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TOTAL FOFANA–GULIYEV SPACES WITH APPLICATIONS TO NORM INEQUALITIES FOR CLASSICAL OPERATORS

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Abstract. We introduce a variant of Fofana spaces called total Fofana–Guliyev spaces. These spaces are closely related to total Morrey spaces. We give some of their basic properties. As applications, we establish norm inequalities for classical operators in these spaces, namely the Hardy–Littlewood maximal operator, fractional integral operators, and fractional maximal operators.

1. Introduction

Let $\mathbb{R}^d$ be the d-dimensional Euclidean space equipped with the Lebesgue measure dx. For any subset E of $\mathbb{R}^d$, we denote by χ_E its characteristic function and by $|E|$ its Lebesgue measure when it is measurable. By $L^0(\mathbb{R}^d)$, we denote the complex vector space of equivalent classes (modulo equality Lebesgue almost everywhere) of Lebesgue measurable complex-valued functions on $\mathbb{R}^d$. Let $1 \leq q,\, p \leq \infty$, and $r > 0$. The classical Lebesgue space is denoted by $L^p(\mathbb{R}^d)$ and $\|f\|_p$ stands for its usual norm. The amalgam space $(L^q,\, L^p)(\mathbb{R}^d)$ is defined as the set of all functions $f \in L^0(\mathbb{R}^d)$ satisfying ${}_r\|f\|_{q,p} < \infty$, where

$$ {}_r\|f\|_{q,p} = \left\| \left[\int_{\mathbb{R}^d} |f\chi_{B(y,r)}|^q(x)dx \right]^{\frac{1}{q}} \right\|_p, \tag{1.1}$$

with the usual modification when $q = \infty$. Here and in the sequel, $B(y,\, r)$ denotes the open ball centered at $y \in \mathbb{R}^d$ with radius r. Amalgam spaces were introduced by Wiener [28] in 1926. However, the first systematic study of these spaces on the

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real line is due to Holland [21]. For more informations about amalgam spaces, we refer to [13] and references cited therein. It is well known (see [5]) that endowed with the norm ${}_r\|\cdot\|_{q,p}$, $(L^q,\, L^p)(\mathbb{R}^d)$ is a complex Banach space. Another relevant norm on the spaces $(L^q,\, L^p)(\mathbb{R}^d)$, which is equivalent to (1.1), is given by

$$ {}_r\widetilde{\|f\|}_{q,p} = \left\| \left\{ \left\| f\chi_{Q_k^r} \right\|_q \right\}_{k\in\mathbb{Z}^d} \right\|_{\ell^p}, $$

where $k = (k_1, \ldots, k_d) \in \mathbb{Z}^d$ and $Q_k^r = \prod_{i=1}^d [rk_i, r(k_i+1))$. This norm has the following properties:

• There exists a constant $C > 0$ such that

$$ C^{-1}\, r^{\frac{d}{p}}\, {}_r\widetilde{\|f\|}_{q,p} \leq {}_r\|f\|_{q,p} \leq C r^{\frac{d}{p}}\, {}_r\widetilde{\|f\|}_{q,p}, \qquad f \in L^0(\mathbb{R}^d); \tag{1.2} $$

• If $1 \leq q_1 \leq q_2 \leq \infty$, $1 \leq p_1 \leq p_2 \leq \infty$ and $1 \leq q,\, p \leq \infty$, then for all $f \in L^0(\mathbb{R}^d)$,

$$ {}_r\widetilde{\|f\|}_{q_1,p} \leq {}_r\widetilde{\|f\|}_{q_2,p}\, r^{d(\frac{1}{q_1}-\frac{1}{q_2})} \tag{1.3} $$

and

$$ {}_r\widetilde{\|f\|}_{q,p_2} \leq 2^d\, {}_r\widetilde{\|f\|}_{q,p_1}. \tag{1.4} $$

We adopt the usual convention $\frac{1}{\infty} = 0$. Recall that the Lebesgue space $L^p(\mathbb{R}^d)$ coincides with the Wiener amalgam space $(L^p,\, L^p)(\mathbb{R}^d)$. However, it is a proper subspace of $(L^q,\, L^p)(\mathbb{R}^d)$ when $q < p$. The dilation operator $\delta_r^q : f \mapsto r^{\frac{d}{q}} f(r\cdot)$ is isometric in Lebesgue spaces. Indeed, this property does not hold in proper amalgam spaces. For $0 < \alpha \leq \infty$, it is easy to see that $f \in (L^q, L^p)(\mathbb{R}^d)$ if and only if

$$ \|\delta_r^\alpha f\|_{q,p} = r^{d(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p})}\, {}_r\|f\|_{q,p} < \infty, $$

for all $r > 0$ and for all $\alpha > 0$. In 1988, Fofana [9] introduced the function spaces $(L^q, L^p)^\alpha(\mathbb{R}^d)$, $1 \leq q \leq \alpha \leq p \leq \infty$, which consists of the set of all functions $f \in (L^q,\, L^p)(\mathbb{R}^d)$ satisfying $\|f\|_{q,p,\alpha} < \infty$, where

$$ \|f\|_{q,p,\alpha} = \sup_{r>0} \|\delta_r^\alpha f\|_{q,p}. $$

It is proved in [9] that for $1 \leq q < \alpha$ fixed and p going from α to ∞, the spaces $(L^q, L^p)^\alpha(\mathbb{R}^d)$ form a chain of distinct Banach spaces beginning with the Lebesgue space $L^\alpha(\mathbb{R}^d)$ and ending by the classical Morrey space $L^{q,d(1-\frac{q}{\alpha})}(\mathbb{R}^d) = (L^q, L^\infty)^\alpha(\mathbb{R}^d)$. Several results about Fourier multipliers and boundedness properties of maximal and fractional integral operators have been established in the framework of Fofana spaces (see, for example, [5, 6, 7, 8, 10, 11, 12, 22]). Feuto [5] proved that classical operators, including the Hardy–Littlewood maximal operator, the Calderón-Zygmund operator, and Riesz potentials which are known to be bounded on Lebesgue and Morrey spaces, are also bounded on classical Fofana spaces $(L^q, L^p)^\alpha(\mathbb{R}^d)$. Recently, Guliyev [16] introduced a new variant of Morrey spaces called total Morrey spaces and denoted by $L^{q,\lambda,\mu}(\mathbb{R}^d)$, $0 < q < \infty$, $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{R}$. Total Morrey spaces generalize classical Morrey spaces so that $L^{q,\lambda,\lambda}(\mathbb{R}^d) = L^{q,\lambda}(\mathbb{R}^d)$. In addition, $L^{q,\lambda,\mu}(\mathbb{R}^d) = L^{q,\lambda}(\mathbb{R}^d) \cap L^{q,\mu}(\mathbb{R}^d)$ with $\mu \leq \lambda$ (see [2] and [19]). He examined the boundedness properties of the maximal commutator operator and the commutator of maximal operator in these spaces. The study of total Morrey spaces and their appplications were also given in [1, 2, 17, 18, 20]. Total Morrey spaces associated with the Dunkl operator on $\mathbb{R}^d$ were investigated in [23, 24, 25].

In this work, we introduce a variant of Fofana spaces called total Fofana–Guliyev spaces, and we give some of their basic properties. As applications, we establish norm inequalities for classical operators, namely the Hardy–Littlewood maximal operator, fractional integral and fractional maximal operators in these spaces. Our results concerning the Hardy–Littlewood maximal operator play a key role in [26], where they are stated without proof. We also note that in [26], total Fofana spaces are exactely total Fofana–Guliyev spaces.

The paper is organized as follows. In Section 2, we define total Fofana–Guliyev spaces, and we give some of their basic properties. Section 3 is devoted to norm inequalities for the Hardy–Littlewood maximal operator in total Fofana–Guliyev spaces. In Section 4, we establish norm inequalities for fractional integral and fractional maximal operators in total Fofana–Guliyev spaces.

Throughout this note, the letter C will be used for positive constants not depending on the relevant variables, and theses constants may change from one occurrence to another. We will use the following abbreviation $A \lesssim B$ for the inequalities $A \leq CB$. If $A \lesssim B$ and $B \lesssim A$, then we write $A \approx B$.

2. Total Fofana–Guliyev spaces

Notation 2.1. For $r > 0$ we put $[r]_1 = \min\{1, r\}$.

