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COHOMOLOGY OF THE (FOURIER) LEBESGUE–FOURIER ALGEBRAS ON COSET SPACES OF LOCALLY COMPACT GROUPS

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Abstract. For a compact subgroup K of a locally compact group G, we describe the multiplier algebra of the (Fourier) Lebesgue–Fourier algebra on the coset space G/K. This allows us to compare and study the multiplier-bounded approximate identity of the Lebesgue–Fourier algebra and the Fourier algebra on G/K. Moreover, characterizing the multiplier algebra of a commutative semisimple regular Banach algebra $\mathcal{A}$ with the Gelfand structure space X, we show that if $\mathcal{A}$ has a multiplier-bounded approximate identity and $\mathcal{A}$ is approximately amenable, then the norm on $\mathcal{A}$ is equivalent to the norm on its multiplier algebra. Additionally, we investigate some cohomological properties of the (Fourier) Lebesgue–Fourier algebra with a multiplier-bounded approximate identity on G/K. Furthermore, we examine certain hereditary properties of amenability, approximate amenability, pseudo-amenability, and weak amenability of the Fourier algebra on G/K.

1. Introduction and preliminaries

Let G be a locally compact group with the left Haar measure λ_G and identity e. The set of all integrable functions on G with respect to λ_G is denoted by $L^1(G)$. The space $L^1(G)$, as defined in [20], with the convolution product, forms a Banach algebra.

The Fourier algebra $A(G)$ and the Fourier–Stieltjes algebra $B(G)$ of G were introduced by Eymard [5]. The set of all multipliers of an algebra $\mathcal{A}$ is denoted by $\mathcal{M}(\mathcal{A})$. It is known that $\mathcal{M}(\mathcal{A})$ is a Banach algebra with respect to the operator norm, called the multiplier algebra of $\mathcal{A}$; see Larsen [23] for the

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theory of multipliers in general Banach algebras and Wang [32] for multipliers of commutative Banach algebras. We have $A(G) \subseteq B(G) \subseteq \mathcal{M}(A(G))$ and $\|\cdot\|_{\mathcal{M}(A(G))} \leq \|\cdot\|_{A(G)} = \|\cdot\|_{B(G)}$.

The Lebesgue–Fourier algebra $S^1A(G) = L^1(G) \cap A(G)$, equipped with the pointwise multiplication and the norm

$$\|\cdot\|_{S^1A(G)} = \|\cdot\|_{L^1(G)} + \|\cdot\|_{A(G)},$$

is an abstract Segal algebra with respect to $A(G)$. In 2002, Ghahramani and Lau [13, 14] introduced and extensively studied the Lebesgue–Fourier algebra of G.

Let K be a compact subgroup of G with the normalized Haar measure λ_K such that $\lambda_K(K) = 1$. There exists a G-invariant Radon measure μ on the coset space G/K, satisfying $\mu(x\dot{E}) = \mu(\dot{E})$ for all Borel subsets E of G and $x \in G$, where $\dot{E}$ is the image of E under the canonical quotient map φ_K. The modular functions of G and K are equal, that is, $\Delta_G|_K = \Delta_K = 1$. This leads to the integral formula:

$$\int_G u(x)\, d\lambda_G(x) = \int_{G/K} \int_K u(xk)\, d\lambda_K(k)\, d\mu(\dot{x}) \quad (u \in C_c(G)),$$

which also holds for all $u \in L^1(G)$. Define the map $P_K : A(G) \to A(G)$ by

$$P_K(u)(x) = \int_K u(xk)\, d\lambda_K(k) \quad (u \in A(G)).$$

The image of $A(G)$ under P_K consists of functions constant on the left cosets of K in G. The map P_K is a contractive projection. The map M_K on the image of $A(G)$ under P_K is defined by

$$M_K(u)(\dot{x}) = u(x) \quad (u \in P_K(A(G))),$$

and is an injective homomorphism with range denoted by $A(G/K)$, the Fourier algebra of the homogeneous space G/K. Both M_K and its inverse are isometric. The linear map $\Gamma_K := M_K \circ P_K : A(G) \to A(G/K)$ is surjective and contractive, leading to

$$A(G/K) = \{\Gamma_K(u) : u \in A(G)\},$$

with

$$\Gamma_K(u)(\dot{x}) = \int_K u(xk)\, d\lambda_K(k) \quad (u \in A(G)).$$

The Fourier algebra $A(G/K)$ is a commutative, semisimple, regular Banach algebra with Gelfand structure space G/K. This algebra was introduced by Forrest, who extended pointwise multiplication of $A(G)$ to $A(G/K)$ and investigated its properties. The Fourier–Stieltjes algebra on G/K, denoted by $B(G/K)$, is a subalgebra of the algebra of continuous functions on G/K. Additionally, we note that $A(G/K)$ is a closed ideal of $B(G/K)$. For more details, see [8, 9, 11, 7, 6, 20, 22, 24, 26].

The algebra $L^1(G/K)$ of the coset space G/K was introduced by Reiter and Stegeman [26]; they transferred convolution multiplication of $L^1(G)$ to $L^1(G/K)$ and studied its properties.

In [4], we defined the Lebesgue–Fourier space $S^1A(G/K)$ on the coset space G/K for a compact subgroup K of a locally compact group G. We proved that

$S^1A(G/K)$, with pointwise multiplication, is a commutative, semisimple, regular Banach algebra with G/K as its Gelfand structure space. When $K = \{e\}$, it corresponds to the Lebesgue–Fourier algebra $S^1A(G)$. We explored properties of $S^1A(G/K)$ and its relationships with $A(G/K)$ and G. Additionally, we studied the algebraic and topological properties of $S^1A(G/K)$ and investigated its approximate amenability, providing necessary conditions based on the properties of G and K.

Forrest and Runde in their paper [10], proved that the Fourier algebra $A(G)$ is amenable if and only if G has an abelian subgroup of finite index. They also demonstrated a hereditary property for weakly amenable $A(G)$ in [10, Lemma 3.1]. Ghahramani and Zhang [18] introduced the notions of pseudo-amenable and pseudo-contractible Banach algebras, while the concept of approximate amenability was introduced by Ghahramani and Loy [15]. Ghahramani and Stokke [17] showed that if G is amenable and discrete, then $A(G)$ is approximately amenable. Furthermore, if G is discrete, then $A(G)$ is pseudo-amenable if and only if it has an approximate identity. In their paper, they also described some hereditary properties of pseudo-amenable $A(G)$.

In this paper, characterizing the multiplier algebras of $S^1A(G/K)$ and $A(G/K)$, we study multiplier-bounded approximate identity of $S^1A(G/K)$. We prove that if $A(G/K)$ has a multiplier-bounded approximate identity, then so has $S^1A(G/K)$. Moreover, we characterize the multiplier algebra of a commutative, semisimple, regular Banach algebra $\mathcal{A}$ with the Gelfand structure space X. We establish that if $\mathcal{A}$ possesses a multiplier-bounded approximate identity and is approximately amenable, then the norm on $\mathcal{A}$ is equivalent to the norm on its multiplier algebra. Additionally, in this work, we extend the aforementioned results from [10, 18, 15, 17] to $A(G/K)$. We characterize the important notions of approximately amenability and pseudo-amenability on $S^1A(G/K)$ and $A(G/K)$. Moreover, we present a number of hereditary properties of notions of amenability for $A(G/K)$ and $S^1A(G/K)$.

