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DAVIS–WIELANDT RADIUS OF 2×2 HILBERT–SCHMIDT OPERATOR MATRICES

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Abstract. We present the formula of the Davis–Wielandt radius for a Hilbert–Schmidt operator and provide an alternative to the classical triangle inequality for the Davis–Wielandt radius of the sum of two Hilbert–Schmidt operators. Additionally, several bounds for the Davis–Wielandt radius of 2×2 Hilbert–Schmidt operator matrices defined on $\mathcal{H} \times \mathcal{H}$ are given. In particular, we focus on the estimation of this radius for specific matrices, such as diagonal and off-diagonal matrices.

1. Introduction

In this paper, $B(\mathcal{H})$ denotes the C*-algebra of all bounded operators on a complex separate Hilbert space, endowed by the inner product $\langle \cdot, \cdot \rangle$ and the corresponding usual norm $\| \cdot \|$. For $T \in B(\mathcal{H})$, the numerical range of T is the convex complex set defined by

$$W(T) := \{\langle Tx, x\rangle : x \in \mathcal{H}, \|x\| = 1\}.$$

The numerical radius of T, denoted by $w(T)$, is defined as $w(T) = \sup\{\lambda \in W(T)\}$. It is well-established that $w(T)$ defines a norm on $\mathcal{H}$, which is equivalent to the usual operator norm. Furthermore, Yamazaki [28] demonstrated that for any operator T in $B(\mathcal{H})$, the following identity holds:

$$w(T) = \sup_{\theta \in \mathbb{R}} \left\| \Re(e^{i\theta} T) \right\|, \tag{1.1}$$

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where $\Re(T)$ and $\Im(T)$ are the real $\frac{1}{2}(T+T^*)$ and imaginary $\frac{1}{2i}(T-T^*)$ parts of T. This elegant representation links spectral properties with geometric interpretations, offering a powerful structure for both theoretical and computational advances.

In recent years, the study of numerical ranges and radius has grown considerably due to their significant applications in mathematics and physics. These concepts are essential tools for analyzing the behavior of bounded linear operators and are particularly useful for understanding the convergence of iterative methods, the analysis of differential and integral equations, and the spectral properties of operator matrices; see [7, 11, 20, 24, 29] and references therein. They are also relevant in applied contexts, such as quantum information theory, signal processing, and control theory. The numerical radius's flexibility and interpretability make it a valuable alternative to the spectral radius in many computational and theoretical frameworks [7, 10, 11]. This growing interest continues to drive advancements in the theoretical development and practical estimation of numerical radius and their extensions [13].

One of the most important generalization of the numerical radius is that given in [1]. Based on (1.1), Abu-Omar and Kittaneh have given the following new concept which called the generalized numerical radius:

$$w_N(T) = \sup_{\theta\in\mathbb{R}} N\left(\Re(e^{i\theta}T)\right),$$

with $N(\cdot)$ is an arbitrary norm on $B(\mathcal{H})$. The reader is referred to [1, 8, 15] for additional details on $w_N(\cdot)$. A similar concept is the A-numerical radius in a semi-Hilbertian space, which was shown in [29] to satisfy a formula analogous to (1.1). For further details, the reader is referred to [18, 17, 20, 22, 26, 29].

Another noteworthy extension is the Davis–Wielandt radius [16, 27], which has been defined as follows:

$$dw(T) = \sup_{x\in\mathcal{H},\|x\|=1} \sqrt{\|\langle Tx, x\rangle\|^2 + \|Tx\|^4}.$$

Moreover, $dw(T)$ is not a norm on $B(\mathcal{H})$ due to its failure to satisfy the triangular inequality. Nevertheless, it has a number of important properties that render it a topic of interest; see, for example, [5, 12, 14, 30, 31]. Indeed, $dw(T)$ includes both the numerical range and the operator norm for T and often provides more precise properties than those obtained from the numerical radius alone. As a result, it serves as a powerful tool in both operator analysis and matrix analysis. In a recent development, the Davis–Wielandt radius has been extended in [6]. This extension, termed the generalized Davis–Wielandt radius, is defined by

$$dw_N(T) = \sup_{\theta\in\mathbb{R}} \sqrt{N^2(\Re(e^{i\theta}T)) + N^4(\Re(e^{i\theta}|T|))}. \tag{1.2}$$

An interesting property of $dw_N(\cdot)$ is that if $N(\cdot)$ is weakly unitarily invariant, then so is $dw_N(\cdot)$, which means that for every T, $U \in B(\mathcal{H})$ such that U is unitary, we have

$$dw_N(UTU^*) = dw_N(T).$$

In addition, several estimates of $dw_N(T)$ have been given in [6]

$$\max\{w_N(T), w_N^2(|T|)\} \leq dw_N(T) \leq \sqrt{w_N^2(T) + w_N^4(|T|)} \tag{1.3}$$

and

$$dw_N(T) \geq \sqrt{\frac{1}{4}N^2(T) + \frac{1}{8}N^4(|T|)}. \tag{1.4}$$

Clearly, $dw_N(T)$ is not a norm and does not satisfy the triangular inequality. Nevertheless, the authors have shown in [23] that

$$\begin{aligned} dw_N^2(T+S) &\leq 8\left(dw_N^2(T) + dw_N^2(S)\right) \\ &\leq 8\left(dw_N(T) + dw_N(S)\right)^2. \end{aligned} \tag{1.5}$$

For any $T, S \in B(\mathcal{H})$, they have also developed several lower bounds, including

$$dw_N(T) \geq \frac{1}{2}\sqrt{N^2(T) + 2N^4(|T|) + 2\left|N^2(\Re(T)) + N^4(|T|) - N^2(\Im(T))\right|}, \tag{1.6}$$

where $|T| = (T^*T)^{\frac{1}{2}}$.

