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PERSISTENT SHADOWING IN FUNCTIONAL DYNAMICS AND ITS MEASURE-THEORETIC IMPLICATIONS

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Abstract. We introduce the persistent shadowing property for continuous group actions $\varphi : G \times X \to X$, establishing its measure-theoretic foundations through compatible Borel probability measures $\mu \in \mathcal{M}_{PSh}(X, \varphi)$. We prove $\mathcal{M}_{PSh}(X, \varphi)$ is an $F_{\sigma\delta}$-set in $\mathcal{M}(X)$ and show that φ has persistent shadowing on $\mathrm{supp}(\mu)$ when $\mu \in \mathcal{M}_{PSh}(X, \varphi)$. For a compact X without isolated points, we characterize when all nonatomic measures imply persistent shadowing. For functional envelopes $\tilde{\varphi} : G \times S(X) \to S(X)$, we demonstrate that topological stability and expansivity are preserved, yielding topological stability for expansive envelopes with weak shadowing. We establish that shadowing properties of $\tilde{\varphi}$ descend to φ, with converses holding when φ is expansive.

1. Introduction

A continuous action is a triple (X, G, φ), where X is a metric space with metric d, G is a finitely generated group with the discrete topology, and φ is a continuous action of G on X. We denote this by $\varphi : G \times X \to X$ and let $Act(G; X)$ represent the set of all continuous actions of G on X.

Given a finite generating set S of G, let $S(X)$ be the space of all continuous self-maps of compact metric space X with the metric $d_0(f, g) = \sup_{x \in X} d(f(x), g(x))$ for all $f, g \in S(X)$. The metric d_S on $Act(G; X)$ is defined by $d_S(\varphi, \psi) = \max_{s \in S} d_0(\varphi_s, \psi_s)$ for $\varphi, \psi \in Act(G; X)$.

For $\delta > 0$, a map $f : G \to X$ is called a δ-pseudo orbit for φ (with respect to S) if $d(f(sg), \varphi(s, f(g))) < \delta$ for all $s \in S$ and $g \in G$. A δ-pseudo orbit $f : G \to X$ is (ϵ, φ)-shadowed by $p \in X$ if $d(f(g), \varphi(g, p)) < \epsilon$ for all $g \in G$. The action

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$\varphi : G \times X \to X$ has the shadowing property (with respect to S) if for every $\epsilon > 0$, there exists $\delta > 0$ such that every δ-pseudo orbit can be (ϵ, φ)-shadowed by some point $p \in X$. This concept was introduced for finitely generated group actions by Osipov and Tikhomirov [5].

The shadowing property depends on the group structure G. For instance, if G is a finitely generated nilpotent group and the action of one element is hyperbolic, this means that $\varphi_g : X \to X$ does have shadowing and it is expansive, then the group action has the shadowing property, which does not necessarily generalize to solvable groups.

The notion of topological stability for finitely generated group actions on compact metric spaces was introduced by Chung and Lee [3]. They provided a group action version of Walter's stability theorem. A continuous action $\varphi : G \times X \to X$ is topologically stable (with respect to S) if for every $\epsilon > 0$, there exists $\delta > 0$ such that for every continuous action $\psi : G \times X \to X$ with $d_S(\varphi, \psi) < \delta$, there is a continuous map $f : X \to X$ satisfying $d_{C^0}(f, id) < \epsilon$ and $\varphi_g f = f \psi_g$ for all $g \in G$.

When f is surjective, φ is called s-topologically stable. If φ is topologically stable, then for every $\epsilon > 0$, there exists $\delta > 0$ such that for every $x \in X$ and every continuous action $\psi : G \times X \to X$ with $d_S(\varphi, \psi) < \delta$, we have $d(\varphi(g, f(x)), \psi(g, x)) < \epsilon$ for all $g \in G$. This property is known as the weak shadowing property (Lemma 3.8). For s-topologically stable actions, if $f(y) = x$, then $d(\varphi(g, x), \psi(g, y)) < \epsilon$ for all $g \in G$, which defines the β-persistent property.

While s-topological stability implies β-persistence (Lemma 3.8), topological stability does not necessarily imply β-persistence. For example, Sakai and Kobayashi [7] observed that the full shift on two symbols is not β-persistent despite being topologically stable.

A continuous action $\varphi : G \times X \to X$ has the shadowing property if and only if for every $\epsilon > 0$, there exists $\delta > 0$ such that every δ-pseudo orbit $f : G \to X$ for a continuous action $\psi : G \times X \to X$ with $d_S(\varphi, \psi) < \delta$ can be ϵ-shadowed by a φ-orbit. This motivates the definition of a stronger version called the persistent shadowing property (Definition 2.1).

The persistent shadowing property is stronger than both the shadowing property and β-persistence. However, for equicontinuous actions, these properties are equivalent. Consequently, every equicontinuous action on the Cantor space X has the persistent shadowing property. This property is independent of the choice of symmetric finitely generating sets and metrics when X is compact. Example 2.2 demonstrates that compactness is essential.

For a subgroup H of G, $\varphi : H \times X \to X$ may have the persistent shadowing property while $\varphi : G \times X \to X$ does not. However, Proposition 2.7 shows that if H is a syndetic subgroup of G, the situation differs. We also study the relationship between the persistent shadowing properties of $\varphi : G \times X \to X$ and $\varphi_g : X \to X$. While there exist systems where $\varphi : G \times X \to X$ has the persistent shadowing property but $\varphi_g : X \to X$ does not, Proposition 2.9 shows that for the free group $F_2 = \langle a, b \rangle$, if $\varphi : F_2 \times X \to X$ has the persistent shadowing property, then $\varphi_{a^{-1}b} : X \to X$ also has this property. Similar results hold for the shadowing, weak shadowing, and β-persistent properties (Remarks 2.8 and 2.10).

In [1], we introduced the notion of measure compatibility with respect to the weak shadowing property. Here, we extend this to the persistent shadowing property, denoting the set of compatible measures by $\mathcal{M}_{PSh}(X,\varphi)$ (Subsection 2.3). We show that $\mathcal{M}_{PSh}(X,\varphi)$ is an $F_{\sigma\delta}$ subset of $\mathcal{M}(X)$, and for $\mu \in \mathcal{M}_{PSh}(X,\varphi)$ and a homeomorphism $f : X \to Y$, we have $f_*(\mu) \in \mathcal{M}_{PSh}(Y, f \circ \varphi \circ f^{-1})$ (Proposition 2.14).

Proposition 2.15 shows that if a measure μ is compatible with the persistent shadowing property for φ, then φ has this property on $\operatorname{supp}(\mu)$. This implies that if every nonatomic probability measure is compatible with the persistent shadowing property for φ, then φ has the persistent shadowing property. We also introduce measure compatibility for the shadowing, weak shadowing, and β-persistent properties, denoted by $\mathcal{M}_{Sh}(X,\varphi)$, $\mathcal{M}_{\alpha}(X,\varphi)$, and $\mathcal{M}_{\beta}(X,\varphi)$, respectively. Results similar to Proposition 2.15 hold for these cases (Remark 2.17).

The functional envelope of a continuous group action $\varphi : G \times X \to X$ is the group action $\tilde{\varphi} : G \times S(X) \to S(X)$ defined by $\tilde{\varphi}(g,h) := \varphi_g \circ h$ for all $h \in S(X)$. In Section 3, we define topological stability within the class of all functional envelopes induced by continuous actions (Definition 3.10) and investigate topological stability, shadowing, weak shadowing, β-persistence, and persistent shadowing properties for these functional envelopes.

