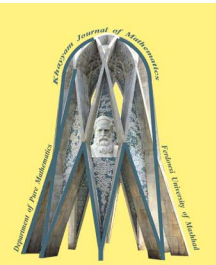
Khayyam J. Math. 11 (2025), no. 2, 318-335
DOI:10.22034/KJM.2025.513329.3488



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ADDITIVE CYCLIC CODES OVER MIXED ALPHABET $\mathbb{Z}_2\mathbb{Z}_2[u^2]\mathbb{Z}_2[u^4]$

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Communicated by B. Mashayekhy

Abstract. Recently, there has been increasing interest in families of mixed alphabet codes that extend the scope of conventional coding theory. In this work, we study mixed alphabet codes over the structure $\mathbb{Z}_2\mathbb{Z}_2[u^2]\mathbb{Z}_2[u^4]$, where $\mathbb{Z}_2[u^4] = \{a_0+ua_1+u^2a_2+u^3a_3 \mid a_i \in \mathbb{Z}_2,\ u^4 = 0\}$. We focus on $\mathbb{Z}_2\mathbb{Z}_2[u^2]\mathbb{Z}_2[u^4]$-additive cyclic codes, first introducing their definition and exploring their algebraic properties. We then derive the standard form of their generator matrices, along with the corresponding generator polynomials and minimal generating sets. Furthermore, we present a detailed characterization of the duals of these codes. The results contribute to the growing body of work on mixed alphabet codes, offering insights into their structural features and potential applications in error correction and related areas of information theory.

1. Introduction

Additive cyclic codes are error-correcting codes with beneficial algebraic properties and can be efficiently implemented in various communication systems. These codes are designed to correct errors introduced during data transmission, ensuring reliable and accurate information retrieval. In the context of codes over mixed alphabets, the code alphabet consists of symbols from more than one domain, which can be binary, quaternary, or any other desired set of elements.

Traditionally, cyclic codes are defined over a finite field (e.g., a specific prime field or its extension). However, in particular scenarios, it is desirable to employ a mixed alphabet that includes both binary and non-binary elements. The construction of additive cyclic codes over mixed alphabets and their properties has been a dynamically explored subject from the last decade of the twentieth

Date: Received: 20 March 2025; Accepted: 14 August 2025.

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2020 *Mathematics Subject Classification.* Primary: 94B05; Secondary: 94B15.

Key words and phrases. Linear codes. cyclic codes, Hamming distance, generator matrix.

century onward. The encoding and decoding algorithms for such codes often leverage the algebraic structure of the underlying fields and take advantage of the cyclic properties to achieve efficient and reliable error correction.

In summary, additive cyclic codes over mixed alphabets combine the advantages of cyclic codes with the flexibility of using symbols from different domains. Such codes find extensive use across various domains due to their versatility and ability to provide highly effective error correction methods. Ongoing research in this area continues to explore novel code constructions, decoding algorithms, and their practical implementations.

Additive codes were introduced in 1973 by Delsarte and Levenshtein [9] as subgroups of the underlying abelian group, using association schemes. These codes possess an intriguing characteristic, their coordinates are divided into two separate components, with each component forming a linear code over its own distinct alphabet. Recently, Borges, Fernández-Córdoba, and Pujol [7] introduced a new code called $\mathbb{Z}_2\mathbb{Z}_4$ additive code. This code is defined as a submodule of $\mathbb{Z}_2^{\alpha} \times \mathbb{Z}_4^{\beta}$, where $\alpha,\ \beta \in \mathbb{Z}^+$. Here, Borges, Fernández-Córdoba, and Pujol [7] explored the duality of $\mathbb{Z}_2\mathbb{Z}_4$-additive codes by introducing a new inner product that differs from the conventional Euclidean inner product. The cyclic structure of $\mathbb{Z}_2\mathbb{Z}_4$-additive codes were investigated by Abualrub, Siap, and Aydin [2] in their pioneering study, which provides a minimal spanning set and determined the generator polynomials of $\mathbb{Z}_2\mathbb{Z}_4$-additive codes as well as their dual codes. This research marked the first exploration of the cyclic properties associated with $\mathbb{Z}_2\mathbb{Z}_4$-additive codes. Aydogdu and Gursoy [3] have investigated $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-cyclic codes, which are an extension of $\mathbb{Z}_2\mathbb{Z}_4$-additive codes and also provide the generator and parity-check polynomial in standard form. Furthermore, Li [12] have examined $\mathbb{Z}_2(\mathbb{Z}_2+u\mathbb{Z}_2)(\mathbb{Z}_2+u\mathbb{Z}_2+u^2\mathbb{Z}_2)$-additive cyclic codes whose gray images are always a binary linear code and some fundamental results are given there. Utilizing the methodologies outlined by Borges, Fernández-Córdoba, and Pujol [7], Aydogdu and Siap [4] conducted a study on $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$-additive codes and also provided the generator and parity-check matrices in standard form. Furthermore, Aydogdu et al. [5, 6] expanded the methods utilized by Borges, Fernández-Córdoba, and Pujol [7] and Abualrub, Siap, and Aydin [2] to investigate cyclic codes over $\mathbb{Z}_2\mathbb{Z}_2[u]$ and $\mathbb{Z}_2\mathbb{Z}_2[u^3]$. Moreover, Borges, Fernández-Córdoba, and Ten-Valls [8] have analyzed $\mathbb{Z}_2$-double cyclic codes. Note that, $\mathbb{Z}_2\mathbb{Z}_2[u^3]$-cyclic codes are expansion of $\mathbb{Z}_2$-double cyclic code and $\mathbb{Z}_2\mathbb{Z}_2[u]$-cyclic code. Srinivasulu and Padmapani [11] have investigated cyclic codes and their dual over the mixed alphabet $\mathbb{Z}_2\mathbb{Z}_2[u^4]$ and gave the generator and parity-check matrix in standard form.

Motivated by these works, this study focuses on investigating the composition of $\mathbb{Z}_2\mathbb{Z}_2[u^2]\mathbb{Z}_2[u^4]$-additive and cyclic codes. The $\mathbb{Z}_2\mathbb{Z}_2[u^2]$-additive codes and $\mathbb{Z}_2\mathbb{Z}_2[u^4]$-additive codes can be seen as a subset or particular case of these additive codes. Our exploration begins by examining the structure of $\mathbb{Z}_2\mathbb{Z}_2[u^2]\mathbb{Z}_2[u^4]$-additive codes and providing the standard generator matrices. We have established a connection between these codes and binary codes by introducing a new gray map. Additionally, we ascertain the generator polynomials and minimal spanning sets for $\mathbb{Z}_2\mathbb{Z}_2[u^2]\mathbb{Z}_2[u^4]$-cyclic codes. Finally, we illustrate several examples of additive and cyclic $\mathbb{Z}_2\mathbb{Z}_2[u^2]\mathbb{Z}_2[u^4]$-codes.

