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MILD SOLUTIONS FOR NONLOCAL SECOND-ORDER PARTIAL FUNCTIONAL AND NEUTRAL FUNCTIONAL PERTURBED EVOLUTION EQUATIONS WITH INFINITE STATE DEPENDENT DELAY

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This paper is dedicated to Professor Noureddine Ghouali
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Abstract. This paper focuses on establishing the existence of mild solutions over a semi-infinite interval for two types of second-order perturbed evolution equations that involve infinite state dependent delays under nonlocal conditions. The approach employs Avramescu's nonlinear alternative for the sum of compact operators and contraction mappings in Fréchet spaces, alongside semigroup theory.

1. Introduction and preliminaries

In this work, we investigate, in a real Banach space $(E, |\cdot|)$, the existence of mild solutions defined on a semi-infinite real interval $J := [0, +\infty)$ for second-order perturbed evolution equations, with infinite state-dependent delay, when the conditions are nonlocal, for two classes of the first partial functional and neutral functional. We will consider the following problem:

$$y''(t) = A(t)y(t) + f\left(t, y_{\rho(t,y_t)}\right) + g\left(t, y_{\rho(t,y_t)}\right), \quad \text{a.e.} \quad t \in J, \tag{1.1}$$

$$y(t) = \phi(t) - h_t(y), \qquad t \in H := (-\infty, 0], \tag{1.2}$$

$$y'(0) = y_1, \tag{1.3}$$

where for an abstract phase space $\mathcal{B}$, which will be defined later, $f, g : J \times \mathcal{B} \to E$, $\phi \in \mathcal{B}$ are given functions, $h_t : \mathcal{B} \to E$: $\rho : J \times \mathcal{B} \to \mathbb{R}$, $y_1 \in E$ and $\{A(t)\}_{t\geq 0}$, is

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a family of linear closed (not necessarily bounded) operators from E into E that generates an unique evolution system of operators $\{U(t,s)\}_{(t,s)\in J\times J}$ for $s \le t$.

For any continuous function y and any $t \in J$, we denote by y_t the element of $\mathcal{B}$ defined by

$$y_t(\theta) = y(t+\theta) \qquad \text{for} \quad \theta \le 0.$$

Here $y_t(\cdot)$ represents the history of the state from time $t \le 0$ up to the present time t. We assume that the histories y_t belong to $\mathcal{B}$. We assume that the histories y_t belongs to some abstract phase space $\mathcal{B}$, to be specified later.

Next, in Section 2, we study the following neutral problem:

$$\frac{d}{dt}[y'(t) - Q(t, y_{\rho(t,y_t)}] = A(t)y(t) + f(t, y_{\rho(t,y_t)}) + g\left(t, y_{\rho(t,y_t)}\right), \text{ a.e. } t \in J, \quad (1.4)$$

$$y(t) = \phi(t) - h_t(y), \qquad t \in H, \quad (1.5)$$

$$y'(0) = y_1, \quad (1.6)$$

where f, g, ρ, h_t, $A(\cdot)$, and ϕ are as in problem (1.1) – (1.3) and $Q : J \times \mathcal{B} \to E$ is a given function.

Finally, we give two examples in Section 3 to illustrate the abstract theory.

The theory of differential functional equations with infinite delay has been the subject of many papers since the end of the twentieth century. Abstract second-order differential equations with infinite delay have been studied in [19, 21]. Differential equations with unbounded delay, albeit of the first order, appear in mathematical modeling of some problems in biology [10].

In the literature, there are a lot of works study the problems of differential equations involving different methods. Among them, the fixed point method is combined by semigroup theory in Fréchet space; see the paper of Baghli and Benchohra [5, 7].

The nonlocal evolution equations in Banach spaces were first studied by Byszewski. The nonlocal conditions were used in many physical problems. For more detailed of nonlocal conditions in different fields we refer to read Byszewski [14, 15, 16, 17], Ntouyas and Tsamatos [25, 26].

Baghli et al. [5, 7, 8, 9, 6] considered existence, uniqueness, and controllability of mild solutions for first-order classes of semilinear partial functional and neutral functional differential and integrodifferential perturbed and nonperturbed evolution equations and inclusions with finite and infinite delay. They gave a global mild solution for evolution inclusions with state-dependent delay in [2, 4, 9, 24] and for control problems of first and second-order with local and nonlocal conditions in [11, 12, 13].

The objective of this paper is to give the sufficient conditions for the existence of mild solutions for the two classes partial functional and neutral functional to the nonlocal second-order perturbed evolution equations with infinite state dependent delay (1.1)–(1.3) and (1.4)–(1.6) using the nonlinear alternative of Avramescu for sum of compact operators and contractions maps in Fréchet spaces [3], combined with semigroup theory [1, 27].

We introduce some notations, definitions and theorems, which are used via the different steps of these paper.

Let $C(\mathbb{R}^+; E)$ be the space of continuous functions from $\mathbb{R}^+$ into E and let $B(E)$ be the space of all bounded linear operators from E into E, with the usual supremum norm

$$\|N\|_{B(E)} = \sup \{ \ |N(y)| \ : \ |y| = 1 \ \}, \quad N \in B(E).$$

A measurable function $y : \mathbb{R}^+ \to E$ is Bochner integrable if and only if $|y|$ is Lebesgue integrable (for the Bochner integral properties, see the classical monograph of Yosida [28]).

Let $L^1(\mathbb{R}^+, E)$ denote the Banach space of measurable functions $y : \mathbb{R}^+ \to E$, which are Bochner integrable normed by

$$\|y\|_{L^1} = \int_0^{+\infty} |y(t)| dt.$$

The nonlocal condition $y(t) + h_t(y) = \phi(t)$ for $t \in (-\infty, 0]$, can be applied in physics with better effect than the classical initial condition $y(0) = y_0$.

For example, $h_t(y)$ may be given by

$$h_t(y) = \sum_{i=1}^{p} c_i y(t_i + t), \ t \in (-\infty, 0],$$

where $c_i, i = 1, \dots, p$ are given constants and $0 < t_1 < \cdots < t_p < +\infty$.

At time $t = 0$, we have

$$h_0(y) = \sum_{i=1}^{p} c_i y(t_i).$$

We shall employ an axiomatic definition of the phase space $\mathcal{B}$ presented by Hale and Kato [18] and follow the terminology used in [22]. Thus, $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ will be a seminormed linear space of functions mapping $\mathbb{R}^-$ into E and satisfying the following axioms:

(A_1) If $y : (-\infty, b) \to E, b > 0$, is continuous on $[0, b]$ and $y_0 \in \mathcal{B}$, then for every $t \in [0, b)$, the following conditions hold:
  - (i) $y_t \in \mathcal{B}$;
  - (ii) There exists a positive constant $\mathcal{D}$ such that $|y(t)| \leq \mathcal{D}\|y_t\|_{\mathcal{B}}$;
  - (iii) There exist two functions $K(\cdot), M(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ independent of y with K continuous and M locally bounded such that
$$\|y_t\|_{\mathcal{B}} \leq K(t) \sup_{0 \leq s \leq t} |y(s)| + M(t)\|y_0\|_{\mathcal{B}}.$$

(A_2) For the function y in axiom (A_1), y_t is a $\mathcal{B}$-valued continuous function on $[0, b]$.

(A_3) The space $\mathcal{B}$ is complete.

Set $K_b = \sup_{t \in [0,b]} K(t) \quad$ and $\quad M_b = \sup_{t \in [0,b]} M(t).$

Remark 1.1.
1. (ii) is equivalent to $|\phi(0)| \leq \mathcal{D}\|\phi\|_{\mathcal{B}}$ for every $\phi \in \mathcal{B}$.
2. Since $\|\cdot\|_{\mathcal{B}}$ is a seminorm, two elements $\phi, \psi \in \mathcal{B}$ can check $\|\phi - \psi\|_{\mathcal{B}} = 0$ without necessarily $\phi(\theta) = \psi(\theta)$ for all $\theta \leq 0$.
3. From the equivalence in the first remark, we can see that for all $\phi, \psi \in \mathcal{B}$ such that $\|\phi - \psi\|_{\mathcal{B}} = 0$, we necessarily have $\phi(0) = \psi(0)$.

Here are some examples of phase spaces from the book of Hino, Murakami, and Naito [22].

