Khayyam J. Math. 11 (2025), no. 2, 269-284
DOI:10.22034/KJM.2025.496717.3416

Khayyam Journal of Mathematics
emis.de/journals/KJM
kjm-math.org

NORMALIZED HYPERSURFACES IN CONTACT LORENTZIAN MANIFOLDS

SUNIL KUMAR CHANYAL

Communicated by F.H. Ghane

Abstract. The aim of the paper is to study normalized null hypersurfaces in Lorentzian manifolds equipped with an almost contact structure using rigging technique. We show that the leaves of integrable screen distribution admit a Sasakian structure. Some theorems for the normalized hypersurfaces are obtained when the ambient manifold is endowed with a Sasakian structure. We also investigate the integrability conditions of various distributions evolving the hypersurface.

1. Introduction

The geometry of null submanifolds is nowadays a well-established area of research because of its sincere applications to general relativity. The null hypersurfaces locally describe the trajectories of light rays that are emitted perpendicular to a spacelike surface of codimension two and are also used to define the boundaries of certain region in spacetime [3, 15]. Hence, the geometry of null hypersurfaces has been developing as a key for the comprehensive study of the physical and mathematical aspects of general relativity. On the other hand, contact geometry helps to study thermodynamic models in curved surfaces in various systems like black holes [14].

The hypersurfaces of a Lorentzian manifolds are classified into three types, namely timelike, spacelike, and lightlike (or null), based on their causal character. As the timelike and spacelike hypersurfaces inherit a nondegenerate metric from the ambient Lorentzian manifold, they can be studied in a similar manner to the submanifolds in Riemannian manifolds, which consist of two orthogonal

Date: Received: 30 December 2024; Accepted: 15 July 2025.
2020 *Mathematics Subject Classification.* Primary: 53C15; Secondary: 53C25.
Key words and phrases. Almost contact Lorentzian manifold, rigging vector field, lightlike hypersurfaces, screen distribution.

complementary bundles (tangent and normal) and canonical projections defined on these bundles. Indeed, in a lightlike hypersurface, the orthogonal (normal) direction is tangent to itself due to degeneracy of the induced metric, and hence canonical projections are not well defined. To overcome this problem, there are different ways to study null hypersurfaces. One way is to consider the quotient vector bundle $TM/TM^{\perp}$ [12]. In [4, 8], the authors considered a null section and a screen distribution to induce some geometric data on the null hypersurface. The authors in [10] introduced rigging technique that depends on a fixed (arbitrary) vector field (called a rigging field). The null section (called rigged vector field) and a screen distribution on the null hypersurface are defined from rigging vector field. In this technique, a Riemannian structure can be induced on the null hypersurface, which is a major advantage.

The rigging technique was explored by many geometers [19, 2, 13, 16, 17, 11, 18] and studied null hypersurfaces in Lorentzian manifolds and pseudo-Riemannian manifolds (some authors use the term semi-Riemannian manifold). Recently, Hamed and Massamba [9] studied invariant null rigged hypersurfaces in indefinite nearly α-Sasakian manifold and proved that leaves of integrable screen distribution carries the (induced) structure of ambient manifold under some conditions.

Next, it is well known that contact metric structures are central objects in the study of Riemannian manifolds and the book [5] contains a detailed survey on the results obtained in this framework. Takahashi [20] initiated the study of contact structures on manifolds endowed with pseudo-Riemannian metric. The systematic study of contact Lorentzian manifolds has been done by Calvaruso [6] and contact pseudo-metric manifolds by Calvaruso and Perrone [7].

In this paper, we study null hypersurfaces in contact Lorentzian manifolds using the rigging technique. The paper is organized as follows. After introduction, section 2 contains some basic results of rigged hypersurfaces as well as contact Lorentzian manifolds. Normalized lightlike hypersurfaces are studied in section 3, and it is proved that the structure reduced to screen distribution is an almost contact metric structure. In section 4, we continue the study of normalized lightlike hypersurfaces when the ambient manifold is Sasakian Lorentzian manifold. The integrability conditions of various distributions evolved has been obtained in section 5.

2. Contact Lorentzian manifolds and null hypersurfaces

2.1. **Contact Lorentzian manifolds.** An almost contact structure $(\overline{\varphi},\, \xi,\, \overline{\eta})$ on a $(2n+1)$-dimensional Lorentzian manifold $(\overline{M},\, \overline{g})$ is formed by a $(1,\, 1)$ tensor $\overline{\varphi}$, a global vector field ξ and a 1-form $\overline{\eta}$ on $\overline{M}$ satisfying [6]

$$\overline{\varphi}^2 = -Id + \overline{\eta}\otimes\xi, \quad \overline{\eta}(\xi) = 1, \quad \overline{\varphi}(\xi) = 0, \quad \overline{\eta}\circ\overline{\varphi} = 0, \tag{2.1}$$

$$\overline{g}(\overline{\varphi}X,\, \overline{\varphi}Y) = \overline{g}(X,\, Y) + \overline{\eta}(X)\overline{\eta}(Y), \qquad \text{for all } X,\, Y \in \Gamma(T\overline{M}). \tag{2.2}$$

The equation (2.2) is metric compatibility of the almost contact structure, and $(\overline{M}, \overline{\varphi}, \xi, \overline{\eta},\, \overline{g})$ is called an almost contact Lorentzian manifold. The following relations hold on an almost contact Lorentzian manifold: [6]:

$$\overline{g}(\overline{\varphi}X,\, Y) = -\overline{g}(X,\, \overline{\varphi}Y), \qquad \overline{g}(X,\, \xi) = -\overline{\eta}(X), \tag{2.3}$$

$$\overline{g}(\xi,\, \xi) = -1. \tag{2.4}$$

Equation (2.4) implies that ξ is a timelike vector field. If the Lorentzian metric $\overline{g}$ satisfies

$$d\overline{\eta} = \omega,$$

then $\overline{\eta}$ is a contact form on $\overline{M}$, and $(\overline{M}, \overline{\varphi}, \xi, \overline{\eta}, \overline{g})$ is called a contact Lorentzian manifold. Here $\omega(X,\, Y) = \overline{g}(X,\, \overline{\varphi}Y)$ and

$$d\overline{\eta}(X,\, Y) = \frac{1}{2}\left(X\left(\overline{\eta}(Y)\right) - Y\left(\overline{\eta}(X)\right) - \overline{\eta}\left([X,\, Y]\right)\right).$$

A contact Lorentzian manifold is said to be Sasakian if it is normal; that is, $[\overline{\varphi},\, \overline{\varphi}] + 2d\overline{\eta}\otimes\xi = 0$, where $[\overline{\varphi},\, \overline{\varphi}]$ denotes the Nijenhius tensor of $\overline{\varphi}$. An almost contact Lorentzian manifold $(\overline{M}, \overline{\varphi}, \xi, \overline{\eta}, \overline{g})$ is a Sasakian Lorentzian manifold if and only if [6, 7]

$$\left(\overline{\nabla}_X\overline{\varphi}\right)Y = \overline{g}(X,\, Y)\xi + \overline{\eta}(Y)X, \qquad \text{for all } X,\, Y \in T\overline{M}. \tag{2.5}$$

