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GENERALIZED SLANT H-TOEPLITZ OPERATORS ON THE BERGMAN SPACE

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ABSTRACT. This paper introduces the concept of generalized slant H-Toeplitz operators on the Bergman space. It entails obtaining the matrix representation of the operator corresponding to the harmonic symbols. Additionally, the paper delves into the algebraic properties of the operator, presents its invariant space, and studies its Berezin transform. Through this study, we provide a deeper understanding of the generalized slant H-Toeplitz operators on the Bergman space, opening avenues for further research and insights.

1. Introduction

Let $\mathbb{D}$ represent the open unit disc in the complex plane $\mathbb{C}$. Let dA be the normalized two dimensional Lebesgue measure on $\mathbb{D}$. Consider the Banach space $L^2(\mathbb{D}, dA)$ comprising complex-valued, measurable functions f on $\mathbb{D}$ that satisfy $\|f\|^2 = \int_{\mathbb{D}} |f(z)|^2 dA(z) < \infty$. The space $L^2(\mathbb{D}, dA)$ forms the Hilbert space with the inner product defined as

$$\langle f, g\rangle = \int_{\mathbb{D}} f(z)\overline{g(z)}dA(z),$$

for all f *and* g in $L^2(\mathbb{D}, dA)$. The Bergman space, denoted by $A^2(\mathbb{D})$, is defined as the set of analytic functions in $L^2(\mathbb{D}, dA)$. An alternative description of the Bergman space is given by

$$A^2(\mathbb{D}) = \Big\{ f(z) = \sum a_n z^n : \sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1} < \infty \Big\}.$$

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$A^2(\mathbb{D})$ being a closed subspace of the Hilbert space $L^2(\mathbb{D}, dA)$, is a Hilbert space itself. It is known that $\{\varphi_i(z)\}_{i=0}^{\infty}$, where $\varphi_i(z) = \sqrt{i+1}\, z^i$ for $z \in \mathbb{D}$, forms an orthonormal basis for the space $A^2(\mathbb{D})$. For a given $z \in \mathbb{D}$, the function defined on $\mathbb{D}$ by $h_z(w) = \dfrac{1}{(1-\overline{z}w)^2}$, is called reproducing kernel for z in $A^2(\mathbb{D})$. Let P denote the orthogonal projection from $L^2(\mathbb{D}, dA)$ onto $A^2(\mathbb{D})$. Given that $P(f) \in A^2(\mathbb{D})$ for $f \in L^2(\mathbb{D}, dA)$, using the property of reproducing kernel h_z, we obtain

$$P(f)(z) = \langle Pf, h_z \rangle = \int_{\mathbb{D}} \frac{f(w)}{(1-z\overline{w})^2} dA(w).$$

The space $L^\infty(\mathbb{D}, dA)$ is the Banach space of Lebesgue measurable functions f on $\mathbb{D}$ with $\|f\|_\infty = ess\,sup\{|f(z)| : z \in \mathbb{D}\} < \infty$. For a function $\psi \in L^\infty(\mathbb{D}, dA)$, the multiplication operator M_ψ is defined as the operator $M_\psi : L^2(\mathbb{D}, dA) \to L^2(\mathbb{D}, dA)$ such that $M_\psi(f) = \psi f$ for all $f \in L^2(\mathbb{D}, dA)$. The Toeplitz operator T_ψ on the Bergman space with the symbol ψ is given by

$$T_\psi(f) = PM_\psi(f) \quad \text{for all} \quad f \in A^2(\mathbb{D}).$$

Consider the flip operator J on $A^2(\mathbb{D})$ defined by $J(\varphi_i(z)) = \overline{\varphi_{i+1}(z)}$ for all nonnegative integers i. For the given symbol ψ, the Hankel operator H_ψ on $A^2(\mathbb{D})$ is defined by $H_\psi(f) = PM_\psi J(f)$ for all $f \in A^2(\mathbb{D})$.

In 1979, McDonald and Sundberg [7] initiated the discussion on Toeplitz operators in the setting of Bergman space. Subsequently, various properties of both Toeplitz and Hankel operators have been studied extensively by several other authors such as Axler, Stroethoff, Zeng [2, 8, 9, 11]. Over the years, mathematicians have extended and generalized these operators, leading to the exploration of intriguing subclasses with unique properties. In this context, the concept of slant Toeplitz operators introduced by Ho [5] in 1996 and the subsequent characterization of slant Hankel operators by Arora and collaborators [1] in 2006 have added valuable insights and applications, particularly in areas like data compression, interpolation problems, and perturbation theory.

This paper builds upon these advancements by introducing and investigating the generalized slant H-Toeplitz operator on the Bergman space. The generalized slant H-Toeplitz operator extends the study of the slant H-Toeplitz operators conducted by Gupta and Singh [4] on the Hardy space. We aim to provide a matrix representation for the harmonic symbol associated with the generalized slant H-Toeplitz operator, offering a concrete mathematical foundation for its study. Our exploration includes the examination of algebraic properties inherent to this operator, shedding light on its behavior within the Bergman space. Moreover, we present an invariant subspace of the operator, where the application of the generalized slant H-Toeplitz operator keeps the fundamental characteristics of the subspace unchanged. Furthermore, the paper delves into the discussion of the Berezin transform for the generalized slant H-Toeplitz operator, connecting our findings to the broader context of operator theory. Through this study, we contribute to the ongoing research in operator theory, providing a deeper understanding of the generalized slant H-Toeplitz operator on the Bergman space.

2. Generalized slant H-Toeplitz operators

In this section, we define the generalized slant H-Toeplitz operator and compute its matrix representation corresponding to the harmonic symbol. Subsequently, we discuss its algebraic properties and find its invariant subspace.

For a positive integer $k \geq 2$, the generalized slant operator, $W_k : A^2(\mathbb{D}) \to A^2(\mathbb{D})$ is defined as follows:

$$W_k(z^j) = \begin{cases} z^i & if\ j = ki \\ 0 & otherwise\ . \end{cases}$$

The operator W_k is a bounded linear operator with $\|W_k\| = \sqrt{k}$. The adjoint of W_k is given by $W_k^*(z^i) = \dfrac{ki+1}{i+1} z^{ki}$, where i is a nonnegative integer.

For any $\psi \in L^\infty(\mathbb{D}, dA)$, the generalized slant Toeplitz operator on the Bergman space $A^2(\mathbb{D})$ is defined as $W_k T_\psi(f) = W_k P M_\psi(f)$ for all $f \in A^2(\mathbb{D})$. Similarly, the generalized slant Hankel operator is defined as $W_k H_\psi(f) = W_k P M_\psi J(f)$ for all $f \in A^2(\mathbb{D})$. To determine the matrix representation of the operators $W_k T_\psi$ and $W_k H_\psi$, the following lemma plays a crucial role.