Example 1.2. Let

BC**:** denote the space of bounded continuous functions defined from $\mathbb{R}^-$ to E;

BUC**:** denote the space of bounded uniformly continuous functions defined from $\mathbb{R}^-$ to E;

C^∞**:** $=\left\{\phi \in BC : \lim_{\theta\to-\infty}\phi(\theta) \text{ exists in } E\right\}$;

C^0**:** $=\left\{\phi \in BC : \lim_{\theta\to-\infty}\phi(\theta) = 0\right\}$, endowed with the uniform norm

$$\|\phi\| = \sup_{\theta\le 0}|\phi(\theta)|.$$

We have that the spaces BUC, C^∞ and C^0 satisfy conditions $(A_1)-(A_3)$. However, BC satisfies axioms $(A_1), (A_3)$ but axiom (A_2) is not satisfied.

Let X be a Fréchet space with a family of seminorms $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$. We suppose that the family of seminorms $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$ verifies

$$\|x\|_1 \le \|x\|_2 \le \ \|x\|_3 \le \cdots \quad \text{for every } x\in X.$$

Let $Y \subset X$. We say that Y is bounded if for every $n\in\mathbb{N}$, there exists $\overline{M}_n > 0$ such that

$$\|y\|_n \le \overline{M}_n \quad \text{for all} \quad y \in Y.$$

In what follows, let $\{A(t), t \le 0\}$ be a family of closed linear operators on the Banach space E with domain $D(A(t))$ that is dense in E and independent of t. The existence of solutions to the problem (1.1)–(1.3) and (1.4)–(1.6) is related to the existence of an evolution operator $U(t,s)$ for the homogeneous problem

$$y''(t) = A(t)y(t), \qquad t\in J. \tag{1.7}$$

This concept of evolution operator has been developed by Kozak [23].

Definition 1.3. A family U of bounded operators $U(t,s): E\to E, (t,s)\in\Delta$, where $\Delta := \{(t,s)\in J\times J : s\le t\}$, is called an evolution operator of (1.7) if the following conditions hold:

$(D1)$ For any $x\in E$, the map $(t,s)\mapsto U(t,s)x$ is continuously differentiable and
- (a) for any $t\in J, U(t,t)=0$;
- (b) for all $(t,s)\in\Delta$ and for any $x\in E$,

$$\frac{\partial}{\partial t}U(t,s)x|_{t=s} = x \text{ and } \frac{\partial}{\partial s}U(t,s)x|_{t=s} = -x.$$

$(D2)$ For all $(t,s)\in\Delta$, if $x\in D(A(t))$, then $\frac{\partial}{\partial s}U(t,s)x \in D(A(t))$, the map $(t,s)\mapsto U(t,s)x$ is of class C^2, and
- (a) $\frac{\partial^2}{\partial t^2}U(t,s)x = A(t)U(t,s)x$;
- (b) $\frac{\partial^2}{\partial s^2}U(t,s)x = U(t,s)A(s)x$;
- (c) $\frac{\partial^2}{\partial s\partial t}U(t,s)x|_{t=s} = 0$.

$(D3)$ For all $(t,s)\in\Delta$, if $x\in D(A(t))$, then
$\frac{\partial}{\partial s}U(t,s)x\in D(A(t))$, $\frac{\partial^3}{\partial t^2\partial s}U(t,s)x$ and $\frac{\partial^3}{\partial s^2\partial t}U(t,s)x$ exist, and

(a) $\frac{\partial^3}{\partial t^2\partial s}U(t,s)x = A(t)\frac{\partial}{\partial s}(t)U(t,s)x$;

(b) $\frac{\partial^3}{\partial s^2\partial t}U(t,s)x = \frac{\partial}{\partial t}U(t,s)A(s)x$.

Moreover, the map $(t,s)\mapsto A(t)\frac{\partial}{\partial s}U(t,s)x$ is continuous.

For the state dependent delay notion, let us set

$$\mathcal{R}(\rho^-) = \{\rho(s,\phi) : (s,\phi)\in J\times\mathcal{B}, \rho(s,\phi)\leq 0\}.$$

We always assume that $\rho : J\times\mathcal{B}\to\mathbb{R}$ is continuous. Additionally, we present the following hypothesis:

(H_ϕ) The function $t\to\phi_t$ is continuous from $\mathcal{R}(\rho^-)$ into $\mathcal{B}$ and there exists a continuous and bounded function $\mathcal{L}^\phi : \mathcal{R}(\rho^-)\to\mathbb{R}_+^\star$ such that for every $t\in\mathcal{R}(\rho^-)$

$$\|\phi_t\|_{\mathcal{B}} \leq \mathcal{L}^\phi\|\phi\|_{\mathcal{B}}.$$

Remark 1.4. The condition (H_ϕ), is frequently checked by continuous and bounded functions. For more details, see for instance [22].

Lemma 1.5. *(see* [20, Lemma 2.4]*) If $y : (-\infty; b]\to E$ is a function such that $y_0=\phi$, then*

$$\|y_s\|_{\mathcal{B}} \leq (M_b+\mathcal{L}^\phi)\|\phi\|_{\mathcal{B}} + K_b|y(\theta)|; \theta\in[0,\max\{0,s\}], s\in\mathcal{R}(\rho^-)\cup J,$$

where $\mathcal{L}^\phi = \sup\limits_{t\in\mathcal{R}(\rho^-)} \mathcal{L}^\phi(t)$.

Proposition 1.6. *(see* [2, Proposition 3.4]*) By (H_ϕ), Lemma 1.5 and the property $(A1)$, we have for each $t\in[0,n]$ and $n\in\mathbb{N}$*

$$\|y_{\rho(t,y_t)}\| \leq K_n|y(t)| + \left(M_n+\mathcal{L}^\phi\right)\|y_0\|_{\mathcal{B}}.$$

Definition 1.7. A function $f : J\times\mathcal{B}\to E$ is said to be an L^1_{loc}-Carathéodory function if it satisfies

(i) for each $t\in J$ the function $f(t,\cdot) : \mathcal{B}\to E$ is continuous;
(ii) for each $y\in\mathcal{B}$ the function $f(\cdot,y) : J\to E$ is measurable;
(iii) for every positive integer k there exists $\hbar_k\in L^1_{loc}(J;\mathbb{R}^+)$ such that

$$|f(t,y)| \leq \hbar_k(t)$$

for all $\|y\|_{\mathcal{B}}\leq k$ and almost every $t\in J$.

The following definition is the appropriate concept of contraction in X.

Definition 1.8. A function $f : X\to X$ is said to be a contraction if for each $n\in\mathbb{N}$ there exists $\alpha_n\in(0,1)$ such that for all $x,y\in X$

$$\|f(x)-f(y)\|_n \leq \alpha_n\,\|x-y\|_n.$$

Theorem 1.9 (Avramescu's nonlinear alternative [3])**.** *Let X be a Fréchet space and let $A,B : X\to X$ be two operators satisfying*

(1) *A is a compact operator,*
(2) *B is a contraction.*

Then either one of the following statements holds:

(Av1) *The operator $A+B$ has a fixed point;*
(Av2) *The set $\left\{x \in X, x = \lambda A(x) + \lambda B\left(\frac{x}{\lambda}\right)\right\}$ is unbounded for some $\lambda \in (0,1)$.*

2. Main results

2.1. **Semilinear nonlocal perturbed evolution equations.** In this section, we provide the existence result for the nonlocal second-order perturbed evolution problem (1.1)–(1.3). Before this, we first present the mild solutions for that problem.