2. $\mathbb{Z}_2\mathbb{Z}_2[u^2]\mathbb{Z}_2[u^4]$-LINEAR CODES AND GENERATOR MATRIX

Suppose that $\mathbb{Z}_2 = \{0, 1\}$ is a binary field and that $R_1 = \frac{\mathbb{Z}_2[u]}{<u^2>} = \{a+ub \mid a, b \in \mathbb{Z}_2 \text{ with } u^2 = 0\}$ and $R_2 = \frac{\mathbb{Z}_2[u]}{<u^4>} = \{a+ub+u^2c+u^3d \mid a, b, c, d \in \mathbb{Z}_2 \text{ where } u^4 = 0\}$ is a finite commutative chain ring. It is easy to verify that $\mathbb{Z}_2$ is a proper subring of R_1 and that R_1 is a proper subring of R_2. Consider the following set:

$$\mathcal{R} = \mathbb{Z}_2R_1R_2 = \{(a_1|b_1|c_1) \mid a_1 \in \mathbb{Z}_2,\ b_1 \in R_1,\ c_1 \in R_2\}.$$

The set $\mathcal{R}$ forms an additive abelian group under componentwise addition. The set $\mathcal{R}$ cannot be made an R_2-module with the multiplication operation directly. For this, we define the multiplication of element $(a_1|b_1|c_1) \in \mathcal{R}$ with the scalar $d \in R_2$ as follows:

$$d \star (a_1|b_1|c_1) = (\delta_1(d)a_1|\delta_2(d)b_1|dc_1),$$

where $\delta_1 : R_2 \longrightarrow \mathbb{Z}_2$ such that $\delta_1(a+ub+u^2c+u^3d) = a$ and $\delta_2 : R_2 \longrightarrow R_1$ such that $\delta_2(a+ub+u^2c+u^3d) = a+ub$. Then, the binary operation $\star$ is well-defined and $\mathcal{R}$ is an R_2-module with the scalar multiplication $\star$. Let us denote $\mathbb{Z}_2^l R_1^m R_2^n$ by $\mathcal{R}_{l,m,n}$ and an element $c = (a_{1,1}, a_{1,2}, \ldots, a_{1,l}|b_{1,1}, b_{1,2}, \ldots, b_{1,m}|c_{1,1}, c_{1,2}, \ldots, c_{1,n}) \in \mathcal{R}_{l,m,n}$ by $c = ((a_{1,i})|(b_{1,j})|(c_{1,k}))_{i,j,k=1}^{l,m,n}$. One can extend the scalar multiplication $\star$ to $\mathcal{R}_{l,m,n}$ such that for an arbitrary scalar $d \in R_2$ and a codeword $c = ((a_{1,i})|(b_{1,j})|(c_{1,k}))_{i,j,k=1}^{l,m,n} \in \mathcal{R}_{l,m,n}$ as follows:

$$d \star c = ((\delta_1(d)a_{1,i})|(\delta_2(d)b_{1,j})|(dc_{1,k}))_{i,j,k=1}^{l,m,n}.$$

This extended operation $\star$ is well-defined and the set $\mathcal{R}_{l,m,n}$ becomes an R_2-module.

Definition 2.1. A subset C of $\mathcal{R}_{l,m,n}$ is called a $\mathbb{Z}_2R_1R_2$-additive code with a block length (l, m, n) if C forms a subgroup of $\mathcal{R}_{l,m,n}$. If $m = n = 0$, then C is a binary additive code of length l. Also, C is referred to as the $\mathbb{Z}_2R_1R_2$-linear code with a block length (l, m, n) if C forms an R_2-submodule of $\mathcal{R}_{l,m,n}$.

Consider the maps $\theta_1 : R_1 \longrightarrow \mathbb{Z}_2^2$ such that $\theta_1(a + ub) = (b, a + b)$ and $\theta_2 : R_2 \longrightarrow \mathbb{Z}_2^4$ such that $\theta_2(a + ub + u^2c + u^3d) = (a + b + c + d, c + d, b + d, d)$ as defined in [1, 10]. We generalizes these maps point-wise to $\theta : \mathbb{Z}_2R_1R_2 \longrightarrow \mathbb{Z}_2^7$ as follows:

$$\theta(a, b_1 + ub_2, c_1 + uc_2 + u^2c_3 + u^3c_4) = (a, \theta_1(b_1 + ub_2), \theta_2(c_1 + uc_2 + u^2c_3 + u^3c_4)).$$

Extending θ componentwise to $\Theta : \mathcal{R}_{l,m,n} \longrightarrow \mathbb{Z}_2^{l+2m+4n}$ as follows:

$$\Theta((a_{1,i})|(b_{1,j})|(c_{1,k}))_{i,j,k=1}^{l,m,n} = ((a_{1,i})|(\theta_1(b_{1,j}))|(\theta_2(c_{1,k})))_{i,j,k=1}^{l,m,n}.$$

It is easy to verify that Θ is a linear map. If C is a $\mathbb{Z}_2R_1R_2$-linear code possessing a block length (l, m, n), then the gray image $\Theta(C)$ is a binary linear code of length $(l + 2m + 4n)$. Note that if $m = n = 0$, then C is a binary additive code with a length l, R_1-linear code with a length m if $l = n = 0$, and R_2-linear code with a length n if $l = m = 0$. By the definition of θ_1 and θ_2, we have R_1 (resp. R_2) is group isomorphic to $\mathbb{Z}_2^2$ (resp., $\mathbb{Z}_2^4$). Thus, as an additive group, a $\mathbb{Z}_2R_1R_2$-linear code possessing a block length (l, m, n) is group isomorphic to $\mathbb{Z}_2^{l_1} \times \mathbb{Z}_2^{2m_1} \times \mathbb{Z}_2^{m_2} \times \mathbb{Z}_2^{4n_1} \times \mathbb{Z}_2^{3n_2} \times \mathbb{Z}_2^{2n_3} \times \mathbb{Z}_2^{n_4}$. Therefore C is of type $(l, m, n; l_1; m_1, m_2; n_1, n_2, n_3, n_4)$.