Despite these advances, the Davis–Wielandt radius under the Hilbert–Schmidt norm ($N(\cdot) = \|\cdot\|_2$) has not been studied. The Hilbert–Schmidt norm is particularly attractive due to its unitarily invariant nature and the Hilbert space structure it induces on the space of operators, facilitating various analytical techniques. This norm has garnered significant attention from researchers in functional analysis and operator theory; see, for instance, [2, 3, 4, 25]. To the best of our knowledge, this paper is the first to study this radius in the Hilbert–Schmidt setting, which we denote by $dw_2(\cdot)$ and refer to it as the Davis–Wielandt radius for Hilbert–Schmidt operators. Let $\mathcal{H}$ be a separable complex Hilbert space. A bounded operator T is said to belong to the Hilbert–Schmidt class $\mathcal{C}_2$ if its Hilbert–Schmidt norm is finite, that is, if the quantity $\|T\|_2 = \sqrt{tr(T^*T)} < \infty$. Also, the Cauchy–Schwarz inequality becomes

$$|tr(TS)| \leq \|T\|_2\|S\|_2,$$

for any $T, S \in \mathcal{C}_2$.

In this study, we begin by giving an explicit formula for the Davis–Wielandt radius of a general Hilbert–Schmidt operator. Building on this, we then specialize to 2×2 operator matrices on $\mathcal{H} \oplus \mathcal{H}$, with a focus on diagonal and off-diagonal block structures. In addition, we present an analogue of the Davis–Wielandt radius bound for the sum of two Hilbert–Schmidt operators.

2. Main results

The section begins with the following result, which gives the formula of $dw_2(T)$ in terms of $\|T\|_2^2$ and $tr(T^2)$ for a Hilbert–Schmidt operator T.

Proposition 2.1. *Let $T \in \mathcal{C}_2$. Then*

$$dw_2(T) = \sup_{\theta\in\mathbb{R}} \sqrt{\frac{1}{2}\|T\|_2^2 + \frac{1}{2}\Re(e^{2i\theta}trT^2) + \cos^4(\theta)\|T\|_2^4}. \tag{2.1}$$

Proof. It is easy to see that for any $T \in \mathcal{C}_2$, we have

$$\begin{aligned}\left\|\Re(e^{i\theta}T)\right\|_2^2 &= tr\left(\Re(e^{i\theta}T)\right)^2\\ &= \frac{1}{4}tr(T^*T + TT^* + 2\Re(e^{2i\theta}T^2))\\ &= \frac{1}{4}\left(2\|T\|_2^2 + 2\Re(e^{2i\theta}trT^2)\right).\end{aligned}$$

So, we have the following identity:

$$\begin{aligned}dw_2(T) &= \sup_{\theta\in\mathbb{R}}\sqrt{\|\Re(e^{i\theta}T)\|_2^2 + \|\Re(e^{i\theta}|T|)\|_2^4}\\ &= \sup_{\theta\in\mathbb{R}}\sqrt{\frac{1}{2}\|T\|_2^2 + \frac{1}{2}\Re(e^{2i\theta}trT^2) + \cos^4(\theta)\|T\|_2^4}.\end{aligned}$$

□

Remark 2.2. It is important to note that if T is self-adjoint, then

$$\begin{aligned}dw_2(T) &= \sqrt{\frac{1}{2}\|T\|_2^2 + \frac{1}{2}trT^2 + \|T\|_2^4}\\ &= \sqrt{\|T\|_2^2 + \|T\|_2^4}.\end{aligned} \tag{2.2}$$

Moreover, if $trT^2 = 0$, then

$$dw_2(T) = \sqrt{\frac{1}{2}\|T\|_2^2 + \|T\|_2^4}. \tag{2.3}$$

Remark 2.3. By (2.1), it is clear that for every $T \in \mathcal{C}_2$, $dw(-T) = dw(T)$. Furthermore

$$\begin{aligned}dw_2(T^*) &= \sup_{\theta\in\mathbb{R}}\sqrt{\frac{1}{2}\|T^*\|_2^2 + \frac{1}{2}\Re\Big(e^{2i\theta}tr(T^*)^2\Big) + \cos^4(\theta)\|T^*\|_2^4}\\ &= \sup_{\alpha\in\mathbb{R}}\sqrt{\frac{1}{2}\|T\|_2^2 + \frac{1}{2}\Re\Big(e^{-2i\alpha}\overline{tr(T^2)}\Big) + \cos^4(-\alpha)\|T\|_2^4}\\ &\qquad\qquad\qquad\text{(by taking } \alpha = -\theta)\\ &= \sup_{\alpha\in\mathbb{R}}\sqrt{\frac{1}{2}\|T\|_2^2 + \frac{1}{2}\Re\Big(\overline{e^{2i\alpha}tr(T^2)}\Big) + \cos^4(\alpha)\|T\|_2^4}\\ &= dw_2(T).\end{aligned}$$

This confirms $dw_2(T^*) = dw_2(T)$, i.e., $dw_2(\cdot)$ is invariant under the adjoint operation.

However, $dw_2(e^{i\theta}T) = dw_2(T)$ is not true for all $\theta \in \mathbb{R}$. To illustrate this, we may consider the example of the matrix $T = \begin{bmatrix} 0 & 1\\ -1 & 0\end{bmatrix}$, and we get $dw_2(T) = 2$ and $dw_2(iT) = \sqrt{6}$.