Definition 2.1. We say that the function $y(\cdot) : \mathbb{R} \to E$ is a mild solution of (1.1)–(1.3) if $y(t) = \phi(t) - h_t(y)$ for all $t \in H$ and y satisfies the following integral equation:

$$y(t) = -\frac{\partial}{\partial s}U(t,0)\phi(0) - \frac{\partial}{\partial s}U(t,0)h_t(y) + U(t,0)y_1 \tag{2.1}$$

$$+ \int_0^t U(t,s)f(s, y_{\rho(s,y_s)})ds \tag{2.2}$$

$$+ \int_0^t U(t,s)g\left(s, y_{\rho(s,y_s)}\right)ds \text{ a.e } t \in J. \tag{2.3}$$

It is necessary to introduce the following hypotheses, which are assumed thereafter:

$(H1)$ $U(t,s)$ is compact for $t-s>0$, and for every $(t,s) \in \Delta$, and there exists a constant $\widehat{M} \geq 1$ such that

$$\|U(t,s)\|_{B(E)} \leq \widehat{M},$$

and there exists a constant $\widetilde{M} \geq 0$ such that

$$\left\|\frac{\partial}{\partial s}U(t,s)\right\|_{B(E)} \leq \widetilde{M}.$$

$(H2)$ There exist a function $p \in L^1_{loc}(J;\mathbb{R}_+)$ and a continuous nondecreasing function
$\psi : \mathbb{R}_+ \to [0,+\infty)$ and such that

$$|f(t,u)| \leq p(t)\ \psi(\|u\|_{\mathcal{B}}).$$

for all $t \in J$ and for each $u \in \mathcal{B}$

$(H3)$ There exists a function $\eta \in L^1(J;\mathbb{R}_+)$ such that

$$|g(t,u) - g(t,v)| \leq \eta(t)\|u-v\|_{\mathcal{B}}$$

for all $t \in J$ and for each $u, v \in \mathcal{B}$.

$(H4)$ There exist a constant $\varpi > 0$ such that

$$\|h_t\|_{\mathcal{B}} \leq \varpi$$

for all $t \in H$.

Corollary 2.2. *By* (H_ϕ)*, Lemma 1.5 and the Proposition 1.6, we have for each* $t \in [0, n]$ *and* $n \in \mathbb{N}$*,*

$$\|y_{\rho(t,y_t)}\| \leq K_n|y(t)| + \left(M_n + \mathcal{L}_h^\phi\right)(\|\phi\|_{\mathcal{B}} + \varpi),$$

where $\mathcal{L}_h^\phi = \sup\limits_{t\in\mathcal{R}(\rho^-)} \mathcal{L}_h^\phi(t)$*.*

Consider the following space:

$$B_{+\infty} = \{y : \mathbb{R} \to E : y|_{[0,T]} \text{ continuous for } T > 0 \text{ and } y_0 \in \mathcal{B}\},$$

where $y|_{[0,T]}$ is the restriction of y to the real compact interval $[0, T]$.

Let us fix $\tau > 1$. For every $n \in \mathbb{N}$, we define in $B_{+\infty}$ the seminorms by

$$\|y\|_n := \sup_{t\in[0,n]} e^{-\tau\ L_n^*(t)}\ |y(t)|,$$

where $L_n^*(t) = \int_0^t \bar{l}_n\, ds\ \bar{l}_n(t) = \widehat{M}\eta(t)K_n$ and η is the function from the hypothesis $(H3)$.

Then $B_{+\infty}$ is a Fréchet space with those family of seminorms $\|\cdot\|_{n\in\mathbb{N}}$.

Theorem 2.3. *Assumed* (H_ϕ) *and* $(H1) - (H4)$ *are satisfied, and moreover for all* $n > 0$*, we have*

$$\int_{\sigma_n}^{+\infty} \frac{ds}{s + \psi(s)} > K_n\widehat{M} \int_0^n \max\left(p(s); \eta(s)\right) ds \tag{2.4}$$

with $\sigma_n = (M_n + \mathcal{L}_h^\phi + K_n\widetilde{M}\mathcal{D})\|\phi\|_{\mathcal{B}} + (M_n + \mathcal{L}_h^\phi + K_n\widetilde{M}\mathcal{D})\varpi + K_n\widehat{M}|y_1| + \widehat{M}K_n \int_0^n |g(s, 0)|ds$*.*
Then, the nonlocal second-order perturbed evolution problem $(1.1) - (1.3)$ *has at least one mild solution on* $\mathbb{R}$*.*

Proof. We transform the problem $(1.1) - (1.3)$ into a fixed-point problem. Consider the operator $N : B_{+\infty} \to B_{+\infty}$ defined by

$$N(y)(t) = \begin{cases} \phi(t) - h_t(y), & \text{if } t \leq 0; \\ -\frac{\partial}{\partial s}U(t,0)\phi(0) - \frac{\partial}{\partial s}U(t,0)h_0(y) + U(t,0)y_1 & \\ +\int_0^t U(t,s)f\left(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\right) ds & \\ +\int_0^t U(t,s)g\left(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\right) ds, & \text{if } t \geq 0. \end{cases}$$

Clearly, fixed points of the operator N are mild solutions of the problem $(1.1) - (1.3)$.

For $\phi - h_t \in \mathcal{B}$, we will define the function

$$x(t) = \begin{cases} \phi(t) - h_t(y) & \text{if } t \leq 0 \\ -\frac{\partial}{\partial s}U(t,0)\phi(0) - \frac{\partial}{\partial s}U(t,0)h_0(y) + U(t,0)y_1 & \text{if} t \geq 0. \end{cases}$$

Then $x_0 = \phi - h_0$. For each function $z \in B_{+\infty}$ with $z(0) = 0$, we denote by $\overline{z}$ the function defined by

$$\overline{z}(t) = \begin{cases} 0, & \text{if } t \in H; \\ z(t), & \text{if } t \in J. \end{cases}$$

If $y(\cdot)$ satisfies (2.1), we can decompose it as $y(t) = z(t) + x(t)$, $t \geq 0$, which implies $y_t = z_t + x_t$, for every $t \in J$ and the function $z(\cdot)$ satisfies

$$\begin{aligned} z(t) &= \int_0^t U(t,s) f\left(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\right) ds \\ &\quad + \int_0^t U(t,s) g\left(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\right) ds. \end{aligned}$$

Let

$$B^0_{+\infty} = \{z \in B_{+\infty} : z_0 = 0 \in \mathcal{B}\}.$$

For any $z \in B^0_{+\infty}$, we have

$$\|z\|_{+\infty} = \|z_0\|_{\mathcal{B}} + \sup\{|z(s)| : 0 \leq s < +\infty\} = \sup\{|z(s)| : 0 \leq s < +\infty\}.$$

Thus $(B^0_{+\infty}, \|.\|_{+\infty})$ is a Banach space.

We define for $t \in J$ the operators $F, G : B^0_{+\infty} \to B^0_{+\infty}$ by

$$F(z)(t) = \int_0^t U(t,s) f\left(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\right) ds$$

and

$$G(z)(t) = \int_0^t U(t,s) g\left(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\right) ds.$$

Obviously the operator N has a fixed points is equivalents to $F + G$ has one, so it turns to prove that $F + G$ has a fixed point. The proof will be given in several steps.

Step 1: We show the continuity of F.

Let$(z_n)_n$ be a sequence in $B^0_{+\infty}$ such that $z_n \to z \in B^0_{+\infty}$. By the hypothesis $(H1)$, we obtain

$$\begin{aligned} |F(z_n)(t) - F(z)(t)| &\leq \int_0^t \|U(t,s)\|_{B(E)} |f(s, z_{n\rho(s,z_{ns}+x_s)} + x_{\rho(s,z_{ns}+x_s)} \\ &\quad - f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}| ds \\ &\leq \widehat{M} \int_0^t |f(s, z_{n\rho(s,z_{ns}+x_s)} + x_{\rho(s,z_{ns}+x_s)} \\ &\quad - f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}| ds. \end{aligned}$$

Since f is continuous, by the dominated convergence theorem of Lebesgue, we get

$$|F(z_n)(t) - F(z)(t)| \to 0 \quad \text{if } \ n \to +\infty.$$

So that F is continuous.

step 2: Show that F transforms any bounded of $B^0_{+\infty}$ in a bounded set.