Theorem 2.2. *Let C be a $\mathbb{Z}_2R_1R_2$-linear code of type $(l, m, n; l_1; m_1, m_2; n_1, n_2, n_3, n_4)$. Then, C is permutation equivalent to a $\mathbb{Z}_2R_1R_2$-linear code having the standard generator matrix as follows:*

$$G_s = \left[\begin{array}{cc|ccc|ccccc} I_{l_1} & A_1 & 0 & 0 & uE_1 & 0 & 0 & 0 & 0 & u^3F_1 \\ 0 & D_1 & I_{m_1} & B_1 & B_2 & 0 & 0 & 0 & u^2F_2 & u^2F_3 \\ 0 & 0 & 0 & uI_{m_2} & uB_3 & 0 & 0 & 0 & 0 & u^3F_4 \\ 0 & D_2 & 0 & E_2 & E_3 & I_{n_1} & C_1 & C_2 & C_3 & C_4 \\ 0 & D_3 & 0 & 0 & uE_4 & 0 & uI_{n_2} & uC_5 & uC_6 & uC_7 \\ 0 & D_4 & 0 & 0 & uE_5 & 0 & 0 & u^2I_{n_3} & u^2C_8 & u^2C_9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u^3I_{n_4} & u^3C_{10} \end{array}\right],$$

where A_1 and D_i for $i \in \{1, \ldots, 4\}$ are matrices over $\mathbb{Z}_2$; B_1, B_2, B_3, and E_j for $j \in \{1, \ldots, 5\}$ are matrices over R_1; and C_i for $i \in \{1, \ldots, 10\}$ and F_j for $j \in \{1, \ldots, 4\}$ are matrices over R_2. Moreover, C has $|C| = 2^{(l_1+2m_1+m_2+4n_1+3n_2+2n_3+n_4)}$, number of codewords.

Proof. For proof follows from [3, Theorem 2.3]. □

Corollary 2.3. *If C_1 is a $\mathbb{Z}_2R_1$-linear code of type $(l, m; l_1; m_1, m_2)$, then C_1 is permutation equivalent to a $\mathbb{Z}_2R_1$-linear code having the standard generator matrix as follows:*

$$G_1 = \left[\begin{array}{cc|ccc} I_{l_1} & A_1 & 0 & 0 & uE_1 \\ 0 & D_1 & I_{m_1} & B_1 & B_2 \\ 0 & 0 & 0 & uI_{m_2} & uB_3 \end{array}\right],$$

where A_1 and D_1 are matrices over $\mathbb{Z}_2$ and B_1, B_2, B_3, and E_1 are matrices over R_1. Moreover, C has $|C| = 2^{(l_1+2m_1+m_2)}$, number of codewords.

Corollary 2.4. *If C_2 is a $\mathbb{Z}_2R_2$-linear code of type $(l, n; l_1; n_1, n_2, n_3, n_4)$, then C_2 is permutation equivalent to a $\mathbb{Z}_2R_2$-linear code having the standard generator matrix as follows:*

$$G_2 = \left[\begin{array}{cc|ccccc} I_{l_1} & A_1 & 0 & 0 & 0 & 0 & u^3F_1 \\ 0 & D_1 & I_{n_1} & C_1 & C_2 & C_3 & C_4 \\ 0 & D_2 & 0 & uI_{n_2} & uC_5 & uC_6 & uC_7 \\ 0 & D_3 & 0 & 0 & u^2I_{n_3} & u^2C_8 & u^2C_9 \\ 0 & 0 & 0 & 0 & 0 & u^3I_{n_4} & u^3C_{10} \end{array}\right],$$

where A_1 and D_i for $i \in \{1, 2, 3\}$ are matrices over $\mathbb{Z}_2$ and C_i for $i \in \{1, \ldots, 10\}$ and F_1 are matrices over R_2. Moreover, C has $|C| = 2^{(l_1+4n_1+3n_2+2n_3+n_4)}$, number of codewords.

Corollary 2.5. *If C_3 is an R_1R_2-linear code of type $(m, n; m_1, m_2; n_1, n_2, n_3, n_4)$, then C_3 is permutation equivalent to an R_1R_2-linear code having the standard*

generator matrix as follows:

$$G_3 = \left[\begin{array}{ccc|ccccc} I_{m_1} & B_1 & B_2 & 0 & 0 & 0 & u^2F_1 & u^2F_2 \\ 0 & uI_{m_2} & uB_3 & 0 & 0 & 0 & 0 & u^3F_3 \\ 0 & E_1 & E_2 & I_{n_1} & C_1 & C_2 & C_3 & C_4 \\ 0 & 0 & uE_3 & 0 & uI_{n_2} & uC_5 & uC_6 & uC_7 \\ 0 & 0 & uE_4 & 0 & 0 & u^2I_{n_3} & u^2C_8 & u^2C_9 \\ 0 & 0 & 0 & 0 & 0 & 0 & u^3I_{n_4} & u^3C_{10} \end{array}\right],$$

where B_1, B_2, B_3, and E_j for $j \in \{1,\ldots,4\}$ are matrices over R_1 and C_i for $i \in \{1,\ldots,10\}$ and F_j for $j \in \{1,2,3\}$ are matrices over R_2. Moreover, C has $|C| = 2^{(2m_1+m_2+4n_1+3n_2+2n_3+n_4)}$, number of codewords.

Example 2.6. Suppose that C is a $\mathbb{Z}_2R_1R_2$-linear code possessing a block length $(2,3,5)$ and generator matrix

$$G = \left[\begin{array}{cc|ccc|ccccc} 1 & 1 & 0 & u & 0 & 0 & u^2 & u^2 & 0 & 0 \\ 0 & 1 & 1 & u & 1+u & 0 & u^2+u^3 & u^2 & 0 & 0 \\ 0 & 1 & 1 & 1+u & u & 0 & u^2+u^3 & u^2+u^3 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1+u^2 & u+u^2+u^3 & 0 & 1+u \\ 0 & 0 & 0 & 1+u & u & 0 & u^3 & 1 & 1 & u^2 \\ 0 & 0 & 0 & 1 & 0 & 0 & u+u^2 & u+u^3 & 0 & u \\ 0 & 0 & 0 & 0 & 0 & 0 & u^2 & u^2+u^3 & 0 & 0 \end{array}\right].$$

Then, C is permutation equivalent to a $\mathbb{Z}_2R_1R_2$-linear code having the standard generator matrix as follows:

$$G_s = \left[\begin{array}{cc|ccc|cccccc} 1 & 1 & 0 & 0 & u & 0 & 0 & 0 & 0 & u^3 \\ 0 & 1 & 1 & 1+u & u & 0 & 0 & 0 & u^2+u^3 & u^2 \\ 0 & 0 & 0 & u & 1 & 0 & 0 & 0 & 0 & u^3 \\ 0 & 1 & 0 & 0 & 1+u & 1 & 0 & 1+u & 1+u^2 & u+u^2 \\ 0 & 0 & 0 & 0 & u & 0 & 1 & u^2 & u^3 & 1+u^3 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & u & u+u^2 & u+u^3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u^2 & u^2+u^3 \end{array}\right].$$

Additionally, C is of type $(2,3,5;1;1,1;2,1,1,0)$, containing $2^1 \cdot 4^1 \cdot 2^1 \cdot 16^2 \cdot 8^1 \cdot 4^1 \cdot 2^0 = 16384$ number of codewords.