In the following, as a first step, we consider the behavior of dw_2 for diagonal operator matrices and for matrices whose second row consisting of null operators.

Theorem 2.4. *Let $A, B \in \mathcal{C}_2$. Then*

(a) $dw_2\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) \leq \sqrt{dw_2^2(A) + dw_2^2(B) + 2\|A\|_2^2\|B\|_2^2}$,

(b) $dw_2\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}\right) \leq \sqrt{dw_2^2(A) + dw_2^2(B) + 2\|A\|_2^2\|B\|_2^2 + \frac{1}{2}|tr(B^2)|}$.

Proof. For diagonal operator matrices of the form $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$, it is easy to check that

$$\left\|\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right\|_2^2 = \|A\|_2^2 + \|B\|_2^2,$$

$$\Re\left(e^{2i\theta} tr \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}^2\right) = \Re\left(e^{2i\theta} tr(A^2)\right) + \Re\left(e^{2i\theta} tr(B^2)\right),$$

and

$$\left\|\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right\|_2^4 = \|A\|_2^4 + \|B\|_2^4 + 2\|A\|_2^2\|B\|_2^2.$$

Then by identity (2.1), we have

$$\begin{aligned} & dw_2^2\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) \\ & = \sup_{\theta\in\mathbb{R}}\left[\frac{1}{2}\left\|\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right\|_2^2 + \frac{1}{2}\Re\left(e^{2i\theta} tr \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}^2\right) + \cos^4(\theta)\left\|\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right\|_2^4\right] \\ & = \sup_{\theta\in\mathbb{R}}\left[\frac{1}{2}\Big(\|A\|_2^2 + \|B\|_2^2 + \Re\left(e^{2i\theta} tr(A^2)\right) + \Re\left(e^{2i\theta} tr(B^2)\right)\Big)\right. \\ & \quad \left. + \cos^4(\theta)\Big(\|A\|_2^4 + \|B\|_2^4 + 2\|A\|_2^2\|B\|_2^2\Big)\right]. \end{aligned}$$

Splitting the supremum into three parts, we get

$$\begin{aligned} dw_2^2\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) & \leq \sup_{\theta\in\mathbb{R}}\left(\frac{1}{2}\|A\|_2^2 + \frac{1}{2}\Re\left(e^{2i\theta} tr(A^2)\right) + \cos^4(\theta)\|A\|_2^4\right) \\ & \quad + \sup_{\theta\in\mathbb{R}}\left(\frac{1}{2}\|B\|_2^2 + \frac{1}{2}\Re\left(e^{2i\theta} tr(B^2)\right) + \cos^4(\theta)\|B\|_2^4\right) \\ & \quad + 2\|A\|_2^2\|B\|_2^2 \\ & = dw_2^2(A) + dw_2^2(B) + 2\|A\|_2^2\|B\|_2^2. \end{aligned}$$

This completes the proof of (a). For part (b), we observe

$$\left\|\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}\right\|_2^2 = \|A\|_2^2 + \|B\|_2^2 \quad \text{and} \quad tr\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}^2 = tr(A^2).$$

Then the identity (2.1) gives us

$$
\begin{aligned}
& dw_2^2\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}\right) \\
& = \sup_{\theta\in\mathbb{R}} \left[\frac{1}{2}\Big(\|A\|_2^2 + \|B\|_2^2 + \Re\left(e^{2i\theta}tr(A^2)\right) + \Re\left(e^{2i\theta}tr(B^2)\right) - \Re\left(e^{2i\theta}tr(B^2)\right)\Big)\right. \\
& \quad \left. + \cos^4(\theta)\Big(\|A\|_2^4 + \|B\|_2^4 + 2\|A\|_2^2\|B\|_2^2\Big)\right].
\end{aligned}
$$

If we divide the supremum into three parts, then we get

$$
\begin{aligned}
dw_2^2\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}\right) &\le \sup_{\theta\in\mathbb{R}} \left[\frac{1}{2}\|A\|_2^2 + \frac{1}{2}\Re\left(e^{2i\theta}tr(A^2)\right) + \cos^4(\theta)\,\|A\|_2^4\right] \\
&\quad + \sup_{\theta\in\mathbb{R}} \left[\frac{1}{2}\|B\|_2^2 + \frac{1}{2}\Re\left(e^{2i\theta}tr(B^2)\right) + \cos^4(\theta)\,\|B\|_2^4\right] \\
&\quad + \sup_{\theta\in\mathbb{R}} \left[2\cos^4(\theta)\,\|A\|_2^2\,\|B\|_2^2 - \frac{1}{2}\Re\left(e^{2i\theta}tr(B^2)\right)\right] \\
&\le dw_2^2\,(A) + dw_2^2\,(B) + 2\,\|A\|_2^2\,\|B\|_2^2 + \frac{1}{2}\left|tr(B^2)\right|,
\end{aligned}
$$

since $\sup_{\theta\in\mathbb{R}}\left(e^{2i\theta}tr(B^2)\right) = |tr(B^2)|$. $\square$

Example 2.5. To illustrate the sharpness of the first inequality in Theorem 2.4 for diagonal operator matrices $T = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$, consider the matrices

$$
A = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}.
$$

The Davis–Wielandt radius of T is given by

$$
\begin{aligned}
dw_2\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) &= \sqrt{\left\|\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right\|_2^2 + \left\|\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right\|_2^4} \qquad \text{(by (2.2))} \\
&= \sqrt{6+36} = 42.
\end{aligned}
$$

Next, since $dw_2(A) = \sqrt{6}$ and $dw_2(B) = \sqrt{20}$. Taking inequality from Theorem 2.4 (a), we have

$$
\sqrt{dw_2^2\,(A) + dw_2^2\,(B) + 2\,\|A\|_2^2\,\|B\|_2^2} = \sqrt{6+20+2\times 2\times 4} = 42.
$$

This is consistent with the direct computation $dw_2^2(T) = 42$, confirming that the inequality in Theorem 2.4 (a) is sharp for this choice of A and B.