2. Multiplier algebras of $A(G/K)$ and $S^1A(G/K)$

Before we study the multiplier algebras of $A(G/K)$ and $S^1A(G/K)$, we need a related result. Let us recall that a Banach algebra $\mathcal{A}$ is said to be without order if for every element $a \in \mathcal{A}$, the condition $a\mathcal{A} = 0$ implies $a = 0$.

Proposition 2.1. *Let K be a compact subgroup of a locally compact group G. Then $S^1A(G/K)$ is a Banach algebra without order.*

Proof. We have to show that if $u \in S^1A(G)$ and $\Gamma_K(u)\Gamma_K(v) = 0$ for all $v \in S^1A(G)$, then $\Gamma_K(u) = 0$. For this, suppose that $\Gamma_K(u) \neq 0$. Then there exists $\dot{x} \in G/K$ such that $\Gamma_K(u)(\dot{x}) \neq 0$. Since $S^1A(G/K)$ is regular by [4, Theorem 2.4], there exists an element $w \in S^1A(G)$ such that $\Gamma_K(w)(\dot{x}) = 1$. Thus $\Gamma_K(u)(\dot{x})\Gamma_K(w)(\dot{x}) \neq 0$. This is a contradiction. □

As we have already seen, for a compact subgroup K of a locally compact group G, the commutative Banach algebra $S^1A(G/K)$ is without order. Therefore, by the closed graph Theorem, every multiplier of $S^1A(G/K)$ is bounded as a linear operator. This holds similarly for $A(G/K)$.

Definition 2.2. Let K be a compact subgroup of a locally compact group G. Set

$$B_b(G/K) := \{\Phi \in C_b(G/K) : \quad \Phi A(G/K) \subseteq A(G/K)\},$$

where $C_b(G/K)$ is the space of all bounded continuous complex-valued functions on G/K with the supremum norm. We define the norm on $B_b(G/K)$ by

$$\| \Phi \|_{B_b(G/K)} := \sup \ \{ \| \Phi\Gamma_K(v) \|_{A(G/K)}: \quad v \in A(G) \text{ and } \| \Gamma_K(v) \|_{A(G/K)} \leq \ 1\}.$$

Also, we define

$$B_b^1(G/K) := \{\Phi \in C_b(G/K) : \quad \Phi S^1 A(G/K) \subseteq S^1 A(G/K)\}$$

and norm on $B_b^1(G/K)$ by

$$\| \Phi \|_{B_b^1(G/K)} := \sup \ \{\|\Phi\Gamma_K(v)\|_{S^1A(G/K)} : \ v \in S^1 A(G) \text{ and } \|\Gamma_K(v)\|_{S^1A(G/K)} \leq 1\}.$$

It is easy to check that $B_b(G/K)$ and $B_b^1(G/K)$ are Banach algebras with pointwise multiplication.

Proposition 2.3. *Let K be a compact subgroup of a locally compact group G. Then*

(a) $B(G/K) \subseteq B_b(G/K) \subseteq B_b^1(G/K)$;
(b) $\| \,.\, \|_{B_b^1(G/K)} \leq \| \,.\, \|_{B_b(G/K)} \leq \| \,.\, \|_{B(G/K)}$;
(c) $S^1A(G/K)$ *is a Banach* $B_b(G/K)$*-module.*

Proof. (a). Since $A(G/K)$ is an ideal of $B(G/K)$, we have $B(G/K) \subseteq B_b(G/K)$. Now, let $\Phi \in B_b(G/K)$. Thus $\Phi \in C_b(G/K)$ and $\Phi S^1A(G/K) \subseteq A(G/K)$. Since

$$\| \Phi\Gamma_K(v) \|_{L^1(G/K)} \leq \| \Phi \|_{L^\infty(G/K)} \| \Gamma_K(v) \|_{L^1(G/K)}$$

for each $v \in S^1A(G)$, then

$$\Phi S^1A(G/K) \subseteq \Phi L^1(G/K) \subseteq L^1(G/K).$$

Therefore $\Phi \in B_b^1(G/K)$.

(b). Let $u \in S^1A(G)$. Using $\| \,.\, \|_{L^\infty(G/K)} \leq \| \,.\, \|_{B_b(G/K)}$, for each $v \in S^1A(G)$ we have

$$\begin{aligned}
\|\Gamma_K(u)\Gamma_K(v)\|_{S^1A(G/K)} &= \| \Gamma_K(u)\Gamma_K(v) \|_{A(G/K)} + \| \Gamma_K(u)\Gamma_K(v) \|_{L^1(G/K)} \\
&\leq \| \Gamma_K(u) \|_{B_b(G/K)} \| \Gamma_K(v) \|_{A(G/K)} \\
&\quad + \| \Gamma_K(u) \|_{L^\infty(G/K)} \| \Gamma_K(v) \|_{L^1(G/K)} \\
&\leq \| \Gamma_K(u) \|_{B_b(G/K)} \| \Gamma_K(v) \|_{A(G/K)} \\
&\quad + \| \Gamma_K(u) \|_{B_b(G/K)} \| \Gamma_K(v) \|_{L^1(G/K)} \\
&= \| \Gamma_K(u) \|_{B_b(G/K)} \|\Gamma_K(v)\|_{S^1A(G/K)}.
\end{aligned}$$

Therefore,

$$\| \Gamma_K(u) \|_{B_b^1(G/K)} \leq \| \Gamma_K(u) \|_{B_b(G/K)} \ .$$

(c). Since

$$\|\Phi\Gamma_K(v)\|_{S^1A(G/K)} \ \leq \| \Phi \|_{B_b(G/K)} \ \|\Gamma_K(v)\|_{S^1A(G/K)}$$

for all $\Phi \in B_b(G/K)$ and $v \in S^1A(G)$, $S^1A(G/K)$ is a Banach $B_b(G/K)$-module. $\square$

For each multiplier T of a semisimple Banach algebra $\mathcal{A}$, there exists a bounded continuous function g on the Gelfand structure space of $\mathcal{A}$ such that

$$Tf = gf$$

for all $f \in \mathcal{A}$. Also, $\| g \|_\infty \leq \| T \|$; see [32, Theorem 3.1]. Since $S^1A(G/K)$ is semisimple with the Gelfand structure space G/K, these facts, together with what we have already seen, lead us to the following result.

Theorem 2.4. *Let K be a compact subgroup of a locally compact group G. Then $\mathcal{M}(S^1A(G/K))$ and $B_b^1(G/K)$ are isometrically isomorphic.*

As a consequence, we have the following result; first, note that for a compact subgroup K of a locally compact group G, the algebras $\mathcal{M}(A(G/K))$ and $B_b(G/K)$ are isometrically isomorphic.

Corollary 2.5. *Let K be a compact subgroup of a locally compact group G. Then the following statements are equivalent.*

(a) *$S^1A(G/K) = B_b^1(G/K)$;*
(b) *G is compact;*
(c) *$A(G/K) = B_b(G/K)$.*

Proof. On the one hand, the algebras $A(G/K)$ and $S^1A(G/K)$ are ideals of $B_b(G/K)$ and $B_b^1(G/K)$ respectively, and the constant one function is the identity element of $B_b(G/K)$ and $B_b^1(G/K)$. On the other hand, any one of $A(G/K)$ and $S^1A(G/K)$ has an identity element if and only if G is compact. $\square$

Combining [4, Proposition 2.5] and Corollary 2.5, we obtain $B_b^1(G/K) = B_b(G/K) = B(G/K)$ when G is also compact.