Lemma 2.1. [6] *For nonnegative-integers i and j, we have*

$$P(z^i \overline{z}^j) = \begin{cases} \dfrac{i-j+1}{i+1} z^{i-j} & if\ \ i \geq j, \\ \\ 0 & if\ \ i < j, \end{cases}$$

where P is an orthogonal projection of $L^2(\mathbb{D}, dA)$ onto $A^2(\mathbb{D})$.

Consider the harmonic symbol, $\psi(z) = \sum_{n=0}^{\infty} \alpha_n z^n + \sum_{m=1}^{\infty} \beta_m \overline{z}^m \in L^\infty(\mathbb{D})$. We find (i,j)th entry of the matrix representation of the generalized slant Toeplitz operator, $W_k T_\psi$ with respect to the orthonormal basis $\{\varphi_i(z)\}_{i=0}^{\infty}$ of $A^2(\mathbb{D})$. If i and j are nonnegative integers, then

$$\begin{aligned} \langle W_k T_\psi(\varphi_j), \varphi_i \rangle &= \langle W_k P M_\psi(\sqrt{j+1} z^j), \sqrt{i+1} z^i \rangle \\ &= \sqrt{i+1}\sqrt{j+1} \Big\langle \Big(\sum_{n=0}^{\infty} \alpha_n z^n + \sum_{m=1}^{\infty} \beta_m \overline{z}^m \Big) z^j, P\, W_k^* z^i \Big\rangle \\ &= (ki+1)\sqrt{\frac{j+1}{i+1}} \Big\langle \Big(\sum_{n=0}^{\infty} \alpha_n z^n + \sum_{m=1}^{\infty} \beta_m \overline{z}^m \Big) z^j, z^{ki} \Big\rangle \\ &= (ki+1)\sqrt{\frac{j+1}{i+1}} \left[\sum_{n=0}^{\infty} \alpha_n \big\langle z^{n+j}, z^{ki} \big\rangle \ + \sum_{m=1}^{\infty} \beta_m \big\langle z^j, z^{m+ki} \big\rangle \right]. \end{aligned} \tag{2.1}$$

Case I: For $ki \geq j$, we get

$$\langle W_k T_\psi(\varphi_j), \varphi_i \rangle \ = \sqrt{\frac{j+1}{i+1}}\ \alpha_{ki-j}. \tag{2.2}$$

Case II: For $ki < j$, we have

$$\langle W_k T_\psi(\varphi_j), \varphi_i\rangle \;= \frac{ki+1}{\sqrt{i+1}\sqrt{j+1}}\,\beta_{j-ki}. \tag{2.3}$$

Using (2.2) and (2.3), we can express the (i, j)th entry $c_{i,j}$ of the generalized slant Toeplitz operator as

$$c_{i,j} = \begin{cases} \sqrt{\dfrac{j+1}{i+1}}\,\alpha_{ki-j} & if \;\; ki \geq j, \\[2ex] \dfrac{ki+1}{\sqrt{i+1}\sqrt{j+1}}\,\beta_{j-ki} & if \;\; ki < j\;. \end{cases}$$

Example 2.2. For $k = 2$, the operator $W_k T_\psi$is called the slant Toeplitz operator. We can represent its matrix as follows:

$$W_2 T_\psi = \begin{bmatrix} \alpha_0 & \frac{1}{\sqrt{2}}\beta_1 & \frac{1}{\sqrt{3}}\beta_2 & \frac{1}{\sqrt{4}}\beta_3 & \frac{1}{\sqrt{5}}\beta_4 & \cdots \\ \frac{1}{\sqrt{2}}\alpha_2 & \alpha_1 & \sqrt{\frac{3}{2}}\alpha_0 & \frac{3}{\sqrt{4}\sqrt{2}}\beta_1 & \frac{3}{\sqrt{5}\sqrt{2}}\beta_2 & \cdots \\ \frac{1}{\sqrt{3}}\alpha_4 & \sqrt{\frac{2}{3}}\alpha_3 & \alpha_2 & \sqrt{\frac{4}{3}}\alpha_1 & \sqrt{\frac{5}{3}}\alpha_0 & \cdots \\ \frac{1}{\sqrt{4}}\alpha_6 & \sqrt{\frac{2}{4}}\alpha_5 & \sqrt{\frac{3}{4}}\alpha_4 & \alpha_3 & \sqrt{\frac{5}{4}}\alpha_2 & \cdots \\ \frac{1}{\sqrt{5}}\alpha_8 & \sqrt{\frac{2}{5}}\alpha_7 & \sqrt{\frac{3}{5}}\alpha_6 & \sqrt{\frac{4}{5}}\alpha_5 & \alpha_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}.$$

Again, to write the (i, j)th entry of generalized slant Hankel operator $W_k H_\psi$,

$$\begin{aligned} &\langle W_k H_\psi(\varphi_j), \varphi_i\rangle \\ &= \langle W_k P M_\psi J(\varphi_j), \varphi_i\rangle \\ &= \langle W_k P M_\psi(\overline{\varphi_{j+1}}), \varphi_i\rangle \\ &= \sqrt{i+1}\sqrt{j+2}\,\Big\langle P\Big(\sum_{n=0}^{\infty}\alpha_n z^n + \sum_{m=1}^{\infty}\beta_m \overline{z}^m\Big)\overline{z}^{j+1}, \frac{ki+1}{i+1}z^{ki}\Big\rangle \\ &= (ki+1)\sqrt{\frac{j+2}{i+1}}\;\Big[\Big\langle P\Big(\sum_{n=0}^{\infty}\alpha_n z^n\;\overline{z}^{j+1}\Big), z^{ki}\Big\rangle + \Big\langle P\sum_{m=1}^{\infty}\beta_m \overline{z}^{m+j+1}, z^{ki}\Big\rangle\Big] \\ &= (ki+1)\sqrt{\frac{j+2}{i+1}}\;\Big\langle \sum_{n=0}^{\infty}\alpha_n z^n, z^{ki+j+1}\Big\rangle \\ &= \frac{ki+1}{ki+j+2}\sqrt{\frac{j+2}{i+1}}\;\alpha_{ki+j+1}. \end{aligned} \tag{2.4}$$

Observe that the operator $W_k H_\psi$ is independent of the choice of co-analytical part of the harmonic symbol ψ. Consequently, two symbols in $L^\infty(\mathbb{D}, A)$ induce the same generalized slant Hankel operator if and only if their Fourier coefficients agree for the nonnegative indices.