2.2. **Null hypersurfaces.** Let $(\overline{M},\, \overline{g})$ be an m-dimensional Lorentzian manifold and let M be a hypersurface of it. We say that M is a null (or lightlike) hypersurface if the induced metric is degenerate on M; that is, there exists a lightlike vector field $E \in \Gamma(TM)$ such that

$$g(X,\, E) = 0, \qquad \text{for all } X \in \Gamma(TM),$$

where $g = i^*\overline{g}$ denotes induced metric on M, $i : M \to \overline{M}$ being the canonical immersion [4, 8]. In this case, the radical distribution $\mathrm{Rad}(TM)$, defined by $\mathrm{Rad}(TM) = TM \cap TM^{\perp}$, is a one-dimensional null distribution spanned by E. The complementary distribution to $\mathrm{Rad}(TM)$ in TM is called screen distribution and is denoted by $\mathrm{S}(TM)$. It is known [8] that there is a unique lightlike one-dimensional distribution $\mathrm{ltr}(TM)$ in $T\overline{M}$ over M, orthogonal to $\mathrm{S}(TM)$ and not contained in TM, such that for any non-zero section E of $\mathrm{Rad}(TM)$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a unique section $\mathcal{N}$ of $\mathrm{ltr}(TM)$ on $\mathcal{U}$ satisfying

$$\overline{g}(\mathcal{N}, E) = 1, \quad \overline{g}(\mathcal{N}, \mathcal{N}) = \overline{g}(\mathcal{N}, W) = 0, \qquad \text{for all } W \in \Gamma(\mathrm{S}(TM)|_{\mathcal{U}}).$$

The distribution $\mathrm{ltr}(TM)$ is called the transverse distribution, and $\mathcal{N}$ is called the null transverse vector field. Since the vector bundle $\mathrm{S}(TM)$ is nondegenerate, one can consider $\mathrm{S}(TM)^{\perp}$, which is orthogonal to $\mathrm{S}(TM)$ in $T\overline{M}$. We have the following decompositions:

$$TM = \mathrm{S}(TM) \perp \mathrm{Rad}(TM)\,, \tag{2.6}$$

$$T\overline{M} = \mathrm{S}(TM) \perp \mathrm{S}(TM)^{\perp}\,, \tag{2.7}$$

$$\mathrm{S}(TM)^{\perp} = \mathrm{Rad}(TM) \oplus \mathrm{ltr}(TM)\,, \tag{2.8}$$

$$\begin{aligned} T\overline{M} &= \mathrm{S}(TM) \perp \{\mathrm{Rad}(TM) \oplus \mathrm{ltr}(TM)\} \\ &= TM \oplus \mathrm{ltr}(TM)\,. \end{aligned} \tag{2.9}$$

Before continuing our discussion, first we define rigging for a null hypersurface.

Definition 2.1 ([10])**.** A rigging vector field (or rigging) for M is a vector field ζ defined in an open set in $(\overline{M},\, \overline{g})$ containing M such that $\zeta_p \notin T_pM$ for all $p \in M$. If it is only defined over M, then it is called a restricted rigging.

Locally, there always exists a rigging for a null hypersurface, but it exists globally for orientable hypersurfaces [10].

From now on, we always assume that there is a rigging ζ for M. From rigging vector ζ, we define a unique lightlike vector field E in M such that $\overline{g}(\zeta,\, E) = 1$, E is called the rigged vector field. The rigging ζ, the rigged vector field E and the null transverse vector field $\mathcal{N}$ are related by

$$\mathcal{N} = \zeta - \frac{1}{2}\overline{g}(\zeta,\, \zeta)E.$$

Let $\overline{\eta}$ be the 1-form defined by

$$\overline{\eta}\,(X) = \overline{g}\,(X,\, \zeta)\,, \qquad \text{for all } X \in T\overline{M}, \tag{2.10}$$

and let $\eta = i^*\overline{\eta}$. Define the tensors $\hat{g} = \overline{g} + \overline{\eta} \otimes \overline{\eta}$ and $\tilde{g} = i^*\hat{g}$. Then $\tilde{g}$ is a Riemannian metric on the hypersurface M, called rigged metric. The rigged vector field E is $\tilde{g}$-metrically equivalent to the 1-form η [8, 11].

From (2.9), we have the local Gauss and Weingarten equations

$$\overline{\nabla}_X Y = \nabla_X Y + B(X,\, Y)\mathcal{N}, \tag{2.11}$$

$$\overline{\nabla}_X \mathcal{N} = -\mathcal{A}_{\mathcal{N}} X + \tau(X)\mathcal{N}, \tag{2.12}$$

for all $X, Y \in \Gamma(TM)$. Here ∇ is the induced connection, $\mathcal{A}$ is called shape operator of M, B is called the second fundamental form of M and τ is a 1-form. Moreover, B is a symmetric tensor and satisfies

$$B(X,\, Y) = -g(\nabla_X E,\, Y),$$

and hence $B(E,\, .) = 0$. We also have $\nabla_E E = -\tau(E)E$. The hypersurface M is called totally geodesic if $B = 0$ and totally umbilic if $B = \rho g$ for some function $\rho \in C^\infty(M)$.

In the view of (2.6), we write

$$X = PX + \theta(X)E,$$

where $P : \Gamma(TM) \to \Gamma(\mathrm{S}(TM)\,)$ is the canonical projection and

$$\theta(X) = \overline{g}(X,\, \mathcal{N}).$$

We observe that ∇ is a torsion free connection on M and satisfies

$$(\nabla_X g)\,(Y,\, Z) = B(X,\, Y)\theta(Z) + B(X,\, Z)\theta(Y).$$

Thus ∇ is a nonmetric connection. For any $X \in TM$ and $PY \in \mathrm{S}(TM)\,$, using the decomposition (2.6), we write

$$\nabla_X PY = \nabla^*_X PY + C(X,\, PY)E, \tag{2.13}$$

$$\nabla_X E = -\mathcal{A}^*_E X - \tau(X)E, \tag{2.14}$$

where ∇^* is the induced connection on $\mathrm{S}(TM)$, $\mathcal{A}^*$ is called shape operator of $\mathrm{S}(TM)$, C is called the second fundamental form of $\mathrm{S}(TM)$ and τ is a 1-form defined by $\tau(X) = \overline{g}(\overline{\nabla}_X\mathcal{N},\, E)$. The shape operator $\mathcal{A}^*$ of $\mathrm{S}(TM)$ satisfies

$$B(X,\, PY) = g(\mathcal{A}^*_E X,\, PY), \tag{2.15}$$

and the tensor C has the property

$$C(X,\, PY) = g(\mathcal{A}_{\mathcal{N}} X,\, PY). \tag{2.16}$$

3. Normalized lightlike hypersurfaces

Let $(M,\, g)$ be a lightlike hypersurface of a $(2n+1)$-dimensional almost contact Lorentzian manifold $(\overline{M}, \overline{\varphi}, \xi, \overline{\eta},\, \overline{g})$. From (2.10) and (2.3), we get $\zeta = -\xi$, which is a timelike vector field. The hypersurface $(M,\, g)$ with rigging $\zeta = -\xi$, is called a normalized lightlike hypersurface of $\overline{M}$ since at any point of M, ξ is transversal to M, and we denote it by $(M,\, \zeta,\, g)$. We recall that locally we have

$$\mathcal{N} + \xi = \frac{1}{2}E, \tag{3.1}$$

where E and $\mathcal{N}$ are, respectively, the rigged vector field and the null transversal vector field associated to the rigging $\zeta = -\xi$, satisfying

$$\overline{g}(E,\, \zeta) = 1 \Leftrightarrow \overline{g}(E,\, \xi) = -1 \qquad \text{and} \qquad \overline{g}(E,\, \mathcal{N}) = 1.$$