We show that topological stability is invariant under functional envelopes (Corollary 3.12), meaning $\tilde{\varphi} : G \times S(X) \to S(X)$ is topologically stable if and only if $\varphi : G \times X \to X$ is topologically stable. Additionally, expansivity is invariant under functional envelopes (Proposition 3.4). Since every uniformly expansive action with the weak shadowing property is topologically stable, every expansive functional envelope with the shadowing property is topologically stable within the class of all functional envelopes.

Proposition 3.6 shows that if the functional envelope $\tilde{\varphi} : G \times X \to X$ induced by φ has the shadowing, weak shadowing, β-persistent, or persistent shadowing properties, then $\varphi : G \times X \to X$ inherits the corresponding properties. Proposition 3.13 establishes that the converse holds when φ is expansive.

2. Persistent shadowing property

In this section, we first extend the notion of persistent shadowing property from [4] to group actions. In Subsection 2.1, we prove that for a continuous action $\varphi : G \times X \to X$ on a compact metric space X, if $\varphi : H \times X \to X$ has the persistent shadowing property for some finite index subgroup H of G, then φ itself has the persistent shadowing property (Proposition 2.7).

Subsection 2.2 establishes that when a continuous action $\varphi : F_2 \times X \to X$ of the free group $F_2 = \langle a, b \rangle$ has the persistent shadowing property, the induced map $\varphi_{a^{-1}b} : X \to X$ also possesses this property.

In Subsection 2.3, we investigate the persistent shadowing property from a measure-theoretic perspective, demonstrating that the set of measures compatible with this property forms an $F_{\sigma\delta}$-subset of $\mathcal{M}(X)$.

Definition 2.1. A continuous action $\varphi : G \times X \to X$ has the persistent shadowing property (with respect to S) if for every $\epsilon > 0$, there exists $\delta > 0$ such that every

δ-pseudo orbit $f : G \to X$ for any continuous action $\psi : G \times X \to X$ with $d_S(\varphi, \psi) < \delta$ can be (ψ, ϵ)-shadowed by some point $p \in X$.

The following example demonstrates that persistent shadowing property dependent of the choice of metric on X.

Example 2.2. Consider the stereographic projection homeomorphism $T : \mathbb{R} \to S^1 \setminus \{(0,1)\}$ defined by

$$T(t) = \left(\frac{2t}{1+t^2}, \frac{t^2-1}{t^2+1} \right), \quad \text{for all } t \in \mathbb{R}. \tag{2.1}$$

This mapping establishes a bijection between the real line and the unit circle minus its north point $(0,1)$. Let $X = T(\mathbb{Z})$ be the image of the integer points under this mapping, that is,

$$X = \left\{ \left(\frac{2n}{1+n^2}, \frac{n^2-1}{n^2+1} \right) : n \in \mathbb{Z} \right\}.$$

We equip X with two distinct metrics:

- d': The metric induced by the standard Riemannian metric on $S^1 \subset \mathbb{R}^2$ (arc-length distance)
- d: The discrete metric, defined by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y, \end{cases}$$

Since $\mathbb{Z}$ is discrete in $\mathbb{R}$ and T is a homeomorphism, $X = T(\mathbb{Z})$ is discrete in $S^1 \setminus \{(0,1)\}$ with the subspace topology. Therefore, there exists $\delta > 0$ such that for any two distinct points $x, y \in X$, $d'(x,y) \geq \delta$.

- Every open ball in the d-topology is open in the d'-topology: For $0 < r < 1$, $B_d(x,r) = \{x\}$, which is open in d' since $\{x\}$ is an isolated point.
- Every open ball in the d'-topology contains a d-open ball: For any $\epsilon > 0$, take $r = \min(\epsilon, \delta/2)$. Then $B_{d'}(x,r) = \{x\} = B_d(x, 1/2)$.

Thus, d and d' induce the same topology on X.

The shift map $f : X \to X$ defined by $f(a_i) = a_{i+1}$ does have the following properties:

- With respect to d, f has the persistent shadowing property since d is discrete.
- With respect to d', f fails to have the persistent shadowing property:
  (1) For any $\epsilon > 0$, choose $\delta > 0$ as in the definition.
  (2) Select $k \in \mathbb{N}$ with $d'(a_k, a_{-k}) < \delta/2$.
  (3) Define $g : X \to X$ by

$$g(a_i) = \begin{cases} a_{i+1} & \text{for } i \in \{-k, \dots, k-1\}, \\ a_{-k} & \text{for } i = k, \\ a_i & \text{otherwise.} \end{cases}$$

  (4) The sequence $\{g^n(y)\}_{n \in \mathbb{Z}}$ forms a δ-pseudo orbit for f

(5) By shadowing, there exists $z \in X$ with $d'(f^n(z), g^n(y)) < \epsilon$ for all $n \in \mathbb{Z}$

(6) However, since $\{f^n(z)\}_{n\in\mathbb{Z}} = X$, we can find n with $d'(f^n(z), g^n(y)) \geq \epsilon$, yielding a contradiction

This shows that f has the persistent shadowing property with respect to d but not with respect to d'.

Definition 2.3. A continuous action $\varphi : G \times X \to X$ satisfies

- The *weak shadowing property* (with respect to S) if for every $\epsilon > 0$, there exists $\delta > 0$ such that for every continuous action $\psi : G \times X \to X$ with $d_S(\varphi, \psi) < \delta$ and every $p \in X$, there exists $q \in X$ satisfying $d(\varphi(g, q), \psi(g, p)) < \epsilon$ for all $g \in G$.
- The *β-persistent property* (with respect to S) if for every $\epsilon > 0$, there exists $\delta > 0$ such that for every continuous action $\psi : G \times X \to X$ with $d_S(\varphi, \psi) < \delta$ and every $p \in X$, there exists $q \in X$ satisfying $d(\varphi(g, p), \psi(g, q)) < \epsilon$ for all $g \in G$.

Proposition 2.4. *For a continuous action $\varphi : G \times X \to X$, the following properties are equivalent:*

(1) *φ has the persistent shadowing property.*

(2) *φ has both the shadowing property and the β-persistent property.*

Proof. The forward implication is straightforward to verify. The converse follows directly from the definitions. □

Corollary 2.5. *For an equicontinuous action $\varphi : G \times X \to X$, the following properties are equivalent:*

(1) *φ has the persistent shadowing property.*

(2) *φ has the shadowing property.*

Proof. This follows from Proposition 2.4 and the observation that any equicontinuous action with the weak shadowing property automatically satisfies the β-persistent property. □

It is known that shadowing and β-persistent are independent of both the choice of symmetric finite generating set S and choice of metric on X when X is compact (see [1, 5]). This and Proposition 2.4 imply that the persistent shadowing property is independent of both followings:

- The choice of symmetric finite generating set S.
- The choice of metric on X when X is compact.