We now compare the norms $\|.\|_{S^1A(G/K)}$ and $\| \, . \, \|_{B_b^1(G/K)}$ which is interest in its own right.

Proposition 2.6. *Let K be a compact subgroup of a locally compact group G. Then the following statements are equivalent.*

(a) *The norms $\|.\|_{S^1A(G/K)}$ and $\| \, . \, \|_{B_b^1(G/K)}$ are equivalent;*
(b) *G is compact.*

Proof. Suppose that the norms $\|.\|_{S^1A(G/K)}$ and $\| \, . \, \|_{B_b^1(G/K)}$ are equivalent. Thus by Proposition 2.3, the norms $\|.\|_{S^1A(G/K)}$ and $\| \, . \, \|_{A(G/K)}$ are equivalent. Since $S^1A(G/K)$ is dense in $A(G/K)$, we conclude that $S^1A(G/K) = A(G/K)$; that is, G is compact by [4, Proposition 2.3]. To show the converse, suppose that G is compact. Then $S^1A(G/K)$ containes the identity element. Thus two norms are equivalent. $\square$

Corollary 2.7. *Let K be a compact subgroup of a locally compact group G. Then the following statements are equivalent.*

(a) *$S^1A(G/K)$ is closed with the norm $\| \, . \, \|_{B_b^1(G/K)}$;*
(b) *G is compact.*

Proof. Suppose that $S^1A(G/K)$ is closed with the norm $\| \, . \, \|_{B^1_b(G/K)}$. Then by the open mapping Theorem, the identity map $I : S^1A(G/K) \to \overline{S^1A(G/K)}^{\|.\|_{B^1_b(G/K)}}$ implies that the norms are equivalent; that is, G is compact. To the converse, assume that $\Gamma_K(u) \in \overline{S^1A(G/K)}^{\|.\|_{B^1_b(G/K)}}$. Then there exists v in $S^1A(G)$ such that $\| \, \Gamma_K(u) - \Gamma_K(v) \, \|_{B^1_b(G/K)} \leq \varepsilon$. Hence $\|\Gamma_K(u) - \Gamma_K(v)\|_{S^1A(G/K)} \leq \varepsilon$. Therefore $\Gamma_K(u) \in S^1A(G/K)$. □

Lemma 2.8. *Let K be a compact subgroup of a locally compact group G. If G is amenable, then $B(G/K) = B_b(G/K)$ and the norms $\| \, . \, \|_{A(G/K)}$ and $\| \, . \, \|_{B_b(G/K)}$ are equivalent.*

Proof. If G is amenable, then $B(G/K) = B_b(G/K)$ by [9, Corollary 4.3]. On the other hand, [9, Theorem 4.2] indicates that $A(G/K)$ has a bounded approximate identity. Let $(e_\alpha)_{\alpha\in\Lambda}$ be a net in $A(G)$ for which $(\Gamma_K(e_\alpha))_{\alpha\in\Lambda}$ is an approximate identity in $A(G/K)$ and is bounded by 1 in the norm of $A(G/K)$. Hence for each $u \in A(G)$, we have

$$\begin{aligned} \| \, \Gamma_K(u) \, \|_{A(G/K)} &= \lim \| \, \Gamma_K(e_\alpha)\Gamma_K(u) \, \|_{A(G/K)} \\ &\leq \lim \| \, \Gamma_K(e_\alpha) \, \|_{A(G/K)} \, \| \, \Gamma_K(u) \, \|_{B_b(G/K)} \\ &\leq \| \, \Gamma_K(u) \, \|_{B_b(G/K)} \, . \end{aligned}$$

On the other hand, $\| \, \Gamma_K(u) \, \|_{B_b(G/K)} \leq \| \, \Gamma_K(u) \, \|_{A(G/K)}$. □

Problem 2.9. Let K be a compact subgroup of a locally compact group G such that the norms $\| \, . \, \|_{A(G/K)}$ and $\| \, . \, \|_{B_b(G/K)}$ are equivalent. Is G amenable? or is G amenable if $B(G/K) = B_b(G/K)$?

While amenability of G ensures the equivalence of the norms on $A(G/K)$ and $B_b(G/K)$, the converse is subtle and non-trivial, as it would imply that a purely functional-analytic condition on the homogeneous space G/K suffices to recover a global algebraic-topological property of the group G.

Remark 2.10. Let $\mathcal{A}$ be a commutative semisimple regular Banach algebra with the Gelfand structure space X. Since $\mathcal{A}$ is regular, it is without order. Hence we conclude that every multiplier of $\mathcal{A}$ is a bounded linear operator. Define

$$B_b(X) := \big\{\Phi \in C_b(X) : \quad \Phi\mathcal{A} \subseteq \mathcal{A}\big\}.$$

We define the norm on $B_b(X)$ by

$$\| \, \Phi \, \|_{B_b(X)} := \sup \, \big\{ \, \| \, \Phi a \, \|_{\mathcal{A}} : \quad a \in \mathcal{A} \text{ and } \| \, a \, \|_{\mathcal{A}} \leq \, 1\big\}.$$

It is straightforward to verify that $B_b(X)$ is a Banach algebra with pointwise multiplication. Additionally, the algebras $\mathcal{M}(\mathcal{A})$ and $B_b(X)$ are isometrically isomorphic. If $\mathcal{A}$ has a bounded approximate identity, then the norms $\| \, . \, \|_{\mathcal{A}}$ and $\| \, . \, \|_{B_b(X)}$ are equivalent. It is not known whether the converse is true. Moreover, $\mathcal{A}$ has an identity if and only if $\mathcal{A} = B_b(X)$. In addition, $\mathcal{A}$ is closed with the norm $\| \, . \, \|_{B_b(X)}$ if and only if $\| \, . \, \|_{\mathcal{A}}$ and $\| \, . \, \|_{B_b(X)}$ are equivalent.

3. Cohomological properties of $A(G/K)$ and $S^1A(G/K)$

A sequence $(a_n)_{n\in\mathbb{N}}$ in a Banach algebra $\mathcal{A}$ is called multiplier-bounded if there exists a constant $R>0$ such that $\sup_{n\in\mathbb{N}}\|a_n a\| \leq R\|a\|$ for all $a\in\mathcal{A}$.

In this section, we compare multiplier-bounded approximate identity of $A(G/K)$ and $S^1A(G/K)$. Also, we investigate cohomology properties for $A(G/K)$ and $S^1A(G/K)$ with a multiplier-bounded approximate identity when K is a compact subgroup of a locally compact group G.