The following equations are easily obtained using (2.1), (2.2), and (3.1):

$$\overline{\varphi}^2(E) = -\left(\mathcal{N} + \frac{1}{2}E\right), \qquad \overline{\varphi}^2(\mathcal{N}) = -\frac{1}{2}\left(\mathcal{N} + \frac{1}{2}E\right), \tag{3.2}$$

$$\overline{\eta}(E) = 1, \qquad \overline{\eta}(\mathcal{N}) = -\frac{1}{2}, \qquad \overline{\varphi}(\mathcal{N}) = \frac{1}{2}\overline{\varphi}(E), \tag{3.3}$$

and

$$\overline{g}(\overline{\varphi}E,\, E) = 0, \qquad \overline{g}(\overline{\varphi}E,\, \mathcal{N}) = 0, \tag{3.4}$$
$$\overline{g}(\overline{\varphi}\mathcal{N},\, E) = 0, \qquad \overline{g}(\overline{\varphi}\mathcal{N},\, \mathcal{N}) = 0. \tag{3.5}$$

From (3.3), (**??**) and (3.5), we find

$$\overline{\varphi}\mathcal{N} = \frac{1}{2}\overline{\varphi}E \in \mathrm{S}(TM). \tag{3.6}$$

The normalized hypersurface in which both $\overline{\varphi}(E)$ and $\overline{\varphi}(\mathcal{N})$ belong to the screen distribution is called a *screen semi-invariant lightlike* hypersurface [1]. From (3.2) and (3.6), we have the following proposition.

Proposition 3.1. *A normalized lightlike hypersurface $(M,\, \zeta,\, g)$ of almost contact Lorentzian manifold $(\overline{M}, \overline{\varphi}, \xi, \overline{\eta},\, \overline{g})$ is a screen semi-invariant lightlike hypersurface.*

Now, for any $X \in \Gamma(TM)$, we put

$$\overline{\varphi}X = \varphi X + u(X)\mathcal{N}, \tag{3.7}$$

where $\varphi X \in TM, u(X) = \overline{g}(\overline{\varphi}X,\, E)$. Let $X, Y \in \Gamma(TM)$. From (2.2) and (3.7), we get

$$\overline{g}\left(\varphi X + u(X)\mathcal{N}, \varphi Y + u(Y)\mathcal{N}\right) = g(X,Y) + \eta(X)\eta(Y), \tag{3.8}$$

where $\eta = i^*\overline{\eta}$. Simplifying (3.8), we obtain

$$g(\varphi X,\, \varphi Y) = g(X,\, Y) + \eta(X)\eta(Y) - u(X)\overline{g}(\mathcal{N},\, \varphi Y) - u(Y)\overline{g}(\mathcal{N},\, \varphi X). \tag{3.9}$$

On the other hand, (2.3) and (3.7) imply

$$\overline{g}(\varphi X,\, \mathcal{N}) = \overline{g}(\overline{\varphi}X,\, \mathcal{N}) = -\overline{g}(X, \overline{\varphi}\mathcal{N}) = \frac{1}{2}u(X). \tag{3.10}$$

Now, we state the following proposition.

Proposition 3.2. *Let $(M,\, \zeta,\, g)$ be a normalized lightlike hypersurface of an almost contact Lorentzian manifold $(\overline{M}, \overline{\varphi}, \xi, \overline{\eta},\, \overline{g})$. Then*

$$g(\varphi X,\, \varphi Y) = g(X,\, Y) + \eta(X)\eta(Y) - u(X)u(Y),$$

for all $X,\, Y \in \Gamma(TM)$ and $\varphi,\, u$ are given by (3.7).

Proof. The proof follows from (3.9) and (3.10). □

The following proposition is obtained from a straightforward calculation.

Proposition 3.3. *If we put $U = \overline{\varphi}\mathcal{N}$ and $e = \frac{1}{2}E$, then the following relations hold on a normalized lightlike hypersurface $(M,\, \zeta,\, g)$:*

$$\varphi^2 = -Id + \eta \otimes e - u \otimes U, \tag{3.11}$$

$$u \circ \varphi = -\eta = -\overline{g}(\cdot,\, \mathcal{N}), \tag{3.12}$$

$$\eta \circ \varphi = \frac{1}{2}u, \tag{3.13}$$

$$\varphi(e) = U, \qquad \varphi(U) = -\frac{1}{2}e, \tag{3.14}$$

and

$$\eta(U) = 0, \qquad \eta(e) = \frac{1}{2}, \tag{3.15}$$

$$u(e) = 0, \qquad u(U) = -\frac{1}{2}. \tag{3.16}$$

Proposition 3.4. *On a normalized lightlike hypersurface $(M,\, \zeta,\, g)$ of an almost contact Lorentzian manifold $(\overline{M}, \overline{\varphi}, \xi, \overline{\eta},\, \overline{g})$, we have*

$$g(\varphi X,\, Y) = -g(X,\, \varphi Y) - \eta(X)u(Y) - u(X)\eta(Y) \tag{3.17}$$

and

$$g(\overline{\varphi}X,\, Y) + g(X,\, \overline{\varphi}Y) = 0, \tag{3.18}$$

where $X,\ Y \in \Gamma(TM)$.

Proof. The proof follows from (3.7) and Proposition 3.2. □

Proposition 3.5. *Let $(M,\, \zeta,\, g)$ be a normalized lightlike hypersurface in $(\overline{M}, \overline{\varphi}, \xi, \overline{\eta},\, \overline{g})$. Then there exists locally a $\overline{\varphi}$-basis $\{e_i,\, \overline{\varphi}e_i,\, \overline{\varphi}E,\, \overline{\varphi}^2E,\, \xi\}$ of $T\overline{M}$, where $\{e_i,\, \overline{\varphi}e_i,\, \overline{\varphi}E\}$ is a pseudo-orthonormal basis of $S(TM)$.*

Proof. The proof is similar to the proof given in [16]. □

Next, we restrict the almost contact Lorentzian structure of $(\overline{M}, \overline{\varphi}, \xi, \overline{\eta}, \overline{g})$ to the screen distribution $S(TM)$ of the M. Let M be a normalized hypersurface, and let φ be a (1, 1)-tensor field on M defined by (3.7). We set

$$\varphi X = \varphi_s X + u_s(X)E, \qquad \text{for all } X \in S(TM) \tag{3.19}$$
$$\text{and} \qquad \eta_s = u|_{S(TM)},\ \xi_s = -2U = -2\overline{\varphi}\mathcal{N}.$$

In (3.19), the decomposition (2.6) is used. Now, for any $X \in S(TM)$, we have

$$u_s(X) = \overline{g}(\varphi X, \mathcal{N}) = \overline{g}\left(\overline{\varphi}X,\ \frac{1}{2}E - \xi\right) = \frac{1}{2}u(X).$$

Equations (3.11) and (3.19) yield

$$\varphi(\varphi_s(X) + u_s(X)E) = -X - u(X)U.$$

The last equation implies

$$\varphi_s^2(X) + u_s(\varphi_s X)E + u_s(X)\varphi_s(E) + u_s(X)u_s(E)E = -X - u(X)U, \tag{3.20}$$

where $\varphi E = \varphi_s E + u_s(E)E$. Collecting screen and radical parts in (3.20), we find

$$\varphi_s^2(X) + u_s(X)\varphi_s(E) \;=\; -X - u(X)U, \tag{3.21}$$
$$u_s(\varphi_s X) \;=\; u_s(X)u_s(E) \;=\; 0. \tag{3.22}$$

Since $\varphi(E) = \overline{\varphi}(E) \in S(TM)$, it follows from (3.19) and (3.14) that

$$\varphi_s(E) = \varphi(E) = \overline{\varphi}(E) = 2U \qquad \text{and} \qquad u_s(E) = 0.$$

Using (3.21) and (3.22), we obtain

$$\varphi_s^2(X) = -X - u(X)(2U) = -X + u(X)(-2U), \tag{3.23}$$

for all $X \in S(TM)$. Thus the expected structure vector field is $-2U$, if it satisfies other properties viz. $\varphi_s(-2U) = 0 = \eta_s \circ \varphi_s$ and $\eta_s(-2U) = 1$. Now, $u(U) = -\frac{1}{2}$ implies

$$\eta_s(\xi_s) = u(-2U) = 1. \tag{3.24}$$

From (3.22), we get

$$\eta_s \circ \varphi_s = u_s \circ \varphi_s = 0. \tag{3.25}$$

Finally, $\varphi(U) = -\frac{1}{4}E$ implies $\varphi_s(U) = 0$ and $u_s(U) = -\frac{1}{4}$. Thus

$$\varphi_s(\xi_s) = -\varphi_s(2U) = 0. \tag{3.26}$$

Now, we have the following theorem.