Definition 2.2. Let $1 \leq q \leq \alpha, \lambda \leq p \leq \infty$. The total Fofana–Guliyev spaces denoted by $(L^q, L^p)^{\alpha,\lambda}(\mathbb{R}^d)$ are defined as

$$(L^q, L^p)^{\alpha,\lambda}(\mathbb{R}^d) = \left\{ \ f \in L^0(\mathbb{R}^d) : \ \|f\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)} < \infty \ \right\},$$

where

$$\|f\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)} = \sup_{r>0} [r]_1^{d(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p})} [1/r]_1^{d(-\frac{1}{\lambda}+\frac{1}{q}+\frac{1}{p})} \ {}_r\|f\|_{q,p}\,.$$

Remark 2.3. Note that

(1) the space $(L^q, L^p)^{\alpha,\lambda}(\mathbb{R}^d)$ is a complex vector subspace of $L^0(\mathbb{R}^d)$;
(2) the map $L^0(\mathbb{R}^d) \ni f \mapsto \ \|f\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}$ defines a norm on the space $(L^q, L^p)^{\alpha,\lambda}(\mathbb{R}^d)$;
(3) according to (1.2), for all $f \in (L^q, L^p)^{\alpha,\lambda}(\mathbb{R}^d)$,

$$\|f\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)} \approx \widetilde{\|f\|}_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)},$$

where

$$\widetilde{\|f\|}_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)} = \sup_{r>0} [r]_1^{d(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p})} [1/r]_1^{d(-\frac{1}{\lambda}+\frac{1}{q}+\frac{1}{p})} r^{\frac{d}{p}} \ {}_r\widetilde{\|f\|}_{q,p};$$

(4) for $1 \leq q \leq \alpha, \lambda < \infty$, the space $(L^q, L^\infty)^{\alpha,\lambda}(\mathbb{R}^d)$ is the total Morrey space $L^{q,d(1-\frac{q}{\alpha}),d(1-\frac{q}{\lambda})}(\mathbb{R}^d)$ defined in [16].

Proposition 2.4. *Let $1 \leq q \leq \alpha, \lambda \leq p \leq \infty$. Then, endowed with the norm $\|\cdot\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}$, $(L^q, L^p)^{\alpha,\lambda}(\mathbb{R}^d)$ is a complex Banach space.*

Proof. Let $(f_m)_{m\in\mathbb{N}}$ be a Cauchy sequence of elements of $(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)$.

Fix $\epsilon>0$. Then there exists an integer N such that for all integers m and n we have

$$m,n>N \implies \|f_m-f_n\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}<\epsilon.$$

This implies that for r fixed and for $m,n>N$,

$$[1/r]_1^{d(-\frac{1}{\lambda}+\frac{1}{q}+\frac{1}{p})}\ {}_r\|f_m-f_n\|_{q,p}<\epsilon.$$

Hence, $(f_m)_{m\in\mathbb{N}}$ is a Cauchy sequence of $(L^q,L^p)_r(\mathbb{R}^d)$. As $(L^q,L^p)_r(\mathbb{R}^d)$ is a complex Banach space, $(f_m)_{m\in\mathbb{N}}$ converges to $f\in(L^q,L^p)_r(\mathbb{R}^d)$. In addition, for all $m,n>N$ we have

$$|\|f_m\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}-\|f_n\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}|\leq\|f_m-f_n\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}<\epsilon.$$

Therefore, $(\|f_n\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)})_{n\in\mathbb{N}}$ is a Cauchy sequence of $\mathbb{R}$. Let us put $M\geq 0$ its limit. It follows that for $m>N$,

$$\begin{aligned}&[r]_1^{d(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p})}[1/r]_1^{d(-\frac{1}{\lambda}+\frac{1}{q}+\frac{1}{p})}\ {}_r\|f\|_{q,p}\\&\leq\|f_m\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}+[r]_1^{d(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p})}[1/r]_1^{d(-\frac{1}{\lambda}+\frac{1}{q}+\frac{1}{p})}\ {}_r\|f-f_m\|_{q,p}.\end{aligned}$$

By letting m go to ∞, we get

$$[r]_1^{d(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p})}[1/r]_1^{d(-\frac{1}{\lambda}+\frac{1}{q}+\frac{1}{p})}\ {}_r\|f\|_{q,p}\leq M.$$

Hence, $\|f\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}\leq M$ and $f\in(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)$.

Let m be an integer such that $m>N$. For all $r>0$ and for all $l\in\mathbb{N}$, we have

$$\begin{aligned}&[r]_1^{d(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p})}[1/r]_1^{d(-\frac{1}{\lambda}+\frac{1}{q}+\frac{1}{p})}\ {}_r\|f_m-f\|_{q,p}\\&\leq\|f_m-f_{m+l}\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}\\&\quad+[r]_1^{d(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p})}[1/r]_1^{d(-\frac{1}{\lambda}+\frac{1}{q}+\frac{1}{p})}\ {}_r\|f_{m+l}-f\|_{q,p}\\&\leq\epsilon+[r]_1^{d(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p})}[1/r]_1^{d(-\frac{1}{\lambda}+\frac{1}{q}+\frac{1}{p})}\ {}_r\|f_{m+l}-f\|_{q,p}.\end{aligned}$$

Since

$$\lim_{l\to\infty}[r]_1^{d(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p})}[1/r]_1^{d(-\frac{1}{\lambda}+\frac{1}{q}+\frac{1}{p})}\ {}_r\|f_{m+l}-f\|_{q,p}=0,$$

we have

$$\|f_m-f\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}\leq\epsilon.$$

□

We will examine the relationship between total Fofana–Guliyev spaces and classical Fofana spaces.

Proposition 2.5. *Let $1\leq q\leq\alpha,\lambda\leq p\leq\infty$. Then*

$$(L^q,L^p)^{\alpha}(\mathbb{R}^d)\cap(L^q,L^p)^{\lambda}(\mathbb{R}^d)\hookrightarrow(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)$$

and for $f\in(L^q,L^p)^{\alpha}(\mathbb{R}^d)\cap(L^q,L^p)^{\lambda}(\mathbb{R}^d)$,

$$\|f\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}\leq\max\{\|f\|_{q,p,\alpha},\|f\|_{q,p,\lambda}\}.$$

Proof. Let $1 \leq q \leq \alpha, \lambda \leq p \leq \infty$ and let $f \in (L^q, L^p)^\alpha(\mathbb{R}^d) \cap (L^q, L^p)^\lambda(\mathbb{R}^d)$. We have

$$\begin{aligned}
&\|f\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)} \\
&= \sup_{r>0} [r]_1^{d(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p})} [1/r]_1^{d(-\frac{1}{\lambda}+\frac{1}{q}+\frac{1}{p})} \;{}_r\|f\|_{q,p} \\
&= \max\left\{ \sup_{0<r\leq 1} r^{d(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p})} \;{}_r\|f\|_{q,p} \;,\; \sup_{r>1} r^{d(\frac{1}{\lambda}-\frac{1}{q}-\frac{1}{p})} \;{}_r\|f\|_{q,p} \right\} \\
&\leq \max\{\|f\|_{q,p,\alpha}, \|f\|_{q,p,\lambda}\}.
\end{aligned}$$

□

Proposition 2.6. *Let $1 \leq q \leq \lambda \leq \alpha \leq p \leq \infty$. Then*

$$(L^q, L^p)^{\alpha,\lambda}(\mathbb{R}^d) = (L^q, L^p)^\alpha(\mathbb{R}^d) \cap (L^q, L^p)^\lambda(\mathbb{R}^d)$$

and for $f \in (L^q, L^p)^{\alpha,\lambda}(\mathbb{R}^d)$,

$$\|f\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)} = \max\{\|f\|_{q,p,\alpha}, \|f\|_{q,p,\lambda}\}.$$

Proof. Let $1 \leq q \leq \lambda < \alpha \leq p \leq \infty$. Let $f \in (L^q, L^p)^{\alpha,\lambda}(\mathbb{R}^d)$. We have on the one hand

$$\begin{aligned}
& \|f\|_{q,p,\alpha} \\
= \;& \max\left\{ \sup_{0<r\leq 1} r^{d(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p})} \;{}_r\|f\|_{q,p}, \sup_{r>1} r^{d(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p})} \;{}_r\|f\|_{q,p} \right\} \\
= \;& \max\left\{ \sup_{0<r\leq 1} r^{d(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p})} \;{}_r\|f\|_{q,p}, \right. \\
& \left. \sup_{r>1} r^{d(\frac{1}{\alpha}-\frac{1}{\lambda})} (1/r)^{d(-\frac{1}{\lambda}+\frac{1}{q}+\frac{1}{p})} \;{}_r\|f\|_{q,p} \right\} \\
\leq \;& \max\left\{ \sup_{0<r\leq 1} r^{d(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p})} \;{}_r\|f\|_{q,p}, \sup_{r>1} (1/r)^{d(-\frac{1}{\lambda}+\frac{1}{q}+\frac{1}{p})} \;{}_r\|f\|_{q,p} \right\}.
\end{aligned}$$

It follows that

$$\|f\|_{q,p,\alpha} \leq \|f\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}. \tag{2.1}$$