Theorem 3.1. *Let K be a compact subgroup of a locally compact group G. If $A(G/K)$ has a multiplier-bounded approximate identity, then $S^1A(G/K)$ has a multiplier-bounded approximate identity.*

Proof. Suppose that $A(G/K)$ has a multiplier-bounded approximate identity; in fact, there exists a net $(e_\alpha)_{\alpha\in\Lambda}$ in $A(G)$ for which $(\Gamma_K(e_\alpha))_{\alpha\in\Lambda}$ is an approximate identity in $A(G/K)$ with a bound R in the multiplier norm of $A(G/K)$. Let Γ be the set of all pairs (α,ε) for which $0<\varepsilon<1$ ordered by

$$(\alpha,\varepsilon)\prec\ (\alpha',\varepsilon')\ \Leftrightarrow\ \alpha\prec\alpha'\ ,\ \varepsilon\geq\varepsilon'.$$

For each $u\in A(G)$ and $\varepsilon>0$, we have

$$\|\ \Gamma_K(e_\alpha)\Gamma_K(u)-\Gamma_K(u)\ \|_{A(G/K)}<\varepsilon.$$

Let $\gamma:=(\alpha,\varepsilon)$. For each $\gamma\in\Gamma$, since $S^1A(G/K)$ is dense in $A(G/K)$, there exists $u_\gamma\in S^1A(G)$ such that

$$\|\ \Gamma_K(u_\gamma)-\Gamma_K(e_\alpha)\ \|_{A(G/K)}<\ \varepsilon.$$

So $(\Gamma_K(u_\gamma))_{\gamma\in\Gamma}$ is a net in $S^1A(G/K)$ such that for each $u\in A(G)$ and $\gamma\in\Gamma$, we have

$$\begin{aligned}\|\ \Gamma_K(u_\gamma)\Gamma_K(u)\ \|_{A(G/K)} &\leq \|\ \Gamma_K(e_\alpha)\Gamma_K(u)\ \|_{A(G/K)}\ +\varepsilon\ \|\ \Gamma_K(u)\ \|_{A(G/K)}\\ &< (R+1)\ \|\ \Gamma_K(u)\ \|_{A(G/K)}\ .\end{aligned}$$

Therefore, $\|\ \Gamma_K(u_\gamma)\ \|_{B_b(G/K)}<\ \ R+1$. Also, by Proposition 2.3, we have $\|\ .\ \|_{B_b^1(G/K)}<R+1$. Hence $(\Gamma_K(u_\gamma))_{\gamma\in\Gamma}$ is a net in $S^1A(G/K)$ that is multiplier-bounded. Now, let $u\in S^1A(G)$ and $\gamma\in\Gamma$. Then

$$\begin{aligned}\|\ \Gamma_K(u_\gamma)\Gamma_K(u)-\Gamma_K(u)\ \|_{A(G/K)} &\leq \|\ \Gamma_K(u)\ \|_{A(G/K)}\|\ \Gamma_K(u_\gamma)-\Gamma_K(e_\alpha)\ \|_{A(G/K)}\\ &\quad+\|\ \Gamma_K(e_\alpha)\Gamma_K(u)-\Gamma_K(u)\ \|_{A(G/K)}\\ &\leq \|\ \Gamma_K(u)\ \|_{A(G/K)}\ \varepsilon+\varepsilon\\ &\leq (\|\ \Gamma_K(u)\ \|_{A(G/K)}\ +1)\varepsilon.\end{aligned}$$

Additionally, there exists a compact subset $\dot{E}\subset G/K$ such that

$$\int_{(G/K)\setminus\dot{E}}|\Gamma_K(u)(\dot{x})|d\mu(\dot{x})<\varepsilon/(R+2).$$

From the regularity of $A(G/K)$, it follows that there exists a function $v \in A(G)$ such that $\Gamma_K(v) = 1$ on $\dot{E}$. Thus for each $\dot{x} \in \dot{E}$,

$$\begin{aligned} |\Gamma_K(u_\gamma)(\dot{x}) - 1| &= |\Gamma_K(u_\gamma)(\dot{x})\Gamma_K(v)(\dot{x}) - \Gamma_K(v)(\dot{x})| \\ &\leq \| \Gamma_K(u_\gamma)\Gamma_K(v) - \Gamma_K(v) \|_{A(\dot{E})} \\ &\leq (\| \Gamma_K(v) \|_{A(\dot{E})} + 1)\varepsilon. \end{aligned}$$

Therefore

$$\int_{\dot{E}} |\Gamma_K(u)(\dot{x})\Gamma_K(u_\gamma)(\dot{x}) - \Gamma_K(u)(\dot{x})| d\mu(\dot{x}) \leq \| \Gamma_K(v) \|_{A(\dot{E})} + 1)\varepsilon \| \Gamma_K(u) \|_{L^1(\dot{E})} .$$

Moreover,

$$\begin{aligned} \int_{G/K} |\Gamma_K(u)(\dot{x})\Gamma_K(u_\gamma)(\dot{x}) - \Gamma_K(v)(\dot{x})| d\mu(\dot{x}) &\leq \varepsilon(\| \Gamma_K(v) \|_{A(\dot{E})} \| \Gamma_K(u) \|_{L^1(\dot{E})} \\ &\quad + \| \Gamma_K(u) \|_{L^1(\dot{E})} + 1). \end{aligned}$$

It follows that $\| \Gamma_K(u_\gamma)\Gamma_K(u) - \Gamma_K(u) \|_{L^1(G/K)} \to 0$ and hence

$$\| \Gamma_K(u_\gamma)\Gamma_K(u) - \Gamma_K(u) \|_{S^1A(G/K)} \to 0$$

which completes the proof. □

A Banach algebra $\mathcal{A}$ is approximately amenable if every continuous derivation $D : \mathcal{A} \to \mathcal{X}^*$ is approximately inner for all Banach $\mathcal{A}$-bimodule $\mathcal{X}$; That is, there exists a net $(x_\alpha)_{\alpha\in\Lambda}$ in $\mathcal{X}$ such that for every $a \in \mathcal{A}$, we have

$$a \cdot x_\alpha - x_\alpha \cdot a \to D(a).$$

Ghahramani and Loy [15] defined and studied the notion of approximate amenability.

We can complete from [4, Corollary 3.5] as follows:

Corollary 3.2. *Let K be a compact subgroup of a locally compact group G such that $S^1A(G/K)$ is approximately amenable. Then the following statements are equivalent.*

(a) *G is compact;*
(b) *G is amenable;*
(c) *$A(G/K)$ has a multiplier-bounded approximate identity;*
(d) *$S^1A(G/K)$ has a multiplier-bounded approximate identity.*

In [4], we proved that for a compact subgroup K of a locally compact group G, if $S^1A(G/K)$ has a multiplier-bounded approximate identity and is approximately amenable, then G is compact. We now generalize this theorem to Banach algebras, with a proof similar to that in [4].