Theorem 3.6. *Let (M, ζ, g) be a normalized lightlike hypersurface in an almost contact Lorentzian manifold $(\overline{M}, \overline{\varphi}, \xi, \overline{\eta}, \overline{g})$. If the screen distribution $S(TM)$ is integrable, then any leaf M_s of $S(TM)$ carries almost contact structure $(\varphi_s, \xi_s, \eta_s)$, where $\eta_s = u|_{S(TM)}$, $\xi_s = -2U = -2\overline{\varphi}\mathcal{N}$ and φ_s given in (3.19). Moreover, if $g_s = g|_{M_s}$ be the induced metric, then $(\varphi_s, \xi_s, \eta_s, g_s)$ is an almost contact metric structure on M_s.*

Proof. Let the screen distribution $\mathrm{S}(TM)$ be integrable, and let M_s be any leaf of $\mathrm{S}(TM)$. From (3.23), (3.24), (3.25), and (3.26), it is easy to verify that the structure $(\varphi_s, \xi_s, \eta_s)$ forms an almost contact structure on M_s. If $g_s = g|_{M_s}$ be the induced metric, then g is Riemannian metric. From Proposition 3.2, we have

$$g_s(\varphi_s(X), \varphi_s(Y)) = g_s(X, Y) - \eta_s(X)\eta_s(Y),$$

for all $X, Y \in \mathrm{S}(TM)$. Equation

$$\overline{g}(X, U) = -\overline{g}(\varphi X, \mathcal{N}) = -\frac{1}{2}u(X), \qquad \text{for all } X \in TM,$$

implies

$$g_s(X, \xi_s) = \eta_s(X).$$

Hence $(\varphi_s, \xi_s, \eta_s, g_s)$ is an almost contact metric structure on M_s. $\square$

Example 3.7. Consider $\overline{M} = \{(x_1, x_2, y_1, y_2, z) \in \mathbb{R}^5 : z \neq 0\}$. The following vector fields are linearly independent at each point of $\overline{M}$:

$$e_1 = z\frac{\partial}{\partial x_1}, \quad e_2 = z\frac{\partial}{\partial x_2}, \quad e_3 = z\frac{\partial}{\partial y_1}, \quad e_4 = z\frac{\partial}{\partial y_2}, \quad e_5 = -z\frac{\partial}{\partial z}.$$

Define Lorentzian metric g on $\overline{M}$ as follows:

$$g(e_1, e_1) = 1, g(e_2, e_2) = 1,\ g(e_3, e_3) = 1,\ g(e_4, e_4) = 1,\ g(e_5, e_5) = -1$$
$$\text{and } g(e_i, e_j) = 0 \text{ for } i \neq j.\ (i, j = 1, 2, 3, 4, 5)$$

Now, we define the structure φ by

$$\varphi(e_1) = -e_2, \quad \varphi(e_2) = e_1, \quad \varphi(e_3) = -e_4, \quad \varphi(e_4) = e_3, \quad \varphi(e_5) = 0.$$

For the vector field $\xi = e_5 = -z\frac{\partial}{\partial z}$ and one form $\eta = -\frac{1}{z}dz$, $(\overline{M}, \varphi, \eta, \xi, g)$ is an almost contact Lorentzian manifold. Consider the hypersurface M given by

$$(u, v, w, t) \to \left(u, v, w, t, -\frac{1}{2}(u + w) + \frac{1}{\sqrt{2}}t\right),$$

then it is easy to see that the tangent bundle TM of M is spanned by

$$\Big\{V_1 = e_1 + \frac{1}{2}e_5 = z\frac{\partial}{\partial x_1} - \frac{1}{2}z\frac{\partial}{\partial z}, V_2 = e_2 + \frac{1}{2}e_5 = z\frac{\partial}{\partial x_2} - \frac{1}{2}z\frac{\partial}{\partial z},$$
$$V_3 = e_3 = z\frac{\partial}{\partial y_1}, V_4 = e_4 - \frac{1}{\sqrt{2}}e_5 = z\frac{\partial}{\partial y_2} + \frac{1}{\sqrt{2}}z\frac{\partial}{\partial z}.\Big\}$$

The rigged vector field E and the transversal vector field $\mathcal{N}$ are

$$E = -\frac{1}{2}e_1 - \frac{1}{2}e_2 + \frac{1}{\sqrt{2}}e_4 - e_5 = -\frac{1}{2}z\frac{\partial}{\partial x_1} - \frac{1}{2}z\frac{\partial}{\partial x_2} + \frac{1}{\sqrt{2}}z\frac{\partial}{\partial y_2} + z\frac{\partial}{\partial z},$$

$$\mathcal{N} = -\frac{1}{4}e_1 - \frac{1}{4}e_2 + \frac{1}{2\sqrt{2}}e_4 + \frac{1}{2}e_5 = -\frac{1}{4}z\frac{\partial}{\partial x_1} - \frac{1}{4}z\frac{\partial}{\partial x_2} + \frac{1}{2\sqrt{2}}z\frac{\partial}{\partial y_2} - \frac{1}{2}z\frac{\partial}{\partial z}.$$

The screen distribution $\mathrm{S}(TM)$ is given by

$$\mathrm{S}(TM) = \mathrm{span}\,(W_1, W_2, W_3),$$

where

$$W_1 = V_1 + \frac{1}{\sqrt{2}}V_4,\ W_2 = V_2 + \frac{1}{\sqrt{2}}V_4,\ W_3 = V_3.$$

Furthermore, we have

$$\varphi(E) = -\frac{1}{2}\,z\frac{\partial}{\partial x_1} + \frac{1}{2}\,z\frac{\partial}{\partial x_2} + \frac{1}{\sqrt{2}}z\frac{\partial}{\partial y_1}$$

and

$$\varphi(\mathcal{N}) = \frac{1}{2}\varphi(E) = \left(-\frac{1}{4}W_1 + \frac{1}{4}W_2 + \frac{1}{2\sqrt{2}}W_3\right) \in \mathrm{S}(TM)\,.$$

Hence M is screen semi-invariant lightlike hypersurface.

4. Normalized lightlike hypersurfaces in Sasakian Lorentzian manifold

We consider $(\overline{M}, \overline{\varphi}, \xi, \overline{\eta}, \overline{g})$ is a Sasakian Lorentzian manifold. We begin with the following proposition.