Example 2.6. In order to verify the sharpness of inequality (b) in Theorem 2.4, let

$$
A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},
$$

and define $T = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}$. Applying the identity from Proposition 2.1, we obtain

$$dw_2^2(A) = 2, \quad dw_2^2(B) = \frac{3}{2} \text{ and } dw_2^2(T) = \frac{11}{2}.$$

By Theorem 2.4 (b), we get

$$dw_2^2(T) \le dw_2^2(A) + dw_2^2(B) + 2\|A\|_2^2\|B\|_2^2 + \frac{1}{2}|\operatorname{tr}(B^2)| = 2 + \frac{3}{2} + 2 + 0 = \frac{11}{2}.$$

Equality holds, confirming that the inequality is sharp.

Remark 2.7. It is easy to conclude from the proof of Theorem 2.4 (a) that

$$dw_2\left(\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}\right) = dw_2\left(\begin{bmatrix} A & 0 \\ 0 & -A \end{bmatrix}\right) = dw_2\left(\sqrt{2}A\right). \tag{2.4}$$

We can also derive the following lower bounds:

$$dw_2\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) \ge \max\{dw_2(A), dw_2(B)\},$$

and

$$dw_2\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}\right) \ge \max\left\{dw_2(A), \sqrt{\frac{1}{2}\|B\|_2^2 + \|B\|_2^4}\right\},$$

for any $A, B \in \mathcal{C}_2$.

Some useful consequences of the above theorem are given below.

Corollary 2.8. *Let A and B be self-adjoint operators in $\mathcal{C}_2$. Then*

(a) $dw_2\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) = \sqrt{dw_2^2(A) + dw_2^2(B) + 2\|A\|_2^2\|B\|_2^2}$,

(b) $dw_2\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}\right) = \sqrt{dw_2^2(A) + dw_2^2(B) + 2\|A\|_2^2\|B\|_2^2 - \frac{1}{2}tr(B^2)}$.

Proof. Given that A and B are self-adjoint, it may be noted that A^2 and B^2 are positive operators. Consequently,

$$\begin{aligned} &dw_2^2\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) \\ &= \sup_{\theta\in\mathbb{R}}\Big(\frac{1}{2}(\|A\|_2^2 + \|B\|_2^2 + \Re\left(e^{2i\theta}tr(A^2)\right) + \Re\left(e^{2i\theta}tr(B^2)\right)) \\ &\quad + \cos^4(\theta)(\|A\|_2^4 + \|B\|_2^4 + 2\|A\|_2^2\|B\|_2^2)\Big) \\ &= \frac{1}{2}(\|A\|_2^2 + \|B\|_2^2 + tr(A^2) + tr(B^2)) + \|A\|_2^4 + \|B\|_2^4 + 2\|A\|_2^2\|B\|_2^2 \\ &= dw_2^2(A) + dw_2^2(B) + 2\|A\|_2^2\|B\|_2^2, \end{aligned}$$

which proves (a), and by an analogous approach, we find (b). $\square$

Corollary 2.9. *Let A and B in $\mathcal{C}_2$ such that $trA^2 = trB^2 = 0$. Then*

(a) $dw_2\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) = \sqrt{dw_2^2(A) + dw_2^2(B) + 2\|A\|_2^2\|B\|_2^2}$,

(b) $dw_2\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}\right) = \sqrt{dw_2^2(A) + dw_2^2(B) + 2\|A\|_2^2\|B\|_2^2}$.

Theorem 2.4 (a) can be naturally generalized to $n \times n$ operator matrices.

Theorem 2.10. *Let $T_i \in \mathcal{C}_2$ for $i = 1, 2, \ldots, n$, and let $T = diag(T_i)$. Then*

$$dw_2(T) = \sqrt{\sum_1^n dw_2^2(T_i) + 2\sum_{1\le i<j\le n} \|T_i\|_2^2\|T_j\|_2^2}.$$

In the next theorem, we present an alternative to the triangular inequality for the Davis–Wielandt radius sum of two Hilbert Schmidt operators A and B, when $tr(A^*B) = 0$.

Theorem 2.11. *Let A and B in $\mathcal{C}_2$ such that $tr(A^*B) = 0$. Then*

$$\begin{aligned} dw_2(A+B) &\le \sqrt{dw_2^2(A) + dw_2^2(B) + |tr(AB)| + 2\|A\|_2^2\|B\|_2^2} \\ &\le \sqrt{dw_2^2(A) + dw_2^2(B) + \frac{\|A\|_2^2 + \|B\|_2^2}{2} + \|A\|_2^4 + \|B\|_2^4}. \end{aligned} \tag{2.5}$$