For any $d > 0$, there exists a positive constant ϱ such that for all $z \in B_d$ for $B_d := \{z \in B^0_{+\infty} : \|z\|_n \leq d\}$, we get $\|F(z)\|_n \leq \varrho$. Let $z \in B_d$. By the hypotheses $(H1)$ and $(H2)$, we have for all $t \in [0, n]$

$$|F(z)(t)| \leq \int_0^t \|U(t,s)\|_{B(E)} |f\left(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\right)| ds,$$

$$\leq \widehat{M} \int_0^t p(s)\psi\left(\left\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\right\|_{\mathcal{B}}\right) ds.$$

From (H_ϕ), Corollary 2.2 and Assumption (A_1), we have for every $t \in [0,n]$

$$\begin{aligned}
\left\|z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)}\right\|_{\mathcal{B}} &\leq \|z_{\rho(t,z_t+x_t)}\|_{\mathcal{B}} + \|x_{\rho(t,z_t+x_t)}\|_{\mathcal{B}} \\
&\leq K_n|z(t)| + (M_n + \mathcal{L}_h^\phi)\|z_0\|_{\mathcal{B}} + K_n|x(t)| \\
&\quad +(M_n + \mathcal{L}_h^\phi s)\|x_0\|_{\mathcal{B}} \\
&\leq K_n|z(t)| + K_n\left\|\frac{\partial}{\partial s}U(t,0)\right\|_{B(E)}|\phi(0) - h_0(y)| \\
&\quad +K_n\|U(t,0)\|_{B(E)}|y_1| + (M_n + \mathcal{L}_h^\phi)\|\phi - h_0\|_{\mathcal{B}} \\
&\leq K_n|z(t)| + K_n\widetilde{M}|\phi(0) - h_0(y)| + K_n\widehat{M}|y_1| \\
&\quad +(M_n + \mathcal{L}_h^\phi)\|\phi - h_0\|_{\mathcal{B}}.
\end{aligned}$$

Using (ii), we get

$$\begin{aligned}
\left\|z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)}\right\|_{\mathcal{B}} &\leq K_n|z(t)| + K_n\widetilde{M}\mathcal{D}\|\phi\|_{\mathcal{B}} + K_n\widetilde{M}\mathcal{D}\varpi \\
&\quad +K_n\widehat{M}\mathcal{D}\|y_1\|_{\mathcal{B}} + (M_n + \mathcal{L}_h^\phi)|\phi\|_{\mathcal{B}} + (M_n + \mathcal{L}_h^\phi)\varpi \\
&\leq K_n|z(t)| + (M_n + \mathcal{L}_h^\phi + K_n\widetilde{M}\mathcal{D})(\|\phi\|_{\mathcal{B}} + \varpi) \\
&\quad +K_n\widehat{M}|y_1|.
\end{aligned}$$

Set $c_n := (M_n + \mathcal{L}_h^\phi + K_n\widetilde{M}\mathcal{D})(\|\phi\|_{\mathcal{B}} + \varpi) + K_n\widehat{M}|y_1|$ and $\varsigma_n := K_n d + c_n$. Then

$$\left\|z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)}\right\|_{\mathcal{B}} \leq K_n d + c_n := \varsigma_n. \tag{2.5}$$

Using the nondecreasing character of ψ, we get for each $t \in [0,n]$

$$\begin{aligned}
|F(z)(t)| &\leq \widehat{M}\int_0^t p(s)\psi(\varsigma_n)ds \\
&\leq \widehat{M}\psi(\varsigma_n)\int_0^t p(s)ds \\
&\leq \widehat{M}\psi(\varsigma_n)\|p\|_{L^1} := \varrho.
\end{aligned}$$

So there exists a positive constant ϱ such that $\|F(z)\|_n \leq \varrho$.
Hence $F(B_d) \subset B_\varrho$.

Step 3 : F maps bounded sets into equicontinuous sets of $B^0_{+\infty}$. We consider B_d as in Step 2, and we show that $F(B_d)$ is equicontinuous. Let $\tau_1, \tau_2 \in J$ with

$\tau_2 > \tau_1$ and $z \in B_d$.

$$\begin{aligned}
&|F(z)(\tau_2) - F(z)(\tau_1)| \\
&\leq \int_0^{\tau_1} \|U(\tau_2, s) - U(\tau_1, s)\|_{B(E)} |f\left(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\right)| \\
&\quad + \int_{\tau_1}^{\tau_2} \|U(\tau_2, s)\|_{B(E)} |f\left(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\right)| ds \\
&\leq \int_0^{\tau_1} \|U(\tau_2, s) - U(\tau_1, s)\|_{B(E)} p(s)\psi\left(\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}}\right) ds \\
&\quad + \widehat{M} \int_{\tau_1}^{\tau_2} p(s)\psi\left(\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\right) \|_{\mathcal{B}}) ds.
\end{aligned}$$

By (2.5) and using the nondecreasing character of ψ, we get

$$\begin{aligned}
|F(z)(\tau_2) - F(z)(\tau_1)| \;\; &\leq \;\; \psi(\varsigma_n) \int_0^{\tau_1} \|U(\tau_2, s) - U(\tau_1, s)\|_{B(E)} p(s) ds \\
&\quad + \widehat{M}\psi(\varsigma_n) \int_{\tau_1}^{\tau_2} p(s) ds.
\end{aligned}$$

Observing that $|F(z)(\tau_2) - F(z)(\tau_1)|$ tends to zero as $\tau_2 - \tau_1 \to 0$ independently of $z \in B_d$. The right-hand side of the above inequality tends to zero as $\tau_2 - \tau_1 \to 0$. Since $U(t, s)$ is a strongly continuous operator and the compactness of $U(t, s)$ for $t > s$ implies the continuity in the uniform operator topology. As a result of Steps 1 to 3 together with the Arzelá–Ascoli theorem it suffices to show that the operator F maps B_d into a pre-compact set in E.

Let $t \in J$ be fixed and let ϵ be a real number such that $0 < \epsilon < t$. For $z \in B_d$, we define

$$F_\epsilon(z)(t) = U(t, t - \epsilon) \int_0^{t-\epsilon} U(t - \epsilon, s) f\left(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\right) ds.$$

Since $U(t, s)$ is a compact operator, the set $Z_\epsilon(t) = \{F_\epsilon(z)(t) : \;\; z \in B_d\}$ is pre-compact in E for every ϵ sufficiently small, $0 < \epsilon < t$. Moreover, using and the nondecreasing character of ψ, we have

$$|F(z)(t) - F_\epsilon(z)(t)| \;\; \leq \;\; \widehat{M}\psi(\varsigma_n) \int_{t-\epsilon}^{t} p(s) ds.$$

Therefore, the set $\{F(z)(t) : z \in B_d\}$ is pre-compact in E. So we deduce from Steps 1, 2, and 3 that F is a continuous compact operator.

Step 4 : We shall show now that the operator G is a contraction.

Indeed, let $z, \overline{z} \in B^0_{+\infty}$. By the hypotheses $(H1)$ and $(H3)$, we get for all $t \in [0, n]$ and $n \in \mathbb{N}$,

$$\begin{aligned}
|G(z)(t) - G(\overline{z})(t)| \;\; &\leq \;\; \int_0^t \|U(t, s)\|_{B(E)} \\
&\quad \times |g(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}) - g(s, \overline{z}_{\rho(s,\overline{z}_s+x_s)} + x_{\rho(s,\overline{z}_s+x_s)}| ds \\
&\leq \;\; \int_0^t \widehat{M}\eta(s) \|z_{\rho(s,z_s+x_s)} - \overline{z}_{\rho(s,\overline{z}_s+x_s)}\|_{\mathcal{B}} ds.
\end{aligned}$$

Using Corollary 2.2, we obtain

$$\begin{aligned}
|G(z)(t)-G(\overline{z})(t)| &\leq \int_0^t \widehat{M}\eta(s)K_n\ |z(s)-\overline{z}(s)|ds \\
&\leq \int_0^t \left[\overline{l}_n(s)\ e^{\tau L_n^*(s)}\right]\ \left[e^{-\tau L_n^*(s)}\ |z(s)-\overline{z}(s)|\right] ds \\
&\leq \int_0^t \left[\frac{e^{\tau\ L_n^*(s)}}{\tau}\right]' ds\ \|z-\overline{z}\|_n \\
&\leq \frac{1}{\tau}\ e^{\tau\ L_n^*(t)}\ \|z-\overline{z}\|_n.
\end{aligned}$$

Therefore,

$$\|G(z)-G(\overline{z})\|_n \leq \frac{1}{\tau}\ \|z-\overline{z}\|_n.$$

So, the operator G is a contraction for all $n\in\mathbb{N}$.