Proposition 4.1. *Let $(\overline{M}, \overline{\varphi}, \xi, \overline{\eta}, \overline{g})$ be a Sasakian Lorentzian manifold, and let (M, ζ, g) be a normalized lightlike hypersurface of $\overline{M}$. Then for X, $Y \in \Gamma(TM)$, we have*

$$(\nabla_X\varphi)(Y) = g(X, Y)e + \eta(Y)X + B(X, Y)U + u(Y)\mathcal{A}_{\mathcal{N}}(X) \tag{4.1}$$

$$(\nabla_X u)(Y) = -g(X, Y) - B(X, \varphi Y) - \tau(X)u(Y). \tag{4.2}$$

Proof. The proof is straightforward from (2.5) and $\overline{\nabla}_X\xi = \overline{\varphi}X$ verified by $\overline{M}$. □

Proposition 4.2. *Let $(\overline{M}, \overline{\varphi}, \xi, \overline{\eta}, \overline{g})$ be a Sasakian Lorentzian manifold, and let (M, ζ, g) be a normalized lightlike hypersurface of $\overline{M}$. Then for $X \in \Gamma(TM)$, we have*

$$\overline{\varphi}(X) = \mathcal{A}_{\mathcal{N}}(X) - \frac{1}{2}\mathcal{A}^*_E(X) - \frac{1}{2}\tau(X)E - \tau(X)\mathcal{N}, \tag{4.3}$$

$$\overline{\varphi}(\mathcal{N}) = \frac{1}{2}\mathcal{A}_{\mathcal{N}}E. \tag{4.4}$$

Proof. From (3.1) and (3.7), we have

$$\frac{1}{2}\overline{\nabla}_X E - \overline{\nabla}_X\mathcal{N} = \varphi X + u(X)\mathcal{N}.$$

Using (2.12) and (2.14), the last equation yields

$$\mathcal{A}_{\mathcal{N}}(X) - \tau(X)\mathcal{N} - \frac{1}{2}\left(\mathcal{A}^*_E(X) + \tau(X)E\right) = \varphi X + u(X)\mathcal{N}.$$

Collecting the tangential and transversal parts, we get

$$\varphi X = \mathcal{A}_{\mathcal{N}}(X) - \frac{1}{2}\mathcal{A}^*_E(X) - \frac{1}{2}\tau(X)E, \tag{4.5}$$

$$u(X) = -\tau(X). \tag{4.6}$$

Equation (4.3) follows from (4.5) and (4.6). Again from (4.5), we obtain $\varphi E = \mathcal{A}_{\mathcal{N}}(E)$, which yields (4.4) in the view of (3.6). □

Corollary 4.3. *Let $(\overline{M}, \overline{\varphi}, \xi, \overline{\eta}, \overline{g})$ be a Sasakian Lorentzian manifold, and let (M, ζ, g) be a normalized lightlike hypersurface of $\overline{M}$. Then*

(I) *The null section E is geodesic in M.*
(II) *The screen distribution $S(TM)$ cannot be conformal.*

Proof. Equation (4.5) gives $\varphi(E) = \mathcal{A}_{\mathcal{N}}(E) - \frac{1}{2}\tau(E)E$. This implies $\tau(E) = 0$ since $\overline{\varphi}(E) = \varphi(E) = \mathcal{A}_{\mathcal{N}}(E)$. Thus we have $\nabla_E E = 0$. Hence the statement of (I).

Next, if the screen distribution is conformal. Then there exists a C^∞-function α on M such that $\mathcal{A}_{\mathcal{N}} = \alpha \mathcal{A}^*_E$. Then it follows that $\overline{\varphi}(E) = \alpha\mathcal{A}^*_E(E) = 0$, which is a contradiction since $\overline{g}(\overline{\varphi}E, \overline{\varphi}E) = 1$. □

In Theorem 3.6, we see that the leaves of integrable screen distribution are equipped with an almost contact metric structure. The next theorem gives a condition that the leaves to be a Sasakian manifold.

Theorem 4.4. *Let $(\overline{M}, \overline{\varphi}, \xi, \overline{\eta}, \overline{g})$ be a Sasakian Lorentzian manifold, and let (M, ζ, g) be a normalized lightlike hypersurface. Assume that the screen distribution $S(TM)$ is integrable. Then the leaves M_s of $S(TM)$ have Sasakian structure if and only if*

$$\mathcal{A}^*_E(PX) = -\varphi_s(PX) - PX, \qquad \textit{for all } X \in \Gamma(TM). \tag{4.7}$$

Proof. Since $\overline{M}$ is a Sasakian Lorentzian manifold, from (2.5) we have for all $X, Y \in \Gamma(TM)$,

$$\left(\overline{\nabla}_{PX}\overline{\varphi}\right) PY = \overline{g}(PX, PY)\xi = \overline{g}(PX, PY)\left(\frac{1}{2}E - \mathcal{N}\right). \tag{4.8}$$

A straightforward computation gives

$$\begin{aligned} \overline{\nabla}_{PX}\overline{\varphi}(PY) &= \overline{\nabla}_{PX}\left(\varphi(PY) + u(PY)\mathcal{N}\right) \\ &= \nabla_{PX}\left(\varphi_s(PY) + u_s(PY)E\right) + B(PX, \varphi(PY))\mathcal{N} \\ &\quad + PX(u(PY))\mathcal{N} + u(PY)\overline{\nabla}_{PX}\mathcal{N} \\ &= \nabla^*_{PX}\left(\varphi_s(PY)\right) + C(PX, \varphi_s(PY))E + PX(u_s(PY))E \\ &\quad + u_s(PY)\left(-\mathcal{A}^*_E(PX) - \tau(PX)E\right) + B(PX, \varphi_s(PY))\mathcal{N} \\ &\quad + PX(u(PY))\mathcal{N} + u(PY)\left(-\mathcal{A}_{\mathcal{N}}(PX) + \tau(PX)\mathcal{N}\right) \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} \overline{\varphi}\left(\overline{\nabla}_{PX}PY\right) &= \overline{\varphi}\left(\nabla_{PX}PY + B(PX, PY)\mathcal{N}\right) \\ &= \varphi\left(\nabla^*_{PX}PY + C(PX, PY)E\right) + u(\nabla_{PX}PY)\mathcal{N} \\ &\quad + B(PX, PY)\overline{\varphi}\mathcal{N} \\ &= \varphi_S\left(\nabla^*_{PX}PY\right) + u_s\left(\nabla^*_{PX}PY\right) + C(PX, PY)\varphi E \\ &\quad + u(\nabla_{PX}PY)\mathcal{N} + B(PX, PY)\overline{\varphi}\mathcal{N}. \end{aligned} \tag{4.10}$$

From (4.9) and (4.10), we have

$$\begin{aligned}\left(\overline{\nabla}_{PX}\overline{\varphi}\right)(PY) \;=\;& \left(\nabla^{*}_{PX}\varphi_s\right)PY - C(PX,\,PY)\varphi E - B(PX,\,PY)\overline{\varphi}\mathcal{N} \\ &+\left\{C(PX,\,\varphi_s(PY)) + \left(\nabla^{*}_{PX}u_s\right)(PY) - u_s(PY)\tau(PX)\right\}E \\ &+\left\{B(PX,\,\varphi_s(PY)) + \left(\nabla_{PX}u\right)(PY) - u(PY)\tau(PX)\right\}\mathcal{N} \\ &-u_s(PY)\mathcal{A}^{*}_{E}(PX) - u(PY)\mathcal{A}_{\mathcal{N}}(PX).\end{aligned} \tag{4.11}$$

Comparing (4.11) and (4.8), we get the following equations:

$$\begin{aligned}\left(\nabla^{*}_{PX}u_s\right)(PY) &= \tfrac{1}{2}\overline{g}(PX,\,PY) - C(PX,\,\varphi_s(PY)) + u_s(PY)\tau(PX),\\ \left(\nabla_{PX}u\right)(PY) &= -\overline{g}(PX,\,PY) - B(PX,\,\varphi_s(PY)) + u(PY)\tau(PX),\\ \left(\nabla^{*}_{PX}\varphi_s\right)(PY) &= u_s(PY)\mathcal{A}^{*}_{E}(PX) + u(PY)\mathcal{A}_{\mathcal{N}}(PX) + C(PX,\,PY)\varphi E\\ &\quad +B(PX,\,PY)\overline{\varphi}\mathcal{N}.\end{aligned}$$