Proof. Take A and B with $tr(A^*B) = 0$, and we get

$$\begin{aligned} \|A+B\|_2^2 &= tr\left((A+B)^*(A+B)\right) \\ &= \|A\|_2^2 + \|B\|_2^2 + tr(A^*B + B^*A) \\ &= \|A\|_2^2 + \|B\|_2^2, \end{aligned}$$

and

$$tr\left((A+B)(A+B)\right) = tr(A^2) + tr(B^2) + 2tr(AB).$$

Thus,

$$\begin{aligned} dw_2^2(A+B) &= \sup_{\theta\in\mathbb{R}}\left[\frac{1}{2}\|A+B\|_2^2 + \frac{1}{2}\Re(e^{2i\theta}tr(A+B)^2) + \cos^4(\theta)\|A+B\|_2^4\right] \\ &= \sup_{\theta\in\mathbb{R}}\left[\frac{1}{2}(\|A\|_2^2 + \|B\|_2^2) + \frac{1}{2}\Re\Big(e^{2i\theta}(tr(A^2) + tr(B^2) + 2tr(AB))\Big)\right. \\ &\quad \left. + \cos^4(\theta)(\|A\|_2^4 + \|B\|_2^4 + 2\|A\|_2^2\|B\|_2^2)\right]. \end{aligned}$$

If we divide the supremum into three parts, then we get

$$\begin{aligned} dw_2^2(A+B) &\le \sup_{\theta\in\mathbb{R}}\left[\frac{1}{2}\|A\|_2^2 + \frac{1}{2}\Re(e^{2i\theta}tr(A^2)) + \cos^4(\theta)\|A\|_2^4\right] \\ &\quad + \sup_{\theta\in\mathbb{R}}\left[\frac{1}{2}\|B\|_2^2 + \frac{1}{2}\Re(e^{2i\theta}tr(B^2)) + \cos^4(\theta)\|B\|_2^4\right] \\ &\quad + |tr(AB)| + 2\|A\|_2^2\|B\|_2^2 \\ &\le dw_2^2(A) + dw_2^2(B) + |tr(AB)| + 2\|A\|_2^2\|B\|_2^2. \end{aligned}$$

Finally, by the application of the Cauchy–Schwarz inequality, we have

$$\begin{aligned} dw_2^2\,(A+B) &\le dw_2^2(A) + dw_2^2(B) + 2\|A\|_2^2\|B\|_2^2 \\ &\qquad + \|A\|_2\|B\|_2 \text{ (by the Cauchy–Schwartz inequality)} \\ &\le dw_2^2(A) + dw_2^2(B) + \frac{\|A\|_2^2 + \|B\|_2^2}{2} + (\|A\|_2^4 + \|B\|_2^4) \\ &\qquad \text{(by the arithmetic-geometric mean inequality).} \end{aligned}$$

□

The following result deals with the Davis–Wielandt radius for an off-diagonal operator matrix.

Theorem 2.12. *Let A and B in $\mathcal{C}_2$. Then*

$$\begin{aligned} dw_2\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) &\le \sqrt{2dw_2^2\left(\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}\right) + 2dw_2^2\left(\begin{bmatrix} 0 & 0 \\ B & 0 \end{bmatrix}\right)} \\ &\le \sqrt{\|A\|_2^2(1+2\|A\|_2^2) + \|B\|_2^2(1+2\|B\|_2^2)}. \end{aligned}$$

Proof. Using the fact that $\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ B & 0 \end{bmatrix}^2 = 0$ and the product $\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}^* \begin{bmatrix} 0 & 0 \\ B & 0 \end{bmatrix}$ is also null, we conclude the following result:

$$\begin{aligned} & dw_2^2\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \\ &= dw_2^2\left(\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ B & 0 \end{bmatrix}\right) \\ &\le dw_2^2\left(\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}\right) + dw_2^2\left(\begin{bmatrix} 0 & 0 \\ B & 0 \end{bmatrix}\right) + \left\|\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}\right\|_2^4 \\ &\quad + \left\|\begin{bmatrix} 0 & 0 \\ B & 0 \end{bmatrix}\right\|_2^4 + \frac{1}{2}\left(\left\|\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}\right\|_2^2 + \left\|\begin{bmatrix} 0 & 0 \\ B & 0 \end{bmatrix}\right\|_2^2\right) \quad \text{(by Theorem 2.5)} \\ &= dw_2^2\left(\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}\right) + dw_2^2\left(\begin{bmatrix} 0 & 0 \\ B & 0 \end{bmatrix}\right) + \frac{1}{2}\left\|\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}\right\|_2^2 \\ &\quad + \left\|\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}\right\|_2^4 + \frac{1}{2}\left\|\begin{bmatrix} 0 & 0 \\ B & 0 \end{bmatrix}\right\|_2^2 + \left\|\begin{bmatrix} 0 & 0 \\ B & 0 \end{bmatrix}\right\|_2^4 \\ &= 2dw_2^2\left(\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}\right) + 2dw_2^2\left(\begin{bmatrix} 0 & 0 \\ B & 0 \end{bmatrix}\right) \quad \text{(by (2.3))} \\ &= \|A\|_2^2 + 2\|A\|_2^4 + \|B\|_2^2 + 2\|B\|_2^4. \end{aligned}$$

So, we deduce the desired result. □

Example 2.13. Consider the matrix $T = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$. We compute the Davis–Wielandt radius $dw_2(T)$ using the identity for self-adjoint operators in (2.2). A direct calculation gives $dw_2(T) = \sqrt{6}$. Now, by Theorem 2.12, we have

$$\begin{aligned} dw_2\left(\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}\right) &\le \sqrt{2dw_2^2\left(\begin{bmatrix} 0 & i \\ 0 & 0 \end{bmatrix}\right) + 2dw_2^2\left(\begin{bmatrix} 0 & 0 \\ -i & 0 \end{bmatrix}\right)} \\ &\le \sqrt{4\left(\frac{1}{2}+1\right)} = \sqrt{6}. \end{aligned}$$

The equality $dw_2(T) = \sqrt{6}$ confirms that the inequality in Theorem 2.12 is sharp for this operator.

