10, we have $\Psi(E \circledcirc I \circledast_\zeta I) = \Psi(2E + 2\zeta E^*) = 2\Psi(E) + 2\zeta\Psi(E^*)$. On the other hand, using Lemma 2.13(*ii*), we have

$$\Psi(E \circledcirc I \circledast_\zeta I) = \Psi(E) \circledcirc I \circledast_\zeta I = 2\Psi(E) + 2\zeta\Psi(E)^*.$$

By comparing the above equations, we find $\Psi(E^*) = \Psi(E)^*$. □

Lemma 2.15. $\Psi(\zeta E) = \zeta\Psi(E)$ *for all* $E \in \Re$.

Proof. For any $E \in \Re$, it follows from $\Psi(I) = 0$ that

$$2\Psi(E) + 2\Psi(\zeta E) = \Psi(I \circledcirc I \circledast_\zeta E) = (2I) \circledast_\zeta \Psi(E) = 2\Psi(E) + 2\zeta\Psi(E).$$

Hence, $\Psi(\zeta E) = \zeta\Psi(E)$ for all $E \in \Re$. □

Now, let $S = \mathcal{P}_1\Psi(\mathcal{P}_1)\mathcal{P}_2 - \mathcal{P}_2\Psi(\mathcal{P}_1)\mathcal{P}_1$. Then $S = -S^*$. Defining a map $\Theta : \Re \longrightarrow \Re$ by $\Theta(E) = \Psi(E) - (ES - SE)$ for all $E \in \Re$. It is easy to verify that for all $E, F, G \in \Re$, $\Theta(E \circledcirc F \circledast_\zeta G) = \Theta(E) \circledcirc F \circledast_\zeta G + E \circledcirc \Theta(F) \circledast_\zeta G + E \circledcirc F \circledast_\zeta \Theta(G)$.

Remark 2.16. Θ has the following properties
(*i*) $\Theta(E^*) = \Theta(E)^*$ for all $E \in \Re$,
(*ii*) Θ is additive,
(*iii*) $\Theta(\mathcal{P}_i) = 0$, for i= 1,2,
(*iv*) $\Theta(I) = 0$,
(*v*) Θ is a $*$ -derivation if and only if Ψ is a $*$-derivation and $\Theta(\zeta E) = \zeta\Theta(E)$ for all $E \in \Re$.

Lemma 2.17. $\Theta(\Re_{ij}) \subseteq \Re_{ij}, i, j = 1, 2$.

Proof. For $E_{ij} \in \Re_{ij}, 1 \leq i \neq j \leq 2$. Using the above Remark 2.16,

$$2\Theta(E_{ij}) = \Theta(I \circledcirc \mathcal{P}_i \circledast_\zeta E_{ij}) = I \circledcirc \mathcal{P}_i \circledast_\zeta \Theta(E_{ij}) = 2\mathcal{P}_i\Theta(E_{ij}) + 2\zeta\Theta(E_{ij})\mathcal{P}_i$$

It gives $\mathcal{P}_i\Theta(E_{ij})\mathcal{P}_i = \mathcal{P}_j\Theta(E_{ij})\mathcal{P}_j = 0$. On the other hand, we have

$$0 = \Theta(I \circledcirc E_{ij} \circledast_\zeta \mathcal{P}_i) = I \circledcirc \Theta(E_{ij}) \circledast_\zeta \mathcal{P}_i = 2\Theta(E_{ij}) \circledast_\zeta \mathcal{P}_i = 2\Theta(E_{ij})\mathcal{P}_i + 2\zeta\mathcal{P}_i\Theta(E_{ij})^*$$

It gives $\mathcal{P}_j\Theta(E_{ij})\mathcal{P}_i = 0$. Now, we get $\Theta(E_{ij}) = \mathcal{P}_i\Theta(E_{ij})\mathcal{P}_i \in \Re_{ij}$, $1 \leq i \neq j \leq 2$.