3. $\mathbb{Z}_2\mathbb{Z}_2[u^2]\mathbb{Z}_2[u^4]$-cyclic codes

In this section, we generalize the definition of cyclic codes over the mixed alphabet $\mathbb{Z}_2R_1R_2$, where $R_1 = \frac{\mathbb{Z}_2[u]}{<u^2>}$ and $R_2 = \frac{\mathbb{Z}_2[u]}{<u^4>}$. A $\mathbb{Z}_2R_1R_2$-linear code C is termed as cyclic If a cyclic shift is applied first to the initial l coordinates, then to the next m coordinates, and finally to the last n coordinates of each codeword, the code remains unchanged. Here, we analyze the structural properties of the $\mathbb{Z}_2R_1R_2$-cyclic code C of length $l+m+n$ (or block length (l,m,n)) by their generator polynomials with odd m and odd n, while l is set to be arbitrary. Let us denote $(a_{1,1}, a_{1,2}, \ldots, a_{1,l}|b_{1,1}, b_{1,2}, \ldots, b_{1,m}|c_{1,1}, c_{1,2}, \ldots, c_{1,n}) \in \mathbb{Z}_2^l R_1^m R_2^n$ by $((a_{1,i})|(b_{1,j})|(c_{1,k}))_{i,j,k=1}^{l,m,n} \in \mathbb{Z}_2^l R_1^m R_2^n$, throughout this paper.

Definition 3.1. A $\mathbb{Z}_2R_1R_2$-linear code C with a block length (l,m,n) is termed as $\mathbb{Z}_2R_1R_2$-cyclic code if $(a_{1,l-1},a_{1,0},\ldots,a_{1,l-2}|b_{1,m-1},b_{1,0},\ldots,b_{1,m-2}|c_{1,n-1},c_{1,0},\ldots,c_{1,n-2})$ belongs to C for any codeword $((a_{1,i})|(b_{1,j})|(c_{1,k}))_{i,j,k=0}^{l-1,m-1,n-1}\in C$.

Suppose that $R_{l,m,n}[y]=\frac{\mathbb{Z}_2[y]}{<y^l-1>}\times\frac{R_1[y]}{<y^m-1>}\times\frac{R_2[y]}{<y^n-1>}$ such that $\mathbb{Z}_{2,l}[y]=\frac{\mathbb{Z}_2[y]}{<y^l-1>}$, $R_{1,m}[y]=\frac{R_1[y]}{<y^m-1>}$, $R_{2,n}[y]=\frac{R_2[y]}{<y^n-1>}$ and $R_{l,m}[y]=\frac{\mathbb{Z}_2[y]}{<y^l-1>}\times\frac{R_1[y]}{<y^m-1>}$. Then, $R_{l,m,n}[y]=\mathbb{Z}_{2,l}[y]\times R_{1,m}[y]\times R_{2,n}[y]$. Therefore, each element $c=((a_{1,i})|(b_{1,j})|(c_{1,k}))_{i,j,k=1}^{l,m,n}\in\mathbb{Z}_2^lR_1^mR_2^n$ is recognized with the polynomial of the form $c(y)=(a_1(y)|b_1(y)|c_1(y))\in R_{l,m,n}[y]$, where $a_1(y)=a_{1,0}+a_{1,1}y+\cdots+a_{1,l-1}y^{l-1}\in\mathbb{Z}_{2,l}[y]$, $b_1(y)=b_{1,0}+b_{1,1}y+\cdots+b_{1,m-1}y^{m-1}\in R_{1,m}[y]$ and $c_1(y)=c_{1,0}+c_{1,1}y+\cdots+c_{1,n-1}y^{n-1}\in R_{2,n}[y]$. Therefore, the correspondence between $\mathbb{Z}_2^lR_1^mR_2^n$ and $R_{l,m,n}[y]$ is one-to-one.

We introduce linear mappings $\delta_1:R_2\longrightarrow\mathbb{Z}_2$ as $\delta_1(a+ub+u^2c+u^3d)=a$ and $\delta_2:R_2\longrightarrow R_1$ as $\delta_2(a+ub+u^2c+u^3d)=a+ub$.

Suppose that $r(y)=\sum_{i=0}^{s}r_iy^i\in R_{2,n}[y]$ and $c(y)=(a_1(y)|b_1(y)|c_1(y))\in R_{l,m,n}[y]$, Then

$$\delta_1(r(y))=\sum_{i=0}^{s}\delta_1(r_i)y^i$$

and

$$\delta_2(r(y))=\sum_{i=0}^{s}\delta_2(r_i)y^i.$$

We establish the multiplication operation $\star$ for the polynomial ring $R_{l,m,n}[y]$ as follows:

$$\begin{aligned}r(y)\star c(y)=&(\delta_1(r(y))\cdot a_1(y)mod(y^l-1)|\delta_2(r(y))\cdot b_1(y)mod(y^m-1)| &(3.1)\\ &r(y)\cdot c_1(y)mod(y^n-1)), &(3.2)\end{aligned}$$

which is simply denoted by $r(y)\star c(y)=(\delta_1(r(y))\cdot a_1(y)|\delta_2(r(y))\cdot b_1(y)|r(y)\cdot c_1(y))\in R_{l,m,n}[y]$. With this notion, $R_{l,m,n}[y]$ is an $R_2[y]$-module.