Corollary 2.14. *Let A and B in $\mathcal{C}_2$. Then*

$$dw_2\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \le \sqrt{2}dw_2\left(\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}\right) + \sqrt{2}dw_2\left(\begin{bmatrix} 0 & 0 \\ B & 0 \end{bmatrix}\right).$$

The following lemma is of considerable utility in our study. For related results regarding the usual and generalized numerical radius, the reader is directed to reference [4, 19].

Lemma 2.15. *Let A and B in $\mathcal{C}_2$. Then*

(a) $dw_2\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) = dw_2\left(\begin{bmatrix} 0 & e^{-i\theta}A \\ e^{i\theta}B & 0 \end{bmatrix}\right)$ *for every $\theta \in \mathbb{R}$,*

(b) $dw_2\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) = dw_2\left(\begin{bmatrix} 0 & B \\ A & 0 \end{bmatrix}\right)$ *and* $dw_2\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) = dw_2\left(\begin{bmatrix} B & 0 \\ 0 & A \end{bmatrix}\right)$,

(c) $dw_2\left(\begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix}\right) = dw_2\left(\sqrt{2}B\right)$,

(d) $dw_2\left(\begin{bmatrix} 0 & B \\ -B & 0 \end{bmatrix}\right) = dw_2\left(i\sqrt{2}B\right)$,

(e) $dw_2\left(\begin{bmatrix} A & B \\ B & A \end{bmatrix}\right) \le \sqrt{dw_2^2(A+B) + dw_2^2(A-B) + 2\|A+B\|_2^2\|A-B\|_2^2}$,

(f) *If A and B are positive operators, then*

$$dw_2^2\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) = \frac{1}{2}\|A+B\|_2^2 + (\|A\|_2^2 + \|B\|_2^2)^2.$$

Proof. **(a)** Let us consider the unitary operator $U = \begin{bmatrix} I & 0 \\ 0 & e^{i\theta}I \end{bmatrix}$, acting on the Hilbert space $\mathcal{H}\oplus\mathcal{H}$. Since Davis–Wielandt radius for Hilbert–Schmidt operators is weakly unitary invariant [6, Proposition 1], we have

$$\begin{aligned} dw_2\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) &= dw_2\left(U\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}U^*\right) \\ &= dw_2\left(\begin{bmatrix} 0 & e^{-i\theta}A \\ e^{i\theta}B & 0 \end{bmatrix}\right), \end{aligned}$$

which demonstrates (a).

(b) For the unitary operator $U = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$, similar to (a), we get (b).

(c) Observe that

$$\left\| \begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix} \right\|_2^2 = 2\|B\|_2^2 \quad \text{and} \quad \Re\left(e^{2i\theta} tr\left(\begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix}^2 \right)\right) = 2\Re\left(e^{2i\theta} tr(B^2)\right).$$

Thus

$$\begin{aligned} dw_2\left(\begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix} \right) &= \sup_{\theta\in\mathbb{R}} \sqrt{\|B\|_2^2 + \Re(e^{2i\theta} tr B^2) + 4\cos^4(\theta)\|B\|_2^4} \quad (\text{by } (2.1)\ ) \\ &= \sup_{\theta\in\mathbb{R}} \sqrt{\frac{1}{2}\|\sqrt{2}B\|_2^2 + \frac{1}{2}\Re\Big(e^{2i\theta} tr((\sqrt{2}B)^2)\Big) + \cos^4(\theta)\|\sqrt{2}B\|_2^4} \\ &= dw_2\left(\sqrt{2}B\right). \end{aligned}$$

(d) For $B \in \mathcal{C}_2$, we have

$$\left\| \begin{bmatrix} 0 & B \\ -B & 0 \end{bmatrix} \right\|_2^2 = 2\|B\|_2^2 \quad \text{and} \quad \Re\left(e^{2i\theta} tr\left(\begin{bmatrix} 0 & B \\ -B & 0 \end{bmatrix}^2 \right)\right) = -2\Re\left(e^{2i\theta} tr(B^2)\right).$$

Similarly to (c), we get

$$dw_2\left(\begin{bmatrix} 0 & B \\ -B & 0 \end{bmatrix} \right) = dw_2\left(i\sqrt{2}B\right).$$

(f) We consider the unitary operator $U = \frac{1}{\sqrt{2}} \begin{bmatrix} I & I \\ -I & I \end{bmatrix}$, acting on the Hilbert space $\mathcal{H} \oplus \mathcal{H}$. So we get

$$\begin{aligned} dw_2\left(\begin{bmatrix} A & B \\ B & A \end{bmatrix} \right) &= dw_2\left(U \begin{bmatrix} A & B \\ B & A \end{bmatrix} U^* \right) \\ &= dw_2\left(\begin{bmatrix} A+B & 0 \\ 0 & A-B \end{bmatrix} \right) \\ &\leq \sqrt{dw_2^2(A+B) + dw_2^2(A-B) + 2\|A+B\|_2^2\|A-B\|_2^2} \\ &\qquad\qquad (\text{by Theorem } 2.4), \end{aligned}$$

as required.