Let $E_{ii} \in \Re_{ii}$, $i = 1, 2$. We have $0 = \Theta(I \circledcirc \mathcal{P}_j \circledast_\zeta E_{ii}) = I \circledcirc \mathcal{P}_j \circledast_\zeta \Theta(E_{ii}) = 2\mathcal{P}_j \circledast_\zeta \Theta(E_{ii}) = 2\mathcal{P}_j\Theta(E_{ii}) + 2\zeta\Theta(E_{ii})\mathcal{P}_j$. It gives $\mathcal{P}_i\Theta(E_{ii})\mathcal{P}_j = \mathcal{P}_j\Theta(E_{ii})\mathcal{P}_i = 0$, $1 \leq i \neq j \leq 2$. Now, we let $\Theta(E_{ii}) = \mathcal{P}_i\Theta(E_{ii})\mathcal{P}_i + \mathcal{P}_j\Theta(E_{ii})\mathcal{P}_j$. For any

$F_{ij} \in \Re_{ij},\ 1 \leq i \neq j \leq 2$, Since $\Theta(F_{ij}) \in \Re_{ij}$, we have $0 = \Theta(I \circledcirc F_{ij} \circledast_\zeta E_{ii}) = I \circledcirc \Theta(F_{ij}) \circledast_\zeta E_{ii} + I \circledcirc F_{ij} \circledast_\zeta \Theta(E_{ii}) = \Theta(F_{ij}) \circledast_\zeta E_{ii} + F_{ij} \circledast_\zeta \Theta(E_{ii}) = F_{ij} \circledast_\zeta \Theta(E_{ii})$ holds true for any $F_{ij} \in \Re_{ij},\ 1 \leq i \neq j \leq 2$. It follows from (2.1) and (2.2) that $\mathcal{P}_j \Theta(E_{ii}) \mathcal{P}_j = 0$. Now we get $\Theta(E_{ii}) = \mathcal{P}_i \Theta(E_{ii}) \mathcal{P}_i \in \Re_{ii},\ i = 1, 2$. □

Lemma 2.18. *For $E_{ii}, F_{ii} \in \Re_{ii}$ and $E_{ij}, F_{ij} \in \Re_{ij},\ 1 \leq i \neq j \leq 2$, we have*

(i) $\Theta(E_{ii}F_{ij}) = \Theta(E_{ii})F_{ij} + E_{ii}\Theta(F_{ij})$,
(ii) $\Theta(E_{ii}F_{ii}) = \Theta(E_{ii})F_{ii} + E_{ii}\Theta(F_{ii})$,
(iii) $\Theta(E_{ij}F_{ji}) = \Theta(E_{ij})F_{ji} + E_{ij}\Theta(F_{ji})$,
(iv) $\Theta(E_{ij}F_{jj}) = \Theta(E_{ij})F_{jj} + E_{ij}\Theta(F_{jj})$.

Proof. (i) By using Remark 2.16 that Θ is additive, we have

$$\Theta(I \circledcirc E_{ii} \circledast_\zeta F_{ij}) = \Theta(2E_{ii} \circledast_\zeta F_{ij}) = 2\Theta(E_{ii}F_{ij}).$$

On the other hand, using Lemma 2.15 and $\Theta(I) = 0$, we have

$$\begin{aligned}\Theta(I \circledcirc E_{ii} \circledast_\zeta F_{ij}) =& I \circledcirc \Theta(E_{ii}) \circledast_\zeta F_{ij} + I \circledcirc E_{ii} \circledast_\zeta \Theta(F_{ij})\\ =& 2\Theta(E_{ii}) \circledast_\zeta F_{ij} + 2E_{ii} \circledast_\zeta \Theta(F_{ij})\\ =& 2\Theta(E_{ii})F_{ij} + 2E_{ii}\Theta(F_{ij}).\end{aligned}$$