2.1. **Action of syndetic subgroups.** Consider the Baumslag–Solitar group $BS(1, n) = \langle a, b \mid ba = a^n b\rangle$ with the continuous action $\varphi : BS(1, n) \times \mathbb{R}^2 \to \mathbb{R}^2$ generated by

$$f_a(x) = Ax \quad \text{and} \quad f_b(x) = Bx \quad \text{where} \quad A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} \lambda & 0 \\ 0 & n\lambda \end{pmatrix}. \tag{2.2}$$

According to [4], for a homeomorphism $f : X \to X$, we have the following properties:

- The persistent shadowing property is equivalent to having both shadowing and β-persistent properties.
- For parameters $n > 1$ and $1 < \lambda < n$, the map f_b has persistent shadowing.

Taking $H = \langle b \rangle \leq BS(1, n)$, we observe that

- $\varphi|_H : H \times \mathbb{R}^2 \to \mathbb{R}^2$ has persistent shadowing.
- By [5, Theorem 4.4(1)], $\varphi : BS(1, n) \times \mathbb{R}^2 \to \mathbb{R}^2$ lacks shadowing.
- Consequently, φ fails to have persistent shadowing on the full group.

For syndetic subgroups (where $H \subseteq G$ satisfies $G = \Lambda H$ for some finite $\Lambda \subseteq G$), the situation differs. Let $||g||_S$ denote the word length with respect to the generating set S. For continuous actions on compact metric spaces, we have the following result.

Lemma 2.6. *Let S be a finite generating set for G and let $\varphi : G \times X \to X$ be a continuous action on a compact metric space (X, d). For any $\epsilon > 0$ and $N \in \mathbb{N}$, there exists $\delta > 0$ such that for any δ-pseudo orbit $f : G \to X$ and any $t \in G$ with $||t||_S < N$, we have*

$$d(f(th), \varphi(t, f(h))) < \epsilon \quad \text{for all } h \in G. \tag{2.3}$$

Proposition 2.7. *Let H be a finite index subgroup of G. If $\varphi : H \times X \to X$ has persistent shadowing, then $\varphi : G \times X \to X$ has persistent shadowing.*

Proof. Since H has finite index, $G = \bigcup_{i=1}^n g_i H$ for some finite set $\{g_i\}$. Extend a symmetric generating set A of H to a symmetric generating set S of G.

Given $\epsilon > 0$, set $N = \max\{||t||_S : t \in \{g_i\}\}$. By continuity and Lemma 2.6, choose $0 < \delta < \epsilon/4$ such that for all $||t||_S < N$,

$$d(x, y) < \delta \Rightarrow d(\varphi(t, x), \psi(t, y)) < \epsilon/4. \tag{2.4}$$

Take $0 < \eta < \delta/2$ from the persistent shadowing property of $\varphi|_H$. For any η-pseudo orbit $F : G \to X$ of ψ with $d_S(\varphi, \psi) < \eta$:

- $F|_H$ is η-pseudo orbit for $\psi|_H$.
- By persistent shadowing, there exists $p \in X$ with $d(F(h), \psi(h, p)) < \delta$ for all $h \in H$.
- For $g = th \in G$ ($t \in \{g_i\}$, $h \in H$),

$$\begin{aligned} d(F(th), \psi(th, p)) &\leq d(F(th), \varphi(t, F(h))) + d(\varphi(t, F(h)), \psi(th, p)) \\ &< \epsilon/4 + \epsilon/4 = \epsilon/2. \end{aligned}$$

□

Remark 2.8. The conclusion of Proposition 2.7 extends to

(1) Shadowing property,
(2) Weak shadowing property,
(3) β-persistent property.

2.2. **Action of free groups.** The proof of Theorem 4.9 in [5] claims that for a finitely generated free group G with standard generating set $S = \{s_1^{\pm 1}, \dots, s_n^{\pm 1}\}$, any uniformly continuous action $\varphi : G \times X \to X$ on a nondiscrete metric space X with the shadowing property satisfies that φ_g has the shadowing property for all $g \in G$. However, we identify a potential issue in their argument.

Consider the normal form $g = s_r \cdots s_1$ and fix $\epsilon > 0$. Let $\delta > 0$ correspond to ϵ via the shadowing property. By uniform continuity, there exists $\eta > 0$ such that for any composition f of the form:

$$f = \varphi_{s_j} \circ \cdots \circ \varphi_{s_1} \quad \text{or} \quad f = \varphi_{s_j}^{-1} \circ \cdots \circ \varphi_{s_r}^{-1}.$$

We have the implication

$$d(x,y) < \eta \Rightarrow d(f(x), f(y)) < \delta \quad \text{for all } x, y \in X.$$

For an η-pseudo orbit $\{x_k\}_{k\in\mathbb{Z}}$, the proof constructs:

$$F(t) = \begin{cases} \varphi(v, z_{rk+j}) & \text{if } t = s_j \cdots s_1 g^k v, \\ \varphi(v, z_{-rk-j}) & \text{if } t = s_{r-j+1}^{-1} \cdots s_r^{-1} g^{-k} v, \end{cases}$$

where $\{z_k\}$ is defined recursively. However, for $t = s_{r-1} \cdots s_2 s_1^{n+1}$, we obtain

$$F(t) = \varphi(s_1^n, z_{r-1}),$$
$$F(s_r t) = \varphi(s_1^n, x_1).$$

This leads to the requirement $d(\varphi(s_1^n, x_1), \varphi(s_1^n g, x_0)) < \eta$ for all $n \in \mathbb{N}$, which generally cannot hold.

Proposition 2.9. *Let $\varphi : F_2 \times X \to X$ be a continuous action of the free group $F_2 = \langle a, b\rangle$ on a compact metric space (X, d). If φ has the persistent shadowing property, then $\varphi_{a^{-1}b} : X \to X$ has the persistent shadowing property.*

Proof. Since $\varphi_b \neq \mathrm{id}_X$, there exists $\epsilon > 0$ such that $d(\varphi_b, \mathrm{id}_X) > \epsilon$. Choose $\epsilon_0 > 0$ corresponding to ϵ via the persistent shadowing property of φ. There exists $\delta > 0$ such that for any continuous action $\psi : F_2 \times X \to X$ with $d_S(\varphi, \psi) < \delta$, we have

$$d(x,y) < \delta \Rightarrow d(\psi(g,x), \psi(g,y)) < \epsilon_0 \quad \text{for } |g|_S \leq 2.$$

Given a δ-pseudo orbit $\{x_n\}_{n\in\mathbb{Z}}$ of a homeomorphism $f : X \to X$ with $d(f, \varphi_{a^{-1}}\varphi_b) < \delta$, define $\psi : F_2 \times X \to X$ by

$$\psi_a = \varphi_a,$$
$$\psi_b = \varphi_a \circ f.$$

Then ψ is ϵ_0-close to φ. Construct $K : F_2 \to X$ via $K(t) = \psi(v, x_k)$, where $t = v(a^{-1}b)^k$ with v minimal.

This K is an ϵ_0-pseudo orbit for ψ. By persistent shadowing, there exists $y \in X$ such that

$$d(K(g), \psi(g,y)) < \epsilon \quad \text{for all } g \in F_2.$$

In particular, for $g = (a^{-1}b)^k$, we have $d(f^k(y), x_k) < \epsilon$ for all $k \in \mathbb{Z}$. □

Remark 2.10. The conclusion of Proposition 2.9 extends to

(1) The shadowing property,
(2) The weak shadowing property,
(3) The β-persistent property.