Suppose that $c=((a_{1,i})|(b_{1,j})|(c_{1,k}))_{i,j,k=1}^{l,m,n}\in\mathbb{Z}_2^lR_1^mR_2^n$ and $e\in\mathbb{Z}^+$. Then, eth-shift of c is denoted and defined as follows:

$$\begin{aligned}c^{(e)}=&((a_{1,i-e})|(b_{1,j-e})|(c_{1,k-e}))_{i,j,k=1}^{l,m,n}\\ =&(a_{1,0-e},a_{1,1-e},\ldots,a_{1,l-1-e}|b_{1,0-e},b_{1,1-e},\ldots,b_{1,m-1-e}|c_{1,0-e},c_{1,1-e},\ldots,c_{1,n-1-e}).\end{aligned}$$

Thus, we have

$$\begin{aligned}y\star c(y)=&(y\cdot a_1(y)|y\cdot b_1(y)|y\cdot c_1(y))\\ =&(a_1^{(1)}(y)|b_1^{(1)}(y)|c_1^{(1)}(y))\\ =&c^{(1)}(y).\end{aligned}$$

Hence, $y\star c(y)$ is recognized with the codeword $c^{(1)}=((a_{1,i-1})|(b_{1,j-1})|(c_{1,k-1}))_{i,j,k=1}^{l,m,n}$. Similarly, for any $e\in\mathbb{Z}^+$, $y^e\star c(y)=c^{(e)}(y)$ is identified with the codeword $c^{(e)}=((a_{1,i-e})|(b_{1,j-e})|(c_{1,k-e}))_{i,j,k=1}^{l,m,n}$. In this setting, the cyclic code C with a block length (l,m,n) over $\mathbb{Z}_2R_1R_2$ is an $R_2[y]$-submodule of $R_{l,m,n}[y]$. In fact, it is a generalization of binary cyclic code, R_1-cyclic code and R_2-cyclic code.

Theorem 3.2. *A $\mathbb{Z}_2R_1R_2$-linear code C with a block length (l,m,n) is cyclic if and only if C forms an $R_2[y]$-submodule of $R_{l,m,n}[y]$.*

Proof. Suppose that C is a cyclic code with a block length (l,m,n) over $\mathbb{Z}_2R_1R_2$. Then, for any codeword $c = ((a_{1,i})|(b_{1,j})|(c_{1,k}))_{i,j,k=1}^{l,m,n}$ belonging to C, its block-wise cyclic shift is $c^{(1)} = ((a_{1,i-1})|(b_{1,j-1})|(c_{1,k-1}))_{i,j,k=1}^{l,m,n} \in C$. Therefore, if $c(y) = (a_1(y)|b_1(y)|c_1(y)) \in R_{l,m,n}[y]$ be the polynomial representation of the codeword c. Then $y \star c(y) = (y \cdot a_1(y)|y \cdot b_1(y)|y \cdot c_1(y)) = (a_1^{(1)}(y)|b_1^{(1)}(y)|c_1^{(1)}(y)) = c^{(1)}(y)$, where $c^{(1)}(y)$ is the polynomial representation of $c^{(1)}$. Similarly, it can be seen that $y^e \star c(y) \in C$ for any positive integer e. Since C is linear, thus for any polynomial $r(y) \in R_2[y]$ and codeword $c(y)$ belonging to C, we have $r(y) \star c(y) \in C$, consequently C forms an $R_2[y]$-submodule of $R_{l,m,n}[y]$. The converse part is easy to prove. □

Suppose that C_l, C_m, and C_n are the canonical projection of the $\mathbb{Z}_2R_1R_2$-linear code C with a block length (l,m,n) on the first l-coordinates, the m-coordinates followed by the last n-coordinates, respectively. Then, C_l is a binary linear code of length l, C_m is an R_1-linear code of length m, and C_n is an R_2-linear code of length n. These codes are studied by Li [12] and Srinivasulu and Seneviratne [11].

By using the reverse projection of C_n in $R_{l,m,n}[y]$, we can get the structure and generator polynomials of $\mathbb{Z}_2R_1R_2$-linear code C. We define the projection map by

$$\varphi : C \longrightarrow R_{2,n}[y]$$

$$(a_1(y)|b_1(y)|c_1(y)) \longmapsto c_1(y). \tag{3.3}$$

It becomes apparent that φ is a module homomorphism. Moreover, the image $\varphi(C)$ forms an $R_2[y]$-submodule of $R_{2,n}[y]$ and $\ker(\varphi)$ forms a submodule of C. Furthermore, the image $\varphi(C)$ is an ideal in $R_{2,n}[y]$. According to [11], we have

$$\varphi(C) = \langle h(y) + ub_1(y) + u^2b_2(y) + u^3b_3(y)\rangle,$$

with $b_j(y)|b_i(y)|h(y)|(y^n-1) \mod u^3$ for $1 \le i < j \le 3$. Suppose that C_l, C_m and C_n are the canonical projection of the $\mathbb{Z}_2R_1R_2$-linear code C with a block length (l,m,n) on the first l-coordinates, the m-coordinates followed by the last n-coordinates, respectively. Then, C_l is a binary linear code of length l, C_m is an R_1-linear code of length m, and C_n is an R_2-linear code of length n. These codes are studied by Li [12], Srinivasulu and Seneviratne [11].

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$$\varphi : C \longrightarrow R_{2,n}[y]$$

$$(a_1(y)|b_1(y)|c_1(y)) \longmapsto c_1(y). \tag{3.4}$$

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Furthermore, the image $\varphi(C)$ is an ideal in $R_{2,n}[y]$. According to [11], we have

$$\varphi(C) = \langle h(y) + ub_1(y) + u^2b_2(y) + u^3b_3(y)\rangle,$$

with $b_j(y)|b_i(y)|h(y)|(y^n-1) \mod u^3$ for $1 \le i < j \le 3$. Assume that

$$\mathbb{A} = \{(f(y), g(y)) \in R_{l,m}[y] | (f(y), g(y), 0) \ker(\varphi)\}.$$

Therefore, $\mathbb{A}$ is an $R_1[y]$-submodule of $R_{l,m}[y]$ leads $\mathbb{A}$ to be a $\mathbb{Z}_2\mathbb{Z}_2[u^2]$-additive cyclic code possessing the block length (l, m). As established in [12], we have

$$\mathbb{A} = \langle (f(y), 0)(f_1(y), g(y))\rangle,$$

where $f(y), f_1(y) \in \mathbb{Z}_2[y]$ such that $f(y)|(y^l-1)$ and $g(y) = g_1(y) + ua_1(y) \in R_1[y]$ such that $a_1(y)|g_1(y)|(y^m-1)$. Assume that $(c_1(y), c_2(y), 0) \in \ker(\varphi)$. Then $(c_1(y), c_2(y)) \in \mathbb{A}$ suggested that there exist polynomials $q_1(y) \in \mathbb{Z}_2[y]$ and $q_2(y) \in R_{1,m}[y]$ such that

$$(c_1(y), c_2(y)) = q_1(y) \star (f(y), 0) + q_2(y) \star (f_1(y), g_1(y) + ua_1(y)).$$