Using (2.15), (2.16) and (3.3) in the last equation, we write

$$\begin{aligned}\left(\nabla^{*}_{PX}\varphi_s\right)(PY) =& g\left(\left[\frac{1}{2}\mathcal{A}^{*}_{E}(PX) + \mathcal{A}_{\mathcal{N}}(PX)\right],\,PY\right)(2U)\\ &+ u(PY)\left[\frac{1}{2}\mathcal{A}^{*}_{E}(PX) + \mathcal{A}_{\mathcal{N}}(PX)\right].\end{aligned} \tag{4.12}$$

Now, assume that the screen distribution $\mathrm{S}(TM)$ is integrable and that M_s denotes the leaves of $\mathrm{S}(TM)$. If the structure $(\varphi_s,\,\xi_s,\,\eta_s,\,g_s)$ on M_s is Sasakian, then for any $X,\,Y\in\mathrm{S}(TM)$, we have

$$\left(\nabla^{*}_{PX}\varphi_s\right)(PY) = g_s(PX,\,PY)\xi_s - \eta_s(PY)PX. \tag{4.13}$$

Comparing (4.12) and (4.13) and using (4.5) and (3.19), we obtain (4.7). Conversely, if (4.7) is satisfied, using (4.7) and (3.19) in (4.12), we have (4.13). Hence $(\varphi_s,\,\xi_s,\,\eta_s,\,g_s)$ is a Sasakian structure on M_s. $\square$

Since $\overline{g}(\overline{\varphi}E,\,\overline{\varphi}E)=1$, so the distribution $\overline{\varphi}(\mathrm{Rad}(TM))$ is nondegenerate. Now, we define the distribution D_0 by setting

$$\mathrm{S}(TM) = D_0 \perp \overline{\varphi}(\mathrm{Rad}(TM)), \tag{4.14}$$

which implies the following decompositions:

$$TM = \{\overline{\varphi}(\mathrm{Rad}(TM)) \perp D_0\} \perp \mathrm{Rad}(TM), \tag{4.15}$$

$$T\overline{M} = \overline{\varphi}(\mathrm{Rad}(TM)) \perp D_0 \perp \{\mathrm{Rad}(TM) \oplus \mathrm{ltr}(TM)\}. \tag{4.16}$$

It is easy to observe that D_0 is $\overline{\varphi}$-invariant. Now we put

$$D = \mathrm{Rad}(TM) \perp D_0 \text{ and } D' = \overline{\varphi}(\mathrm{Rad}(TM)). \tag{4.17}$$

Then we have

$$TM = D \oplus D'.$$

Theorem 4.5. *Let $(\overline{M},\overline{\varphi},\xi,\overline{\eta},\,\overline{g})$ be a $(2n+1)$-dimensional Sasakian Lorentzian manifold and let $(M,\,\zeta,\,g)$ be a normalized lightlike hypersurface. Then M is* *totally geodesic if and only if the following are satisfied for all $X \in \Gamma(TM)$ and for all $Y \in \Gamma(D)$:*

$$(\nabla_X \varphi) Y = g(X, Y)e + \eta(Y)X, \tag{4.18}$$

$$\frac{1}{2}\mathcal{A}_{\mathcal{N}}(X) = g(X, U)e + \frac{1}{2}(\nabla_X e) + \varphi(\nabla_X U). \tag{4.19}$$

Proof. First, consider M is a totally geodesic submanifold in $(\overline{M}, \overline{\varphi}, \xi, \overline{\eta}, \overline{g})$. Since $Y \in \Gamma(D)$ implies $u(Y) = \overline{g}(\overline{\varphi}Y, E) = 0$, thus (4.1) yields (4.18). Again from (4.1), we get

$$(\nabla_X \varphi) U = g(X, U)e + \eta(U)X - \frac{1}{2}\mathcal{A}_{\mathcal{N}}(X),$$

which implies (4.19) using (3.15).

Conversely, suppose that the given equations are satisfied. For any $X, Y \in TM$, using $TM = D \oplus D'$, we write $Y = Y_D + \alpha U$ for some $\alpha \in C^\infty(M)$. Therefore we have

$$B(X, Y) = B(X, Y_D) + \alpha B(X, U).$$

Now, from (4.1) for $Y = Y_D$, we have

$$(\nabla_X \varphi)(Y_D) = g(X, Y_D)e + \eta(Y_D)X + B(X, Y_D)U + u(Y_D)\mathcal{A}_{\mathcal{N}}(X).$$

In the view of (4.18), the last equation yields

$$B(X, Y_D)U + u(Y_D)\mathcal{A}_{\mathcal{N}}(X) = 0.$$

The last equation implies

$$B(X, Y_D) = 0. \tag{4.20}$$

Here we have used $u(Y_D) = -\overline{g}(Y_D, 2U) = 0$ as $U = \frac{1}{2}\overline{\varphi}E \in \overline{\varphi}(\mathrm{Rad}(TM))$.

On the other hand, from (3.15), (3.16), and (4.1), we obtain

$$(\nabla_X \varphi) U = g(X, U)e + B(X, U)U - \frac{1}{2}\mathcal{A}_{\mathcal{N}}(X).$$

Using (4.19) in the last equation, we get

$$B(X, U) = 0. \tag{4.21}$$

From (4.20) and (4.21), we have

$$B(X, Y) = 0, \qquad \text{for all } X, Y \in TM.$$

□

5. Integrability of distributions.

We begin with the following lemma.

Lemma 5.1. *Let (M, ζ, g) be a normalized lightlike hypersurface of a Sasakian Lorentzian manifold $(\overline{M}, \overline{\varphi}, \xi, \overline{\eta}, \overline{g})$. Then*

$$\overline{g}\left(\left(\overline{\nabla}_X \overline{\varphi}\right) Y, E\right) + \overline{g}(X, Y) = 0 \qquad \textit{for all } X, Y \in TM.$$

Proof. From (2.5), we get

$$\overline{g}\left(\left(\overline{\nabla}_X\overline{\varphi}\right)Y,\, E\right) = \overline{g}\left(\overline{g}(X,\, Y)\xi + \overline{\eta}(Y)X,\, E\right) = -\overline{g}(X,\, Y).$$

□

First, we consider the distribution D_0 defined in (4.14) as follows:

$$\mathrm{S}(TM) = D_0 \perp \overline{\varphi}(\mathrm{Rad}(TM)).$$

We denote it by $\beta = \mathrm{Rad}(TM) \perp \overline{\varphi}(\mathrm{Rad}(TM))$; then (4.15) implies

$$TM = D_0 \perp \beta.$$

Therefore, for any $X \in TM$, $Y \in D_0$ and $Z \in \beta$, we have

$$\nabla_X Y = \mathring{\nabla}_X Y + \mathring{h}(X,\, Y) \text{ and } \nabla_X Z = -\mathring{A}_Z(X) + \nabla^{\beta}_X Z, \tag{5.1}$$

where $\mathring{\nabla}$ is a linear connection on D_0, $\mathring{h} : TM \times \Gamma(D_0) \to \Gamma(\beta)$ is a bilinear map, $\mathring{A}_Z$ is $\Gamma(D_0)$-valued $C^\infty(M)$-linear operator on TM and ∇^β is a linear connection on β. From the definition of the distribution β, we write

$$\mathring{h}(X,\, Y) = \alpha_1(X,\, Y)\overline{\varphi}E + \alpha_2(X,\, Y)E, \tag{5.2}$$

for all $X \in TM$, $Y \in \Gamma(D_0)$, where

$$\alpha_1(X,\, Y) = \overline{g}(\mathring{h}(X,\, Y),\, \overline{\varphi}E) \quad \text{and} \quad \alpha_2(X,\, Y) = \overline{g}(\mathring{h}(X,\, Y),\, \mathcal{N}).$$