Step 5 : To apply Theorem 1.9, we must check the statement $(Av2)$; that is, it remains to show that the following set:

$$\Gamma=\left\{z\in B^0_{+\infty}:\ z=\lambda\ F(z)+\lambda\ G\left(\frac{z}{\lambda}\right)\ \text{ for some } 0<\lambda<1\right\}.$$

is bounded. Let $z\in\Gamma$. By $(H1),(H2)$, and $(H3)$, Proposition 2.2 and inequality (2.5), we have for each $t\in[0,n]$,

$$\begin{aligned}
\frac{|z(t)|}{\lambda} &\leq \int_0^t \|U(t,s)\|_{B(E)}|f\left(s,z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)}\right)|ds \\
&\quad+\int_0^t \|U(t,s)\|_{B(E)}\left|g(s,\frac{z_{\rho(S,z_s+x_s)}+x_{\rho(s,z_s+x_s)}}{\lambda})\right|ds \\
&\leq \widehat{M}\int_0^t p(s)\psi(\|z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)}\|_B)ds \\
&\quad+\widehat{M}\int_0^t \left|g(s,\frac{z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)}}{\lambda})-g(s,0)+g(s,0)\right|ds \\
&\leq \widehat{M}\int_0^t p(s)\psi\left(K_n|z(s)|+c_n\right)ds+\widehat{M}\int_0^t \eta(s)\left\|\frac{z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)}}{\lambda}\right\|_B \\
&\quad+\widehat{M}\int_0^n |g(s,0)|ds \\
&\leq \widehat{M}\int_0^t p(s)\psi\left(K_n|z(s)|+c_n\right)ds+\widehat{M}\int_0^t \eta(s)\left(\frac{K_n}{\lambda}|z(s)|+c_n\right)ds \\
&\quad+\widehat{M}\int_0^n |g(s,0)|ds.
\end{aligned}$$

We consider the function $u(t) := \sup |z(\theta)|$ defined for $t \in J$ with the fact that $0 < \lambda < 1$. Thus

$$\begin{aligned}\frac{K_n}{\lambda}u(t) + c_n &\leq c_n + K_n\widehat{M}\int_0^t p(s)\psi\left(\frac{K_n}{\lambda}u(s) + c_n\right)ds \\ &\quad + K_n\widehat{M}\int_0^t \eta(s)\left(\frac{K_n}{\lambda}u(s) + c_n\right)ds + K_n\widehat{M}\int_0^n |g(s,0)|ds.\end{aligned}$$

Set $\sigma_n := c_n + K_n\widehat{M}\int_0^n |g(s,0)|ds$. Then, we have

$$\begin{aligned}\frac{K_n}{\lambda}u(t) + c_n &\leq \sigma_n + K_n\widehat{M}\int_0^t p(s)\ \psi\left(\frac{K_n}{\lambda}u(s) + c_n\right)ds \\ &\quad + K_n\widehat{M}\int_0^t \eta(s)\left(\frac{K_n}{\lambda}u(s) + c_n\right)ds.\end{aligned}$$

We consider the function $\mu(t)$ defined by

$$\mu(t) = \sup_{s\in[0,t]} \frac{K_n|u(s)|}{\lambda} + c_n, \quad 0 \leq t \leq +\infty.$$

Let $t^\star \in [0,t]$ be such that $\mu(t^\star) = \dfrac{K_n|u(t^\star)|}{\lambda} + c_n$. By the previous inequality and the nondecreasing character of ψ, we have $t \in [0,n]$

$$\mu(t) \leq \sigma_n + K_n\widehat{M}\int_0^t p(s)\psi(\mu(s))\ ds + K_n\widehat{M}\int_0^t \eta(s)\mu(s)ds.$$

Let us take the right-hand side of the above inequality as $v(t)$. Then, we have

$$\mu(t) \leq v(t) \qquad for\ \ all\ \ t \in [0,n].$$

From the definition of v, we have

$$v'(t) = K_n\widehat{M}\left(p(t)\psi(v(t)) + \eta(t)v(t)\right) \quad a.e. \quad t \in [0,n].$$

This implies that for all $t \in [0,n]$ and by using Theorem 2.3, we get

$$\begin{aligned}\int_{\sigma_n}^{v(t)} \frac{ds}{s+\psi(s)} &\leq K_n\widehat{M}\int_0^t \max\left(p(s);\eta(s)\right)ds \\ &\leq K_n\widehat{M}\int_0^n \max\left(p(s);\eta(s)\right)ds \\ &< \int_{\sigma_n}^{+\infty} \frac{ds}{s+\psi(s)}.\end{aligned}$$

Thus, for every $t \in [0,n]$, there exists a constant Φ_n such that $v(t) \leq \Phi_n$, and hence $\mu(t) \leq \Phi_n$. Since $\|z\|_n \leq \mu(t)$, we have$\|z\|_n \leq \Phi_n$. This shows that the set Γ is bounded. Then the statement (Av2) in Theorem 1.9 does not hold. The nonlinear alternative of Avramescu implies that (Av1) is satisfied. We deduce that the operator $F + G$ has a fixed point z^*. Then $y^*(t) = z^*(t) + x(t),\ t \in \mathbb{R}$ is the fixed point of the operator N, which is a mild solution of the problem (1.1)–(1.3). □

2.2. **Neutral perturbed evolution equations.** In this section, we give an existence result for the problem (1.4)–(1.6). First, we define the concept of the mild solution for that problem.

Definition 2.4. We say that the function $y(\cdot) : \mathbb{R} \to E$ is a mild solution of the problem (1.4)–(1.6) if $y(t) = \phi(t)$ for all $t \leq 0$ and y satisfies the following integral equation:

$$\begin{aligned} y(t) = \quad & -\frac{\partial}{\partial s} U(t,0)\phi(0) + U(t,0)[y_1 - Q(0,\phi)] \\ & - \int_0^t \frac{\partial}{\partial s} U(t,s) Q(s, y_{\rho(s,y_s)}) ds \\ & + \int_0^t U(t,s) f(s, y_{\rho(s,y_s)}) ds + \int_0^t U(t,s) g(s, y_{\rho(s,y_s)}) ds \quad \text{ for all } t \in J. \end{aligned} \tag{2.6}$$

We consider the hypotheses $(H_\phi), (H1) - (H4)$, and we will need the following assumptions:

$(H5)$ There exists a constant $\overline{M_0} > 0$ such that
$$\|A^{-1}(t)\|_{B(E)} \leq \overline{M_0}$$
for all $t \in J$.

$(H6)$ There exists a constant $0 < L$ such that
$$|A(t)Q(t,\phi))| \leq L(\|\phi\|_{\mathcal{B}} + 1)$$
for all $t \in J$ for every $\phi \in \mathcal{B}$.

$(H7)$ There exists a constant $L_* > 0$ such that
$$|A(s)Q(s,\phi) - A(\overline{s})Q(\overline{s},\overline{\phi})| \leq L_* \left(|s - \overline{s}| + (\|\phi - \overline{\phi}\|_{\mathcal{B}}.\right)$$
for all $s, \overline{s} \in J$ for every $\phi, \overline{\phi} \in \mathcal{B}$.

Theorem 2.5. *Assume that $(H\phi)$, $(H1) - (H7)$ are satisfied, and moreover for all $n \geq 0$,*
$$\int_{\widetilde{\sigma_n}}^{+\infty} \frac{ds}{s + \psi(s)} > K_n \int_0^n \max\left(\widehat{M} p(s), \widetilde{M}\ \overline{M_0} L + \widehat{M}\eta(s)\right) ds \tag{2.7}$$
with $\widetilde{\sigma_n} := K_n \chi_n + c_n$, and
$$c_n = (K_n \widetilde{M} \mathcal{D} + M_n + \mathcal{L}_h^\phi)(\|\phi\|_{\mathcal{B}} + \varpi) + K_n \widehat{M} |y_1|$$
and
$$\chi_n = \overline{M_0} L(\widehat{M} + \widetilde{M} n) + \widehat{M}\overline{M_0} L \|\phi\|_{\mathcal{B}} + \widehat{M} \int_0^n |g(s,0)| ds.$$
Then, the nonlocal second-order neutral perturbed evolution problem (1.4)–(1.6) has at least one mild solution on $\mathbb{R}$.