2.3. **Persistent shadowing property and related measures.** For a continuous action $\varphi : G \times X \to X$, $\epsilon > 0$, $\delta > 0$, and a generating set S, we denote by $PSh_{\varphi,S}(\epsilon, \delta)$ the set of all $x \in X$ such that every δ-pseudo orbit $f : G \to X$ of a continuous action ψ with $d_S(\varphi, \psi) < \delta$ and $f(e) = x$ can be (ϵ, ψ)-shadowed by a point in X.

The following properties are clear:

(1) Let S, S' be generating sets for G and let $\epsilon > 0$. Then for every $\delta > 0$, there exists $\eta > 0$ such that $PSh_{\varphi,S}(\epsilon, \eta) \subseteq PSh_{\varphi,S'}(\epsilon, \delta)$.
(2) A continuous action $\varphi : G \times X \to X$ has the persistent shadowing property with respect to the generating set S if and only if for every $\epsilon > 0$, there exists $\delta > 0$ such that $PSh_{\varphi,S}(\epsilon, \delta) = X$.

A σ-algebra on a space X is a collection of subsets of X that is closed under complements and countable unions. The smallest σ-algebra containing all open subsets of X is called the Borel σ-algebra and is denoted by $\mathcal{B}(X)$. Every $A \in \mathcal{B}(X)$ is called a Borel set.

Proposition 2.11. *Let $\varphi : G \times X \to X$ be a continuous action of a finitely generated group G on a compact metric space X. Then for every $\epsilon, \delta > 0$:*

(1) *$PSh_{\varphi,S}(\epsilon, \delta)$ is a closed subset of X;*
(2) *$PSh_{\varphi,S}(\epsilon, \delta)$ is a Borel set;*
(3) *If $PSh_{\varphi,S}(\epsilon^+) = \bigcap_{n\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} PSh_{\varphi,S}(\epsilon + \frac{1}{n}, \frac{1}{k})$, then $PSh_{\varphi,S}(\epsilon^+)$ is a Borel set.*

Proof. Since every closed set is a Borel set and $\mathcal{B}(X)$ is closed under countable intersections and unions, it is sufficient to show that $PSh_{\varphi,S}(\epsilon, \delta)$ is closed in X.

Let $d_S(\varphi, \psi) < \delta$ and let $\{z_n\}_{n\in\mathbb{N}} \subseteq PSh_{\varphi,S}(\epsilon, \delta)$ such that $z_n \to z$. Let $F : G \to X$ be a δ-pseudo orbit for ψ with $F(e) = z$. Define $F_n : G \to X$ by $F_n(e) = z_n$ and $F_n(g) = F(g)$ for $g \neq e$. For sufficiently large n, F_n is also a δ-pseudo orbit. Since $z_n \in PSh_{\varphi,S}(\epsilon, \delta)$, there exists $y_n \in X$ such that $d(F_n(g), \psi(g, y_n)) < \epsilon$ for all $g \in G$, which implies $d(F(g), \psi(g, y_n)) < \epsilon$ for all $g \neq e$.

By compactness, there is a convergent subsequence $y_{n_j} \to y$.
Then $d(F(g), \psi(g, y)) \leq \epsilon$ for all $g \in G$. Hence, $z \in PSh_{\varphi,S}(\epsilon, \delta)$. □

A measure μ is a nonnegative function defined on a σ-algebra such that $\mu(\emptyset) = 0$ and μ is countably additive: $\mu\left(\bigcup A_j\right) = \sum \mu(A_j)$ for disjoint sets A_j. A measure on $\mathcal{B}(X)$ is called a Borel measure. Thus, by Proposition 2.11, we have the following result.

Corollary 2.12. *Let X be a metric space and let μ be a measure on $\mathcal{B}(X)$. Then for every $\epsilon, \delta > 0$, both $PSh_{\varphi,S}(\epsilon^+)$ and $PSh_{\varphi,S}(\epsilon, \delta)$ are μ-measurable sets.*

A Borel probability measure is a measure μ on $\mathcal{B}(X)$ such that $\mu(X) = 1$. We denote the set of all such measures by $\mathcal{M}(X)$, which is convex and compact metrizable under the weak* topology: $\mu_n \to \mu$ if and only if $\int f\, d\mu_n \to \int f\, d\mu$ for all continuous $f : X \to \mathbb{R}$.

Definition 2.13. A measure $\mu \in \mathcal{M}(X)$ is said to be compatible with the persistent shadowing property for the continuous action $\varphi : G \times X \to X$ with generating set S, denoted $\mu \in \mathcal{M}_{PSh}(X,\varphi)$, if for every $\epsilon > 0$, there exists $\delta > 0$ such that for every measurable set A with $\mu(A) > 0$, we have

$$A \cap PSh_{\varphi,S}(\epsilon,\delta) \neq \emptyset.$$

Let $\mu \in \mathcal{M}_{PSh}(X,\varphi)$ and let $h : (X,d) \to (Y,\rho)$ be a homeomorphism. We show that $h_*(\mu) \in \mathcal{M}_{PSh}(Y, h\circ\varphi\circ h^{-1})$.

For $\epsilon > 0$, there exists $\epsilon_0 > 0$ such that

$$d(a,b) < \epsilon_0 \Rightarrow \rho(h(a),h(b)) < \epsilon.$$

For this ϵ_0, there exists $\epsilon_1 > 0$ such that if $\mu(h^{-1}(B)) > 0$, then

$$h^{-1}(B) \cap PSh_{\varphi}(\epsilon_0,\epsilon_1) \neq \emptyset.$$

There exists $\delta > 0$ such that

$$\rho(c,d) < \delta \Rightarrow d(h^{-1}(c), h^{-1}(d)) < \epsilon_1.$$

Fix $x \in h^{-1}(B) \cap PSh_{\varphi}(\epsilon_0,\epsilon_1)$. Let $F' : G \to Y$ be a δ-pseudo orbit of a continuous action $\psi' : G \times Y \to Y$ with $\rho_S(h\circ\varphi\circ h^{-1}, \psi') < \delta$ and $F'(e) = h(x)$. Define $F(g) = h^{-1}(F'(g))$. Then F is an ϵ_1-pseudo orbit for $h^{-1}\circ\psi'\circ h$ and $F(e) = x$.

Since $x \in PSh_{\varphi}(\epsilon_0,\epsilon_1)$, there exists $p \in X$ such that $d(F(g), h^{-1}\circ\psi'_g\circ h(p)) < \epsilon_0$, and hence $\rho(F'(g), \psi'_g(h(p))) < \epsilon$ for all $g \in G$. Therefore, $h(x) \in B \cap PSh_{h\circ\varphi\circ h^{-1}}(\epsilon,\delta)$.

- If $h : (X,d) \to (Y,\rho)$ is a homeomorphism and $\mu \in \mathcal{M}_{PSh}(X,\varphi)$, then $h_*(\mu) \in \mathcal{M}_{PSh}(Y, h\circ\varphi\circ h^{-1})$.
- If G is an abelian group and $\mu \in \mathcal{M}_{PSh}(X,\varphi)$, then $(\varphi_g)_*(\mu) \in \mathcal{M}_{PSh}(X,\varphi)$ for all $g \in G$.

It is known that every continuous action of a countable abelian group on a compact metric space admits a φ-invariant measure. Moreover, $\mathcal{M}_{PSh}(X,\varphi)$ is a convex subset of $\mathcal{M}(X)$. Hence, by [1, Proposition 2.12], we have the following property:

- If G is abelian, then $\overline{\mathcal{M}_{PSh}(X,\varphi)}$ contains a φ-invariant measure.