Note that $\alpha_1(X,\, Y) = \overline{g}(\nabla_X Y,\, \overline{\varphi}E)$. Using (2.3), (2.5), (2.11) and (2.14), we calculate

$$\begin{aligned}
\alpha_1(X,\, Y) &= \overline{g}(\nabla_X Y,\, \overline{\varphi}E) = -\overline{g}(\overline{\nabla}_X(\overline{\varphi}Y) - (\overline{\nabla}_X\overline{\varphi})Y,\, E) \\
&= -\overline{g}(\overline{\nabla}_X(\overline{\varphi}Y),\, E) - \overline{g}(X,\, Y) \\
&= \overline{g}(\overline{\varphi}Y,\, \overline{\nabla}_X E) - \overline{g}(X,\, Y) \\
&= \overline{g}(\overline{\varphi}E,\, -\mathcal{A}^*_E(X) - \tau(X)E) - \overline{g}(X,\, Y) \\
&= -\overline{g}(X,\, Y) - B(X,\, \overline{\varphi}Y);
\end{aligned}$$

that is,

$$\alpha_1(X,\, Y) = -\overline{g}(X,\, Y) - B(X,\, \overline{\varphi}Y). \tag{5.3}$$

In similar way, from (2.13), we obtain

$$\alpha_2(X,\, Y) = \overline{g}(\nabla_X Y,\, \mathcal{N}) = \overline{g}(\mathring{\nabla}_X Y + C(X,\, Y)E,\, \mathcal{N}) = C(X,\, Y) \tag{5.4}$$

Therefore, applying (5.3) and (5.4) in (5.2) and (5.1), we have

$$\nabla_X Y = \mathring{\nabla}_X Y - \{g(X,\, Y) + B(X,\, \overline{\varphi}Y)\}\,\overline{\varphi}E + \{C(X,\, Y)\}\,E. \tag{5.5}$$

Theorem 5.2. *Let $(M,\, \zeta,\, g)$ be a normalized lightlike hypersurface of a Sasakian Lorentzian manifold $(\overline{M}, \overline{\varphi}, \xi, \overline{\eta}, \overline{g})$. Then the distribution D_0 is integrable if and only if*

$$C(X,\, Y) = C(Y,\, X) \text{ and } B(X,\, \overline{\varphi}Y) = B(\overline{\varphi}X,\, Y), \qquad \text{for all } X,\, Y \in \Gamma(D_0).$$

Proof. Let $X,\, Y \in \Gamma(D_0)$. Since the connection $\overline{\nabla}$ is torsion-free. Thus

$$[X,\, Y] = \overline{\nabla}_X Y - \overline{\nabla}_Y X.$$

Using (5.5), we find

$$[X,\, Y] = \overset{\circ}{\nabla}_X Y - \overset{\circ}{\nabla}_Y X + \{B(\overline{\varphi}X,\, Y) - B(X,\, \overline{\varphi}Y)\}\overline{\varphi}E + \{C(X,\, Y) - C(Y,\, X)\}E.$$

The statement of the theorem follows from the last equation. □

Corollary 5.3. *Let $(M,\, \zeta,\, g)$ be a normalized lightlike hypersurface of a Sasakian Lorentzian manifold $(\overline{M}, \overline{\varphi}, \xi, \overline{\eta},\, \overline{g})$. Then the distribution D_0 is integrable if and only if $\overset{\circ}{h}$ is symmetric on D_0.*

Proof. From (5.3) and (5.4), we get

$$\overset{\circ}{h}(X,\, Y) = \{C(X,\, Y)\}E - \{g(X,\, Y) + B(X,\, \overline{\varphi}Y)\}\overline{\varphi}E, \tag{5.6}$$

for any $X \in TM$ and $Y \in D_0$. Hence the statement follows from (5.6) and Theorem 5.2. □

Theorem 5.4. *Let $(M,\, \zeta,\, g)$ be a normalized lightlike hypersurface of a $(2n+1)$-dimensional Sasakian Lorentzian manifold $(\overline{M}, \overline{\varphi}, \xi, \overline{\eta},\, \overline{g})$. Let the distribution D_0 be integrable and let*

$$C(\overline{\varphi}X,\, \overline{\varphi}Y) = -C(X,\, Y),$$

for any $X,\, Y \in D_0$. Then $trace(\overset{\circ}{h}) = 2(1-n)\overline{\varphi}E$.

Proof. First, we note that $rank(D_0) = 2(n-1)$. Now, we consider an orthonormal $\overline{\varphi}$-basis $\{e_i,\, \overline{\varphi}e_i\}$ of D_0, $i = 1,\, 2,\, \ldots, (n-1)$. Then

$$trace(\overset{\circ}{h}) = \sum_{i=1}^{n-1} \left[\overset{\circ}{h}(e_i,\, e_i) + \overset{\circ}{h}(\overline{\varphi}e_i,\, \overline{\varphi}e_i)\right].$$

Using the symmetry of B in (5.6), we have

$$\overset{\circ}{h}(e_i,\, e_i) + \overset{\circ}{h}(\overline{\varphi}e_i,\, \overline{\varphi}e_i) = -2\overline{\varphi}E + \{C(e_i,\, e_i) + C(\overline{\varphi}e_i,\, \overline{\varphi}e_i)\}E.$$

□

Next, the decomposition given in (4.16) can be written as

$$T\overline{M} = D_0 \perp \nu,$$

where $\nu = \overline{\varphi}(\mathrm{Rad}(TM)) \perp \{\mathrm{Rad}(TM) \oplus \mathrm{ltr}(TM)\}$.

From this decomposition, for any $X \in TM$, $Y \in D_0$ and $Z \in \Gamma(\nu)$, we have

$$\overline{\nabla}_X Y = \hat{\nabla}_X Y + \hat{h}(X,\, Y) \text{ and } \overline{\nabla}_X Z = -\hat{A}_Z(X) + \nabla^{\nu}_X Z, \tag{5.7}$$

where $\hat{\nabla}$ is a linear connection on D_0, $\hat{h} : TM \times \Gamma(D_0) \to \Gamma(\nu)$ is a bilinear map, $\hat{A}_Z$ is $\Gamma(D_0)$-valued $C^{\infty}(M)$-linear operator on TM and ∇^{ν} is a linear connection on ν. From the definition of the distribution ν, we put

$$\hat{h}(X,\, Y) = \beta_1(X,\, Y)\overline{\varphi}E + \beta_2(X,\, Y)E + \beta_3(X,\, Y)\mathcal{N}, \tag{5.8}$$

for all $X,\, Y\in\Gamma(D_0)$. Then it is easy to see that

$$\begin{array}{rclcl}\beta_1(X,\,Y) &=& \overline{g}(\hat{h}(X,\,Y),\,\overline{\varphi}E) &=& -\overline{g}(X,\,Y)-B(X,\,\overline{\varphi}Y),\\ \beta_2(X,\,Y) &=& \overline{g}(\hat{h}(X,\,Y),\,\mathcal{N}) &=& C(X,\,Y),\\ \beta_3(X,\,Y) &=& \overline{g}(\hat{h}(X,\,Y),\,E) &=& B(X,\,Y).\end{array} \tag{5.9}$$

Applying (5.9) in (5.8) and using (5.7), we get

$$\begin{aligned}\hat{h}(X,\,Y)=&-\{g(X,\,Y)+B(X,\,\overline{\varphi}Y)\}\overline{\varphi}E+\{C(X,\,Y)\}E\\ &+\{B(X,\,Y)\}\mathcal{N},\end{aligned} \tag{5.10}$$

and

$$\begin{aligned}\overline{\nabla}_XY=&\hat{\nabla}_XY-\{g(X,\,Y)+B(X,\,\overline{\varphi}Y)\}\overline{\varphi}E+\{C(X,\,Y)\}E\\ &+\{B(X,\,Y)\}\mathcal{N}.\end{aligned} \tag{5.11}$$

Now, we have the following theorem.