The space consisting of all harmonic functions in $L^2(\mathbb{D}, A)$ is referred to as the Harmonic Bergman space and is represented by $B^2(\mathbb{D})$. The Harmonic Bergman space is a closed subspace of $L^2(\mathbb{D}, A)$ and forms Hilbert space with the orthonormal basis $\{\sqrt{n+1}z^n\}_{n=0}^{\infty} \cup \{\sqrt{n+1}\ \overline{z}^n\}_{n=1}^{\infty}$. Let $Q : L^2(\mathbb{D}, A) \to B^2(\mathbb{D})$ denote the orthogonal projection of $L^2(\mathbb{D}, A)$ onto $B^2(\mathbb{D})$.

Lemma 2.3. *The orthogonal projection Q of $L^2(\mathbb{D}, A)$ onto $B^2(\mathbb{D})$ satisfies the following:*

$$Q(\overline{z}^i z^j) = \begin{cases} \dfrac{i-j+1}{i+1}\overline{z}^{i-j} & if \;\; i > j, \\ \\ \dfrac{j-i+1}{j+1} z^{j-i} & if \;\; j \geq i, \end{cases}$$

for nonnegative integers i and j.

Proof. The orthogonal projection Q of $L^2(\mathbb{D}, A)$ onto $B^2(\mathbb{D})$ can be represented as $Q(f) = P(f) + \overline{P(\overline{f})} - Pf(0) \quad for \quad f \in L^2(\mathbb{D}, A)$. From this, we obtain

$$Q(\overline{z}^i z^j) = P(\overline{z}^i z^j) + \overline{P(z^i \overline{z}^j)}, \tag{2.5}$$

for nonnegative integers i and j.
Case I: Let $i > j$, then by Lemma 2.1, we get $P(\overline{z}^i z^j) = 0$ and $P(z^i \overline{z}^j) = \dfrac{i-j+1}{i+1} z^{i-j}$. This implies $Q(\overline{z}^i z^j) = \dfrac{i-j+1}{i+1}\overline{z}^{i-j}$.
Case II: Let $j \geq i$, then $P(\overline{z}^i z^j) = \dfrac{j-i+1}{j+1} z^{j-i}$ and $P(z^i \overline{z}^j) = 0$. Thus, using (2.5), we obtain $Q(\overline{z}^i z^j) = \dfrac{j-i+1}{j+1} z^{j-i}$. □

Define the operator, $K : A^2(\mathbb{D}) \to B^2(\mathbb{D})$ as follows:

$$K(\varphi_{2i})(z) = \varphi_i(z) \;\; and \;\; K(\varphi_{2i+1})(z) = \overline{\varphi_{i+1}(z)},$$

where i is a nonnegative integer and $z \in \mathbb{D}$. The operator K is a bounded linear operator on the Bergman space with $\|K\| = 1$. The adjoint K^* of the operator K is given by : $K^* : B^2(\mathbb{D}) \to A^2(\mathbb{D})$ by $K^*(\varphi_i)(z) = \varphi_{2i}(z)$ and $K^*(\overline{\varphi_{i+1}(z)}) = \varphi_{2i+1}(z)$, for all nonnegative integers i. Thus, from the definitions of K and K^*, we get that $KK^* = I_{B^2(\mathbb{D})}$ and $K^*K = I_{A^2(\mathbb{D})}$.

Inspired by the definitions of the generalized slant Toeplitz operator and generalized slant Hankel operator, we define the generalized slant H-Toeplitz operator as follows:

Definition 2.4. For $\psi \in L^\infty(\mathbb{D}, A)$, the generalized slant H-Toeplitz operator is defined on the Bergman space as the operator, $S_\psi^k : A^2(\mathbb{D}) \to A^2(\mathbb{D})$ such that $S_\psi^k(f) = W_k P M_\psi K(f)$ for all $f \in A^2(\mathbb{D})$.

From the definition of the generalized slant H-Toeplitz operator, it follows that the operator S_ψ^k is a bounded linear operator on $A^2(\mathbb{D})$ and is also linear with respect to the corresponding symbol.

Theorem 2.5. *The operator S_ψ^k satisfies the following properties:*
(i) S_ψ^k is a bounded linear operator on $A^2(\mathbb{D})$ with $\|S_\psi^k\| \leq \sqrt{k}\|\psi\|_\infty$, for $\psi \in$

$L^\infty(\mathbb{D}, dA)$.
(ii) For any scalars, λ, λ' *and functions* ψ, ψ' *in* $L^\infty(\mathbb{D}, dA)$, $S^k_{\lambda\psi+\lambda'\psi'} = \lambda S^k_\psi + \lambda' S^k_{\psi'}$.

Remark 2.6. For the harmonic symbol, $\psi(z) = \sum_{n=0}^{\infty} \alpha_n z^n + \sum_{m=1}^{\infty} \beta_m \overline{z}^m \in L^\infty(\mathbb{D})$, we find the (i, j)th entry of the matrix of generalized slant H-Toeplitz operator as follows:
(i) For $j = 2l$, $S^k_\psi(\varphi_j) = W_k P M_\psi K(\varphi_{2l}) = W_k P M_\psi(\varphi_l) = W_k T_\psi(\varphi_l)$.
Using (2.2) and (2.3), we get that
$\langle S^k_\psi(\varphi_j), \varphi_i\rangle = \sqrt{\frac{l+1}{i+1}}\alpha_{ki-l}$ if $ki \geq l$ and $\langle S^k_\psi(\varphi_j), \varphi_i\rangle = \frac{ki+1}{\sqrt{i+1}\sqrt{l+1}}\beta_{l-ki}$ if $ki < l$.
(ii) For $j = 2l+1$, $S^k_\psi(\varphi_j) = W_k P M_\psi K(\varphi_{2l+1}) = W_k P M_\psi(\overline{\varphi_{l+1}}) = W_k P M_\psi J(\varphi_l) = W_k H_\psi(\varphi_l)$.

Therefore, from (2.4) we obtain $\langle S^k_\psi(\varphi_j), \varphi_i\rangle = \frac{ki+1}{ki+l+2}\sqrt{\frac{l+2}{i+1}}\alpha_{ki+l+1}$.