Theorem 3.3. *Let $\mathcal{A}$ be a commutative semisimple regular Tauberian Banach algebra with the Gelfand structure space X. If $\mathcal{A}$ has a multiplier-bounded approximate identity and $\mathcal{A}$ is approximately amenable, then the norms $\| \,.\, \|_{\mathcal{A}}$ and $\| \,.\, \|_{B_b(X)}$ are equivalent.*

Proof. Let $(a_\alpha)_{\alpha\in\Lambda}$ be an approximate identity in $\mathcal{A}$ which is multiplier-bounded by R. Define Γ as the set of pairs (α, ε) with $0 < \varepsilon < 1$ ordered by

$$(\alpha,\varepsilon) \prec (\alpha',\varepsilon') \Leftrightarrow \alpha \prec \alpha' , \varepsilon \geq \varepsilon'.$$

For each $\gamma = (\alpha,\varepsilon) \in \Gamma$, since $\mathcal{A}$ is Tauberian, there exists $b_\gamma \in \mathcal{A}$ such that

$$\| b_\gamma - a_\alpha \|_{\mathcal{A}} < \varepsilon$$

and b_γ has compact support. The net $(b_\gamma)_{\gamma\in\Gamma}$ satisfies:

$$\| b_\gamma b - b \|_{\mathcal{A}} < \varepsilon \,(\| b \|_{\mathcal{A}} + 1)$$

and

$$\begin{aligned} \| b_\gamma b \|_{\mathcal{A}} &\leq \| a_\alpha b \|_{\mathcal{A}} + \varepsilon \| b \|_{\mathcal{A}} \\ &< (R+1) \| b \|_{\mathcal{A}} . \end{aligned}$$

Thus $(b_\gamma)_{\gamma\in\Gamma}$ is a multiplier-bounded approximate identity in $\mathcal{A}$ with compact supports. For a compact subset H of $\Delta(\mathcal{A})$, let $\mathcal{I}(H)$ be the $\| \,.\, \|_{B_b(X)}$-closure of elements in $\mathcal{A}$ with compact support vanishing on H, and define

$$A_b(H) := \frac{B_b(X)}{\mathcal{I}(H)}$$

with the quotient norm. Since $\mathcal{A}$ is regular, there exists c_H in $\mathcal{A}$ with compact support such that $c_H \mid_H = 1$. Thus

$$c_H b_\gamma - b_\gamma \in \mathcal{I}(H)$$

for all γ. Therefore,

$$\begin{aligned} \| c_H \|_{A(H)} &= \lim_\gamma \| c_H b_\gamma \|_{A_b(H)} \\ &= \lim_\gamma \| b_\gamma \|_{A_b(H)} \\ &< R+1. \end{aligned}$$

We can choose d_H in $\mathcal{I}(H)$ with compact support such that

$$\| c_H + d_H \|_{B_b(X)} < 2(R+1).$$

Define

$$b_H := c_H + d_H,$$

then b_H has compact support,

$$b_H \mid_H = 1 \quad \text{and} \quad \| b_H \|_{B_b(X)} < 2(R+1).$$

Let Σ be the collection of all compact subsets of $\Delta(\mathcal{A})$ which is partially ordered by set inclusion. The net $(b_H)_{H\in\Sigma}$ is multiplier-bounded and $b_H b = b$ for $b \in \mathcal{A}$ with support$(b) \subseteq H$. Since $\mathcal{A}$ is approximately amenable, $(b_H)_{H\in\Sigma}$ is bounded in $\mathcal{A}$; see [1, Theorem 2.3]. Therefore,

$$C := \sup \{ \| b_H \|_{\mathcal{A}} : \quad H \in \Sigma \} < \infty,$$

and

$$\begin{aligned} \| b \|_{\mathcal{A}} &= \| b_H \, b \|_{\mathcal{A}} \\ &\leq \| b_H \|_{\mathcal{A}} \| b \|_{B_b(X)} \\ &\leq C \| b \|_{B_b(X)}; \end{aligned}$$

that is, the norms $\| \,.\, \|_{\mathcal{A}}$ and $\| \,.\, \|_{B_b(X)}$ are equivalent. □

If $\mathcal{A}$ is approximately amenable, we wish to see whether $\mathcal{A}$ has a multiplier-bounded approximate identity if and only if $\mathcal{A}$ has a bounded approximate identity.

Our next result is an application of the previous theorem and also a necessary condition for approximate amenability of $A(G/K)$.

Theorem 3.4. *Let K be a compact subgroup of a locally compact group G such that $A(G/K)$ has a multiplier-bounded approximate identity. If $A(G/K)$ is approximately amenable, then the norms $\| \, . \, \|_{A(G/K)}$ and $\| \, . \, \|_{B_b(G/K)}$ are equivalent.*

Theorem 3.5. *Let K be an open compact subgroup of an amenable locally compact group G. Then $A(G/K)$ is approximately amenable.*

Proof. It is sufficient to note that every commutative semisimple Banach algebra with a bounded approximate identity, consisting of elements with compact support and a discrete Gelfand structure space, is approximately amenable, see [16, Proposition 4.2]. Additionally, $A(G/K)$ is a commutative semisimple Banach algebra with the Gelfand structure space G/K. If G is amenable, then $A(G/K)$ has a bounded approximate identity consisting of elements with compact support in G/K; see [9, Theorem 4.2]. Therefore, $A(G/K)$ is approximately amenable. $\square$

A Banach algebra $\mathcal{A}$ is approximately biflat if there are bounded $\mathcal{A}$-bimodule morphisms

$$\theta_\gamma : (\mathcal{A}\hat{\otimes}\mathcal{A})^* \to \mathcal{A}^* \quad (\gamma \in \Gamma)$$

such that $W^* \circ T - \lim_\gamma \theta_\gamma \circ \pi^* = id_{\mathcal{A}^*}$, where $(W^* \circ T)$ is the weak* operator topology on $B(\mathcal{A}^*)$, the Banach algebra of bounded linear operators from $\mathcal{A}^*$ to itself; see [29, Section 2].

We recall that an approximate diagonal for a Banach algebra $\mathcal{A}$ is a net $(m_\gamma)_{\gamma\in\Gamma}$ in the projective tensor product $\mathcal{A}\hat{\otimes}\mathcal{A}$ such that $a \cdot m_\gamma - m_\gamma \cdot a \to 0$ and $\pi(m_\gamma)a \to a$ for all $a \in \mathcal{A}$ and $\gamma \in \Gamma$, where π denotes the product morphism from $\mathcal{A}\hat{\otimes}\mathcal{A}$ into $\mathcal{A}$ given by $\pi(a \otimes b) = ab$ for all $a, b \in \mathcal{A}$. In fact, $\mathcal{A}$ is amenable if and only if it has a bounded approximate diagonal. This notion was first introduced in [21]. Now, a Banach algebra $\mathcal{A}$ is pseudo-amenable if it has an approximate diagonal, [13].

Theorem 3.6. *Let K be a compact subgroup of a locally compact group G. If $A(G/K)$ is approximately biflat and $S^1A(G/K)$ has a multiplier-bounded approximate identity, then $S^1A(G/K)$ is pseudo-amenable.*

Proof. Assume that $S^1A(G/K)$ has an approximate identity $(\Gamma_K(u_\gamma))_{\gamma\in\Gamma}$ which is bounded in the multiplier norm. Thus $\big((\Gamma_K(u_\gamma))^2\big)_{\gamma\in\Gamma}$ is an approximate identity for $S^1A(G/K)$. Any abstract Segal algebra with respect to an approximately biflat Banach algebra, which has a net in its center such that the elements of this net, when squared, form an approximate identity for it, is approximately biflat; see [29, Proposition 2.5]. Hence $S^1A(G/K)$ is approximately biflat. Since for every Banach algebra with an approximate identity, the concept of pseudo-amenability is weaker than approximate biflatness, we conclude that $S^1A(G/K)$ is pseudo-amenable; see [29, Theorem 2.4]. $\square$

4. Some hereditary properties on open subgroups

In this section of the paper, we intend to study the interaction between cohomological properties of $A(G/K)$ and $A(H/K)$ for some open subgroup H of G including K.