Theorem 5.5. *Let $(M,\,\zeta,\,g)$ be a normalized lightlike hypersurface of a $(2n+1)$-dimensional Sasakian Lorentzian manifold $(\overline{M},\overline{\varphi},\xi,\overline{\eta},\,\overline{g})$. Let the distribution D_0 be integrable and let*

$$B(\overline{\varphi}X,\,\overline{\varphi}Y)=-B(X,\,Y);\qquad C(\overline{\varphi}X,\,\overline{\varphi}Y)=-C(X,\,Y),$$

for any $X,\,Y\in D_0$. Then on D_0 we have

$$trace(\hat{h})=2(1-n)\overline{\varphi}E.$$

Proof. The proof follows from (5.10) and (5.11) in a similar manner to the proof of Theorem 5.4. □

Next, we consider the distribution D defined in (4.17) and recall that

$$D'=\overline{\varphi}(\mathrm{Rad}(TM)\,),\qquad TM=D\perp D'.$$

For $X\in TM$, $Y\in\Gamma(D)$ and $Z\in\Gamma(D')$, we have

$$\nabla_XY=\check{\nabla}_XY+\check{h}(X,\,Y)\text{ and }\nabla_XZ=-\check{A}_Z(X)+\nabla^{D'}_XZ, \tag{5.12}$$

where $\check{\nabla}$ is a linear connection on D, $\check{h}:TM\times\Gamma(D)\to\Gamma(D')$ is a bilinear map, $\check{A}_Z$ is $\Gamma(D)$-valued $C^\infty(M)$-linear operator, and $\nabla^{D'}$ is a linear connection on D'. From the definition of the distribution D', we set

$$\check{h}(X,\,Y)=\sigma_1(X,\,Y)\overline{\varphi}E. \tag{5.13}$$

Then it is easy to note that

$$\begin{aligned}\sigma_1(X,\,Y)=\overline{g}(\overline{\nabla}_XY,\,\overline{\varphi}E)&=\overline{g}\left(\left(\overline{\nabla}_X\overline{\varphi}\right)Y-\overline{\nabla}_X(\overline{\varphi}Y),\,E\right)\\ &=-g(X,\,Y)-B(X,\,\overline{\varphi}Y).\end{aligned} \tag{5.14}$$

Putting (5.14) and (5.13) on (5.12), we have

$$\nabla_XY=\check{\nabla}_XY-\{g(X,\,Y)+B(X,\,\overline{\varphi}Y)\}\,\overline{\varphi}E,$$

from which we obtain for any $X\in TM$, $Y\in\Gamma(D)$ and $Z\in\Gamma(D')$

$$\nabla_XY-\nabla_YX=\check{\nabla}_XY-\check{\nabla}_YX-\{B(X,\,\overline{\varphi}Y)-B(\overline{\varphi}X,\,Y)\}\,\overline{\varphi}E. \tag{5.15}$$

From (5.15), we are able to state the following theorem.

Theorem 5.6. *Let (M, ζ, g) be a normalized lightlike hypersurface of a Sasakian Lorentzian manifold $(\overline{M}, \overline{\varphi}, \xi, \overline{\eta}, \overline{g})$. Then the distribution D is integrable if and only if*

$$B(X, \overline{\varphi}Y) = B(\overline{\varphi}X, Y), \qquad \text{for all } X, Y \in \Gamma(D).$$

Acknowledgement. The author is grateful to the referees for their comments to improve the paper.

References

1. B.E. Acet, S.Y. Perktas and E. Kiliç *On Lightlike Geometry of Para-Sasakian manifolds*, Sci. World J. (2014), article ID 696231, 1–12.
2. C. Atindogbe, M. Gutierrez, and R. Hounnonkpe *New properties on normalized null hypersurfaces*, Mediterr. J. Math. **15** (2018), no. 166, 1–19.
3. C. Atindogbe and T.K. Mbiakop *Null hypersurfaces in Brinkmann Spacetime*, Bull. Math. Anal. Appl. **12** (2020), no. 3, 60–75.
4. A. Bejancu and K.L. Duggal *Lightlike submanifolds of semi-Riemannian manifolds*, Acta Appl. Math. **38** (1995), 197–215.
5. D.E. Blair *Riemannian geometry of contact and symplectic manifolds*, Progress in Math., Birkhauser, 2002.
6. G. Calvaruso *Contact Lorentzian manifolds*, Diff. Geo. Appl. **29** (2011), 541–551.
7. G. Calvaruso and D. Perrone *Contact pseudo-metric manifolds*, Diff. Geo. Appl. **28** (2010), 615–634.
8. K.L. Duggal and A. Bejancu *Lightlike Submanifolds of Semi-Riemannian Manifolds And Applications*, vol. 364 of Mathematics and Its applications, Kluwer Academic Publishers, 1996.
9. M.H.A. Hamed and F. Massamba *Invariant null rigged hypersurfaces of indefinite nearly α-Sasakian manifolds*, Commun. Korean Math. Soc. **39** (2024), no. 2, 493–511.
10. M. Gutierrez and B. Olea *Induced Riemannian structures on null hypersurfaces*, Math. Nachr. **289** (2016), no. 10, 1219–1236.
11. M. Gutierrez and B. Olea *Rigging technique for null hypersurfaces*, axioms **10** (2021), no. 284, 1–35.
12. D.N. Kupeli *On null submanifolds in spacetimes*, Geom. Dedicata **23** (1987), 33–51.
13. C.B. Meli, F. Ngakeu and B. Olea *Statistical structures arising in null submanifolds*, Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat. **117** (2023), no. 48, 1–29.
14. R. Mrugala *On a Riemannian metric on contact thermodynamic spaces*, Rep. Math. Phys. **38** (1996), no. 3, 339–348.
15. M. Nararro, O. Palmas and D.A. Solis *Null hypersurfaces in generalized Robertson-Walker spacetimes*, J. Geom. Phys. **106** (2016), 256–267.
16. F. Ngakeu and H.F. Testing *α-Associated metrics on rigged null hypersurfaces*, Turkish J. Math. **43** (2019), 1161–1181.
17. F. Ngakeu, A.C. Djomako and B. Ndombol *Rigged null hypersurfaces in almost paracontact metric manifolds*, Gulf J. Math. **7** (2019), no. 2, 37–56.
18. S.Y. Perktas, E. Kiliç and B.E. Acet *Lightlike Hypersurfaces of a Para-Sasakian Space Form*, Gulf J. Math. **2** (2014), 7–18.
19. A.P. Singh, C. Atindogbe, R. Kumar and V. Jain *normalized null hypersurfaces of indefinite Kähler manifolds*, Inter. Elect. J. Geom. **16** (2023), no. 1, 417–434.
20. T. Takahashi *Sasakian manifold with pseudo-Riemannian metrics*, Tohoku Math. J. **21** (1969), 271–290.

Department of Mathematics, D. S. B. Campus, Kumaun University Nainital, Nainital-263001, Uttarakhand, India.

Email address: `skchanyal.math@gmail.com; skchanyal@kunainital.ac.in`