(e) Let take now, A and B positive operators. Then

$$\left\| \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right\|_2^2 = \|A\|_2^2 + \|B\|_2^2 = tr(A^2) + tr(B^2)$$

and

$$tr\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}^2 \right) = tr(AB + BA) = 2tr(A^{\frac{1}{2}} B A^{\frac{1}{2}}) \geq 0.$$

By using the identity (2.2), we obtain

$$\begin{aligned}
dw_2^2\left(\begin{bmatrix}0 & A\\ B & 0\end{bmatrix}\right) &= \frac{1}{2}\left\|\begin{bmatrix}0 & A\\ B & 0\end{bmatrix}\right\|_2^2 + \frac{1}{2}tr\left(\begin{bmatrix}0 & A\\ B & 0\end{bmatrix}^2\right) + \left\|\begin{bmatrix}0 & A\\ B & 0\end{bmatrix}\right\|_2^4\\
&= \frac{1}{2}(tr(A^2) + tr(B^2) + tr(AB+BA)) + (tr(A^2)+tr(B^2))^2\\
&= \frac{1}{2}tr(A+B)^2 + (tr(A^2)+tr(B^2))^2\\
&= \frac{1}{2}\|A+B\|_2^2 + (\|A\|_2^2 + \|B\|_2^2)^2,
\end{aligned}$$

from which the result is derived. □

Based on Lemma 2.15, we establish two-sided bounds for the Davis–Wielandt radius of an off-diagonal Hilbert–Schmidt operator matrices.

Theorem 2.16. *Let $A, B \in \mathcal{C}_2$. Then*

$$dw_2\left(\begin{bmatrix}0 & A\\ B & 0\end{bmatrix}\right) \geq \frac{1}{4}\max\left\{dw_2\left(\sqrt{2}(A+B)\right), dw_2\left(i\sqrt{2}(A-B)\right)\right\}$$

and

$$dw_2\left(\begin{bmatrix}0 & A\\ B & 0\end{bmatrix}\right) \leq 2\sqrt{2}\left(dw_2\left(\frac{A+B}{\sqrt{2}}\right) + dw_2\left(i\frac{A-B}{\sqrt{2}}\right)\right).$$

Proof. Let $A, B \in \mathcal{C}_2$. By Lemma 2.15 (c), we obtain

$$\begin{aligned}
dw_2\left(\sqrt{2}(A+B)\right) &= dw_2\left(\begin{bmatrix}0 & A+B\\ A+B & 0\end{bmatrix}\right)\\
&= dw_2\left(\begin{bmatrix}0 & A\\ B & 0\end{bmatrix} + \begin{bmatrix}0 & B\\ A & 0\end{bmatrix}\right)\\
&\leq 2\sqrt{2}\sqrt{dw_2^2\left(\begin{bmatrix}0 & A\\ B & 0\end{bmatrix}\right) + dw_2^2\left(\begin{bmatrix}0 & B\\ A & 0\end{bmatrix}\right)} \quad \text{(by (1.5))}\\
&= 4dw_2\left(\begin{bmatrix}0 & A\\ B & 0\end{bmatrix}\right) \quad \text{(by Lemma 2.15 (b))}.
\end{aligned}$$

If we consider the property (a) of Lemma 2.15, then we have also

$$dw_2\left(\begin{bmatrix}0 & -iA\\ iB & 0\end{bmatrix}\right) \geq \frac{dw_2\left(\sqrt{2}(A+B)\right)}{4}.$$

Thus

$$dw_2\left(\begin{bmatrix}0 & A\\ B & 0\end{bmatrix}\right) \geq \frac{1}{4}\max\left\{dw_2\left(\sqrt{2}(A+B)\right), dw_2\left(i\sqrt{2}(A-B)\right)\right\}.$$

For the second inequality, we consider the unitary operator $U = \frac{1}{\sqrt{2}} \begin{bmatrix} I & I \\ -I & I \end{bmatrix}$.

$$\begin{aligned}
dw_2\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) &= dw_2\left(U\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}U^*\right) \\
&= dw_2\left(\frac{1}{2}\begin{bmatrix} A+B & A-B \\ -(A-B) & -(A+B) \end{bmatrix}\right) \\
&\le dw_2\left(\frac{1}{2}\begin{bmatrix} A+B & 0 \\ 0 & -(A+B) \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & A-B \\ -(A-B) & 0 \end{bmatrix}\right) \\
&\le 2\sqrt{2}dw_2\left(\frac{1}{2}\begin{bmatrix} A+B & 0 \\ 0 & -(A+B) \end{bmatrix}\right) \\
&\quad + 2\sqrt{2}dw_2\left(\frac{1}{2}\begin{bmatrix} 0 & A-B \\ -(A-B) & 0 \end{bmatrix}\right) \qquad \text{(by (1.5))} \\
&\le 2\sqrt{2}\left(dw_2\left(\frac{\sqrt{2}}{2}(A+B)\right) + dw_2\left(i\frac{\sqrt{2}}{2}(A-B)\right)\right) \\
&\qquad \text{(by Remark 2.7 and Lemma 2.15 (d))},
\end{aligned}$$

as required. □

Corollary 2.17. *Let $T \in \mathcal{C}_2$ with the Cartesian decomposition $T = A+iB$. Then*

$$dw_2\left(\begin{bmatrix} 0 & \sqrt{2}A \\ i\sqrt{2}B & 0 \end{bmatrix}\right) \ge \frac{1}{4}\max\{dw_2(2T), dw_2(2iT^*)\}$$

and

$$dw_2\left(\begin{bmatrix} 0 & \sqrt{2}A \\ i\sqrt{2}B & 0 \end{bmatrix}\right) \le 2\sqrt{2}\left(dw_2(T) + dw_2(iT^*)\right).$$

Remark 2.18. It should be noted that for any $A \in \mathcal{C}_2$, the unitary operator $U = \frac{1}{\sqrt{2}} \begin{bmatrix} I & I \\ -I & I \end{bmatrix}$ yields the result that

$$\left\|\begin{bmatrix} A & A \\ -A & -A \end{bmatrix}\right\|_2 = \left\|U\begin{bmatrix} A & A \\ -A & -A \end{bmatrix}U^*\right\|_2 = \left\|\begin{bmatrix} 0 & 0 \\ -2A & 0 \end{bmatrix}\right\|_2 = 2\|A\|_2.$$