This derivation serves as motivation for introducing the matrix of the generalized slant H-Toeplitz operator as follows: For a harmonic symbol $\psi(z) = \sum_{n=0}^{\infty} \alpha_n z^n + \sum_{m=1}^{\infty} \beta_m \overline{z}^m \in L^\infty(\mathbb{D})$, the matrix representation of generalized slant H-Toeplitz has (i, j)th entry $s_{i,j}$ satisfying

$$s_{i,j} = \begin{cases} \dfrac{ki+1}{\sqrt{i+1}\sqrt{l+1}}\beta_{l-ki} & for\ j = 2l,\ l > ki, \\[2ex] \sqrt{\dfrac{l+1}{i+1}}\alpha_{ki-l} & for\ j = 2l,\ l \leq ki, \\[2ex] \dfrac{ki+1}{ki+l+2}\sqrt{\dfrac{l+2}{i+1}}\alpha_{ki+l+1} & for\ j = 2l+1. \end{cases}$$

Example 2.7. For $k = 2$, the operator S^k_ψ is called slant H-Toeplitz operator and its matrix with respect to the orthonormal basis $\{\varphi_i(z)\}_{i=0}^\infty$ of $A^2(\mathbb{D})$ for the symbol $\psi(z) = \sum_{n=0}^{\infty} \alpha_n z^n + \sum_{m=1}^{\infty} \beta_m \overline{z}^m$ is represented as

$$S^2_\psi = \begin{bmatrix} \alpha_0 & \frac{1}{\sqrt{2}}\alpha_1 & \frac{1}{\sqrt{2}}\beta_1 & \frac{1}{\sqrt{3}}\alpha_2 & \frac{1}{\sqrt{3}}\beta_2 & \cdots \\ \frac{1}{\sqrt{2}}\alpha_2 & \frac{3}{4}\alpha_3 & \alpha_1 & \frac{3}{5}\sqrt{\frac{3}{2}}\alpha_4 & \sqrt{\frac{3}{2}}\alpha_0 & \cdots \\ \frac{1}{\sqrt{3}}\alpha_4 & \frac{5}{6}\sqrt{\frac{2}{3}}\alpha_5 & \sqrt{\frac{2}{3}}\alpha_3 & \frac{5}{7}\alpha_6 & \alpha_2 & \cdots \\ \frac{1}{\sqrt{4}}\alpha_6 & \frac{7}{8}\sqrt{\frac{2}{4}}\alpha_7 & \sqrt{\frac{2}{4}}\alpha_5 & \frac{7}{9}\sqrt{\frac{3}{4}}\alpha_8 & \sqrt{\frac{3}{4}}\alpha_4 & \cdots \\ \frac{1}{\sqrt{5}}\alpha_8 & \frac{9}{10}\sqrt{\frac{2}{5}}\alpha_9 & \sqrt{\frac{2}{5}}\alpha_7 & \frac{9}{11}\sqrt{\frac{3}{5}}\alpha_{10} & \sqrt{\frac{3}{5}}\alpha_6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}.$$

The next theorem shows that for the harmonic symbol ψ, the mapping $\psi \to S_\psi^k$ is one to one, where $\mathfrak{B}(A^2(\mathbb{D}))$ denotes the set of all bounded linear operators on $A^2(\mathbb{D})$.

Theorem 2.8. *The mapping $\Gamma : B^2(\mathbb{D}) \to \mathfrak{B}(A^2(\mathbb{D}))$ defined by $\Gamma(\psi) = S_\psi^k$ is one-one.*

Proof. Let $\psi(z) = \sum_{n=0}^{\infty} \alpha_n z^n + \sum_{m=1}^{\infty} \beta_m \overline{z}^m$ and $\psi'(z) = \sum_{n=0}^{\infty} \alpha'_n z^n + \sum_{m=1}^{\infty} \beta'_m \overline{z}^m$ be two harmonic functions. Assume that $S_\psi^k = S_{\psi'}^k$, which implies that $S_{\psi-\psi'}^k = 0$. Consequently, $S_{\psi-\psi'}^k(\varphi_i) = 0$, for each $i \geq 0$. This leads to $\langle S_{\psi-\psi'}^k(\varphi_j), \varphi_i\rangle = 0$ for nonnegative integers i, j. In particular,
$\langle S_{\psi-\psi'}^k(\varphi_{2j}), \varphi_0\rangle = 0$ for arbitrary $j > 0$. Thus, $\frac{1}{\sqrt{j+1}}(\beta_j - \beta'_j) = 0$ for each $j > 0$ which implies that $\beta_j = \beta'_j$ for $j \geq 1$. Again, for odd numbered columns, we have $\langle S_{\psi-\psi'}^k(\varphi_{2j+1}), \varphi_0\rangle = 0$ for arbitrary $j > 0$. That is $\frac{1}{\sqrt{j+2}}(\alpha_j - \alpha'_j) = 0$ or $\alpha_j = \alpha'_j$ for all $j > 0$. Also, $\langle S_{\psi-\psi'}^k(\varphi_0), \varphi_0\rangle = 0$ implies that $\alpha_0 = \alpha'_0$. Therefore, $\psi = \psi'$ demonstrating that the map Γ is one-one. □

An operator A is termed a diagonal operator if $\langle A\varphi_i, \varphi_j\rangle = 0$ for all nonnegative integers i, j such that $i \neq j$. It is observed that for the harmonic symbol ψ, the operator S_ψ^k is a diagonal operator only when its symbol ψ is zero.

Theorem 2.9. *The operator S_ψ^k is a diagonal operator if and only if the harmonic function ψ is identically zero.*

Proof. For $\psi = 0$, the operator S_ψ^k is trivially a diagonal operator. Let S_ψ^k be the diagonal operator on the Bergman space with the symbol
$\psi(z) = \sum_{n=0}^{\infty} \alpha_n z^n + \sum_{m=1}^{\infty} \beta_m \overline{z}^m \in L^\infty(\mathbb{D})$. Then $\langle S_\psi^k(\varphi_i), \varphi_j\rangle = 0$ for all nonnegative integers i, j such that $i \neq j$.
Case I: For $i = 2l$, we get

$$\langle S_\psi^k(\varphi_{2l}), \varphi_j\rangle = \begin{cases} \dfrac{kj+1}{\sqrt{j+1}\sqrt{l+1}}\beta_{l-kj} & for\ l > kj, \\[2ex] \sqrt{\dfrac{l+1}{j+1}}\alpha_{kj-l} & for\ l \leq kj. \end{cases}$$

Case II: For $i = 2l+1$, we have $\langle S_\psi^k(\varphi_{2l+1}), \varphi_j\rangle = \frac{kj+1}{kj+l+2}\sqrt{\frac{l+2}{j+1}}\alpha_{kj+l+1}$.