For every function $u : G \to \mathbb{C}$ and $x \in G$, the left translation is defined by

$$L_x u(y) = u(x^{-1}y) \qquad (y \in G),$$

also, the right translation is defined by

$$R_x u(y) = u(yx) \qquad (y \in G).$$

For every $\Phi \in L^\infty(G/K)$ and $x \in G$, the left translation is defined by

$$L_x\Phi(\dot{y}) = \Phi(x^{-1}\dot{y}) \qquad (\mu\text{-locally almost all } \dot{y} \in G/K),$$

where the left translation is induced by the natural action of G on the left coset space G/K and

$$L^\infty(G/K) = \{T_K^\infty(u) \ : \ u \in L^\infty(G)\},$$

$T_K^\infty : L^\infty(G) \to L^\infty(G/K)$ is defined by

$$T_K^\infty(u)(\dot{x}) = \int_K u(xk)d\lambda_K(k)$$

for μ-almost all $\dot{x} \in G/K$ and $u \in L^\infty(G)$; it is a surjective norm decreasing linear map; see [25].

For every $v \in L^1(G)$ and $x \in G$, the left translation is defined by

$$L_x T_K^1(v)(\dot{y}) = T_K^1(v)(x^{-1}\dot{y}) \qquad (\mu\text{-locally almost all } \dot{y} \in G/K)$$

where the left translation is induced of the natural action of G on the left coset space G/K and

$$L^1(G/K) = \{T_K^1(u) \ : \ u \in L^1(G)\},$$

$T_K^1 : L^1(G) \to L^1(G/K)$ is defined by

$$T_K^1(u)(\dot{x}) = \int_K u(xk)d\lambda_K(k)$$

for μ-locally almost all $\dot{x} \in G/K$ and $u \in L^1(G)$; it is a surjective norm decreasing linear map; see [6].

We need the following Lemma for the description of the notions of amenability, pseudo-amenability and weak amenability of $A(G/K)$ in this section. We have also provided the following two lemmas in another case, which is still pending acceptance [3].

Lemma 4.1. *Let K be a compact subgroup of a locally compact group G. Then*

(a) *$A(G/K)$ is closed under the left translation;*
(b) *The map $x \mapsto L_x\Gamma_K(u) : G \to A(G/K)$ is continuous for all $u \in A(G)$;*
(c) *$\| L_x\Gamma_K(u) \|_{A(G/K)} = \| \Gamma_K(u) \|_{A(G/K)}$ for all $u \in A(G)$ and $x \in G$.*

Proposition 4.2. *Let K be a compact subgroup of a locally compact group G. If H is an open subgroup of finite index in G such that $K \subseteq H$, then the following statements are equivalent.*

(a) *$A(G/K)$ is amenable;*
(b) *$A(H/K)$ is amenable.*

Proof. (a) $\Rightarrow$ (b). Recall from [24, Lemma 3.1 (i)] that $A(H/K)$ is a quotient of $A(G/K)$; indeed, $A(G/K)/\mathcal{I}(H/K)$ is isometrically isomorphic to $A(H/K)$, where $\mathcal{I}(H/K)$ is the closed ideal $\{\Gamma_K(u) \in A(G/K) : \Gamma_K(u) = 0 \text{ on } H/K\}$ in $A(G/K)$. So, the result follows from [28, Corollary 2.3.2], which states that the quotient algebra $\mathcal{A}/\mathcal{I}$ of every amenable Banach algebra $\mathcal{A}$ by a two-sided closed ideal $\mathcal{I}$ of $\mathcal{A}$ is amenable.

(b) $\Rightarrow$ (a). Assume that $A(H/K)$ is amenable. By [24, Lemma 3.1(ii)], the map $\Gamma_K(u) \mapsto \Gamma_K(u)\chi_{H/K}$ is an isometric isomorphism of $A(H/K)$ onto $A(G/K)\chi_{H/K}$. Additionally, Lemma 4.1 implies that for each $x \in G$, the left translation map $\Gamma_K(u)\chi_{H/K} \mapsto \Gamma_K(u)\chi_{xH/K} : A(G/K)\chi_{H/K} \to A(G/K)\chi_{xH/K}$ is an isometric isomorphism. Thus for each $x \in G$, $A(G/K)\chi_{xH/K}$ is amenable. Now, suppose that $x_1 := e, x_2, \dots, x_n$ are distinct left coset representatives for H in G. Since $\Gamma_K(u) = \Gamma_K(u)\chi_{x_1H/K} + \cdots + \Gamma_K(u)\chi_{x_nH/K}$, we have $A(G/K) = \oplus_{i=1}^n A(G/K)\chi_{x_iH/K}$. So, we only need to recall that any direct sum of finitely many amenable Banach algebras is amenable [21]. □

Theorem 4.3. *Let K be a compact subgroup of a locally compact group G. If H is an open subgroup of G such that $K \subseteq H$, then the following statements are equivalent.*

(a) *$A(G/K)$ is pseudo-amenable;*
(b) *$A(H/K)$ is pseudo-amenable and $A(G/K)$ has an approximate identity.*

Proof. For (a) $\Rightarrow$ (b), from [18, Proposition 2.2], the proof is similar to the proof the part (a) $\Rightarrow$ (b) of Proposition 4.2. Also, every pseudo-amenable Banach algebra has an approximate identity. Thus it suffices to show that (b) implies (a). To see this, choose a set C which contains distinct left coset representatives for H in G and let e be in C. We order the collection Σ consisting of all finite subsets of C by inclusion and for each $\beta \in \Sigma$ let $E_\beta^K := \cup_{x\in\beta} xH/K$ and note that $E_{\{x\}}^K = xH/K$. By [24, Lemma 3.1(ii)] $\Gamma_K(u) \mapsto \Gamma_K(u)\chi_{H/K}$ is an isometric isomorphism of $A(H/K)$ onto $A(G/K)\chi_{H/K}$. Also, by Lemma 4.1 for each $x \in G$ the left translation map $\Gamma_K(u)\chi_{H/K} \mapsto \Gamma_K(u)\chi_{xH/K} : A(G/K)\chi_{H/K} \to A(G/K)\chi_{xH/K}$ is an isometric isomorphism. Thus $A(G/K)\chi_{xH/K}$ is pseudo-amenable. Now, using [18, Proposition 2.1] pseudo-amenable holds for $\oplus_{x\in\beta} A(G/K)\chi_{xH/K}$ and consequently $A(G/K)\chi_{E_\beta^K} = \oplus_{x\in\beta} A(G/K)\chi_{xH/K}$ is pseudo-amenable. Let $(e_\alpha)_{\alpha\in\Lambda}$ be a net in $A(G)$ for which $(\Gamma_K(e_\alpha))_{\alpha\in\Lambda}$ is an approximate identity in $A(G/K)$. Let Γ be the set of all pairs (α, ϵ) for which $\epsilon > 0$ directed by

$$(\alpha, \epsilon) \prec (\alpha', \epsilon') \Leftrightarrow \alpha \prec \alpha', \ \epsilon \geq \epsilon'.$$