Henceforth, $\ker(\varphi)$ forms a submodule of C generated by $(f(y), 0, 0)$ and $(f_1(y), g_1(y)+ua_1(y), 0)$. In other words, $\ker(\varphi) = \langle (f(y), 0, 0), (f_1(y), g_1(y)+ua_1(y), 0)\rangle$. Hence, applying the First isomorphism theorem, we have $\frac{C}{\ker(\varphi)} \cong \varphi(C) = \langle h(y)+ub_1(y)+u^2b_2(y)+u^3b_3(y)\rangle$. Consider $(f(y), g(y), h(y)+ub_1(y)+u^2b_2(y)+u^3b_3(y)) \in C$. Then

$$\varphi(f(y), g(y), h(y)+ub_1(y)+u^2b_2(y)+u^3b_3(y)) = h(y)+ub_1(y)+u^2b_2(y)+u^3b_3(y).$$

Moreover, any $\mathbb{Z}_2R_1R_2$-additive cyclic code C can be generated as an $R_2[y]$-submodule of $R_{l,m,n}[y]$ as follows:

$$C = \langle (f(y), 0, 0), (f_1(y), g_1(y) + ua_1(y), 0), (f_2(y), g_2(y), h(y) + ub_1(y) + u^2b_2(y) + u^3b_3(y))\rangle.$$

Theorem 3.3. *Let* $C = \langle (f(y), 0, 0), (f_1(y), g_1(y)+ua_1(y), 0), (f_2(y), g_2(y), h(y)+ub_1(y) + u^2b_2(y) + u^3b_3(y))\rangle$ *be a* $\mathbb{Z}_2R_1R_2$*-additive cyclic code of length* (l, m, n). *Then*

(1) $f(y)|(\frac{y^m-1}{a_1(y)})f_1(y)$ *and* $f(y)|(\frac{y^m-1}{a_1(y)})\gcd(f_1(y), f(y))$,
(2) $f_1(y)|(\frac{y^n-1}{b_3(y)})f_2(y)$ *and there exist some polynomials* $q(y) \in R_{1,m}[y]$ *such that* $(\frac{y^n-1}{b_3(y)})g_2(y) = (g_1(y) + ua_1(y))q(y)$ *and*
(3) $f(y)|(q(y)f_1(y) + \big(\frac{y^n-1}{b_3(y)}\big)f_2(y))$.

Proof. Suppose φ is a module homomorphism as defined in 3.4. Then $\varphi(C) = h(y) + ub_1(y) + u^2b_2(y) + u^3b_3(y)$. Consider

$$\begin{aligned}
&((\frac{y^m-1}{a_1(y)}) \star (f_1(y), g_1(y) + ua_1(y), 0))\\
&\quad = (\delta_1(\frac{y^m-1}{a_1(y)})f_1(y), \delta_2(\frac{y^m-1}{a_1(y)})(g_1(y) + ua_1(y)), 0)\\
&\quad = (\delta_1(\frac{y^m-1}{a_1(y)})f_1(y), 0, 0), \varphi((\frac{y^m-1}{a_1(y)}) \star (f_1(y), g_1(y) + ua_1(y), 0))\\
&\quad = 0.
\end{aligned}$$

This implies that $(\delta_1(\frac{y^m-1}{a_1(y)})f_1(y),0,0)=((\frac{y^m-1}{a_1(y)})f_1(y),0,0)\in \ker(\varphi)\subseteq C$. Also, we have $(f(y),0,0)\in C$. Hence, $f(y)|(\frac{y^m-1}{a_1(y)})f_1(y)$, which meets our requirements.

Since $b_j(y)|b_i(y)|h(y)|(y^n-1) \mod u^3$ for $1\leq i<j\leq 3$. Moreover, we have $\delta_1(\frac{y^n-1}{b_3(y)})f_2(y)=(\frac{y^n-1}{b_3(y)})f_2(y)$ and $\delta_2(\frac{y^n-1}{b_3(y)})g_2(y)=(\frac{y^n-1}{b_3(y)})g_2(y)$. Consider

$$\begin{aligned}(\frac{y^n-1}{b_3(y)})\star(f_2(y),g_2(y),h(y)+ub_1(y)+u^2b_2(y)+u^3b_3(y))&\\ =(\delta_1(\frac{y^n-1}{b_3(y)})f_2(y),\delta_2(\frac{y^n-1}{b_3(y)})g_2(y),0)&\\ =((\frac{y^n-1}{b_3(y)})f_2(y),(\frac{y^n-1}{b_3(y)})g_2(y),0),&\\ \Longrightarrow \varphi((\frac{y^n-1}{b_3(y)})\star(f_2(y),g_2(y),h(y)+ub_1(y)+u^2b_2(y)+u^3b_3(y)))=\ 0.&\end{aligned}$$

Thus, we have $((\frac{y^n-1}{b_3(y)})f_2(y),(\frac{y^n-1}{b_3(y)})g_2(y),0)\in\ker(\varphi)\subseteq C$. Since $(f_1(y),g_1(y)+ua_1(y),0))\in C$. Therefore, $f_1(y)|(\frac{y^n-1}{b_3(y)})f_2(y)$ and $(g_1(y)+ua_1(y))|(\frac{y^n-1}{b_3(y)})g_2(y)$ as a consequence, there exists polynomial $q(y)\in R_{1,m}[y]$ such that $(\frac{y^n-1}{b_3(y)})g_2(y)=(g_1(y)+ua_1(y))q(y)$. Since R is a commutative ring, and C is an additive cyclic code. Thus, we have

$$\begin{aligned}&q(y)(f_1(y),g_1(y)+ua_1(y),0))\\ &\quad-(\frac{y^n-1}{b_3(y)})(f_2(y),g_2(y),h(y)+ub_1(y)+u^2b_2(y)+u^3b_3(y))\\ &\quad=(q(y)f_1(y)+(\frac{y^n-1}{b_3(y)})f_2(y),0,0).\end{aligned}$$

This expression belongs to $\ker(\varphi)\subseteq C$. Additionally, we have $(f(y),0,0)\in C$. As a result, we can conclude that $f(y)$ divides $q(y)f_1(y)+(\frac{y^n-1}{b_3(y)})f_2(y)$. □