Since $\langle S_\psi^k(\varphi_i), \varphi_j\rangle = 0$, from Cases I and II, we conclude that the values of the coefficients α_n and β_m of the harmonic symbol ψ must be zero. This result establishes that ψ is identically zero. □

Theorem 2.10. *For the harmonic symbol* $\psi(z) = \sum_{n=0}^{\infty} \alpha_n z^n + \sum_{m=1}^{\infty} \beta_m \overline{z}^m \in L^\infty(\mathbb{D})$, *the adjoint of the generalized slant H-Toeplitz operator* S_ψ^k *is given by* $S_\psi^{k^*} = K^* Q M_{\overline{\psi}} W_k^*$, *where* Q *is the orthogonal projection* $L^2(\mathbb{D}, A)$ *onto* $B^2(\mathbb{D})$.

Proof. Computing the (i,j)th entry of the matrix of the operator $K^* Q M_{\overline{\psi}} W_k^*$ for the symbol $\psi(z) = \sum_{n=0}^{\infty} \alpha_n z^n + \sum_{m=1}^{\infty} \beta_m \overline{z}^m \in L^\infty(\mathbb{D})$, we obtain the following:

$$\begin{aligned}
\langle K^* Q M_{\overline{\psi}} W_k^* \varphi_j, \varphi_i \rangle &= \big\langle K^* Q M_{\overline{\psi}} W_k^* \sqrt{j+1} z^j, \sqrt{i+1} z^i \big\rangle \\
&= \sqrt{(i+1)(j+1)} \big\langle K^* Q M_{\overline{\psi}} z^{kj}, z^i \big\rangle \\
&= \frac{(kj+1)\sqrt{i+1}}{\sqrt{j+1}} \Big\langle K^* Q \big(\sum_{n=0}^{\infty} \overline{\alpha_n} z^{kj} \overline{z}^n + \sum_{m=1}^{\infty} \overline{\beta_m} z^{m+kj} \big), z^i \Big\rangle \\
&= \frac{(kj+1)\sqrt{i+1}}{\sqrt{j+1}} \Big\langle \sum_{n=0}^{kj} \frac{kj-n+1}{kj+1} \overline{\alpha_n} z^{kj-n} \\
&\quad + \sum_{n=kj+1}^{\infty} \frac{n-kj+1}{n+1} \overline{\alpha_n}\, \overline{z}^{n-kj} + \sum_{m=1}^{\infty} \overline{\beta_m} z^{m+kj}, K z^i \Big\rangle \\
&= \begin{cases} \sqrt{\dfrac{l+1}{j+1}} \overline{\alpha}_{kj-l}, & i = 2l,\ l \le kj, \\[2ex] \dfrac{kj+1}{\sqrt{(j+1)(l+1)}} \overline{\beta}_{l-kj}, & i = 2l,\ l > kj, \\[2ex] \dfrac{kj+1}{kj+l+2} \sqrt{\dfrac{l+2}{j+1}} \overline{\alpha}_{kj+l+1}, & i = 2l+1, \end{cases}
\end{aligned}$$

which is same as the (i,j)th entry of the matrix of the operator $S_\psi^{k^*}$. □

Proposition 2.11. *Let* $\psi \in L^\infty(\mathbb{D})$ *be a co-analytic function; then the generalized slant H-Toeplitz operator on the Bergman space satisfies* $S_\psi^k S_\psi^{k^*} = \frac{kl+1}{l+1} M_{|\psi|^2}$.

Proof. Using Theorem 2.10, we obtain the following expression for a co-analytic function ψ in the space $L^\infty(\mathbb{D})$ and a nonnegative integer l:

$$\begin{aligned}
S_\psi^k S_\psi^{k^*}(z^l) &= W_k P M_\psi K K^* Q M_{\overline{\psi}} W_k^*(z^l) \\
&= W_k P M_\psi Q M_{\overline{\psi}} W_k^*(z^l) \\
&= \frac{kl+1}{l+1} W_k P M_\psi Q M_{\overline{\psi}}(z^{kl}) \\
&= \frac{kl+1}{l+1} W_k P M_{|\psi|^2}(z^{kl}) \\
&= \frac{kl+1}{l+1} M_{|\psi|^2}(z^l).
\end{aligned}$$

□

An operator T is called a Hilbert–Schmidt operator if its Hilbert–Schmidt norm is finite. The following theorem shows that there is no nonzero generalized H-Toeplitz operator that is Hilbert–Schmidt.

Theorem 2.12. *For an essentially bounded harmonic function ψ on $\mathbb{D}$, the operator S^k_ψ is Hilbert–Schmidt operator if and only if $\psi = 0$.*

Proof. Consider $\psi(z) = \sum_{n=0}^{\infty}\alpha_n z^n + \sum_{m=1}^{\infty}\beta_m \overline{z}^m \in L^\infty(\mathbb{D})$. Let S^k_ψ be Hilbert–Schmidt operator. Using the definition of the operator, we get

$$\begin{aligned}
\sum_{j=0}^{\infty}\left|\left|S^k_\psi\varphi_j\right|\right|^2 =& ||S^k_\psi\varphi_0||^2 + \sum_{j=1}^{\infty}\left|\left|S^k_\psi\varphi_{2j}\right|\right|^2 + \sum_{j=0}^{\infty}\left|\left|S^k_\psi\varphi_{2j+1}\right|\right|^2\\
=&\left|\left|W_k P M_\psi\varphi_0\right|\right|^2 + \sum_{j=1}^{\infty}\left|\left|W_k P M_\psi\varphi_j\right|\right|^2 + \sum_{j=0}^{\infty}\left|\left|W_k P M_\psi\overline{\varphi}_{j+1}\right|\right|^2\\
=&\left|\left|W_k P\sum_{n=0}^{\infty}\alpha_n z^n\right|\right|^2\\
&+\sum_{j=1}^{\infty}(j+1)\left|\left|W_k\sum_{n=0}^{\infty}\alpha_n z^{n+j} + W_k P\sum_{m=1}^{\infty}\beta_m\overline{z}^m z^j\right|\right|^2\\
&+\sum_{j=0}^{\infty}(j+2)\left|\left|W_k P\sum_{n=0}^{\infty}\alpha_n\overline{z}^{j+1}z^n\right|\right|^2\\
=&\left|\left|W_k\sum_{n=0}^{\infty}\alpha_n z^n\right|\right|^2\\
&+\sum_{j=1}^{\infty}(j+1)\left|\left|W_k\sum_{n=j}^{\infty}\alpha_{n-j}\, z^n + W_k\sum_{m=1}^{j}\frac{j-m+1}{j+1}\beta_m z^{j-m}\right|\right|^2\\
&+\sum_{j=0}^{\infty}(j+2)\left|\left|W_k\sum_{n=j+1}^{\infty}\frac{n-j}{n+1}\alpha_n z^{n-j-1}\right|\right|^2\\
=&||\sum_{n=0}^{\infty}\alpha_{kn}z^n||^2\\
&+\sum_{j=1}^{\infty}(j+1)\left|\left|W_k\sum_{n=j}^{\infty}\alpha_{n-j}\, z^n + W_k\sum_{m=0}^{j-1}\frac{m+1}{j+1}\beta_{j-m}\, z^m\right|\right|^2\\
&+\sum_{j=0}^{\infty}(j+2)\left|\left|W_k\sum_{n=0}^{\infty}\frac{n+1}{n+j+2}\alpha_{n+j+1}\, z^n\right|\right|^2\\
=&\frac{|\alpha_{kn}|^2}{n+1} + \sum_{j=1}^{\infty}(j+1)\left|\left|W_k\sum_{n=j}^{\infty}\alpha_{n-j}\, z^n + W_k\sum_{m=0}^{j-1}\frac{m+1}{j+1}\beta_{j-m}\, z^m\right|\right|^2
\end{aligned}$$