On the other hand, we have

$$\begin{aligned}
&\|f\|_{q,p,\lambda} \\
&= \max\left\{ \sup_{0<r\leq 1} r^{d(\frac{1}{\lambda}-\frac{1}{q}-\frac{1}{p})} \;{}_r\|f\|_{q,p}, \sup_{r>1} r^{d(\frac{1}{\lambda}-\frac{1}{q}-\frac{1}{p})} \;{}_r\|f\|_{q,p} \right\} \\
&= \max\left\{ \sup_{0<r\leq 1} r^{d(\frac{1}{\lambda}-\frac{1}{\alpha})} r^{d(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p})} \;{}_r\|f\|_{q,p}, \sup_{r>1} (1/r)^{d(-\frac{1}{\lambda}+\frac{1}{q}+\frac{1}{p})} \;{}_r\|f\|_{q,p} \right\} \\
&\leq \max\left\{ \sup_{0<r\leq 1} r^{d(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p})} \;{}_r\|f\|_{q,p}, \sup_{r>1} (1/r)^{d(-\frac{1}{\lambda}+\frac{1}{q}+\frac{1}{p})} \;{}_r\|f\|_{q,p} \right\}.
\end{aligned}$$

It follows that

$$\|f\|_{q,p,\lambda} \leq \|f\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}. \tag{2.2}$$

From (2.1) and (2.2) we deduce that

$$\max\{\|f\|_{q,p,\alpha}, \|f\|_{q,p,\lambda}\} \leq \|f\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}.$$

We end the proof by using Proposition 2.5. □

Remark 2.7. Total Fofana–Guliyev spaces are generalizations of classical Fofana spaces since for $1 \leq q \leq \alpha \leq p \leq \infty$, Proposition 2.6 asserts that

$$(L^q, L^p)^{\alpha,\alpha}(\mathbb{R}^d) = (L^q, L^p)^{\alpha}(\mathbb{R}^d).$$

The following result shows that the family of spaces $(L^q, L^p)^{\alpha,\lambda}(\mathbb{R}^d)$ is decreasing with respect to the λ power and increasing with respect to the α power.

Proposition 2.8. *Let $1 \leq q \leq \alpha_1 \leq \alpha_2 \leq p \leq \infty$ and let $1 \leq q \leq \lambda_2 \leq \lambda_1 \leq p \leq \infty$. Then*

$$\|f\|_{(L^q,L^p)^{\alpha_1,\lambda_1}(\mathbb{R}^d)} \leq \|f\|_{(L^q,L^p)^{\alpha_2,\lambda_2}(\mathbb{R}^d)}, \qquad f \in L^0(\mathbb{R}^d)$$

and consequently, $(L^q, L^p)^{\alpha_2,\lambda_2}(\mathbb{R}^d) \subset (L^q, L^p)^{\alpha_1,\lambda_1}(\mathbb{R}^d)$.

Proof. Let $1 \leq q \leq \alpha_1 \leq \alpha_2 \leq p \leq \infty$, let $1 \leq q \leq \lambda_2 \leq \lambda_1 \leq p \leq \infty$, and let $f \in L^0(\mathbb{R}^d)$. We have

$$\begin{aligned}
&\|f\|_{(L^q,L^p)^{\alpha_1,\lambda_1}(\mathbb{R}^d)} \\
= \ & \max\left\{\sup_{0<r\leq 1} r^{d(\frac{1}{\alpha_1}-\frac{1}{q}-\frac{1}{p})}\ {}_r\|f\|_{q,p}, \sup_{r>1}(1/r)^{d(-\frac{1}{\lambda_1}+\frac{1}{q}+\frac{1}{p})}\ {}_r\|f\|_{q,p}\right\} \\
= \ & \max\left\{\sup_{0<r\leq 1} r^{d(\frac{1}{\alpha_1}-\frac{1}{\alpha_2})} r^{d(\frac{1}{\alpha_2}-\frac{1}{q}-\frac{1}{p})}\ {}_r\|f\|_{q,p},\right. \\
& \left.\sup_{r>1}(1/r)^{d(-\frac{1}{\lambda_1}+\frac{1}{\lambda_2})}(1/r)^{d(-\frac{1}{\lambda_2}+\frac{1}{q}+\frac{1}{p})}\ {}_r\|f\|_{q,p}\right\} \\
\leq \ & \max\left\{\sup_{0<r\leq 1} r^{d(\frac{1}{\alpha_2}-\frac{1}{q}-\frac{1}{p})}\ {}_r\|f\|_{q,p}, \sup_{r>1}(1/r)^{d(-\frac{1}{\lambda_2}+\frac{1}{q}+\frac{1}{p})}\ {}_r\|f\|_{q,p}\right\}.
\end{aligned}$$

Hence, $\|f\|_{(L^q,L^p)^{\alpha_1,\lambda_1}(\mathbb{R}^d)} \leq \|f\|_{(L^q,L^p)^{\alpha_2,\lambda_2}(\mathbb{R}^d)}$, and we conclude that

$$(L^q, L^p)^{\alpha_2,\lambda_2}(\mathbb{R}^d) \subset (L^q, L^p)^{\alpha_1,\lambda_1}(\mathbb{R}^d).$$

□

The family of spaces $(L^q, L^p)^{\alpha,\lambda}(\mathbb{R}^d)$ is decreasing with respect to the q power and increasing with respect to the p power. More precisely, we have the following result.

Proposition 2.9. *Let $1 \leq q_1 \leq q_2 \leq \alpha, \lambda \leq p_1 \leq p_2 \leq \infty$. Then*

(1) $\|f\|_{(L^{q_1},L^p)^{\alpha,\lambda}(\mathbb{R}^d)} \leq \|f\|_{(L^{q_2},L^p)^{\alpha,\lambda}(\mathbb{R}^d)}, \qquad f \in L^0(\mathbb{R}^d)$
and consequently, $(L^{q_2}, L^p)^{\alpha,\lambda}(\mathbb{R}^d) \subset (L^{q_1}, L^p)^{\alpha,\lambda}(\mathbb{R}^d)$;

(2) $\|f\|_{(L^q,L^{p_2})^{\alpha,\lambda}(\mathbb{R}^d)} \lesssim \|f\|_{(L^q,L^{p_1})^{\alpha,\lambda}(\mathbb{R}^d)}, \qquad f \in L^0(\mathbb{R}^d)$
and consequently, $(L^q, L^{p_1})^{\alpha,\lambda}(\mathbb{R}^d) \subset (L^q, L^{p_2})^{\alpha,\lambda}(\mathbb{R}^d)$.

Proof. Let $1 \le q_1 \le q_2 \le \alpha, \lambda \le p_1 \le p_2 \le \infty$ and let $f \in L^0(\mathbb{R}^d)$.

(1) We have

$$\begin{aligned} & \|f\|_{(L^{q_1},L^p)^{\alpha,\lambda}(\mathbb{R}^d)} \\ = \; & \max\left\{ \sup_{0<r\le 1} r^{d(\frac{1}{\alpha}-\frac{1}{q_1}-\frac{1}{p})} \, {}_r\|f\|_{q_1,p} \, , \, \sup_{r>1} (1/r)^{d(-\frac{1}{\lambda}+\frac{1}{q_1}+\frac{1}{p})} \, {}_r\|f\|_{q_1,p} \right\} \\ \le \; & \max\left\{ \sup_{0<r\le 1} r^{d(\frac{1}{\alpha}-\frac{1}{q_2}-\frac{1}{p})} \, {}_r\|f\|_{q_2,p} \, , \, \sup_{r>1} (1/r)^{d(-\frac{1}{\lambda}+\frac{1}{q_2}+\frac{1}{p})} \, {}_r\|f\|_{q_2,p} \right\} \\ & \text{(according to (1.3))}. \end{aligned}$$

Hence, $\|f\|_{(L^{q_1},L^p)^{\alpha,\lambda}(\mathbb{R}^d)} \le \|f\|_{(L^{q_2},L^p)^{\alpha,\lambda}(\mathbb{R}^d)}$ and so

$$(L^{q_2}, L^p)^{\alpha,\lambda}(\mathbb{R}^d) \subset (L^{q_1}, L^p)^{\alpha,\lambda}(\mathbb{R}^d).$$

(2) We have

$$\begin{aligned} & \widetilde{\|f\|}_{(L^q,L^{p_2})^{\alpha,\lambda}(\mathbb{R}^d)} \\ & = \max\left\{ \sup_{0<r\le 1} r^{d(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p_2})} r^{\frac{d}{p_2}} \, {}_r\widetilde{\|f\|}_{q,p_2}, \sup_{r>1} (1/r)^{d(-\frac{1}{\lambda}+\frac{1}{q}+\frac{1}{p_2})} r^{\frac{d}{p_2}} \, {}_r\widetilde{\|f\|}_{q,p_2} \right\} \\ & \le 2^d \max\left\{ \sup_{0<r\le 1} r^{d(\frac{1}{\alpha}-\frac{1}{q})} \, {}_r\widetilde{\|f\|}_{q,p_1}, \sup_{r>1} r^{d(\frac{1}{\lambda}-\frac{1}{q})} \, {}_r\widetilde{\|f\|}_{q,p_1} \right\} \\ & \le 2^d \max\left\{ \sup_{0<r\le 1} r^{d(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p_1})} r^{\frac{d}{p_1}} \, {}_r\widetilde{\|f\|}_{q,p_1}, \sup_{r>1} (1/r)^{d(-\frac{1}{\lambda}+\frac{1}{q}+\frac{1}{p_1})} r^{\frac{d}{p_1}} \, {}_r\widetilde{\|f\|}_{q,p_1} \right\} \\ & \le 2^d \widetilde{\|f\|}_{(L^q,L^{p_1})^{\alpha,\lambda}(\mathbb{R}^d)} \; \text{(according to (1.4))}. \end{aligned}$$