Example 3.4. Let C be a $\mathbb{Z}_2R_1R_2$-additive cyclic code with a block length $(15,7,7)$ in $R_{15,7,7}[y]=\frac{\mathbb{Z}_2[y]}{<y^{15}-1>}\times\frac{R_1[y]}{<y^7-1>}\times\frac{R_2[y]}{<y^7-1>}$ generated by

$$\langle(f(y),0,0),(f_1(y),g_1(y)+ua_1(y),0)(f_2(y),g_2(y),h(y)+ub_1(y)+u^2b_2(y)+u^3b_3(y))\rangle.$$

Then, we have $y^{15}-1=(y+1)(y^2+y+1)(y^4+y+1)(y^4+y^3+1)(y^4+y^3+y^2+y+1)$ and $y^7-1=(y+1)(y^3+y+1)(y^3+y^2+1)$ over $\mathbb{Z}_2$. Assume that $f(y)=(y+1)(y^4+y+1)\in\mathbb{Z}_{2,15}[y]$, $g_1(y)=(y+1)(y^3+y+1)$, $a_1(y)=(y^3+y+1)\in\mathbb{Z}_{2,7}[y]$, $g_2(y)=(y+1+u)\in R_{1,7}[y]$, $h(y)=y^4+y^2+y+1$, $b_1(y)=b_2(y)=b_3(y)=y^3+y^2+1$. Then, by applying the Theorem 3.3, we get $f_1(y)=f_2(y)=y^4+y+1$ and $(\frac{y^7-1}{b_3(y)})g_2(y)=(g_1(y)+ua_1(y))q(y)$ implies that $q(y)=y+1$.

Example 3.5. Let C be a $\mathbb{Z}_2R_1R_2$-additive cyclic code with a block length $(15,9,9)$ in $R_{15,9,9}[y]=\frac{\mathbb{Z}_2[y]}{<y^{15}-1>}\times\frac{R_1[y]}{<y^9-1>}\times\frac{R_2[y]}{<y^9-1>}$ generated by

$$\langle(f(y),0,0),(f_1(y),g_1(y)+ua_1(y),0)(f_2(y),g_2(y),h(y)+ub_1(y)+u^2b_2(y)+u^3b_3(y))\rangle.$$

Then, we have $y^{15}-1=(y+1)(y^2+y+1)(y^4+y+1)(y^4+y^3+1)(y^4+y^3+y^2+y+1)$ and $y^9-1=(y+1)(y^2+y+1)(y^6+y^3+1)$ over $\mathbb{Z}_2$. Assume that $f(y)=$

$(y+1)(y^4+y+1)(y^4+y^3+1) \in \mathbb{Z}_{2,15}[y]$, $g_1(y) = (y+1)(y^6+y^3+1)$, $a_1(y) = (y^6+y^3+1) \in \mathbb{Z}_{2,9}[y]$, $g_2(y) = (y+1+u) \in R_{1,9}[y]$, $h(y) = (y^2+y+1)(y^6+y^3+1)$, $b_1(y) = b_2(y) = b_3(y) = y^2+y+1$. Then, by applying the Theorem 3.3, we get $f_1(y) = f_2(y) = (y^4+y+1)(y^4+y^3+1)$ and $(\frac{y^7-1}{b_3(y)})g_2(y) = (g_1(y)+ua_1(y))q(y)$ suggested that $q(y) = y+1$.

Theorem 3.6. *Let* $C = \langle (f(y),0,0), (f_1(y), g_1(y)+ua_1(y), 0), (f_2(y), g_2(y), h(y)+ub_1(y)+u^2b_2(y)+u^3b_3(y))\rangle$ *be a* $\mathbb{Z}_2R_1R_2$*-additive cyclic code possessing the block length* (l,m,n) *with the properties* $f(y)p_f(y) \equiv (y^l-1) \mod 2$, $g_1(y)p_1(y) \equiv (y^m-1) \mod u^2$, $g_1(y) = a_1(y)p_2(y) \mod u^2$, $h(y)q(y) \equiv (y^n-1) \mod u^4$,$h(y) = b_1(y)q_1(y)$,$b_1(y) = b_2(y)q_2(y)$ *and* $b_2(y) = b_3(y)q_3(y)$, *where* $p_f(y), p_i(y), q_j(y)| \in \mathbb{Z}_2[y]$, $i = 1,2$*;* $j = 1,2,3,4$. *Define the sets*

$$E_1 = \bigcup_{i=0}^{\deg(p_f(y))-1} y^i \star (f(y),0,0),$$

$$E_2 = \bigcup_{i=0}^{\deg(p_1(y))-1} y^i \star (f_1(y), g_1(y)+ua_1(y), 0),$$

$$E_3 = \bigcup_{i=0}^{\deg(p_2(y))-1} y^i \star (f_1(y)p_1(y), ua_1(y)p_1(y), 0),$$

$$E_4 = \bigcup_{i=0}^{\deg(q(y))-1} y^i \star (f_2(y), g_2(y), h(y)+ub_1(y)+u^2b_2(y)+u^3b_3(y)),$$

$$E_5 = \bigcup_{i=0}^{\deg(q_1(y))-1} y^i \star (f_2(y)q(y), g_2(y)q(y), u(b_1(y)+ub_2(y)+u^2b_3(y))q(y)),$$

$$E_6 = \bigcup_{i=0}^{\deg(q_2(y))-1} y^i \star (f_2(y)q(y)q_1(y), g_2(y)q(y)q_1(y), u^2(b_2(y)+ub_3(y))q(y)q_1(y)),$$

$$E_7 = \bigcup_{i=0}^{\deg(q_3(y))-1} y^i \star (f_2(y)q(y)q_1(y)q_2(y), g_2(y)q(y)q_1(y)q_2(y), u^3b_3(y)q(y)q_1(y)q_2(y)).$$

Then $G_M = \bigcup_{i=1}^{7} E_i$ *forms a minimal generating set for* C*, and the code* C *has* $2^{\deg(p_f(y))}4^{\deg(p_1(y))}2^{\deg(p_2(y))}16^{\deg(q(y))}8^{\deg(q_1(y))}4^{\deg(q_2(y))}2^{\deg(q_3(y))}$*, number of codewords.*