Let $\gamma := (\alpha, \epsilon)$. Using the fact that $A(G/K)$ is a Tauberian algebra, we conclude that for each $\gamma \in \Gamma$ there exists $u_\gamma \in A(G)$ such that $\| \Gamma_K(u_\gamma) - \Gamma_K(e_\alpha) \|_{A(G/K)} < \epsilon$ and $\Gamma_K(u_\gamma)$ has compact support. Hence for each $u \in A(G)$ and $\gamma \in \Gamma$, we have

$$\| \Gamma_K(u_\gamma)\Gamma_K(u) - \Gamma_K(u) \|_{A(G/K)} < \epsilon \, (\| \Gamma_K(u) \|_{A(G/K)} + 1).$$

Thus $(\Gamma_K(u_\gamma))_{\gamma\in\Gamma}$ is an approximate identity in $A(G/K)$ and each $\Gamma_K(u_\gamma)$ has compact support. Since H is open in G and support$(\Gamma_K(u_\gamma)) \subseteq \cup_{x\in G} xH/K$, we have $\Gamma_K(u_\gamma) \in \cup_\beta A(G/K)\chi_{E^K_\beta}$ for all γ. Since $\cup_\beta A(G/K)\chi_{E^K_\beta}$ is an ideal in $A(G/K)$, $\Gamma_K(u_\gamma)\Gamma_K(v) \in \cup_\beta A(G/K)\chi_{E^K_\beta}$ for all $v \in A(G)$ and $\gamma \in \Gamma$. Thus from [17, Theorem 2.3] $A(G/K)$ is pseudo-amenable. □

Let $\mathcal{R}(G)$ denote the ring of subsets of G generated by all cosets of subgroups of G, where a ring of subsets of G is a collection of subsets of G that is closed under union and set difference. $\mathcal{R}(G)$ is said the coset ring of G. Also, the closed coset ring is denoted by $\mathcal{R}_c(G)$ and is defined by

$$\mathcal{R}_c(G) = \big\{E \in \mathcal{R}(G) \ : \ E \text{ is closed in } G\big\};$$

see [30].

As a consequence of Theorem 4.3, we have the following result.

Corollary 4.4. *Let K be a compact subgroup of a locally compact group G such that $A(G/K)$ has an approximate identity. If H is an open abelian subgroup of G such that $K \subseteq H$ and $\Delta(K\times K) \in \mathcal{R}_c(H\times H)$, then $A(G/K)$ is pseudo-amenable.*

Proof. Since H is an abelian subgroup of G, $A(H)$ is amenable [10, Theorem 2.3]. Hence $A(H)$ has bounded approximate identity and so H is amenable. Since by [31, Corollary 3.9] $A(H/K)$ is operator amenable if and only if H is amenable and $\Delta(K \times K) \in \mathcal{R}_c(H \times H)$, we conclude that $A(H/K)$ is operator amenable; that is, every completely bounded derivation $D : A(H/K) \to \mathcal{V}^*$, where $\mathcal{V}^*$ is the operator dual space to completely bounded $A(H/K)$-bimodule, is inner. Also, if H has an abelian subgroup of finite index, then $A(H/K)$ and $MAX(A(H/K))$ are completely isomorphic [24, Theorem 3.3]. In particular, from [27, Theorem 2.5] amenability of a Banach algebra $\mathcal{A}$ coincides with operator amenability of $MAX(\mathcal{A})$ as a completely contractive Banach algebra. Thus $A(H/K)$ is amenable and hence is pseudo-amenable. □

Theorem 4.5. *Let K be a compact subgroup of a locally compact group G such that G is amenable or $A(G/K)$ has an approximate identity and every approximate identity of $A(G/K)$ is a one-sided approximate identity for any Banach $A(G/K)-$bimodule. If H is an open subgroup of G such that $K \subseteq H$, then the following statements are equivalent.*

(a) *$A(G/K)$ is approximately amenable;*
(b) *$A(H/K)$ is approximately amenable.*

Proof. (a) ⇒ (b). This follows using the same argument as in the proof of Proposition 4.2 by [15, Corollary 2.1].

(b) ⇒ (a). According to [18, Corollary 3.4], any approximately amenable commutative Banach algebra is pseudo-amenable. Additionally, any pseudo-amenable commutative Banach algebra is approximately amenable when every approximate identity of $A(G/K)$ is a one-sided approximate identity for any Banach $A(G/K)-$bimodule, or when $A(G/K)$ has bounded approximate identity; see [18, Propositions 3.5 and 3.2]. Since G is amenable, from [9, Theorem 4.2] indicates that $A(G/K)$ has bounded approximate identity. The concepts approximately amenability and pseudo-amenability are thus equivalent; see [18, Proposition 3.2]. Thus by Theorem 4.3, $A(G/K)$ is approximately amenable. $\square$

Corollary 4.6. *Let K be a compact subgroup of an amenable locally compact group G. If H is an open abelian subgroup of G such that $K \subseteq H$ and $\Delta(K \times K) \in \mathcal{R}_c(H \times H)$, then $A(G/K)$ is approximately amenable.*

If $\mathcal{A}$ is a Banach algebra, a Banach $\mathcal{A}$-bimodule $\mathcal{X}$ is symmetric if

$$a \cdot x = x \cdot a \qquad (a \in \mathcal{A}, x \in \mathcal{X}).$$

A Banach algebra $\mathcal{A}$ is weakly amenable if there is no non-zero derivation from $\mathcal{A}$ into a symmetric Banach $\mathcal{A}$-bimodule; see [2].

Theorem 4.7. *Let K be a compact subgroup of a locally compact group G. If H is an open subgroup of G such that $K \subseteq H$, then the following statements are equivalent.*

(a) *$A(G/K)$ is weakly amenable;*
(b) *$A(H/K)$ is weakly amenable.*

Proof. (a) $\Rightarrow$ (b). The proof follows directly from Proposition 4.2 in [19]. For the converse, let $\mathcal{X}$ be a symmetric Banach $A(G/K)$-bimodule and let $D : A(G/K) \to \mathcal{X}$ be a derivation. Consider $u \in A(G)$ such that $\Gamma_K(u)$ has compact support. Thus there exist $x_1, \dots, x_n$ in G such that support$(\Gamma_K(u)) \subseteq \cup_{j=1}^n x_j H/K$. By [24, Lemma 3.1(ii)], the map $\Gamma_K(u) \mapsto \Gamma_K(u)\chi_{H/K}$ is an isometric isomorphism of $A(H/K)$ onto $A(G/K)\chi_{H/K}$. Additionally, by Lemma 4.1 for each $x \in G$, the left translation map $\Gamma_K(u)\chi_{H/K} \mapsto \Gamma_K(u)\chi_{xH/K} : A(G/K)\chi_{H/K} \to A(G/K)\chi_{xH/K}$ is an isometric isomorphism. Thus $A(H/K)$ is isometrically isomorphic to $A(G/K)\chi_{xH/K}$. Hence $D \mid_{A(G/K)\chi_{x_j H/K}} = 0$ for $j = 1, \dots, n$. Therefore $D(\Gamma_K(u)\chi_{x_j H/K}) = 0$ for $j = 1, \dots, n$. Since $\Gamma_K(u) = \Gamma_K(u)\chi_{x_1 H/K} + \cdots + \Gamma_K(u)\chi_{x_n H/K}$, we conclude that $D(\Gamma_K(u)) = 0$. Since $A(G/K)$ is Tauberian, we have $D \mid_{A(G/K)} = 0$, completing the proof. $\square$

The following lemma is proved in [3], an accepted but unpublished work.