Define:

$$\mathcal{C}_{PSh(\varphi)}(\epsilon,\delta) = \{\mu \in \mathcal{M}(X) : \mu(PSh_{\varphi}(\epsilon,\delta)) = 1\}.$$

A subset of a topological space is F_σ if it is a countable union of closed sets, and G_δ if it is a countable intersection of open sets. An $F_{\sigma\delta}$ set is a countable intersection of F_σ sets.

Proposition 2.14. *Let $\varphi : G \times X \to X$ be a continuous action. Then*

1. *$\mathcal{C}_{PSh(\varphi)}(\epsilon,\delta)$ is a convex and closed subset of $\mathcal{M}(X)$ for every $\epsilon,\delta > 0$;*
2. *$\mathcal{M}_{PSh}(X,\varphi) = \bigcap_{n\in\mathbb{N}} \bigcup_{m\in\mathbb{N}} \bigcap_{l\in\mathbb{N}} \mathcal{C}_{PSh(\varphi)}(n^{-1}+l^{-1}, m^{-1})$;*
3. *The subset $\mathcal{M}_{PSh}(X,\varphi)$ is an $F_{\sigma\delta}$ subset of $\mathcal{M}(X)$.*

Proof. (1) Suppose $\mu_n \in \mathcal{C}_{PSh(\varphi)}(\epsilon,\delta)$ and $\mu_n \to \mu$ in the weak* topology. Since $PSh_{\varphi}(\epsilon,\delta)$ is closed, we have

$$\mu(PSh_{\varphi}(\epsilon,\delta)) \geq \limsup_{n\to\infty} \mu_n(PSh_{\varphi}(\epsilon,\delta)) = 1.$$

Hence, $\mu \in \mathcal{C}_{PSh(\varphi)}(\epsilon,\delta)$. Convexity is clear.

(2) Let $\mu \in \mathcal{M}_{PSh}(X,\varphi)$, and fix n. There exists $\delta > 0$ such that if $\mu(A) > 0$, then $A \cap PSh_\varphi(\frac{1}{n},\delta) \neq \emptyset$. Choose m such that $1/m < \delta$. Then $\mu \in \bigcup_m \bigcap_l \mathcal{C}_{PSh(\varphi)}(\frac{1}{n} + 1/l, 1/m)$. Taking the intersection over n, we get $\mu \in \mathcal{M}_{PSh}(X,\varphi)$.

Conversely, if μ lies in the right-hand side, then for each n there exists m such that $\mu \in \bigcap_l \mathcal{C}_{PSh(\varphi)}(\frac{1}{n}+1/l, 1/m)$. Hence $\mu(PSh_\varphi(\frac{1}{n}+1/l, 1/m)) = 1$. This implies $\mu \in \mathcal{M}_{PSh}(X,\varphi)$.

(3) Follows directly from (2), since the countable union of closed sets is F_σ and the countable intersection of F_σ is $F_{\sigma\delta}$.

□

The **spectrum** or **support** of a measure $\mu \in \mathcal{M}(X)$ is a closed set C_μ satisfying:

- $\mu(C_\mu) = 1$, and
- if A is any closed set with $\mu(A) = 1$, then $C_\mu \subseteq A$.

It is known that every measure $\mu \in \mathcal{M}(X)$ in a separable metric space has a unique support C_μ. Moreover, C_μ is the set of all points $x \in X$ such that $\mu(U) > 0$ for every open set U containing x. In this paper, the support of μ is denoted by $\mathrm{supp}(\mu)$.

Proposition 2.15. *Let $\varphi : G \times X \to X$ be a continuous action on a compact metric space (X, d) and let $\mu \in \mathcal{M}_{\mathrm{PSh}}(\varphi)$. Then φ has the persistent shadowing property on* $\mathrm{supp}(\mu)$.

Proof. Let $\epsilon > 0$ be given. Choose $0 < \delta < \frac{\epsilon}{2}$ such that

$$\mu(A) > 0 \Rightarrow A \cap \mathrm{PSh}_\varphi\left(\frac{\epsilon}{2}, \delta\right) \neq \emptyset.$$

For $\delta > 0$, there exists $0 < \eta < \frac{\delta}{2}$ such that for every continuous action $\psi : G \times X \to X$ with $d_S(\varphi, \psi) < \eta$, we have

$$d(a, b) < \eta \Rightarrow d(\psi_s(a), \psi_s(b)) < \frac{\delta}{2} \quad \text{for all } s \in S.$$

We claim that if $d_S(\varphi, \psi) < \eta$ and $F : G \to X$ is an η-pseudo orbit for $\psi : G \times X \to X$ with $F(e) = p \in \mathrm{supp}(\mu)$, then there exists $z \in X$ such that $d(F(g), \psi(g, z)) < \epsilon$ for all $g \in G$.

Since $p \in \mathrm{supp}(\mu)$, we have $\mu(B_\eta(p)) > 0$, which implies that $B_\eta(p) \cap \mathrm{PSh}_\varphi\left(\frac{\epsilon}{2}, \delta\right) \neq \emptyset$. Take $q \in B_\eta(p) \cap \mathrm{PSh}_\varphi\left(\frac{\epsilon}{2}, \delta\right)$, and define $f : G \to X$ by $f(e) = q$ and $f(g) = F(g)$ for all $g \neq e$. It is easy to see that $f : G \to X$ is a δ-pseudo orbit of ψ. Since $f(e) \in \mathrm{PSh}_\varphi(\epsilon, \delta)$, there exists $y \in X$ with $d(f(g), \psi(g, y)) < \frac{\epsilon}{2}$ for all $g \in G$. This implies that $d(F(g), \psi(g, y)) < \epsilon$ for all $g \in G$. □

It is known that the set of Borel probability measures of a compact metric space X with support equal to X is a dense G_δ subset of $\mathcal{M}(X)$; see [2, Lemma 3.6]. Also, if X has no isolated points, then the nonatomic Borel probability measures form a dense G_δ subset of $\mathcal{M}(X)$; see [6, Corollary 8.2].

Since the intersection of two G_δ sets is a G_δ set, if X is a compact space without isolated points, then the set of nonatomic Borel probability measures with support equal to X is dense in $\mathcal{M}(X)$. By Proposition 2.15, we have the following result.

Corollary 2.16. *Let $\varphi : G\times X \to X$ be a continuous action on a compact metric space X without isolated points. If every nonatomic Borel probability measure μ is compatible with the persistent shadowing property for $\varphi : G \times X \to X$, then φ has the persistent shadowing property.*

For continuous actions $\varphi, \psi : G \times X \to X$ and $x \in X$, we denote

$$\Gamma_\epsilon^{\varphi,\psi}(x) = \bigcap_{g\in G} \varphi\left(g^{-1}, B[\psi(g,x),\epsilon]\right) = \{y \in X : d(\varphi(g,y),\psi(g,x)) \leq \epsilon \text{ for every } g \in G\}$$

and

$$B(\epsilon,\varphi,\psi) = \{x \in X : \Gamma_\epsilon^{\varphi,\psi}(x) \neq \emptyset\}.$$

It is easy to see that $B(\epsilon,\varphi,\psi)$ is a compact subset of X.