Hence, $\|f\|_{(L^q,L^{p_2})^{\alpha,\lambda}(\mathbb{R}^d)} \lesssim \|f\|_{(L^q,L^{p_1})^{\alpha,\lambda}(\mathbb{R}^d)}$ and so

$$(L^q, L^{p_1})^{\alpha,\lambda}(\mathbb{R}^d) \subset (L^q, L^{p_2})^{\alpha,\lambda}(\mathbb{R}^d).$$

□

3. Norm inequalities for the Hardy–Littlewood maximal operator

Given γ, $0 < \gamma < d$, the fractional maximal operator M_γ is defined by

$$M_\gamma f(x) = \sup_{r>0} |B(x,r)|^{\frac{\gamma}{d}-1} \int_{B(x,r)} |f(y)|\, dy, \qquad x \in \mathbb{R}^d, \tag{3.1}$$

for locally integrable functions f on $\mathbb{R}^d$. In the limiting case $\gamma = 0$, the fractional maximal operator reduces to the Hardy–Littlewood maximal operator M. It is proved in [5] that M is bounded on classical Fofana spaces. It is also bounded on total Morrey spaces (see [16]). In the setting of total Fofana–Guliyev spaces, we have the following norm inequalities for the Hardy–Littlewood maximal operator.

Theorem 3.1. *(1) Let $1 < q \le \alpha, \lambda < p < \infty$ such that $\frac{1}{q} + \frac{2}{p} < \frac{1}{\alpha} + \frac{1}{\lambda}$. Then*

$$\|Mf\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)} \lesssim \|f\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}, \quad f \in (L^q, L^p)^{\alpha,\lambda}(\mathbb{R}^d).$$

(2) Let $q = 1 < \alpha, \lambda < p < \infty$. Then

$$\|Mf\|_{(L^{1,\infty},L^p)^{\alpha,\lambda}(\mathbb{R}^d)} \lesssim \|f\|_{(L^1,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}, \quad f \in (L^1, L^p)^{\alpha,\lambda}(\mathbb{R}^d),$$

where

$$\begin{aligned}&\|f\|_{(L^{1,\infty},L^p)^{\alpha,\lambda}(\mathbb{R}^d)}\\ &:= \sup_{r>0}[r]_1^{d(\frac{1}{\alpha}-1-\frac{1}{p})}[1/r]_1^{d(-\frac{1}{\lambda}+1+\frac{1}{p})}\left[\int_{\mathbb{R}^d}\left(\left\|f\chi_{B(y,r)}\right\|_{1,\infty}^{*}\right)^p dy\right]^{\frac{1}{p}}\end{aligned}$$

with

$$\left\|f\chi_{B(y,r)}\right\|_{1,\infty}^{*} = \sup_{r>0} r\left|\{x\in B(y,r) : |f(x)|>r\}\right|.$$

Proof. (1) Let $1<q\le\alpha,\lambda\le p<\infty$ such that $\frac{1}{q}+\frac{2}{p}<\frac{1}{\alpha}+\frac{1}{\lambda}$. Assume that $f\in(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)$. Let $r>0$ and $y\in\mathbb{R}^d$. Following [14, Theorem 2.12], we may write

$$\int_{\mathbb{R}^d}(Mf)^q(x)\chi_{B(y,r)}(x)dx \lesssim \int_{\mathbb{R}^d}|f(x)|^q(M\chi_{B(y,r)})(x)dx. \tag{3.2}$$

Then, proceeding as in the proof of Theorem 1 in [3], we obtain

$${}_{r}\left\|Mf\right\|_{q,p}^{q} \lesssim {}_{2r}\left\|f\right\|_{q,p}^{q} + \sum_{i=1}^{\infty}\frac{1}{(2^{i-1})^d}\,{}_{2^{i+1}r}\left\|f\right\|_{q,p}^{q}.$$

On the one hand, we have

$$\begin{aligned}{}_{2r}\left\|f\right\|_{q,p}^{q} &= \frac{[2r]_1^{dq(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p})}[1/2r]_1^{dq(-\frac{1}{\lambda}+\frac{1}{q}+\frac{1}{p})}}{[2r]_1^{dq(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p})}[1/2r]_1^{dq(-\frac{1}{\lambda}+\frac{1}{q}+\frac{1}{p})}}\,{}_{2r}\left\|f\right\|_{q,p}^{q}\\ &\le [2r]_1^{dq(-\frac{1}{\alpha}+\frac{1}{q}+\frac{1}{p})}[1/2r]_1^{dq(\frac{1}{\lambda}-\frac{1}{q}-\frac{1}{p})}\left\|f\right\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}^{q}\\ &\le (2[r]_1)^{dq(-\frac{1}{\alpha}+\frac{1}{q}+\frac{1}{p})}\left(\frac{1}{2}[1/r]_1\right)^{dq(\frac{1}{\lambda}-\frac{1}{q}-\frac{1}{p})}\left\|f\right\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}^{q}.\end{aligned}$$

Hence,

$${}_{2r}\|f\|_{q,p}^{q} \lesssim [r]_1^{dq(-\frac{1}{\alpha}+\frac{1}{q}+\frac{1}{p})}[1/r]_1^{dq(\frac{1}{\lambda}-\frac{1}{q}-\frac{1}{p})}\left\|f\right\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}^{q}. \tag{3.3}$$

On the other hand,

$$\begin{aligned}&\sum_{i=1}^{\infty}\frac{1}{(2^{i-1})^d}\,{}_{2^{i+1}r}\left\|f\right\|_{q,p}^{q}\\ &=\sum_{i=1}^{\infty}\frac{1}{(2^{i-1})^d}\frac{[2^{i+1}r]_1^{dq(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p})}[1/2^{i+1}r]_1^{dq(-\frac{1}{\lambda}+\frac{1}{q}+\frac{1}{p})}}{[2^{i+1}r]_1^{dq(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p})}[1/2^{i+1}r]_1^{dq(-\frac{1}{\lambda}+\frac{1}{q}+\frac{1}{p})}}\,{}_{2^{i+1}r}\left\|f\right\|_{q,p}^{q}\\ &\le \left\|f\right\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}^{q}\sum_{i=1}^{\infty}\frac{1}{(2^{i-1})^d}[2^{i+1}r]_1^{dq(-\frac{1}{\alpha}+\frac{1}{q}+\frac{1}{p})}[1/2^{i+1}r]_1^{dq(\frac{1}{\lambda}-\frac{1}{q}-\frac{1}{p})}\\ &\le [r]_1^{dq(-\frac{1}{\alpha}+\frac{1}{q}+\frac{1}{p})}[1/r]_1^{dq(\frac{1}{\lambda}-\frac{1}{q}-\frac{1}{p})}\left\|f\right\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}^{q}\sum_{i=1}^{\infty}(2^i)^{dq(-\frac{1}{\alpha}-\frac{1}{\lambda}+\frac{1}{q}+\frac{2}{p})}.\end{aligned}$$

Since $-\frac{1}{\alpha}-\frac{1}{\lambda}+\frac{1}{q}+\frac{2}{p}<0$ then $\sum_{i=1}^{\infty}(2^i)^{dq(-\frac{1}{\alpha}-\frac{1}{\lambda}+\frac{1}{q}+\frac{2}{p})}<\infty$. Therefore,

$$\sum_{i=1}^{\infty}\frac{1}{(2^{i-1})^d}\ {}_{2^{i+1}r}\left\|f\right\|_{q,p}^q\lesssim [r]_1^{dq(-\frac{1}{\alpha}+\frac{1}{q}+\frac{1}{p})}[1/r]_1^{dq(\frac{1}{\lambda}-\frac{1}{q}-\frac{1}{p})}\left\|f\right\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}^q \tag{3.4}$$

From (3.3) and (3.4), we deduce that

$$_r\left\|Mf\right\|_{q,p}^q\lesssim [r]_1^{dq(-\frac{1}{\alpha}+\frac{1}{q}+\frac{1}{p})}[1/r]_1^{dq(\frac{1}{\lambda}-\frac{1}{q}-\frac{1}{p})}\left\|f\right\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}^q.$$

It follows that

$$[r]_1^{dq(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p})}[1/r]_1^{dq(-\frac{1}{\lambda}+\frac{1}{q}+\frac{1}{p})}\ {}_r\left\|Mf\right\|_{q,p}^q\lesssim\left\|f\right\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}^q. \tag{3.5}$$

We obtain the desired result by taking the supremum over all $r>0$ in the left-hand side of (3.5).