Lemma 4.8. *Let K be a compact subgroup of a locally compact group G. Then*

(a) *$S^1A(G/K)$ is closed under the left translation;*
(b) *The map $x \mapsto L_x\Gamma_K(u) : G \to S^1A(G/K)$ is continuous for all $u \in S^1A(G)$;*
(c) *$\|L_x\Gamma_K(u)\|_{S^1A(G/K)} = \|\Gamma_K(u)\|_{S^1A(G/K)}$ for all $u \in S^1A(G)$ and $x \in G$.*

Theorem 4.9. *Let K be a compact subgroup of a locally compact group G such that $S^1A(G/K)$ has an approximate identity. If H is an open subgroup of G such that $K \subseteq H$ and $S^1A(H/K)$ is pseudo-amenable, then $S^1A(G/K)$ is pseudo-amenable.*

Proof. Let C be a set containing distinct left coset representatives for H in G, and let e be an element of C. We order the collection Σ of all finite subsets of C by inclusion. For each $\beta \in \Sigma$, define $E_\beta^K := \cup_{x\in\beta} xH/K$ and note that $E_{\{x\}}^K = xH/K$. According to [24, Lemma 3.1(ii)], the map $\Gamma_K(u) \mapsto \Gamma_K(u)\chi_{H/K}$

is an isometric isomorphism of $A(H/K)$ onto $A(G/K)\chi_{H/K}$. It suffices to show that the map $T^1_K(v) \mapsto T^1_K(v)\chi_{H/K}$ is an isometric isomorphism of $L^1(H/K)$ onto $L^1(G/K)\chi_{H/K}$. Since $(T^1_K(v) \circ \varphi_K)\chi_H = T^1_K(v)\chi_{H/K} \circ \varphi_K$, the function $T^1_K(v) \circ \varphi_K$ is constant on left cosets of K if and only if $T^1_K(v)\chi_{H/K} \circ \varphi_K$ is so. Thus $T^1_K(v) \in L^1(H/K)$ if and only if $T^1_K(v)\chi_{H/K} \in L^1(G/K)$. Moreover, for $v \in L^1(G)$, we obtain

$$\| T^1_K(v)\chi_{H/K} \|_{L^1(G/K)} = \int_{G/K} | T^1_K(v)\chi_{H/K}(\dot{x}) | \, d\mu(\dot{x}) = \| T^1_K(v) \|_{L^1(H/K)} .$$

Therefore, $\Gamma_K(u) \mapsto \Gamma_K(u)\chi_{H/K}$ is an isometric isomorphism of $S^1A(H/K)$ onto $S^1A(G/K)\chi_{H/K}$. Additionally, by Lemma 4.8, for each $x \in G$, the left translation map $\Gamma_K(u)\chi_{H/K} \mapsto \Gamma_K(u)\chi_{xH/K}$ is an isometric isomorphism from $S^1A(G/K)\chi_{H/K}$ to $S^1A(G/K)\chi_{xH/K}$. Using [18, Proposition 2.1], the pseudo-amenability holds for $\oplus_{x\in\beta}S^1A(G/K)\chi_{xH/K}$. Thus

$$S^1A(G/K)\chi_{E^K_\beta} = \oplus_{x\in\beta}S^1A(G/K)\chi_{xH/K}$$

is pseudo-amenable.

Let $(e_\alpha)_{\alpha\in\Lambda}$ be a net in $S^1A(G)$ such that $(\Gamma_K(e_\alpha))_{\alpha\in\Lambda}$ forms an approximate identity in $S^1A(G/K)$. Define Γ as the set of all pairs (α, ϵ) with $\epsilon > 0$ ordered by

$$(\alpha, \epsilon) \prec (\alpha', \epsilon') \Leftrightarrow \alpha \prec \alpha' , \ \epsilon \geq \epsilon'.$$

Let $\gamma := (\alpha, \epsilon)$. Since $S^1A(G/K)$ is a Tauberian algebra, for each $\gamma \in \Gamma$, there exists $u_\gamma \in S^1A(G)$ such that $\|\Gamma_K(u_\gamma) - \Gamma_K(e_\alpha)\|_{S^1A(G/K)} < \epsilon$ and $\Gamma_K(u_\gamma)$ has compact support. Hence for each $u \in S^1A(G)$ and $\gamma \in \Gamma$,

$$\|\Gamma_K(u_\gamma)\Gamma_K(u) - \Gamma_K(u)\|_{S^1A(G/K)} < \epsilon \ (\|\Gamma_K(u)\|_{S^1A(G/K)} + 1).$$

Thus $(\Gamma_K(u_\gamma))_{\gamma\in\Gamma}$ is an approximate identity in $S^1A(G/K)$ and each $\Gamma_K(u_\gamma)$ has compact support. Since H is open in G and support$(\Gamma_K(u_\gamma)) \subseteq \cup_{x\in G}xH/K$, we have $\Gamma_K(u_\gamma) \in \cup_\beta S^1A(G/K)\chi_{E^K_\beta}$ for all γ. Also, $\cup_\beta S^1A(G/K)\chi_{E^K_\beta}$ is an ideal in $S^1A(G/K)$. Thus $\Gamma_K(u_\gamma)\Gamma_K(v) \in \cup_\beta S^1A(G/K)\chi_{E^K_\beta}$ for all $v \in S^1A(G)$ and $\gamma \in \Gamma$. Therefore, by [17, Theorem 2.3], $S^1A(G/K)$ is pseudo-amenable. □

We end the work by a comparison result and an extension result.

Proposition 4.10. *Let K and H be compact subgroups of a locally compact group G. Then the following statements are equivalent.*

(a) *$K \subseteq H$;*
(b) *If a function in $S^1A(G)$ is constant on the left cosets of H in G, then it is constant on the left cosets of K in G.*

Proof. To prove (b) ⇒ (a), suppose that there is an element $x_0 \in K$ such that $x_0 \notin H$. Then

$$\dot{x}_0 := \varphi_H(x_0) \in G/H.$$

If $\dot{e} := \varphi_H(e)$, then $\dot{x}_0 \neq \dot{e}$. Thus there exists an open set $\dot{U} \subset G/H$ with $\dot{e} \in \dot{U}$ and $\dot{x}_0 \notin \dot{U}$. If $U = {\varphi_H}^{-1}(\dot{U})$, then U is an open neighborhood of H such that

$x_0 \notin U$. Since $S^1A(G)$ is regular, there exists a function $u \in S^1A(G)$ such that $u\mid_H = 1$ and $u(x) = 0$ outside of U. Set

$$v := \Gamma_H(u) \circ \varphi_H.$$

Then $v(e) = 1$ and $v(x_0) = 0$. Hence v is not constant on cosets of K. This is a contradiction. Hence $K \subseteq H$. The converse is trivial. □

Proposition 4.11. *Let K be a compact subgroup of a locally compact group G and let H be an open subgroup of G such that $K \subseteq H$. Then every function in $S^1A(H/K)$ extends to a function in $S^1A(G/K)$.*

Proof. Recall from [12, Section 3, p. 10] that $S^1A(G)\mid_H = S^1A(H)$. So, the result follows from the fact that $\Gamma_K(u)\mid_{H/K} = \Gamma_K(u\mid_H)$ for all $u \in S^1A(G)$. □

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