Proof. We consider the operator $\widetilde{N} : B_{+\infty} \to B_{+\infty}$ defined by

$$\widetilde{N}(y)(t) = \begin{cases} \phi(t) - h_t(y), & \text{if } t \leq 0; \\ -\dfrac{\partial}{\partial s}U(t,0)[\phi(0) - h_0(y)] + U(t,0)[y_1 - Q(0,\phi)] \\ - \int_0^t \dfrac{\partial}{\partial s}U(t,s)Q(s, y_{\rho(s,y_s)})ds + \int_0^t U(t,s)f(s, y_{\rho(s,y_s)})ds \\ + \int_0^t U(t,s)g(s, y_{\rho(s,y_s)})ds & \text{if } t \geq 0. \end{cases}$$

Then, fixed points of the operator $\widetilde{N}$ are mild solutions of the problem (1.4) – (1.6).

For $\phi - h_t \in \mathcal{B}$, we consider the function $x(\cdot) : R \to E$ defined as below by

$$x(t) = \begin{cases} \phi(t) - h_t(y), & \text{if } t \leq 0; \\ -\dfrac{\partial}{\partial s}U(t,0)[\phi(0) - h_0(y)] + U(t,0)y_1, & \text{if } t \in J; \end{cases}$$

Then $x_0 = \phi - h_0$, and for each function $z \in B^0_{+\infty}$ set $y(t) = z(t) + x(t)$. we denote by $\overline{z}$ the function defined by

$$\overline{z}(t) = \begin{cases} 0, & \text{if } t \in H; \\ z(t), & \text{if } t \in J. \end{cases}$$

If $y(\cdot)$ satisfies (2.1), then we can decompose it as $y(t) = z(t) + x(t)$, $t \geq 0$, which implies $y_t = z_t + x_t$, for every $t \in J$ and the function $z(\cdot)$ satisfies

$$\begin{aligned} z(t) &= -U(t,0)Q(0,\phi) - \int_0^t \frac{\partial}{\partial s}U(t,s)Q\left(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\right)ds \\ &\quad + \int_0^t U(t,s)f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})ds \\ &\quad + \int_0^t U(t,s)g\left(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\right)ds. \end{aligned}$$

Let $B^0_{+\infty} = \{z \in B_{+\infty} : z_0 = 0\}$. Define the operator $F, \widetilde{G} : B^0_{+\infty} \longrightarrow B^0_{+\infty}$ by

$$F(z)(t) = \int_0^t U(t,s)f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})ds,$$

and

$$\begin{aligned} \widetilde{G}(z)(t) &= -U(t,0)Q(0,\phi) - \int_0^t \frac{\partial}{\partial s}U(t,s)Q\left(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\right)ds \\ &\quad + \int_0^t U(t,s)g\left(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\right)ds. \end{aligned}$$

Obviously the operator $\widetilde{N}$ having a fixed points is equivalent to $F + \widetilde{G}$ having one, so it turns to prove that $F + \widetilde{G}$ has a fixed point. We have shown that the operator F is continuous and compact as in section 3. It remains to show that the operator $\widetilde{G}$ is a contraction.

Let $z,\overline{z}\in B_0^{+\infty}$. By $(H1)$, $(H3)$, $(H5)$ and $(H7)$, we have for each $t\in[0,n]$ and $n\in\mathbb{N}$,

$$\begin{aligned}
&|\widetilde{G}(z)(t)-\widetilde{G}(\overline{z})(t)|\\
&\leq \int_0^t \left\|\frac{\partial}{\partial s}U(t,s)\right\|_{B(E)} \|A^{-1}(s)\|_{B(E)}\\
&\quad\times|A(s)Q\left(s,z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)}\right)-A(s)Q\left(s,\overline{z}_{\rho(s,\overline{z}_s+x_s)}+x_{\rho(s,\overline{z}_s+x_s)}\right)|ds\\
&\quad+\int_0^t\|U(t,s)\|_{B(E)}\left|g\left(s,z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)}\right)-g\left(s,\overline{z}_{\rho(s,\overline{z}_s+x_s)}+x_{\rho(s,\overline{z}_s+x_s)}\right)\right|ds\\
&\leq \int_0^t \widetilde{M}\overline{M_0}L_*\|z_{\rho(s,z_s+x_s)}-\overline{z}_{\rho(s,z_s+x_s)}\|_{\mathcal{B}}ds+\int_0^t\widehat{M}\eta(s)\|z_{\rho(s,z_s+x_s)}-\overline{z}_{\rho(s,z_s+x_s)}\|_{\mathcal{B}}ds.
\end{aligned}$$

By (2.5), we have

$$\begin{aligned}
|\widetilde{G}(z)(t)\widetilde{G}(\overline{z})(t)| &\leq \int_0^t \widetilde{M}\ \overline{M_0}L_*K_n|z(s)-\overline{z}(s)|ds\\
&\quad+\int_0^t\widehat{M}\eta(s)K_n|z(s)-\overline{z}(s)|ds\\
&\leq \int_0^t K_n\left[\widetilde{M}\ \overline{M_0}L_*+\widehat{M}\eta(s)\right]|z(s)-\overline{z}(s)|ds.
\end{aligned}$$

Set $\overline{l_n}(t)=K_n\left[\widetilde{M}\ \overline{M_0}L_*+\widehat{M}\eta(t)\right]$ for the family of seminorms $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$. Then

$$\begin{aligned}
|\widetilde{G}(z)(t)-\widetilde{G}(\overline{z})(t)| &\leq \int_0^t\left(\overline{l_n}(s)e^{\tau L_n^*(s)}\right)\left(e^{-\tau L_n^*(s)}|z(s)-\overline{z}(s)|\right)ds\\
&\leq \int_0^t\left[\frac{e^{\tau L_n^*(s)}}{\tau}\right]'ds\|z-\overline{z}\|_n\\
&\leq \frac{1}{\tau}e^{\tau L_n^*(t)}\|z-\overline{z}\|_n.
\end{aligned}$$

Therefore,

$$\|\widetilde{G}(z)-\widetilde{G}(\overline{z})\|_n\leq\frac{1}{\tau}\|z-\overline{z}\|_n.$$

Assume that $\tau>1$. Then the operator $\widetilde{G}$ is a contraction for all $n\in\mathbb{N}$.

For applying Theorem 1.9, we must check the statement $(Av2)$; that is, it remains to show that the set

$$\widetilde{\Gamma}=\left\{z\in B_{+\infty}^0:\ z=\lambda\ F(z)+\lambda\widetilde{G}\left(\frac{z}{\lambda}\right)\quad\text{for some } 0<\lambda<1\right\}$$

is bounded.

Let $z \in \widetilde{\Gamma}$. By $(H1)-(H3),(H5)-(H7)$, we have for each $t \in [0,n]$,

$$\begin{aligned}
&\frac{|z(t)|}{\lambda}\\
&\leq \widehat{M}\|A^{-1}(0)\|_{B(E)}|A(0)Q(0,\phi)|\\
&\quad + \int_0^t \left\|\frac{\partial}{\partial s}U(t,s)\right\|_{B(E)} \|A^{-1}(s)\|_{B(E)} \times |A(s)Q\left(s, z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)}\right)|ds\\
&\quad +\widehat{M}\int_0^t p(s)\psi(\|z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}})ds\\
&\quad +\widehat{M}\int_0^t \left|g\left(s, \frac{z_{\rho(s,z_s+x_s)}}{\lambda}+x_{\rho(s,z_s+x_s)}\right)-g(s,0)\right|ds+\widehat{M}\int_0^n |g(s,0)|ds\\
&\leq \widehat{M}\overline{M_0}L(\|\phi\|_{\mathcal{B}}+1)+\widetilde{M}\ \overline{M_0}L\int_0^t \|z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}}+1\ ds\\
&\quad +\widehat{M}\int_0^t p(s)\psi(\|z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}})ds+\widehat{M}\int_0^n |g(r,0)|drds\\
&\quad +\widehat{M}\int_0^t \eta(s)\left\|\frac{z_{\rho(s,\frac{z_s}{\lambda}+x_s)}}{\lambda}+x_{\rho(s,\frac{z_s}{\lambda}+x_s)}\|_{\mathcal{B}}\right\|ds\\
&\leq \overline{M_0}L(\widehat{M}+\widetilde{M}n)+\widehat{M}\overline{M_0}L\|\phi\|_{\mathcal{B}}+\widehat{M}\int_0^n |g(s,0)|ds\\
&\quad +\widetilde{M}\ \overline{M_0}L\int_0^t \|z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}}ds\\
&\quad +\widehat{M}\int_0^t p(s)\psi(\|z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}})ds\\
&\quad +\widehat{M}\int_0^t \eta(s)\left\|\frac{z_{\rho(s,\frac{z_s}{\lambda}+x_s)}}{\lambda}+x_{\rho(s,\frac{z_s}{\lambda}+x_s)}\|_{\mathcal{B}}\right\|ds.
\end{aligned}$$