Since $\begin{bmatrix} A & A \\ -A & -A \end{bmatrix}^2 = 0$, by the identity (2.3), we have

$$dw_2^2\left(\begin{bmatrix} A & A \\ -A & -A \end{bmatrix}\right) = \frac{1}{2}\|2A\|_2^2 + \|2A\|_2^4.$$

Theorem 2.19. *Let $A, B \in \mathcal{C}_2$. Then*

$$\begin{aligned}
\frac{1}{4}dw_2^2\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) &\le \|A+B\|_2^2 + 2\|A+B\|_2^4 \\
&\quad + 2\min\left\{dw_2^2(i\sqrt{2}A), dw_2^2(i\sqrt{2}B)\right\},
\end{aligned} \tag{2.6}$$

and

$$\frac{1}{4}dw_2^2\left(\begin{bmatrix}0 & A\\ B & 0\end{bmatrix}\right) \leq \|A-B\|_2^2 + 2\|A-B\|_2^4 + 2\min\left\{dw_2^2(\sqrt{2}A), dw_2^2(\sqrt{2}B)\right\}. \tag{2.7}$$

Proof. Let $U = \frac{1}{\sqrt{2}}\begin{bmatrix}I & I\\ -I & I\end{bmatrix}$. Since $dw_2(\cdot)$ is weakly unitary invariant, we have

$$\begin{aligned}
&dw_2^2\left(\begin{bmatrix}0 & A\\ B & 0\end{bmatrix}\right)\\
&= dw_2^2\left(U\begin{bmatrix}0 & A\\ B & 0\end{bmatrix}U^*\right)\\
&= dw_2^2\left(\frac{1}{2}\begin{bmatrix}A+B & A-B\\ -(A-B) & -(A+B)\end{bmatrix}\right)\\
&\leq dw_2^2\left(\frac{1}{2}\begin{bmatrix}A+B & A+B\\ -(A+B) & -(A+B)\end{bmatrix} + \begin{bmatrix}0 & -B\\ B & 0\end{bmatrix}\right)\\
&\leq 8dw_2^2\left(\frac{1}{2}\begin{bmatrix}A+B & A+B\\ -(A+B) & -(A+B)\end{bmatrix}\right) + 8dw_2^2\left(\begin{bmatrix}0 & -B\\ B & 0\end{bmatrix}\right) \quad \text{(by (1.5))}.
\end{aligned}$$

Using Remark 2.18 and Lemma 2.15 (d), we get

$$dw_2^2\left(\frac{1}{2}\begin{bmatrix}A+B & A+B\\ -(A+B) & -(A+B)\end{bmatrix}\right) \leq \frac{1}{2}\|A+B\|_2^2 + \|A+B\|_2^4$$

and

$$dw_2^2\left(\begin{bmatrix}0 & -B\\ B & 0\end{bmatrix}\right) = dw_2^2(i\sqrt{2}B).$$

Thus, we obtain

$$\begin{aligned}
dw_2^2\left(\begin{bmatrix}0 & A\\ B & 0\end{bmatrix}\right) &\leq 8\left(\frac{1}{2}\|A+B\|_2^2 + \|A+B\|_2^4 + dw_2^2(i\sqrt{2}B)\right)\\
&= 4\left(\|A+B\|_2^2 + 2\|A+B\|_2^4 + 2dw_2^2(i\sqrt{2}B)\right).
\end{aligned} \tag{2.8}$$

Moreover, using Lemma 2.15 (a), we find

$$\begin{aligned}
dw_2^2\left(\begin{bmatrix}0 & A\\ B & 0\end{bmatrix}\right) &= dw_2^2\left(\begin{bmatrix}0 & iA\\ -iB & 0\end{bmatrix}\right)\\
&\leq 4\left(\|iA-iB\|_2^2 + 2\|iA-iB\|_2^4 + 2dw_2^2(\sqrt{2}B)\right),
\end{aligned}$$

so, we conclude that

$$dw_2^2\left(\begin{bmatrix}0 & A\\ B & 0\end{bmatrix}\right) \leq 4\left(\|A-B\|_2^2 + 2\|A-B\|_2^4 + 2dw_2^2(\sqrt{2}B)\right). \tag{2.9}$$

Now, by lemma 2.15 (b), we get

$$dw_2^2\left(\begin{bmatrix}0 & A\\ B & 0\end{bmatrix}\right) \leq 4\left(\|A+B\|_2^2 + 2\|A+B\|_2^4 + 2dw_2^2(i\sqrt{2}A)\right), \tag{2.10}$$

and

$$dw_2^2\left(\begin{bmatrix}0 & A\\ B & 0\end{bmatrix}\right) \leq 4\left(\|A-B\|_2^2 + 2\,\|A-B\|_2^4 + 2dw_2^2(\sqrt{2}A)\right). \tag{2.11}$$

Consequently, the upper bound in (2.6) results from combining (2.8) and (2.10), whereas (2.7) derived from (2.9) and (2.11). $\square$

As an intuitive consequence of Theorem 2.19, we have the following result.

Corollary 2.20. *Let $A, B \in \mathcal{C}_2$. Then*

$$dw_2\left(\begin{bmatrix}0 & A\\ B & 0\end{bmatrix}\right) \leq 2\sqrt{\|A+B\|_2^2 + 2\,\|A+B\|_2^4 + dw_2^2(i\sqrt{2}B) + dw_2^2(i\sqrt{2}A)}.$$

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