(2) For $q=1<\alpha,\lambda<p<\infty$ and $f\in(L^1,L^p)^{\alpha,\lambda}(\mathbb{R}^d)$, the proof is the same as in part (1) by replacing (3.2) by the following weak-type inequality:

$$|\{x\in B(y,r):Mf(x)>t\}|\lesssim\frac{1}{t}\int_{\mathbb{R}^d}|f(x)|(M\chi_{B(y,r)})(x)dx,\quad t>0.$$

$\square$

Remark 3.2. Note that Theorem 3.1 in the case where $\alpha=\lambda$, was established in [5, Proposition 4.2]. However, for $q>1$, the proof given in [5] does not use the hypothesis $\frac{1}{q}+\frac{2}{p}<\frac{2}{\alpha}$.

4. Norm inequalities for fractional integral and fractional maximal operators

In what follows, for $1\leq q\leq\infty$, we shall denote by q' its conjugate exponent: $\frac{1}{q}+\frac{1}{q'}=1$. For $0<\gamma<d$, define the fractional integral operator (also known as the Riesz potential) of order γ by

$$I_\gamma f(x)=\int_{\mathbb{R}^d}\frac{f(y)}{|x-y|^{d-\gamma}}dy,\quad x\in\mathbb{R}^d,$$

for suitable functions f on $\mathbb{R}^d$. It follows from the definitions that

$$M_\gamma f(x)\leq C_d I_\gamma(|f|)(x),\quad x\in\mathbb{R}^d, \tag{4.1}$$

where C_d is the Lebesgue measure of the unit ball in $\mathbb{R}^d$. Necessary and sufficient conditions for the boundedness of the fractional maximal operator M_γ in total Morrey spaces are provided in [18].

Now, recall the boundedness properties of the fractional integral operator in classical Fofana spaces. Let $1<q\leq\alpha\leq p\leq\infty$. Put $\frac{1}{\alpha^*}=\frac{1}{\alpha}-\frac{\gamma}{d}$. It was proved in [4] that if $0<\gamma<d(\frac{1}{\alpha}-\frac{1}{p})$, then

$$\|I_\gamma f\|_{(L^{q^*},L^p)^{\alpha^*}(\mathbb{R}^d)}\lesssim\|f\|_{(L^q,L^p)^{\alpha}(\mathbb{R}^d)},$$

where $\frac{1}{q^*}=\frac{1}{q}-\frac{\gamma}{d}$.

By using a different method, Feuto [5] established that if $0<\gamma<d$, then

$$\|I_\gamma f\|_{(L^{\tilde{q}},L^{\tilde{p}})^{\alpha^*}(\mathbb{R}^d)}\lesssim\|f\|_{(L^q,L^p)^{\alpha}(\mathbb{R}^d)},$$

where $\frac{1}{\bar{q}} = \frac{1}{q} - \frac{\alpha}{q}\frac{\gamma}{d}$ and $\frac{1}{\bar{p}} = \frac{1}{p} - \frac{\alpha}{p}\frac{\gamma}{d}$.
In the framework of total Fofana–Guliyev spaces, we have the following result.

Theorem 4.1. *Let* $1 < q \leq \alpha, \lambda < p < \infty$ *such that* $\frac{1}{q} + \frac{2}{p} < \frac{1}{\alpha} + \frac{1}{\lambda}$ *and* $0 < \gamma < d(\frac{1}{\alpha} + \frac{1}{\lambda} - \frac{1}{q})$*. Put*

$$\frac{1}{\alpha^*} = \frac{1}{\alpha} - \frac{q\lambda\gamma}{d(q\lambda + q\alpha - \alpha\lambda)}, \quad \frac{1}{\lambda^*} = \frac{1}{\lambda} - \frac{q\alpha\gamma}{d(q\lambda + q\alpha - \alpha\lambda)},$$

$$\frac{1}{\bar{p}} = \frac{1}{p}\left(1 - \frac{q\alpha\lambda\gamma}{d(q\lambda + q\alpha - \alpha\lambda)}\right) \text{ and } \frac{1}{\bar{q}} = \frac{1}{q} - \frac{\lambda\alpha\gamma}{d(q\lambda + q\alpha - \alpha\lambda)}.$$

Then, for all $f \in (L^q, L^p)^{\alpha,\lambda}(\mathbb{R}^d)$*, we have*

$$\|I_\gamma f\|_{(L^{\bar{q}}, L^{\bar{p}})^{\alpha^*,\lambda^*}(\mathbb{R}^d)} \lesssim \|f\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}^{1-\frac{q\alpha\lambda\gamma}{d(q\lambda+q\alpha-\alpha\lambda)}} \|f\|_{(L^q,L^\infty)^{\alpha,\lambda}(\mathbb{R}^d)}^{\frac{q\alpha\lambda\gamma}{d(q\lambda+q\alpha-\alpha\lambda)}}$$

and

$$\|I_\gamma f\|_{(L^{\bar{q}}, L^{\bar{p}})^{\alpha^*,\lambda^*}(\mathbb{R}^d)} \lesssim \|f\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}.$$

Proof. Let $1 < q \leq \alpha, \lambda < p < \infty$ such that $\frac{1}{q}+\frac{2}{p} < \frac{1}{\alpha}+\frac{1}{\lambda}$ and $0 < \gamma < d(\frac{1}{\alpha}+\frac{1}{\lambda}-\frac{1}{q})$. Assume that $f \in (L^q, L^p)^{\alpha,\lambda}(\mathbb{R}^d) \setminus \{0\}$ since the case $f = 0$ is obvious. Let $r > 0$ and $x \in \mathbb{R}$. We have

$$I_\gamma f(x) = \int_{B(x,r)} \frac{f(y)}{|x-y|^{d-\gamma}} dy + \int_{\mathbb{R}^d \setminus B(x,r)} \frac{f(y)}{|x-y|^{d-\gamma}} dy = A(x) + C(x).$$

Following the proof of Theorem 2 in [3], we may write

$$|A(x)| = \left| \int_{|x-y|\leq r} \frac{f(y)}{|x-y|^{d-\gamma}} dy \right| \leq C r^\gamma M f(x). \tag{4.2}$$

On the other hand, we have

$$\begin{aligned} |C(x)| &\leq \sum_{i=0}^{\infty} (2^i r)^{-d+\gamma} |B(x, 2^{i+1}r)|^{\frac{d}{q'}} \left(\int_{B(x,2^{i+1}r)} |f(y)|^q dy \right)^{\frac{1}{q}} \\ &\leq C r^{\gamma - \frac{d}{q}} \|f\|_{(L^q,L^\infty)^{\alpha,\lambda}(\mathbb{R}^d)} \sum_{i=0}^{\infty} (2^i)^{\frac{-d}{q}+\gamma} [2^{i+1}r]_1^{d(-\frac{1}{\alpha}+\frac{1}{q})} [1/2^{i+1}r]_1^{d(\frac{1}{\lambda}-\frac{1}{q})}. \end{aligned}$$

Since

$$[2^{i+1}r]_1^{d(-\frac{1}{\alpha}+\frac{1}{q})} [1/2^{i+1}r]_1^{d(\frac{1}{\lambda}-\frac{1}{q})} \leq (2^{i+1}r)^{d(-\frac{1}{\alpha}+\frac{1}{q})} \left(\frac{1}{2^{i+1}} [1/r]_1 \right)^{d(\frac{1}{\lambda}-\frac{1}{q})},$$

then

$$\begin{aligned} &\sum_{i=0}^{\infty} (2^i)^{\frac{-d}{q}+\gamma} (2^{i+1}r)^{d(-\frac{1}{\alpha}+\frac{1}{q})} \left(\frac{1}{2^{i+1}} [1/r]_1 \right)^{d(\frac{1}{\lambda}-\frac{1}{q})} \\ &= 2^{d(-\frac{1}{\alpha}-\frac{1}{\lambda}+\frac{2}{q})} [1/r]_1^{d(\frac{1}{\lambda}-\frac{1}{q})} r^{d(-\frac{1}{\alpha}+\frac{1}{q})} \sum_{i=0}^{\infty} (2^i)^{\gamma - d(\frac{1}{\alpha}+\frac{1}{\lambda}-\frac{1}{q})}. \end{aligned}$$

In addition, $\sum_{i=0}^{\infty}(2^i)^{\gamma-d(\frac{1}{\alpha}+\frac{1}{\lambda}-\frac{1}{q})}<\infty$. So,

$$|C(x)|\leq Cr^{\gamma-\frac{d}{\alpha}}[1/r]_1^{d(\frac{1}{\lambda}-\frac{1}{q})}\|f\|_{(L^q,L^\infty)^{\alpha,\lambda}(\mathbb{R}^d)}. \tag{4.3}$$