Using Corollary 2.2 and inequality (2.5) yields

$$\begin{aligned}
&\left\|\frac{z_{\rho(s,\frac{z_s}{\lambda}+x_s)}}{\lambda}+x_{\rho(s,\frac{z_s}{\lambda}+x_s)}\right\|_{\mathcal{B}}\\
&\leq \frac{1}{\lambda}\|z_{\rho(s,\frac{z_s}{\lambda}+x_s)}\|_{\mathcal{B}}+\|x_{\rho(s,\frac{z_s}{\lambda}+x_s)}\|_{\mathcal{B}}\\
&\leq \frac{K_n}{\lambda}|z(s)|+\frac{M_n+\mathcal{L}^{\phi}}{\lambda}\|z_0\|_{\mathcal{B}}+K_n|x_s|+(M_n+\mathcal{L}^{\phi})\|x_0\|_{\mathcal{B}}\\
&\leq \frac{K_n}{\lambda}|z(s)|+K_n\left[\left\|\frac{\partial}{\partial s}U(s,0)\right\|_{B(E)}|\phi(0)|+\|U(s,0)\|_{B(E)}|y_1|\right]\\
&\quad +(M_n+\mathcal{L}^{\phi})\|\phi\|_{\mathcal{B}}\\
&\leq \frac{K_n}{\lambda}|z(s)|+(K_n\widetilde{M}\mathcal{D}+M_n+\mathcal{L}_h^{\phi})(\|\phi\|_{\mathcal{B}}+\varpi)+K_n\widehat{M}|y_1|.
\end{aligned}$$

Hence

$$\left\|\frac{z_{\rho(s,\frac{z_s}{\lambda}+x_s)}}{\lambda}+x_{\rho(s,\frac{z_s}{\lambda}+x_s)}\right\|_{\mathcal{B}} \leq \frac{K_n}{\lambda}|z(s)|+c_n.$$

Use the function $u(\cdot)$ and the nondecreasing character of ψ to get

$$\begin{aligned}
\frac{u(t)}{\lambda} \leq\ & \overline{M_0}L(\widehat{M}+\widetilde{M}n)+\widehat{M}\overline{M_0}L\|\phi\|_{\mathcal{B}}+\widehat{M}\int_0^n |g(r,0)|drds+\\
&+\widetilde{M}\ \overline{M_0}L\int_0^t K_nu(s)+c_nds+\widehat{M}\int_0^t p(s)\psi(K_nu(s)+c_n)ds+\\
&+\widehat{M}\int_0^t \eta(s)\left(\frac{K_n}{\lambda}u(s)+c_n\right)ds.
\end{aligned}$$

Set $\chi_n = \overline{M_0}L(\widehat{M} + \widetilde{M}n) + \widehat{M}\overline{M_0}L\|\phi\|_{\mathcal{B}} + \widehat{M}\int_0^n |g(r,0)|ds$.
Then

$$\begin{aligned}\frac{u(t)}{\lambda} \leq\ & \chi_n + \widetilde{M}\ \overline{M_0}L\int_0^t \frac{K_n}{\lambda}u(s) + c_n ds + \widehat{M}\int_0^t p(s)\psi\left(\frac{K_n}{\lambda}u(s) + c_n\right)ds \\ & + \widehat{M}\int_0^t \eta(s)\left(\frac{K_n}{\lambda}u(s) + c_n\right)ds.\end{aligned}$$

So

$$\begin{aligned}\frac{K_n}{\lambda}u(t) + c_n \leq\ & c_n + K_n\chi_n + K_n\widetilde{M}\ \overline{M_0}L\int_0^t \frac{K_n}{\lambda}u(s) + c_n ds \\ & + K_n\widehat{M}\int_0^t p(s)\psi\left(\frac{K_n}{\lambda}u(s) + c_n\right)ds \\ & + K_n\widehat{M}\int_0^t \eta(s)\left(\frac{K_n}{\lambda}u(s) + c_n\right)ds.\end{aligned}$$

Set $\widetilde{\sigma_n} := K_n\chi_n + c_n$. Then

$$\begin{aligned}\frac{K_n}{\lambda}u(t) + c_n \leq\ & \widetilde{\sigma_n} + K_n\int_0^t \left(\widetilde{M}\ \overline{M_0}L + \widehat{M}\eta(s)\right)\left(\frac{K_n}{\lambda}u(s) + c_n\right)ds \\ & + K_n\widehat{M}\int_0^t p(s)\psi\left(\frac{K_n}{\lambda}u(s) + c_n\right)ds.\end{aligned}$$

We consider the function μ defined by

$$\mu(t) = \sup_{0\leq t\leq +\infty}\left\{\frac{K_n}{\lambda}u(t) + c_n : 0 \leq s \leq t\right\}.$$

Let $t^* \in [0,t]$ be such that $\mu(t) = \dfrac{K_n}{\lambda}u(t^*) + c_n$. By the previous inequality, we have for $t \in [0,n]$

$$\begin{aligned}\mu(t) \leq\ & \widetilde{\sigma_n} + K_n\int_0^t \left(\widetilde{M}\ \overline{M_0}L + \widehat{M}\eta(s)\right)\mu(s)ds \\ & + K_n\int_0^t \widehat{M}p(s)\psi(\mu(s))ds.\end{aligned}$$

Let us take the right-hand side the above inequality as $v(t)$. Then we have

$$\mu(t) \leq v(t) \qquad \text{for all } t \in [0,n].$$

From the definition of v, we get $v(0) = \widetilde{\sigma_n}$ and

$$v'(t) = K_n\left(\widetilde{M}\ \overline{M_0}L + \widehat{M}\eta(t)v(t) + \widehat{M}p(t)\psi(v(t)\right),$$

$$\begin{aligned}\int_{\widetilde{\sigma_n}}^{v(t)} \frac{ds}{s+\psi(s)} &\leq K_n \int_0^t \max\left(\widehat{M}p(s), \widetilde{M}\overline{M_0}L + \widehat{M}\eta(s)\right) ds \\ &\leq K_n \int_0^n \max\left(\widehat{M}p(s), \widetilde{M}\ \overline{M_0}L + \widehat{M}\eta(s)\right) ds \\ &< \int_{\widetilde{\sigma_n}}^{+\infty} \frac{ds}{s+\psi(s)}.\end{aligned}$$

Thus, for every $t \in [0, n]$, there exists a constant $\widetilde{\Phi_n}$ such that $v(t) \leq \widetilde{\Phi_n}$, and hence $\mu(t) \leq \widetilde{\Phi_n}$. Since $\|z\|_n \leq \mu(t)$, we have $\|z\|_n \leq \widetilde{\Phi_n}$. This shows that the set $\widetilde{\Gamma}$ is bounded. Then the statement (Av2) in Theorem 1.9 does not hold. The nonlinear alternative of Avramescu implies that (Av1) is satisfied, we deduce that the operator $F + \widetilde{G}$ has a fixed point z^*. Then $y^*(t) = z^*(t) + x(t)$, $t \in \mathbb{R}$ is the fixed point of the operator $\widetilde{N}$, which is a mild solution of the problem $(1.4) - (1.6)$. □

3. Examples

To illustrate the previous results, we give in this section two examples.