$$+\sum_{j=0}^{\infty}(j+2)\Big|\Big|\sum_{n=0}^{\infty}\frac{kn+1}{kn+j+2}\alpha_{kn+j+1}\,z^n\Big|\Big|^2. \tag{2.6}$$

Case I: j is a multiple of k. Then from (2.6), we get the following:

$$\begin{aligned}\sum_{j=0}^{\infty}\big|\big|S_\psi^k\varphi_j\big|\big|^2 =&\frac{|\alpha_{kn}|^2}{n+1}+\sum_{j=1}^{\infty}(j+1)\Big|\Big|\sum_{n=\frac{j}{k}}^{\infty}\alpha_{kn-j}\,z^n+\sum_{m=0}^{\frac{j}{k}-1}\frac{km+1}{j+1}\beta_{j-km}\,z^m\Big|\Big|^2\\ &+\sum_{j=0}^{\infty}(j+2)\Big|\Big|\sum_{n=0}^{\infty}\frac{kn+1}{kn+j+2}\alpha_{kn+j+1}\,z^n\Big|\Big|^2\\ =&\sum_{j=0}^{\infty}(j+1)\sum_{n=\frac{j}{k}}^{\infty}\frac{\big|\alpha_{kn-j}\big|^2}{n+1}+\sum_{j=1}^{\infty}(j+1)\sum_{m=0}^{\frac{j}{k}-1}\Big(\frac{km+1}{j+1}\Big)^2\frac{\big|\beta_{j-km}\big|^2}{m+1}\\ &+\sum_{j=0}^{\infty}(j+2)\sum_{n=0}^{\infty}\Big(\frac{kn+1}{kn+j+2}\Big)^2\frac{\big|\alpha_{kn+j+1}\big|^2}{n+1}.\end{aligned} \tag{2.7}$$

Case II: j is not a multiple of k. Again using (2.6), we obtain

$$\begin{aligned}\sum_{j=0}^{\infty}\big|\big|S_\psi^k\varphi_j\big|\big|^2 =&\sum_{j=0}^{\infty}\Bigg[\sum_{n=\left\lfloor\frac{j}{k}\right\rfloor+1}^{\infty}\frac{j+1}{n+1}\big|\alpha_{kn-j}\big|^2+\sum_{m=0}^{\left\lfloor\frac{j}{k}\right\rfloor}\frac{(km+1)^2\big|\beta_{j-km}\big|^2}{(m+1)(j+1)}\\ &+\sum_{n=0}^{\infty}\Big(\frac{kn+1}{kn+j+2}\Big)^2\,\frac{j+2}{n+1}\big|\alpha_{kn+j+1}\big|^2\Bigg].\end{aligned} \tag{2.8}$$

Since $\sum_{j=0}^{\infty}||S_\psi^k\varphi_j||^2<\infty$, from (2.7) and (2.8), we deduce that the jth term of the series on the right hand side (in either case) tends to zero. This implies that $\alpha_n=0$ and $\beta_m=0$ for $n\geq 0, m\geq 1$. Consequently, the symbol $\psi=0$. □

We know that a subspace $\mathcal{V}$ is said to be invariant under a linear operator T on a Hilbert space $\mathcal{H}$, if $T(\mathcal{V})\subseteq\mathcal{V}$. If both $\mathcal{V}$ and $\mathcal{V}^{\perp}$ are invariant under T, then $\mathcal{V}$ is termed a reducing subspace for T. In the upcoming theorem, we find an invariant subspace of S_ψ^k, where the symbol ψ takes the form $\psi(z)=\sum_{i=-\infty}^{(k-1)N}c_iz^i$, with N as a nonnegative integer. Let $V=\text{Span}\,\{z^l:0\leq l\leq N\}$. This subspace V is a closed subspace of $A^2(\mathbb{D})$. We demonstrate that V is invariant under S_ψ^k but does not qualify as a reducing subspace for it.

Theorem 2.13. *For the symbol* $\psi(z) = \sum_{i=-\infty}^{(k-1)N} c_i z^i \in L^\infty(\mathbb{D}, dA)$*, the subspace* $V = Span\{z^l : 0 \le l \le N\}$ *of* $A^2(\mathbb{D})$ *is invariant under* S_ψ^k*.*