It follows from (4.2) and (4.3) that

$$\begin{aligned}|I_\gamma f(x)| &\leq C\left(r^\gamma Mf(x)+r^{\gamma-\frac{d}{\alpha}}[1/r]_1^{d(\frac{1}{\lambda}-\frac{1}{q})}\|f\|_{(L^q,L^\infty)^{\alpha,\lambda}(\mathbb{R}^d)}\right)\\ &\leq C\min\Big\{\ r^\gamma Mf(x)+r^{\gamma-d(\frac{1}{\alpha}+\frac{1}{\lambda}-\frac{1}{q})}\|f\|_{(L^q,L^\infty)^{\alpha,\lambda}(\mathbb{R}^d)}\ ,\\ &\qquad r^\gamma Mf(x)+r^{\gamma-\frac{d}{\alpha}}\|f\|_{(L^q,L^\infty)^{\alpha,\lambda}(\mathbb{R}^d)}\ \Big\}.\end{aligned}$$

Minimizing with respect to

$$r=\left(\frac{\|f\|_{(L^q,L^\infty)^{\alpha,\lambda}(\mathbb{R}^d)}}{Mf(x)}\right)^{\frac{q\alpha\lambda}{d(q\lambda+q\alpha-\alpha\lambda)}}\quad\text{or}\quad r=\left(\frac{\|f\|_{(L^q,L^\infty)^{\alpha,\lambda}(\mathbb{R}^d)}}{Mf(x)}\right)^{\frac{\alpha}{d}},$$

we obtain

$$\begin{aligned}|I_\gamma f(x)| &\leq C\min\Big\{\ \|f\|^{\frac{q\alpha\lambda\gamma}{d(q\lambda+q\alpha-\alpha\lambda)}}_{(L^q,L^\infty)^{\alpha,\lambda}(\mathbb{R}^d)}(Mf(x))^{1-\frac{q\lambda\alpha\gamma}{d(q\lambda+q\alpha-\alpha\lambda)}}\ ,\\ &\qquad (Mf(x))^{1-\frac{\alpha\gamma}{d}}\|f\|^{\frac{\alpha\gamma}{d}}_{(L^q,L^\infty)^{\alpha,\lambda}(\mathbb{R}^d)}\ \Big\}\end{aligned}$$

for all $x\in\mathbb{R}^d$.

This implies that

$$|I_\gamma f(x)|\leq C\|f\|^{\frac{q\alpha\lambda\gamma}{d(q\lambda+q\alpha-\alpha\lambda)}}_{(L^q,L^\infty)^{\alpha,\lambda}(\mathbb{R}^d)}(Mf(x))^{1-\frac{q\lambda\alpha\gamma}{d(q\lambda+q\alpha-\alpha\lambda)}}. \tag{4.4}$$

From (4.4), we have

$${}_r\|I_\gamma f\|_{\bar{q},\bar{p}}\ \leq\ C\|f\|^{\frac{q\alpha\lambda\gamma}{d(q\lambda+q\alpha-\alpha\lambda)}}_{(L^q,L^\infty)^{\alpha,\lambda}(\mathbb{R}^d)}\,{}_r\|Mf\|^{1-\frac{q\lambda\alpha\gamma}{d(q\lambda+q\alpha-\alpha\lambda)}}_{q,p}. \tag{4.5}$$

Then, multiplying both sides of (4.5) by $[r]_1^{d(\frac{1}{\alpha^*}-\frac{1}{\bar{q}}-\frac{1}{\bar{p}})}[1/r]_1^{d(-\frac{1}{\lambda^*}+\frac{1}{\bar{q}}+\frac{1}{\bar{p}})}$, we get

$$\begin{aligned}&[r]_1^{d(\frac{1}{\alpha^*}-\frac{1}{\bar{q}}-\frac{1}{\bar{p}})}[1/r]_1^{d(-\frac{1}{\lambda^*}+\frac{1}{\bar{q}}+\frac{1}{\bar{p}})}\,{}_r\|I_\gamma f\|_{\bar{q},\bar{p}}\\ &\leq C\|f\|^{\frac{q\alpha\lambda\gamma}{d(q\lambda+q\alpha-\alpha\lambda)}}_{(L^q,L^\infty)^{\alpha,\lambda}(\mathbb{R}^d)}\left([r]_1^{d(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p})}\right)^{1-\frac{q\lambda\alpha\gamma}{d(q\lambda+q\alpha-\alpha\lambda)}}\\ &\quad\times\left([1/r]_1^{d(-\frac{1}{\lambda}+\frac{1}{q}+\frac{1}{p})}\right)^{1-\frac{q\lambda\alpha\gamma}{d(q\lambda+q\alpha-\alpha\lambda)}}\,{}_r\|Mf\|^{1-\frac{q\lambda\alpha\gamma}{d(q\lambda+q\alpha-\alpha\lambda)}}_{q,p}.\end{aligned}$$

Hence,

$$\|I_\gamma f\|_{(L^{\bar{q}},L^{\bar{p}})^{\alpha^*,\lambda^*}(\mathbb{R}^d)}\leq C\|f\|^{\frac{q\alpha\lambda\gamma}{d(q\lambda+q\alpha-\alpha\lambda)}}_{(L^q,L^\infty)^{\alpha,\lambda}(\mathbb{R}^d)}\|Mf\|^{1-\frac{q\lambda\alpha\gamma}{d(q\lambda+q\alpha-\alpha\lambda)}}_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}.$$

We deduce from Theorem 3.1 (1) that

$$\|I_\gamma f\|_{(L^{\bar{q}},L^{\bar{p}})^{\alpha^*,\lambda^*}(\mathbb{R}^d)}\lesssim\|f\|^{\frac{q\alpha\lambda\gamma}{d(q\lambda+q\alpha-\alpha\lambda)}}_{(L^q,L^\infty)^{\alpha,\lambda}(\mathbb{R}^d)}\|f\|^{1-\frac{q\lambda\alpha\gamma}{d(q\lambda+q\alpha-\alpha\lambda)}}_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}. \tag{4.6}$$

We end the proof by combining Proposition 2.9 (1) with the fact that

$$\|f\|_{(L^q,L^\infty)^{\alpha,\lambda}(\mathbb{R}^d)} \lesssim \|f\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}.$$

□

For $q = 1$, we have the following result.

Theorem 4.2. *Let $1 < \alpha, \lambda < p < \infty$ such that $1 + \frac{2}{p} < \frac{1}{\alpha} + \frac{1}{\lambda}$ and $0 < \gamma < d(\frac{1}{\alpha} + \frac{1}{\lambda} - 1)$. Put*

$$\frac{1}{\alpha^*} = \frac{1}{\alpha} - \frac{\lambda\gamma}{d(\lambda+\alpha-\alpha\lambda)}, \quad \frac{1}{\lambda^*} = \frac{1}{\lambda} - \frac{\alpha\gamma}{d(\lambda+\alpha-\alpha\lambda)},$$

$$\frac{1}{\bar{p}} = \frac{1}{p}\left(1 - \frac{\alpha\lambda\gamma}{d(\lambda+\alpha-\alpha\lambda)}\right) \text{ and } \frac{1}{\bar{q}} = 1 - \frac{\lambda\alpha\gamma}{d(\lambda+\alpha-\alpha\lambda)}.$$

Then, for all $f \in (L^1, L^p)^{\alpha,\lambda}(\mathbb{R}^d)$, we have

$$\|I_\gamma f\|_{(L^{\bar{q},\infty},L^{\bar{p}})^{\alpha^*,\lambda^*}(\mathbb{R}^d)} \lesssim \|f\|_{(L^1,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}^{1-\frac{\alpha\lambda\gamma}{d(\lambda+\alpha-\alpha\lambda)}} \|f\|_{(L^1,L^\infty)^{\alpha,\lambda}(\mathbb{R}^d)}^{\frac{\alpha\lambda\gamma}{d(\lambda+\alpha-\alpha\lambda)}}$$

and

$$\|I_\gamma f\|_{(L^{\bar{q},\infty},L^{\bar{p}})^{\alpha^*,\lambda^*}(\mathbb{R}^d)} \lesssim \|f\|_{(L^1,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}, \qquad f \in (L^1, L^p)^{\alpha,\lambda}(\mathbb{R}^d).$$

Proof. Let $1 < \alpha, \lambda < p < \infty$ such that $1 + \frac{2}{p} < \frac{1}{\alpha} + \frac{1}{\lambda}$ and $0 < \gamma < d(\frac{1}{\alpha} + \frac{1}{\lambda} - 1)$. Assume that $f \in (L^1, L^p)^{\alpha,\lambda}(\mathbb{R}^d) \setminus \{0\}$ since the case $f = 0$ is obvious. Inequality (4.4) and the same arguments as in the proof of Theorem 4.1 lead to the desired results. □

Combining Theorem 4.1 with Theorem 4.2 in the case $\alpha = \lambda$, we get the following result.