(1) A measure $\mu \in \mathcal{M}(X)$ is **compatible with the shadowing property** for a continuous action $\varphi : G \times X \to X$, denoted $\mu \in \mathcal{M}_{\mathrm{Sh}}(X,\varphi)$, if for every $\epsilon > 0$, there exists $\delta > 0$ such that if $\mu(A) > 0$, then
$$A \cap \mathrm{Sh}_\varphi(\epsilon,\delta) \neq \emptyset.$$

(2) (see [1]) A measure $\mu \in \mathcal{M}(X)$ is **compatible with the weak shadowing property** for a continuous action $\varphi : G \times X \to X$, denoted $\mu \in \mathcal{M}_\alpha(X,\varphi)$, if for every $\epsilon > 0$, there exists $\delta > 0$ such that if $\mu(A) > 0$, then
$$A \cap B(\epsilon,\varphi,\psi) \neq \emptyset,$$
for every continuous action $\psi : G \times X \to X$ with $d_S(\varphi,\psi) < \delta$.

(3) A measure $\mu \in \mathcal{M}(X)$ is **compatible with the β-persistent shadowing property** for a continuous action $\varphi : G \times X \to X$, denoted $\mu \in \mathcal{M}_\beta(X,\varphi)$, if for every $\epsilon > 0$, there exists $\delta > 0$ such that if $\mu(A) > 0$, then
$$A \cap B(\epsilon,\psi,\varphi) \neq \emptyset,$$
for every continuous action $\psi : G \times X \to X$ with $d_S(\varphi,\psi) < \delta$.

Remark 2.17. With a proof similar to that of Proposition 2.15, one can verify that $\mathcal{M}_{\mathrm{Sh}}(X,\varphi)$, $\mathcal{M}_\alpha(X,\varphi)$, and $\mathcal{M}_\beta(X,\varphi)$ are $F_{\sigma\delta}$ subsets of $\mathcal{M}(X)$.

Also, for a homeomorphism $h : (X,d) \to (Y,\rho)$, if $\mu \in \mathcal{M}_{\mathrm{Sh}}(X,\varphi)$, $\mu \in \mathcal{M}_\alpha(X,\varphi)$, or $\mu \in \mathcal{M}_\beta(X,\varphi)$, then

- $f_*(\mu) \in \mathcal{M}_{\mathrm{Sh}}(Y, h\circ\varphi\circ h^{-1})$,
- $f_*(\mu) \in \mathcal{M}_\alpha(Y, h\circ\varphi\circ h^{-1})$,
- $f_*(\mu) \in \mathcal{M}_\beta(Y, h\circ\varphi\circ h^{-1})$.

Using techniques similar to those in Proposition 2.15, we can show that if $\mu \in \mathcal{M}_{\mathrm{Sh}}(X,\varphi)$, $\mu \in \mathcal{M}_\alpha(X,\varphi)$, or $\mu \in \mathcal{M}_\beta(X,\varphi)$, then $\varphi : G \times X \to X$ has the shadowing property, weak shadowing property, and β-persistent shadowing property on $\mathrm{supp}(\mu)$, respectively.

3. Function envelope of a continuous group action

The function envelope of continuous group action $\varphi : G \times X \to X$ is the group action $\tilde{\varphi} : G \times S(X) \to S(X)$ defined by $\tilde{\varphi}(g, h) := \varphi_g \circ h$ for all $h \in S(X)$. In addition, distance between function envelops $\tilde{\varphi}(g, h) := \varphi_g \circ h$ and $\tilde{\psi}(g, h) := \psi_g \circ h$ for all $h \in S(X)$, defined by

$$D(\tilde{\varphi}, \tilde{\psi}) := \max_{s \in S}\{ \sup_{f \in S(X)} d_0(\varphi_s \circ f, \psi_s \circ f)\}. \tag{3.1}$$

Let $D(\tilde{\psi}, \tilde{\varphi}) < \delta$. Take $x \in X$ and $s \in S$. Since $d_0(\varphi_s \circ Const_x, \psi_s \circ Const_x) = d(\varphi_s(x), \psi_s(x))$, hence we have the following lemma.

Lemma 3.1. *If $\tilde{\psi}$ is δ-close to $\tilde{\varphi}$, then ψ is δ-close to φ.*

Conversely assume that $d_S(\varphi, \psi) < \delta$. Then for $f \in S(X)$, we have $d(\varphi(s, f(x)), \psi(s, f(x))) < \delta$. This implies that $D(\tilde{\varphi}, \tilde{\psi}) < \delta$. This by Lemma 3.1 implies the following proposition.

Proposition 3.2. *$d_S(\varphi, \psi) < \delta$ if and only if $D(\tilde{\varphi}, \tilde{\psi}) < \delta$.*

For continuous action $\varphi : G \times X \to X$, define dynamical ball by

$$\Gamma_c(x, \varphi) := \{y : d(\varphi_g(x), \varphi_g(y)) < c, \text{ for all } g \in G\}. \tag{3.2}$$

We say that continuous action $\varphi : G \times X \to X$ is expansive in $p \in X$, with expansive constant $c > 0$, if $\Gamma_c(p, \varphi) = \{p\}$. Also, $\varphi : G \times X \to X$ is called expansive action with expansive constant $c > 0$, if φ is expansive in every point with the same expansive constant $c > 0$. Take $y \in \Gamma_c(x, \varphi)$, by $\varphi_g \circ Const_y = \varphi_g(y)$, we have $Const_y \in \Gamma_c(Const_x, \tilde{\varphi})$. Hence, we have the following proposition.

Proposition 3.3. *If function envelope $\tilde{\varphi} : G \times S(X) \to S(X)$ is expansive in every constant function with the same expansive constant, then $\varphi : G \times X \to X$ is expansive.*

Take $f \in S(X)$. Assume that $h \in \Gamma_c(f, \tilde{\varphi})$. Then $d_0(\varphi_g \circ f, \varphi_g \circ h) < c$ for all $g \in G$. This implies that $h(x) \in \Gamma_c(f(x), \varphi)$. Hence by Proposition 3.3, we have the following proposition.

Proposition 3.4. *Continuous action $\varphi : G \times X \to X$ is expansive if and only if function envelope $\tilde{\varphi} : G \times S(X) \to S(X)$ is expansive.*

Continuous action $\varphi : G \times X \to X$ is called c-uniformly expansive, if for every $\epsilon > 0$ there is finite set $F \subseteq G$ such that

$$\sup_{g \in F} d(\varphi_g(x), \varphi_g(y)) < c \Rightarrow d(x, y) < \epsilon. \tag{3.3}$$

By Lemma, if for every $\epsilon > 0$ there is a compact set $C \subseteq (X, d)$ such that $d^{-1}([0, \epsilon) \cup C \times C = X \times X$, then continuous action $\varphi : G \times (X, d) \times (X, d)$ is expansive if and only if it is uniformly expansive. This implies that the following corollary.

Corollary 3.5. *Every expansive continuous action $\varphi : G \times X \to X$ on compact metric space is uniformly expansive.*

In the following we show that if function envelope $\tilde{\varphi} : G \times S(X) \to S(X)$ does have shadowing property, weak shadowing property and β-persistent, then $\varphi : G \times X \to X$ does have shadowing property, weak shadowing property and β-persistent, respectively.