3.1. **Example 1.** Consider the partial nonlocal second-order perturbed evolution equations

$$\begin{cases} \frac{\partial^2}{\partial t^2} v(t,\xi) = \frac{\partial^2}{\partial \xi^2} v(t,\xi) + a_0(t,\xi)v(t,\xi) \\ \qquad\qquad + \int_{-\infty}^{0} a_1(s-t)v\left[s - \rho_1(t)\rho_2\left(\int_0^{2\pi} a_2(\theta)|v(t,\theta)|^2 d\theta\right), \xi\right] ds \\ \qquad\qquad + \int_{-\infty}^{0} a_3(s-t)v\left[s - \rho_1(t)\rho_2\left(\int_0^{2\pi} a_2(\theta)|v(t,\theta)|^2 d\theta\right), \xi\right] ds, \\ \qquad\qquad\qquad\qquad\qquad\qquad t \geq 0,\ \xi \in [0, 2\pi], \\ v(t,0) = v(t,2\pi) = 0, \qquad\qquad t \geq 0, \\ v(\theta,\xi) + \sum_{j=1}^{p} c_j v(\theta + t_j, \xi) = v_0(\theta,\xi), \qquad -\infty < \theta \leq 0,\ \xi \in [0,2\pi], \\ \frac{\partial}{\partial t} v(0,\xi) = \varphi(\xi) \qquad -\infty < \theta \leq 0,\ \xi \in [0,2\pi], \end{cases} \tag{3.1}$$

where $a_0 : \mathbb{R}^+ \times [0, 2\pi] \to \mathbb{R}$ is a continuous function, $a_1, a_3 : \mathbb{R}_- \to \mathbb{R}$ and $a_2 : [0, 2\pi] \to \mathbb{R}$, $\rho_i : \mathbb{R}^+ \to \mathbb{R}$ are given functions for $i = 1, 2$.

Let $E = L^2(\mathbb{R}, \mathbb{C})$ be the space of 2π-periodic square-integrable functions from $\mathbb{R}$ *into* $\mathbb{C}$, and let $H^2(\mathbb{R}, \mathbb{C})$ denote the Sobolev space of 2π-periodic functions $x : \mathbb{R} \to \mathbb{C}$ such that $x'' \in L^2(\mathbb{R}, \mathbb{C})$.

We consider the operator $A_1 y(\xi) = y''(\xi)$ with domain $D(A_1) = H^2(\mathbb{R}, \mathbb{C})$. In addition, we take $A^2(t)y(s) = a(t)y'(s)$ defined on $H^1(\mathbb{R}, \mathbb{C})$, and consider the closed linear operator $A(t) = A_1 + A_2(t)$, which generates an evolution operator U defined by

$$U(t,s) = \sum_{n \in \mathbb{Z}} z_n(t,s) < x, w_n > w_n,$$

where z_n is a solution to the scalar initial value problem

$$\begin{cases} z''(t) = -n^2 z(t) + ina(t)z(t), \\ z(s) = 0, \quad z'(s) = z_1. \end{cases} \tag{3.2}$$

Theorem 3.1. *Let $\mathcal{B} = B \cup C(\mathbb{R}_-; E)$ and $\phi - h_t \in \mathcal{B}$. Assume that condition (H_ϕ) holds. suppose that the functions $a_1, a_3 : \mathbb{R}^- \to \mathbb{R}$, $a_2 : [0, \pi] \to \mathbb{R}^+$, and $\rho_i : \mathbb{R}^+ \to \mathbb{R}$, $i = 1, 2$, are continuous. Then there exists a mild solution of partial nonlocal second-order perturbed evolution equation with state dependent delay (3.1) on $\mathbb{R}$.*

Proof. From the assumptions, we have

$$f(t,\psi)(\xi) = \int_{-\infty}^{0} a_1(s)\psi(s,\xi)ds, \quad t \geq 0,\ \xi \in [0, 2\pi],$$

$$\rho(t,\psi)(\xi) = t - \rho_1(t)\rho_2\left(\int_0^\pi a_2(\theta)|\psi(0,\xi)|^2 d\theta\right), \qquad t \geq 0,\ \xi \in [0, 2\pi],$$

$$g(t,\psi) = \int_{-\infty}^{0} a_3(s)\psi(s,\xi)ds, \quad t \geq 0,\ \xi \in [0, 2\pi]$$

$$\phi(t)(\xi) = v_0(t,\xi), \quad t \leq 0,\ \xi \in [0, 2\pi],$$

and

$$h_t(v)(\xi) = \sum_{j=1}^{p} c_j v(t + t_j, \xi), \qquad t \leq 0,\ \xi \in [0, 2\pi],$$

$$\frac{\partial}{\partial t} v(0,\xi) = \frac{\partial}{\partial t} y(0,\xi)$$

are well defined functions, which permit to transform system (3.1) into the abstract system (1.1)–(1.3). Moreover, the functions f and g are bounded and linear. Now, the existence of mild solutions can be deduced from a direct application of Theorem 2.3. □

Corollary 3.2. *Let $\phi - h_t \in \mathcal{B}$ be continuous and bounded. Then the problem (3.1) has a mild solution on $\mathbb{R}$.*

3.2. **Example 2.** Consider the neutral second-order perturbed evolution equation

$$\begin{cases} \dfrac{\partial}{\partial t}\left[\dfrac{\partial}{\partial t} v(t,\xi) - \int_{-\infty}^{0} a_4(s-t) v(s - \rho_1(t)\rho_2\left(\int_0^\pi a_2(\theta)|v(t,\theta)|^2 d\theta\right), \xi)\, ds\right] \\ = \frac{\partial^2 v}{\partial \xi^2}(t,\xi) + a_0(t,\xi)v(t,\xi) \\ \quad + \int_{-\infty}^{0} a_1(s-t) v\left[s - \rho_1(t)\rho_2\left(\int_0^{2\pi} a_2(\theta)|v(t,\theta)|^2 d\theta\right), \xi\right] ds \\ \quad + \int_{-\infty}^{0} a_3(s-t) v\left[s - \rho_1(t)\rho_2\left(\int_0^{2\pi} a_2(\theta)|v(\tau,\theta)|^2 d\theta\right), \xi\right] ds, t \geq 0,\ \xi \in [0, 2\pi] \\ v(t,0) = v(t,2\pi) = 0, \hfill t \geq 0, \\ v(\theta,\xi) \displaystyle\sum_{j=1}^{p} c_j v(\theta + t_j, \xi) = v_0(\theta,\xi), \hfill -\infty < \theta \leq 0,\ \xi \in [0, 2\pi], \\ \quad \frac{\partial}{\partial t} v(0,\xi) = \varphi(\xi) \hfill -\infty < \theta \leq 0,\ \xi \in [0, 2\pi], \end{cases} \tag{3.3}$$

where $a_4 : R_- \to R$ is a continuous function.

Theorem 3.3. *Let $\mathcal{B} = B \cup C(R_-, E)$ and $\phi - h_t \in \mathcal{B}$. Assume that the condition (H_ϕ) holds. Suppose that the functions $a_1, a_3, a_4 : \mathbb{R}_- \to \mathbb{R}$, $a_2 : [0, 2\pi] \to \mathbb{R}$, and $\rho_i : \mathbb{R}_- \to \mathbb{R}$ for $i = 1, 2$ are continuous. Then there exists a mild solution of (3.3) on $\mathbb{R}$.*

Proof. From the assumptions, we have

$$f(t,\psi)(\xi) = \int_{-\infty}^{0} a_1(s)\psi(s,\xi)ds, \quad t \geq 0,\ \xi \in [0,\pi],$$

$$\rho(t,\psi)(\xi) = t - \rho_1(t)\rho_2\left(\int_0^\pi a_2(\theta)|\psi(0,\xi)|^2 d\theta\right), \qquad t \geq 0,\ \xi \in [0,\pi],$$

$$Q(t,\psi) = \int_{-\infty}^{0} a_4(s)\psi(s,\xi)ds, \quad t \geq 0,\ \xi \in [0,\pi],$$

$$g(t,\psi) = \int_{-\infty}^{0} a_3(s)\psi(s,\xi)ds, \quad t \geq 0,\ \xi \in [0,\pi],$$

$$h_t(v)(\xi) = \sum_{j=1}^{p} c_j v(t + t_j, \xi), \qquad t \leq 0,\ \xi \in [0, 2\pi],$$

$$\phi(t)(\xi) = v_0(t,\xi), \quad t \leq 0,\ \xi \in [0, 2\pi],$$

and

$$\frac{\partial}{\partial t} v(0,\xi) = \frac{\partial}{\partial t} y(0,\xi), \quad t \leq 0,\ \xi \in [0, 2\pi]$$

are well defined functions, which permit to transform system (3.3) into the abstract system (1.4) − (1.6). Now, the existence of mild solutions can be deduced from a direct application of Theorem 2.5. □

Corollary 3.4. *Let $\phi - h_t \in \mathcal{B}$ be continuous and bounded. Then there exists a mild solution of (3.3) on $\mathbb{R}$.*

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