Corollary 4.3. *Let $1 < q \leq \alpha < p < \infty$ such that $\frac{1}{q} + \frac{2}{p} < \frac{2}{\alpha}$ and $0 < \gamma < d(\frac{2}{\alpha} - \frac{1}{q})$. Put*

$$\frac{1}{\alpha^*} = \frac{1}{\alpha} - \frac{\gamma}{d}\frac{1}{\alpha(\frac{2}{\alpha} - \frac{1}{q})}, \quad \frac{1}{\bar{p}} = \frac{1}{p} - \frac{\gamma}{d}\frac{1}{p(\frac{2}{\alpha} - \frac{1}{q})} \text{ and } \frac{1}{\bar{q}} = \frac{1}{q} - \frac{\gamma}{d}\frac{1}{q(\frac{2}{\alpha} - \frac{1}{q})}.$$

(1) If $1 < q$, then for all $f \in (L^q, L^p)^\alpha(\mathbb{R}^d)$ we have

$$\|I_\gamma f\|_{(L^{\bar{q}},L^{\bar{p}})^{\alpha^*}(\mathbb{R}^d)} \lesssim \|f\|_{(L^q,L^p)^\alpha(\mathbb{R}^d)}^{1-\frac{\gamma}{d}\frac{1}{\frac{2}{\alpha}-\frac{1}{q}}} \|f\|_{(L^q,L^\infty)^\alpha(\mathbb{R}^d)}^{\frac{\gamma}{d}\frac{1}{\frac{2}{\alpha}-\frac{1}{q}}}$$

and

$$\|I_\gamma f\|_{(L^{\bar{q}},L^{\bar{p}})^{\alpha^*}(\mathbb{R}^d)} \lesssim \|f\|_{(L^q,L^p)^\alpha(\mathbb{R}^d)}.$$

(2) If $q = 1 < \alpha$, then for all $f \in (L^1, L^p)^\alpha(\mathbb{R}^d)$ we have

$$\|I_\gamma f\|_{(L^{\bar{q},\infty},L^{\bar{p}})^{\alpha^*,\lambda^*}(\mathbb{R}^d)} \lesssim \|f\|_{(L^1,L^p)^\alpha(\mathbb{R}^d)}^{1-\frac{\gamma}{d}\frac{1}{\frac{2}{\alpha}-1}} \|f\|_{(L^1,L^\infty)^\alpha(\mathbb{R}^d)}^{\frac{\gamma}{d}\frac{1}{\frac{2}{\alpha}-1}}$$

and

$$\|I_\gamma f\|_{(L^{\bar{q},\infty},L^{\bar{p}})^{\alpha^*,\lambda^*}(\mathbb{R}^d)} \lesssim \|f\|_{(L^1,L^p)^\alpha(\mathbb{R}^d)}, \qquad f \in (L^1, L^p)^\alpha(\mathbb{R}^d).$$

From part (1) of Corollary 4.3, we get the Hardy–Littlewood–Sobolev theorem stating the boundedness of I_γ between Lebesgue spaces.

Corollary 4.4 (see [27])**.** *Let* $1<\alpha<\infty$ *and* $0<\gamma<\frac{d}{\alpha}$. *Put* $\frac{1}{\alpha^*}=\frac{1}{\alpha}-\frac{\gamma}{d}$. *Then, for all* $f\in L^\alpha(\mathbb{R}^d)$, *we have*

$$\|I_\gamma f\|_{L^{\alpha^*}(\mathbb{R}^d)}\lesssim\|f\|_{L^\alpha(\mathbb{R}^d)}.$$

Proof. Let $1<\alpha=q$ and let $p=2\alpha+2$ in Corollary 4.3. Then, the desired result follows from the fact that for $1\le s\le t\le\infty$ we have $(L^s,L^t)^s(\mathbb{R}^d)=L^s(\mathbb{R}^d)$ (see [9] or [12, Proposition 1.5]). □

As an immediate consequence of Theorem 4.1 and inequality (4.1), we have the following corollary.

Corollary 4.5. *Let* $1<q\le\alpha,\lambda<p<\infty$ *such that* $\frac{1}{q}+\frac{2}{p}<\frac{1}{\alpha}+\frac{1}{\lambda}$ *and* $0<\gamma<d(\frac{1}{\alpha}+\frac{1}{\lambda}-\frac{1}{q})$. *Put*

$$\frac{1}{\alpha^*}=\frac{1}{\alpha}-\frac{q\lambda\gamma}{d(q\lambda+q\alpha-\alpha\lambda)},\qquad\frac{1}{\lambda^*}=\frac{1}{\lambda}-\frac{q\alpha\gamma}{d(q\lambda+q\alpha-\alpha\lambda)},$$

$$\frac{1}{\bar{p}}=\frac{1}{p}\left(1-\frac{q\alpha\lambda\gamma}{d(q\lambda+q\alpha-\alpha\lambda)}\right)\ \textit{and}\ \frac{1}{\bar{q}}=\frac{1}{q}-\frac{\lambda\alpha\gamma}{d(q\lambda+q\alpha-\alpha\lambda)}.$$

Then, for all $f\in(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)$, *we have*

$$\|M_\gamma f\|_{(L^{\bar{q}},L^{\bar{p}})^{\alpha^*,\lambda^*}(\mathbb{R}^d)}\lesssim\|f\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}^{1-\frac{q\alpha\lambda\gamma}{d(q\lambda+q\alpha-\alpha\lambda)}}\|f\|_{(L^q,L^{+\infty})^{\alpha,\lambda}(\mathbb{R}^d)}^{\frac{q\alpha\lambda\gamma}{d(q\lambda+q\alpha-\alpha\lambda)}}$$

and

$$\|M_\gamma f\|_{(L^{\bar{q}},L^{\bar{p}})^{\alpha^*,\lambda^*}(\mathbb{R}^d)}\lesssim\|f\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}.$$

As an immediate consequence of Theorem 4.2 and inequality (4.1), we have the following result.

Corollary 4.6. *Let* $1<\alpha,\lambda<p<\infty$ *such that* $1+\frac{2}{p}<\frac{1}{\alpha}+\frac{1}{\lambda}$ *and* $0<\gamma<d(\frac{1}{\alpha}+\frac{1}{\lambda}-1)$. *Put*

$$\frac{1}{\alpha^*}=\frac{1}{\alpha}-\frac{\lambda\gamma}{d(\lambda+\alpha-\alpha\lambda)},\qquad\frac{1}{\lambda^*}=\frac{1}{\lambda}-\frac{\alpha\gamma}{d(\lambda+\alpha-\alpha\lambda)},$$

$$\frac{1}{\bar{p}}=\frac{1}{p}\left(1-\frac{\alpha\lambda\gamma}{d(\lambda+\alpha-\alpha\lambda)}\right)\ \textit{and}\ \frac{1}{\bar{q}}=1-\frac{\lambda\alpha\gamma}{d(\lambda+\alpha-\alpha\lambda)}.$$

Then, for all $f\in(L^1,L^p)^{\alpha,\lambda}(\mathbb{R}^d)$, *we have*

$$\|M_\gamma f\|_{(L^{\bar{q},\infty},L^{\bar{p}})^{\alpha^*,\lambda^*}(\mathbb{R}^d)}\lesssim\|f\|_{(L^1,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}^{1-\frac{\alpha\lambda\gamma}{d(\lambda+\alpha-\alpha\lambda)}}\|f\|_{(L^1,L^\infty)^{\alpha,\lambda}(\mathbb{R}^d)}^{\frac{\alpha\lambda\gamma}{d(\lambda+\alpha-\alpha\lambda)}}$$

and

$$\|M_\gamma f\|_{(L^{\bar{q},\infty},L^{\bar{p}})^{\alpha^*,\lambda^*}(\mathbb{R}^d)}\lesssim\|f\|_{(L^1,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}.$$

By Corollary 4.3 and inequality (4.1), we have the following result.

Corollary 4.7. *Let $1 < q \leq \alpha < p < \infty$ such that $\frac{1}{q}+\frac{2}{p} < \frac{2}{\alpha}$ and $0 < \gamma < d(\frac{2}{\alpha}-\frac{1}{q})$. Put*

$$\frac{1}{\alpha^*} = \frac{1}{\alpha} - \frac{\gamma}{d}\frac{1}{\alpha(\frac{2}{\alpha}-\frac{1}{q})}, \quad \frac{1}{\bar{p}} = \frac{1}{p} - \frac{\gamma}{d}\frac{1}{p(\frac{2}{\alpha}-\frac{1}{q})} \text{ and } \frac{1}{\bar{q}} = \frac{1}{q} - \frac{\gamma}{d}\frac{1}{q(\frac{2}{\alpha}-\frac{1}{q})}.$$

(1) If $1 < q$, then for all $f \in (L^q, L^p)^\alpha(\mathbb{R}^d)$ we have

$$\|M_\gamma f\|_{(L^{\bar{q}}, L^{\bar{p}})^{\alpha^*}(\mathbb{R}^d)} \lesssim \|f\|_{(L^q,L^p)^\alpha(\mathbb{R}^d)}^{1-\frac{\gamma}{d}\frac{1}{\frac{2}{\alpha}-\frac{1}{q}}} \|f\|_{(L^q,L^\infty)^\alpha(\mathbb{R}^d)}^{\frac{\gamma}{d}\frac{1}{\frac{2}{\alpha}-\frac{1}{q}}}$$

and

$$\|M_\gamma f\|_{(L^{\bar{q}}, L^{\bar{p}})^{\alpha^*}(\mathbb{R}^d)} \lesssim \|f\|_{(L^q,L^p)^\alpha(\mathbb{R}^d)}.$$