By comparing the above two equations, we get

$$\Theta(E_{ii}F_{ij}) = \Theta(E_{ii})F_{ij} + E_{ii}\Theta(F_{ij}).$$

(ii) Now, for any $G_{ij} \in \Re_{ij}$, we have from Lemma 2.17, that

$$\begin{aligned}\Theta(E_{ii}F_{ii})G_{ij} + E_{ii}F_{ii}\Theta(G_{ij}) =& \Theta(E_{ii}F_{ii}G_{ij})\\ =& \Theta(E_{ii})F_{ii}G_{ij} + E_{ii}\Theta(F_{ii}G_{ij})\\ =& \Theta(E_{ii})F_{ii}G_{ij} + E_{ii}\Theta(F_{ii})G_{ij} + E_{ii}F_{ii}\Theta(G_{ij}).\end{aligned}$$

Hence $(\Theta(E_{ii}F_{ii}) - \Theta(E_{ii})F_{ii} - E_{ii}\Theta(F_{ii}))G_{ij} = 0$, for any $G_{ij} \in \Re_{ij}$, It follows that

$$\Theta(E_{ii}F_{ii}) = \Theta(E_{ii})F_{ii} + E_{ii}\Theta(F_{ii}).$$

(iii) By using Remark 2.16 that Θ is additive, we have

$$\Theta(I \circledcirc E_{ij} \circledast_\zeta F_{ji}) = \Theta(2E_{ij} \circledast_\zeta F_{ji}) = 2\Theta(E_{ij}F_{ji}).$$

On the other hand, using Lemma 2.15 and $\Theta(I) = 0$, we have

$$\begin{aligned}\Theta(I \circledcirc E_{ij} \circledast_\zeta F_{ji}) =& I \circledcirc \Theta(E_{ij}) \circledast_\zeta F_{ji} + I \circledcirc E_{ij} \circledast_\zeta \Theta(F_{ji})\\ =& 2\Theta(E_{ij}) \circledast_\zeta F_{ji} + 2E_{ij} \circledast_\zeta \Theta(F_{ji})\\ =& 2\Theta(E_{ij})F_{ji} + 2E_{ij}\Theta(F_{ji}).\end{aligned}$$

By comparing the above two equations, we get

$$\Theta(E_{ij}F_{ji}) = \Theta(E_{ij})F_{ji} + E_{ij}\Theta(F_{ji}).$$

(iv) For any $G_{ji} \in \Re_{ji}$ $(i \neq j)$, we have from Lemma 2.17, that

$$\begin{aligned}G_{ji}\Theta(E_{ij}F_{jj}) + \Theta(G_{ji})E_{ij}F_{jj} =& \Theta(G_{ji}E_{ij}F_{jj})\\ =& \Theta(G_{ji}E_{ij})F_{jj} + G_{ji}E_{ij}\Theta(F_{jj})\\ =& \Theta(G_{ji})E_{ij}F_{jj} + G_{ji}\Theta(E_{ij})F_{jj} + G_{ji}E_{ij}\Theta(F_{jj}).\end{aligned}$$

Hence $G_{ji}(\Theta(E_{ij}F_{jj}) - \Theta(E_{ij})F_{jj} - E_{ij}\Theta(F_{jj})) = 0$, for any $G_{ji} \in \Re_{ji}$. It follows that

$$\Theta(E_{ij}F_{jj}) = \Theta(E_{ij})F_{jj} + E_{ij}\Theta(F_{jj}).$$

□

Lemma 2.19. Ψ *is an additive derivation.*

Proof. It follows from Lemma 2.17 that $0 = \Theta(E_{ij}F_{kl}) = \Theta(E_{ij})F_{kl} + E_{ij}\Theta(F_{kl})$, if $i, j, k, l \in \{1, 2\}$, $j \neq k$. By Lemma 2.18 and additivity of Θ, it is easy to show that Θ is an additive derivation. Hence Ψ is an additive derivation. □

Proof of Theorem 2.1. Now, by the Remark 2.16 and Lemma 2.19, we get that Ψ is an additive $*$-derivation and $\Psi(\zeta E) = \zeta\Psi(E)$ for all $E \in \Re$.

□

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