Proof. **Case I**: Let $0 < i < \infty$. We will compute $S_{\overline{z}^i}^k(z^l)$, for $0 \le l \le N$.
If $l = 2m$. Then, $S_{\overline{z}^i}^k(z^l) = S_{\overline{z}^i}^k(z^{2m}) = W_k P M_{\overline{z}^i} K(z^{2m}) = W_k P M_{\overline{z}^i}(\sqrt{\frac{m+1}{2m+1}} z^m)$ $= \sqrt{\frac{m+1}{2m+1}} W_k P(\overline{z}^i z^m)$. Using Lemma 2.1, we find that $S_{\overline{z}^i}^k(z^{2m}) = 0$ for $i > m$. For $i \le m$, when k divides $m - i$, we have $S_{\overline{z}^i}^k(z^{2m}) = \frac{m-i+1}{\sqrt{(2m+1)(m+1)}} z^{\frac{m-i}{k}}$ and zero otherwise.
Also for $l = 2m+1$, $S_{\overline{z}^i}^k(z^{2m+1}) = W_k P M_{\overline{z}^i} K(z^{2m+1}) = \sqrt{\frac{m+2}{2(m+1)}} W_k P(\overline{z}^i \overline{z}^{m+1}) =$ 0. Therefore, for both values of l, even or odd, $S_{\overline{z}^i}^k(z^l) \in V$ for $0 < i < \infty$ and $0 \le l \le N$.
Case II: For $0 \le i \le (k-1)N$, we have
$S_{z^i}^k(z^{2m+1}) = \sqrt{\frac{m+2}{2m+2}} W_k P(z^i \overline{z}^{m+1}) = \frac{i-m}{i+1}\sqrt{\frac{m+2}{2m+2}} z^{\frac{i-(m+1)}{k}}$ if $i \ge m+1, k$ divides $i-(m+1)$ and 0 otherwise. Also, $S_{z^i}^k(z^{2m}) = \sqrt{\frac{m+1}{2m+1}} z^{\frac{i+m}{k}}$ if $i+m$ is divisible by k, and zero otherwise. Thus, in any case, $S_{z^i}^k(z^l) \in V$ for $-\infty < i < (k-1)N$, $0 \le l \le N$.
Since the generalized slant H-Toeplitz is linear in its symbol, using Theorem 2.5, we get that $S_\psi^k(z^l) = W_k P M_\psi K(z^l) = \sum_{i=-\infty}^{(k-1)N} c_i W_k P M_{z^i} K(z^l) = \sum_{i=-\infty}^{(k-1)N} c_i S_{z^i}^k(z^l) \in$ V for $0 \le l \le N$. As S_ψ^k is a linear operator, we get that $S_\psi^k(f) \in V$ for any $f \in V$ and hence the subspace V is invariant under the generalized slant H-Toeplitz operator. □

The following example shows that $S_\psi^k(V^\perp) \nsubseteq V^\perp$ and hence the space V is not a reducing subspace for the operator S_ψ^k.

Example 2.14. Consider S_ψ^2, for $\psi = \overline{z}^{(N-2)} \in L^\infty(\mathbb{D}, dA)$, where $N > 2$. Then, for $z^{2N} \in V^\perp$, we get that

$$\begin{aligned} S_{\overline{z}^{(N-2)}}^2(z^{2N}) &= W_2 P M_{\overline{z}^{(N-2)}} K(z^{2N}) \\ &= \sqrt{\frac{N+1}{2N+1}} W_2 P M_{\overline{z}^{(N-2)}}(z^N) = \sqrt{\frac{3}{(N+1)(2N+1)}} W_2(z^2) \\ &= \sqrt{\frac{3}{(N+1)(2N+1)}}\, z \notin V^\perp. \end{aligned}$$

This implies that $S_\psi^2(V^\perp) \nsubseteq V^\perp$.

We know that, a closed subspace $\mathcal{V}$ is invariant under a linear operator T if and only if $\mathcal{V}^{\perp}$ is invariant under T^{*}. This leads to the following corollary:

Corollary 2.15. *For the symbol* $\psi(z) = \sum_{i=-\infty}^{(k-1)N} c_i z^i \in L^{\infty}(\mathbb{D}, dA)$, *the subspace* $V^{\perp}$ $\big($*for given* $V = Span\,\{z^l : 0 \leq l \leq N\}$ $\big)$ *is invariant under* $S_{\psi}^{k\,*}$.

In particular, for $\psi = z^N$, the following example demonstrates that the subspace $V^{\perp}$ is invariant under ${S_{z^N}^{k}}^{*}$, the adjoint of the operator $S_{z^N}^{k}$.

Example 2.16. The subspace $V^{\perp}$ of the Bergman space $A^2(\mathbb{D})$ is invariant under the operator ${S_{z^N}^{k}}^{*}$.

For $l > N$, ${S_{z^N}^{k}}^{*}(z^l) = \dfrac{kl+1}{l+1} K^{*} Q M_{\overline{z}^N}(z^{kl})$. Since $kl > N$, we have

$$\begin{aligned} {S_{z^N}^{k}}^{*}(z^l) &= \frac{kl+1}{l+1} K^{*} Q(z^{kl}\overline{z}^N) \\ &= \frac{kl-N+1}{l+1} K^{*}(z^{kl-N}) \\ &= \frac{kl-N+1}{l+1}\sqrt{\frac{2(kl-N)+1}{kl-N+1}}\, z^{2(kl-N)} \\ &= C z^{2(kl-N)}, \qquad\qquad \textit{where } C = \frac{\sqrt{2(kl-N)+1}\sqrt{kl-N+1}}{l+1}. \end{aligned}$$

For $f \in V^{\perp}$, using this relation for nonnegative integer $i \leq N$, we get

$$\left\langle {S_{z^N}^{k}}^{*} f(z), z^i \right\rangle = \left\langle {S_{z^N}^{k}}^{*}\Big(\sum_{l=N+1}^{\infty} c_l z^l\Big), z^i \right\rangle = \Big\langle \sum_{l=N+1}^{\infty} C z^{2(kl-N)}, z^i \Big\rangle = 0.$$

This shows that ${S_{z^N}^{k}}^{*}(V^{\perp}) \subseteq V^{\perp}$ establishing that $V^{\perp}$ is invariant under the operator ${S_{z^N}^{k}}^{*}$.

3. Berezin transform of generalized slant H-Toeplitz operators

The Berezin transform has proven to be a valuable tool in the examination of Toeplitz and Hankel operators in the setting of the Bergman spaces. Researchers such as Zhao and Zheng [12] have explored the positivity of Toeplitz operators using the Berezin transform, while others like Zeng [11], Axler and Dechao [3], and Stroethoff [10] have investigated commutativity and compactness of Toeplitz operators through this lens. Motivated by these significant studies, we extend our exploration to the Berezin transform of the generalized slant H-Toeplitz operator.

Let $z_0 \in \mathbb{D}$. Then, for any arbitrary $\zeta \in \mathbb{D}$, normalizing the reproducing kernel, we get a family of the unit vectors k_{z_0}, as follows:

$$k_{z_0}(\zeta) = \frac{h_{z_0}(\zeta)}{\|h_{z_0}\|} = \frac{1-|z_0|^2}{(1-\overline{z_0}\zeta)^2} = (1-|z_0|^2)\sum_{n=0}^{\infty}(n+1)\overline{z_0}^{\,n}\zeta^n. \tag{3.1}$$

Definition 3.1. Let B be a bounded linear operator on a Hilbert space H. The Berezin transform of the function B, denoted by $\tilde{B}$, is defined as follows:

$$\tilde{B}(z) = \langle Bk_z, k_z \rangle \qquad \text{for all } z \in \mathbb{D}.$$

The Berezin transform $\tilde{B}$, is a bounded function on $\mathbb{D}$. Stroethoff [10] demonstrated that $B = 0$ if and only if $\tilde{B} = 0$. In the next theorem, we discuss the Berezin transform of the generalized slant operator acting on the Bergman space.