Proposition 3.6. *Assume that $\varphi : G \times X \to X$ is a continuous action and that $\tilde{\varphi} : G \times S(X) \to S(X)$ is defined by $\tilde{\varphi}(g, h)(x) = \varphi(g, h(x))$.*

(1) *If $\tilde{\varphi} : G \times S(X) \to S(X)$ has shadowing property, then $\varphi : G \times X \to X$ has shadowing property.*
(2) *If $\tilde{\varphi} : G \times S(X) \to S(X)$ has weak shadowing property, then $\varphi : G \times X \to X$ has weak shadowing property.*
(3) *If $\tilde{\varphi} : G \times S(X) \to S(X)$ is β-persistent, then $\varphi : G \times X \to X$ is β-persistent.*

Proof. (1) Firstly, we assume that $\tilde{\varphi} : G \times S(X) \to S(X)$ has shadowing property. Then we show that $\varphi : G \times X \to X$ has shadowing property. Let $\epsilon > 0$ be given. Choose $\delta > 0$ corresponding to $\epsilon > 0$ by shadowing of $\tilde{\varphi}$. We show that every δ-chain $f : G \to X$ for $\varphi : G \times X \to X$ can be ϵ-shadowed by an orbit of φ. Define $F : G \to S(X)$ by $F(g) = Const_{f(g)}$. It is easy to see that $F : G \to S(X)$ is δ-chain for $\tilde{\varphi} : G \times S(X) \to S(X)$. By shadowing of $\tilde{\varphi}$, there is $h \in S(X)$ such that $D(\tilde{\varphi}(g, h), F(g)) < \epsilon$ for all $g \in G$. This means that $d(\varphi(g, h(x)), f(g)) < \epsilon$ for all $g \in G$.

(2) Assume that $\tilde{\varphi} : G \times S(X) \to S(X)$ has weak shadowing property. Then we show that $\varphi : G \times X \to X$ has weak shadowing property. Let $\epsilon > 0$ be given. Choose $\delta > 0$ corresponding to $\epsilon > 0$ by weak shadowing of $\tilde{\varphi}$. We show that every $p \in X$ and every continuous action $\psi : G \times X \to X$ with $d_0(\psi, \varphi) < \delta$, there is $q \in X$ such that $d(\varphi(g, q), \psi(g, p)) < \epsilon$. Since $d_S(\varphi, \psi) < \delta$, by proposition, we have $D(\tilde{\psi}, \tilde{\varphi}) < \delta$. Hence, by shadowing of $\tilde{\varphi}$, for $Const_p \in S(X)$ there is $h \in S(X)$ such that $d_0(\varphi_g \circ h, \psi_g \circ Const_p) < \epsilon$ for all $g \in G$. This implies that $d(\varphi(g, h(x)), \psi(g, p)) < \epsilon$ for all $g \in G$.

(3) The proof is similar item (2).

$\square$

By proposition, peristen shadowing is equivalence with shadowing and β-persistent. Hence, by proposition, we have the following corollary.

Corollary 3.7. *Assume that $\tilde{\varphi} : G \times S(X) \to S(X)$ is function envelope of continuous action $\varphi : G \times X \to X$. If $\tilde{\varphi} : G \times S(X) \to S(X)$ has persistent shadowing property, then $\varphi : G \times X \to X$ has persistent shadowing property.*

We say that $\varphi : G \times X \to X$ is topologically stable, if for every $\epsilon > 0$ there is $\delta > 0$ such that for every continuous action $\psi : G \times X \to X$ with $d_S(\varphi, \psi) < \delta$ there is $h \in S(X)$ such that $d(h, id_X) < \epsilon$ and $h \circ \psi_g = \varphi_g \circ h$ for all $g \in G$. Moreover we say that $\varphi : G \times X \to X$ is s-topologically stable if $h : X \to X$ is surjective.

Lemma 3.8. (1) *If continuous action φ is topologically stable, then it does have weak shadowing property.*

(2) *If continuous action φ is s-topologically stable, then it is β-persistent.*

Proof. The proof is clear and it left to reader. □

Proposition 3.9. *Assume that $\varphi : G\times X \to X$ on metric space(X,d) is uniformly expansive.*

(1) *If φ does have weak shadowing property, then it is topologically stable.*
(2) *If φ does have persistent shadowing property, then it is s-topologically stable.*

Proof. Assume that $\varphi : G \times X \to X$ does have weak shadowing property and it is c-uniformly continuous. For $0 < \epsilon < \frac{c}{9}$ choose $\delta > 0$ in definition of weak shadowing property φ. Take continuous action $\psi : G\times X \to X$ with $d_S(\varphi,\psi) < \delta$. For $p \in X$ there is $q_p \in X$ such that $d(\varphi(g,q_p),\psi(g,p)) < \epsilon$ for all $g \in G$. If $d(\varphi(g,q'_p),\psi(g,p)) < \epsilon$ for all $g \in G$, then $d(\varphi(g,q_p),\varphi(g,q'_p)) < c$ for all $g \in G$. By expansivity of φ, we have $q_p = q'_p$. This implies that $h : X \to X$ by $h(p) = q_p$ is well defined. We claim that $h \in S(X)$. Let $\eta > 0$ be given. By uniformly expansivity of $\varphi : G \times X \to X$, there is finite set $F \subseteq G$ such that

$$\sup_{g\in F} d(\varphi_g(x),\varphi_g(y)) < c \Rightarrow d(x,y) < \eta. \tag{3.4}$$

For $\eta > 0$ and finite set $F \subseteq G$, there is $\lambda > 0$ such that

$$d(a,b) < \lambda \Rightarrow d(\psi(g,a),\psi(g,b)) < \eta, \text{ for all } g \in F. \tag{3.5}$$

Since $d(\varphi(g,h(z)),\psi(g,z)) < \epsilon$ for all $g \in G$, then by (3.5), for $x,y \in X$ with $d(x,y) < \lambda$ we have $d(\varphi(g,h(x)),\varphi(g,h(y)) < c$ for all $g \in F$. Hence, by (3.4), we have $d(h(x),h(y)) < \eta$ $h \in S(X)$. Also, since $d(h(x),x) < \epsilon$, we have $d_S(\varphi_g \circ h, h\circ\psi_g) < 3\epsilon$ for all $g \in G$. By this relation, we have $d(\varphi_g(h(\psi_g(x))), h\circ\psi_{g^2}(x)) < 3\epsilon$ and $d(\varphi_{g^2}(h(x)), h\circ\psi_{g^2}(x)) < \epsilon$ for all $g \in G$. This implies that $d(\varphi_g(h(\psi_g(x))),\varphi_g(\varphi_g\circ h(x))) < c$ for all $g \in G$. By expansivity of φ, we have $h\circ\psi_g = \varphi_g\circ h$ for all $g \in G$.