(2) If $q = 1 < \alpha$, then for all $f \in (L^1, L^p)^\alpha(\mathbb{R}^d)$ we have

$$\|M_\gamma f\|_{(L^{\bar{q},\infty}, L^{\bar{p}})^{\alpha^*,\lambda^*}(\mathbb{R}^d)} \lesssim \|f\|_{(L^1,L^p)^\alpha(\mathbb{R}^d)}^{1-\frac{\gamma}{d}\frac{1}{\frac{2}{\alpha}-1}} \|f\|_{(L^1,L^\infty)^\alpha(\mathbb{R}^d)}^{\frac{\gamma}{d}\frac{1}{\frac{2}{\alpha}-1}}$$

and

$$\|M_\gamma f\|_{(L^{\bar{q},\infty}, L^{\bar{p}})^{\alpha^*,\lambda^*}(\mathbb{R}^d)} \lesssim \|f\|_{(L^1,L^p)^\alpha(\mathbb{R}^d)}, \qquad f \in (L^1, L^p)^\alpha(\mathbb{R}^d).$$

From Corollary 4.4 and inequality (4.1), we get the following well known Hardy–Littlewood–Sobolev theorem for the boundedness of M_γ between Lebesgue spaces.

Corollary 4.8. *Let $1 < \alpha < \infty$ and $0 < \gamma < \frac{d}{\alpha}$. Put $\frac{1}{\alpha^*} = \frac{1}{\alpha} - \frac{\gamma}{d}$. Then, for all $f \in L^\alpha(\mathbb{R}^d)$, we have*

$$\|M_\gamma f\|_{L^{\alpha^*}(\mathbb{R}^d)} \lesssim \|f\|_{L^\alpha(\mathbb{R}^d)}.$$

The next theorem gives a norm equivalence of fractional integral and its corresponding fractional maximal function when we deal with nonnegative measurable functions. A similar result has been established in Morrey spaces (see [15, Theorem 7.1]) and in classical Fofana spaces (see [5, Theorem 4.7]).

Theorem 4.9. *Let $1 \leq q \leq \alpha, \lambda < p \leq \infty$ such that $\frac{1}{q}+\frac{2}{p} < \frac{1}{\alpha}+\frac{1}{\lambda}$ and $0 < \gamma < d$. For every positive function $f \in L^q_{loc}(\mathbb{R}^d)$, we have*

$$\|I_\gamma f\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)} \approx \|M_\gamma f\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}.$$

Proof. Let $1 \leq q \leq \alpha, \lambda < p \leq \infty$ such that $\frac{1}{q} + \frac{2}{p} < \frac{1}{\alpha} + \frac{1}{\lambda}$ and $0 < \gamma < d$. In view of inequality (4.1) it suffices to prove that

$$\|I_\gamma f\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)} \lesssim \|M_\gamma f\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}.$$

Let $r > 0$ an $y \in \mathbb{R}^d$. By Lemma 4.2 of [15], we have

$$\|(I_\gamma f)\chi_{B(y,r)}\|_q \approx \|(M_\gamma f)\chi_{B(y,r)}\|_q + |B(y,r)|^{\frac{1}{q}} \int_{\mathbb{R}^d \setminus B(y,r)} \frac{f(x)}{|x-y|^{d-\gamma}}dx. \tag{4.7}$$

Furthermore, we have

$$\begin{aligned}
&|B(y,r)|^{\frac{1}{q}}\int_{\mathbb{R}^d\setminus B(y,r)}\frac{f(x)}{|x-y|^{d-\gamma}}dx\\
&=\sum_{i=0}^{\infty}|B(y,r)|^{\frac{1}{q}}\int_{B(y,\,2^{i+1}r)\setminus B(y,\,2^{i}r)}\frac{f(x)}{|x-y|^{d-\gamma}}dx\\
&\lesssim\sum_{i=0}^{\infty}(2^i)^{-\frac{d}{q}}|B(y,r)|^{\frac{\gamma}{d}-(1-\frac{1}{q})}\|f\chi_{B(y,2^{i+1}r)}\|_1.
\end{aligned}$$

So, thanks to [15, Theorem 5.2], we get

$$|B(y,r)|^{\frac{1}{q}}\int_{\mathbb{R}^d\setminus B(y,r)}\frac{f(x)}{|x-y|^{d-\gamma}}dx\lesssim\sum_{i=0}^{\infty}(2^i)^{-\frac{d}{q}}\|(M_\gamma f)\chi_{B(y,2^{i+1}r)}\|_q.$$

Taking into account this inequality and (4.7), the L^p-norm of both sides with respect to y of the inequality leads to

$${}_r\|I_\gamma f\|_{q,p}\lesssim {}_r\|M_\gamma f\|_{q,p}+\sum_{i=0}^{\infty}(2^i)^{-\frac{d}{q}}\ {}_{2^{i+1}r}\|M_\gamma f\|_{q,p}.$$

We have, on the one hand,

$${}_r\|M_\gamma f\|_{q,p}\ =\ \frac{[r]_1^{d(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p})}[1/r]_1^{d(-\frac{1}{\lambda}+\frac{1}{q}+\frac{1}{p})}}{[r]_1^{d(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p})}[1/r]_1^{d(-\frac{1}{\lambda}+\frac{1}{q}+\frac{1}{p})}}\ {}_r\|M_\gamma f\|_{q,p}.$$

Hence,

$${}_r\|M_\gamma f\|_{q,p}\le[r]_1^{d(-\frac{1}{\alpha}+\frac{1}{q}+\frac{1}{p})}[1/r]_1^{d(\frac{1}{\lambda}-\frac{1}{q}-\frac{1}{p})}\|M_\gamma f\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}\,. \tag{4.8}$$

On the other hand,

$$\begin{aligned}
&\sum_{i=0}^{\infty}(2^i)^{-\frac{d}{q}}\ {}_{2^{i+1}r}\|M_\gamma f\|_{q,p}\\
&=\sum_{i=0}^{\infty}(2^i)^{-\frac{d}{q}}\frac{[2^{i+1}r]_1^{d(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p})}[1/2^{i+1}r]_1^{d(-\frac{1}{\lambda}+\frac{1}{q}+\frac{1}{p})}}{[2^{i+1}r]_1^{d(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p})}[1/2^{i+1}r]_1^{d(-\frac{1}{\lambda}+\frac{1}{q}+\frac{1}{p})}}\ {}_{2^{i+1}r}\|M_\gamma f\|_{q,p}\\
&\le\|M_\gamma f\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}\sum_{i=0}^{\infty}(2^i)^{-\frac{d}{q}}[2^{i+1}r]_1^{d(-\frac{1}{\alpha}+\frac{1}{q}+\frac{1}{p})}[1/2^{i+1}r]_1^{d(\frac{1}{\lambda}-\frac{1}{q}-\frac{1}{p})}\\
&\lesssim[r]_1^{d(-\frac{1}{\alpha}+\frac{1}{q}+\frac{1}{p})}[1/r]_1^{d(\frac{1}{\lambda}-\frac{1}{q}-\frac{1}{p})}\|M_\gamma f\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}\\
&\quad\times\sum_{i=0}^{\infty}(2^i)^{d(-\frac{1}{\alpha}-\frac{1}{\lambda}+\frac{1}{q}+\frac{2}{p})}.
\end{aligned}$$

Since $-\frac{1}{\alpha}-\frac{1}{\lambda}+\frac{1}{q}+\frac{2}{p}<0$, then $\sum_{i=0}^{\infty}(2^i)^{d(-\frac{1}{\alpha}-\frac{1}{\lambda}+\frac{1}{q}+\frac{2}{p})}<\infty$.

Hence,

$$\sum_{i=0}^{\infty}(2^i)^{-\frac{d}{q}}\ {}_{2^{i+1}r}\left\|M_\gamma f\right\|_{q,p} \lesssim [r]_1^{d(-\frac{1}{\alpha}+\frac{1}{q}+\frac{1}{p})}[1/r]_1^{d(\frac{1}{\lambda}-\frac{1}{q}-\frac{1}{p})}\left\|M_\gamma f\right\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}. \quad (4.9)$$

From (4.8) and (4.9), we deduce that

$${}_r\left\|I_\gamma f\right\|_{q,p} \lesssim [r]_1^{d(-\frac{1}{\alpha}+\frac{1}{q}+\frac{1}{p})}[1/r]_1^{d(\frac{1}{\lambda}-\frac{1}{q}-\frac{1}{p})}\left\|M_\gamma f\right\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}.$$

This implies that

$$[r]_1^{d(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p})}[1/r]_1^{d(-\frac{1}{\lambda}+\frac{1}{q}+\frac{1}{p})}\ {}_r\left\|I_\gamma f\right\|_{q,p} \lesssim \left\|M_\gamma f\right\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}.$$

We end the proof by taking the supremum over all $r > 0$ in the left-hand side of the previous relation. □

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