Proposition 3.2. *The Berezin transform of the generalized slant operator, W_k is given by the following:*

$$\tilde{W_k}(z_0) = (1 - |z_0|^2)^2 \sum_{n=0}^{\infty} (kn+1) {z_0}^n \overline{z_0}^{kn},$$

where $z_0 \in \mathbb{D}$.

Proof. For an arbitrary $\zeta \in \mathbb{D}$ using the expression (3.1), we get the following:

$$\begin{aligned} W_k(k_{z_0})(\zeta) &= (1 - |z_0|^2) W_k \sum_{n=0}^{\infty} (n+1) \overline{z_0}^n \zeta^n \\ &= (1 - |z_0|^2) \sum_{n=0}^{\infty} (kn+1) \overline{z_0}^{kn} \zeta^n \\ &= (1 - |z_0|^2) \frac{1 + (k-1)\zeta \overline{z_0}^k}{(1 - \zeta \overline{z_0}^k)^2}. \end{aligned}$$

Using the above expression, the Berezin transform of the generalized slant operator is given by

$$\begin{aligned} \tilde{W_k}(z_0) &= \langle W_k k_{z_o}(\zeta), k_{z_o}(\zeta) \rangle \\ &= (1 - |z_0|^2) \langle \sum_{n=0}^{\infty} (kn+1) \overline{z_0}^{kn} \zeta^n, k_{z_o}(\zeta) \rangle \\ &= (1 - |z_0|^2) \sum_{n=0}^{\infty} (kn+1) \langle \overline{z_0}^{kn} \zeta^n, k_{z_o}(\zeta) \rangle \\ &= (1 - |z_0|^2) \sum_{n=0}^{\infty} (kn+1) \langle \overline{z_0}^{kn} \zeta^n, (1 - |z_0|^2) h_{z_o}(\zeta) \rangle \\ &= (1 - |z_0|^2)^2 \sum_{n=0}^{\infty} (kn+1) \overline{z_0}^{kn} {z_0}^n. \end{aligned}$$

□

The following proposition shows that the Berezin transform $\tilde{S_\psi^k}$, is bounded by $\sqrt{k} ||\psi||_\infty$.

Proposition 3.3. *For a harmonic function ψ in the space $L^\infty(\mathbb{D}, dA)$, $||\tilde{S_\psi^k}|| \leq \sqrt{k} ||\psi||_\infty$.*

Proof. Let $z_0 \in \mathbb{D}$; then using the definition of the Berezin transform, we have

$$|\tilde{S^k_\psi}(z_0)| = |\langle S^k_\psi k_{z_0}, k_{z_0}\rangle| = |\langle S^k_\psi \frac{h_{z_0}}{\|h_{z_0}\|}, \frac{h_{z_0}}{\|h_{z_0}\|}\rangle| \leq ||S^k_\psi|| \leq \sqrt{k}\|\psi\|_\infty.$$

$\square$

Applying the operator K to the normalized reproducing kernel, it is a straightforward exercise to verify the following.

Lemma 3.4. *The operator $K : A^2(\mathbb{D}) \to B^2(\mathbb{D})$ satisfies the following:*

$$K(k_{z_0}(\zeta))$$
$$= (1-|z_0|^2)\sum_{m=0}^{\infty}\Big(\sqrt{(2m+1)(m+1)}\,\overline{z_0}^m\zeta^m + \sqrt{2(m+1)(m+2)}\,\overline{z_0}^{(2m+1)}\overline{\zeta}^{m+1}\Big),$$

for z_0 and $\zeta \in \mathbb{D}$.

Using $G(z_0,\zeta)$ to denote the expression of $K(k_{z_0}(\zeta))$ in Lemma 3.4, we compute the Berezin transform of the generalized slant H-Toeplitz operator.

Theorem 3.5. *For any $L^\infty(\mathbb{D}, dA)$ function ψ, the Berezin transform of S^k_ψ can be expressed as*

$$\tilde{S^k_\psi}(z_0) = (1-|z_0|^2)\int_{\mathbb{D}} \psi(\zeta)\, G(z_0,\zeta)\Big[\frac{1+(k-1)z_0\overline{\zeta}^k}{(1-z_0\,\overline{\zeta}^k)^2}\Big]dA(\zeta),$$

for $z_0 \in \mathbb{D}$.

Proof. Given $z_0 \in \mathbb{D}$, the Berezin transform of S^k_ψ can be computed as follows:

$$\tilde{S^k_\psi}(z_0) = \langle S^k_\psi k_{z_o}, k_{z_o}\rangle = \langle W_k P M_\psi K(k_{z_0}), k_{z_0}\rangle = \langle P M_\psi K(k_{z_0}), W_k^*(k_{z_0})\rangle.$$

The adjoint of the generalized slant operator is given by

$$\begin{aligned}
W_k^*(k_{z_0})(\zeta) &= (1-|z_0|^2)W_k^*\Big(\sum_{n=0}^{\infty}(n+1)\overline{z_0}^n\zeta^n\Big)\\
&= (1-|z_0|^2)\sum_{n=0}^{\infty}(kn+1)\overline{z_0}^n\zeta^{kn}\\
&= (1-|z_0|^2)\frac{1+(k-1)\overline{z_0}\zeta^k}{(1-\overline{z_0}\zeta^k)^2}. \qquad (3.2)
\end{aligned}$$

Applying Lemma 3.4 and using (3.2), we get

$$\begin{aligned}
\tilde{S^k_\psi}(z_0) &= \langle \psi K(k_{z_0}), P(W_k^*(k_{z_0}))\rangle\\
&= \int_{\mathbb{D}} \psi(\zeta)K(k_{z_0})(\zeta)\overline{W_k^*(k_{z_0})(\zeta)}dA(\zeta)\\
&= (1-|z_0|^2)\int_{\mathbb{D}} \psi(\zeta)\, G(z_0,\zeta)\Big[\frac{1+(k-1)z_0\overline{\zeta}^k}{(1-z_0\overline{\zeta}^k)^2}\Big]dA(\zeta).
\end{aligned}$$

$\square$

Remark 3.6. For any function $\psi \in L^\infty(\mathbb{D}, dA)$, the Berezin transform $\tilde{S^k_\psi}(z_0)$ tends to zero as $|z_o| \to 1^-$.

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