We know that if expansive continuous action $\varphi : G\times X \to X$ on compact metric space(X,d) does have weak shadowing property, then for every continuous action $\psi : G \times X \to X$ with $d_S(\varphi,\psi) < \delta$, there is $h \in S(X)$ such that $d_S(h, id_X) < \epsilon$ and $h\circ\psi_g = \varphi_g\circ h$. We claim that $\varphi : G \times X \to X$ is β-persistent if and only if $h : X \to X$ is surjective. Firstly, assume that φ is β-persistent. Then we show that $h : X \to X$ is surjective. By contradiction, we assume that f is not surjective. Then there is $y \in X$ such that $y \notin f(X)$. By β-persistent of φ, for $y \in X$ there is $z \in X$ such that $d(\varphi(g,y),\psi(g,z)) < \epsilon$ for all $g \in G$. Also, by weak shadowing of φ, for $z \in X$ we have $d(\varphi_g(h(z)),\psi_g(z)) < \epsilon$ for all $g \in G$. Hence $d(\varphi_g(h(z)),\varphi_g(y)) < c$ for all $g \in G$. By expansivity of φ, we have $h(z) = y$ which is a contradiction. Also, if $h : X \to X$ is surjective, then for $x \in X$, there is $y \in X$ such that $h(y) = x$. By $h\circ\psi_g = \varphi_g\circ h$, we have $h(\psi_g(y)) = \varphi_g(x)$ for all $g \in G$. This implies that $d(\varphi_g(x),\psi_g(y)) < \epsilon$ for all $g \in G$. □

Definition 3.10. We say that the functional envelope $\tilde{\varphi}$ is topologically stable in the class of all functional envelopes of $S(X)$ induced by continuous action $\varphi : G \times X \to X$ if for every $\epsilon > 0$, there exists $\delta > 0$ with the property that for each functional envelope $\tilde{\psi}$ induced by continuous action $\psi : G \times X \to X$ with $D(\tilde{\varphi}, \tilde{\psi}) < \delta$ there is a continuous map $H : S(X) \to S(X)$ such that

- $\tilde{\varphi}_g \circ H = H \circ \tilde{\psi}_g$ for all $g \in G$,
- $D(H, id_{S(X)}) < \epsilon$.

Assume that $\varphi : G \times X \to X$ is topologically stable. We show that $\tilde{\varphi}$ is topologically stable. Let $\epsilon > 0$ be given. Choose $\delta > 0$ corresponding to $\epsilon > 0$ by topological stability of φ. Choose functional envelope $\tilde{\psi}$ with $D(\tilde{\varphi}, \tilde{\psi}) < \delta$. Then $d_S(\varphi, \psi) < \delta$. By topological stability of φ there is $h : X \to X$ with $d_S(h, id_X) < \delta$ and $\varphi_g \circ h = h \circ \psi_g$ for all $g \in G$. Define $H : S(X) \to S(X)$ by $f \mapsto h \circ f$. Since $h : X \to X$ is continuous, H is continuous. Also, it is easy to see that $D(H, id_{S(X)}) < \delta$ and for $f \in S(X)$, we have $\tilde{\varphi}_g \circ H(f) = \varphi_g \circ h \circ f = h \circ \psi_g \circ f = h \circ \tilde{\psi}_g(f) = H \circ \tilde{\psi}_g(f)$. Hence, we have the following proposition.

Proposition 3.11. *If $\varphi : G \times X \to X$ is topologically stable, then $\tilde{\varphi}$ is topologically stable*

Conversely, assume that $\tilde{\varphi}$ is topologically stable. We claim that $\varphi : G \times X \to X$ is topologically stable. For $\epsilon > 0$ choose $\delta > 0$ by definition of topological stability of $\tilde{\varphi}$. Take continuous action $\psi : G \times X \to X$ with $d_S(\varphi, \psi) < \delta$. Then for functional envelope $\tilde{\psi}$ induces by ψ we have $D(\tilde{\psi}, \tilde{\varphi}) < \delta$. By topological stability of $\tilde{\varphi}$, there is $H : S(X) \to S(X)$ such that $D(H, id_{S(X)}) < \epsilon$ and $\tilde{\varphi}_g \circ H = H \circ \tilde{\psi}_g$ for all $g \in G$. Define $h : X \to X$ by $h := H(id_X)$. Then $d(h(x), x) = d(H \circ id_X(x), id_X(x)) \leq D(H, id_{S(X)}) < \epsilon$. By $\tilde{\varphi}_g \circ H = H \circ \tilde{\psi}_g$ for all $g \in G$ and $\tilde{\varphi}_g \circ H(id_X) = \varphi_g \circ h$, we have $\varphi_g \circ h = h \circ \psi_g$, that is, $\varphi : G \times X \to X$ is topologically stable. Hence by Proposition 3.11, we have the following corollary.

Corollary 3.12. *Continuous action $\varphi : G \times X \to X$ is topologically stable if and only if functional envelope $\tilde{\varphi}$ induces by φ is topologically stable.*

By Proposition 3.6, if functional envelope $\tilde{\varphi} : G \times X \to X$ induces by φ does have shadowing property, weak shadowing property, β-persistent and persistent shadowing property, then $\varphi : G \times X \to X$ does have shadowing property, weak shadowing property, β-persistent and persistent shadowing property respectively. In the following, we show that the converse does hold if φ is expansive.

Proposition 3.13. *Assume that $\varphi : G \times X \to X$ is uniformly expansive.*

1. *Functional envelope $\tilde{\varphi}$ induces by φ does have shadowing property if and only if φ does have shadowing property.*
2. *Functional envelope $\tilde{\varphi}$ induces by φ does have weak shadowing property if and only if φ does have weak shadowing property.*
3. *Functional envelope $\tilde{\varphi}$ induces by φ is both β-persistent and weak shadowable if and only if φ is both β-persistent and weak shadowable.*
4. *Functional envelope $\tilde{\varphi}$ induces by φ does have persistent shadowing property if and only if φ does have persistent shadowing property.*

Proof. (1) Assume that φ does have shadowing property and that $\epsilon > 0$ is given. Choose $\delta > 0$ corresponding to $\epsilon > 0$ in definition of shadowing property for φ. Let $F : G \to S(X)$ be a δ-chain for $\tilde{\varphi}$. We show that there is $h \in S(X)$ such that $d_S(\varphi_g \circ h, F(g)) < \delta$ for all $g \in G$. For $p \in X$, define $f_p : G \to X$ by $f_p(g) := F(g)(p)$. Then $f_p : G \to X$ is δ-chain for φ. By shadowing property, there is $q_p \in X$ such that $d(\varphi(g, q_p), f_p(g)) < \epsilon$ for all $g \in G$. By expansivity of φ, if $h : X \to X$ defined by $h(p) = q_p$, then h is well defined and one can check that $h \in S(X)$. Hence $d(\varphi(g, h(p)), F(g)(p)) < \epsilon$, that is, $d_S(\varphi_g \circ h, F(g)(h)) < \epsilon$. This implies that for δ-chain $F : G \to S(X)$ there is $h \in S(X)$ such that $d_S(\tilde{\varphi}_g(h), F(g)) < \epsilon$ for all $g \in G$.

(2) Assume that φ does have weak shadowing property and that $\epsilon > 0$ is given. By Proposition 3.9, φ is topologically stable, hence $\tilde{\varphi}$ is topologically stable by corollary. By Lemma 3.8, we can say that $\tilde{\varphi}$ does have weak shadowing property.

(3) If φ is both β-persistent and weak shadowable, then φ is s-topologically stable, hence by Proposition 3.9, functional envelope $\tilde{\varphi}$ is s-topologically stable, hence by Lemma 3.8, $\tilde{\varphi}$ is both weak shadowing property and β-persistent.

(4) If φ does have persistent shadowing property, then φ is both β-persistent and shadowable. By item (1), $\tilde{\varphi}$ does have shadowing property and by item (3) $\tilde{\varphi}$ is β-persistent, hence it is both β-persistent and shadowable, that is, $\tilde{\varphi}$ does have persistent shadowing property.

□

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