Proof. Suppose that $c(y)$ is a codeword in C. We can assert that there exist polynomials $v_1(y)$, $v_2(y)$, and $v_3(y)$ in $R_{2,n}[y]$ such that $c(y)$ can be expressed as follows:

$$\begin{aligned} c(y) =& v_1(y) \star (f(y),0,0) + v_2(y) \star (f_1(y), g_1(y)+ua_1(y), 0) \\ &+ v_3(y) \star (f_2(y), g_2(y), h(y)+ub_1(y)+u^2b_2(y)+u^3b_3(y)). \end{aligned}$$

If $\deg(\delta_1(v_1(y))) < l - \deg(f(y))$. Then $v_1(y) \star (f(y),0,0) \in span(E_1)$. On the other hand, by the division algorithm, there exist polynomials $r_1(y)$, $s_1(y) \in \mathbb{Z}_2[y]$, which satisfy that $\delta_1(v_1(y)) = s_1(y)p_f(y)+r_1(y)$, where $r_1(y) = 0$ or $\deg(r_1(y)) =$

$l - \deg(f(y))$ and $f(y).p_f(y) \equiv (y^l - 1)$. Then, we get

$$\begin{aligned} v_1(y) \star (f(y), 0, 0) =& s_1(y)p_f(y) \star (f(y), 0, 0) + r_1(y) \star (f(y), 0, 0) \\ =& r_1(y) \star (f(y), 0, 0) \in span(E_1). \end{aligned}$$

If $\deg(\delta_2(v_2(y))) < m - \deg(g_1(y))$, then $v_2(y) \star (f_1(y), g_1(y) + ua_1(y), 0) \in span(E_2)$. Contrarily, by the division algorithm, there exist polynomials $r_2(y)$, $s_2(y) \in \mathbb{Z}_2[y]$ such that $v_2(y) = s_2(y)p_1(y)+r_2(y)$, where $r_2(y) = 0$ or $\deg(r_2(y)) < m - \deg(g_1(y))$. Then, we have

$$\begin{aligned} v_2(y) \star (f_1(y), g_1(y) + ua_1(y), 0) =& s_2(y) \star (f_1(y)p_1(y), ua_1(y)p_1(y), 0) \\ &+ r_2(y) \star (f_1(y), g_1(y) + ua_1(y), 0). \end{aligned}$$

We can see that the polynomial $r_2(y) \star (f_1(y), g_1(y) + ua_1(y), 0) \in span(E_2)$. If $\deg(s_2(y)) < \deg(p_2(y))$, then $s_2(y) \star (f_1(y)p_1(y), ua_1(y)p_1(y), 0) \in span(E_3)$. Otherwise, applying the division algorithm, we can find polynomials $r_3(y)$, $s_3(y) \in \mathbb{Z}_2[y]$ such that $s_2(y) = s_3(y)p_2(y) + r_3(y)$, where $r_3(y) = 0$ or $\deg(r_3(y)) < \deg(p_2(y))$. Then, we obtain that

$$\begin{aligned} s_2(y) \star (f_1(y)p_1(y), ua_1(y)p_1(y), 0) =& s_3(y) \star (f_1(y)p_1(y)p_2(y), ua_1(y)p_1(y)p_2(y), 0) \\ &+ r_3(y) \star (f_1(y)p_1(y), ua_1(y)p_1(y), 0). \end{aligned}$$

Obviously, $r_3(y) \star (f_1(y)p_1(y), ua_1(y)p_1(y), 0) \in span(E_3)$. Moreover, we have $f_1(y)p_1(y)p_2(y) = f_1(y)\big(\frac{y^m-1}{a_1(y)}\big)$, which is divisible by $f(y)$ and $a_1(y)p_1(y)p_2(y) \equiv 0 \mod (y^m - 1)$. Thus, we have $s_3(y) \star (f_1(y)p_1(y)p_2(y), ua_1(y)p_1(y)p_2(y), 0) \in span(E_1)$. Henceforth, we conclude that

$$v_2(y) \star (f_1(y), g_1(y) + ua_1(y), 0) \in span(E_2 \cup E_3).$$

Consider the term $v_3(y) \star (f_2(y), g_2(y), h(y) + ub_1(y) + u^2b_2(y) + u^3b_3(y))$. If $\deg(v_3(y)) < n - \deg(h(y))$. Then $v_3(y) \star (f_2(y), g_2(y), h(y) + ub_1(y) + u^2b_2(y) + u^3b_3(y)) \in span(E_4)$. Contrarily, applying the division algorithm, we can find polynomials $r_4(y)$, $s_4(y) \in \mathbb{Z}_2[y]$ such that $v_3(y) = s_4(y)q(y)+r_4(y)$ with $r_4(y) = 0$ or $\deg(r_4(y)) < n - \deg(h(y))$. Then, we get

$$\begin{aligned} & v_3(y) \star (f_2(y), g_2(y), h(y) + ub_1(y) + u^2b_2(y) + u^3b_3(y)) \\ =& s_4(y) \star (f_2(y)q(y), g_2(y)q(y), u(b_1(y) + ub_2(y) + u^2b_3(y))q(y)) \\ &+ r_4(y) \star (f_2(y), g_2(y), h(y) + ub_1(y) + u^2b_2(y) + u^3b_3(y)), \end{aligned}$$

where $r_4(y) \star (f_2(y), g_2(y), h(y) + ub_1(y) + u^2b_2(y) + u^3b_3(y)) \in span(E_4)$. If $\deg(s_4(y)) < \deg(q_1(y))$, then $s_4(y) \star (f_2(y)q(y), g_2(y)q(y), u(b_1(y) + ub_2(y) + u^2b_3(y))q(y)) \in span(E_5)$. If not so, then applying the division algorithm, we can find polynomials $r_5(y)$, $s_5(y) \in \mathbb{Z}_2[y]$, which satisfy $s_4(y) = s_5(y)q_1(y)+r_5(y)$ where $r_5(y) = 0$ or $\deg(r_5(y)) < \deg(q_1(y))$. Then, we have

$$\begin{aligned} & s_4(y) \star (f_2(y)q(y), g_2(y)q(y), u(b_1(y) + ub_2(y) + u^2b_3(y))q(y)) \\ =& s_5(y) \star (f_2(y)q(y), g_2(y)q(y), u^2(b_2(y) + ub_3(y))q(y))q_1(y) \\ &+ r_5(y) \star (f_2(y), g_2(y), u(b_1(y) + ub_2(y) + u^2b_3(y)))q(y), \end{aligned}$$

where $r_5(y) \star (f_2(y), g_2(y), u(b_1(y) + ub_2(y) + u^2b_3(y)))q(y) \in span(E_5)$. In the case when $\deg(s_5(y)) < \deg(q_2(y))$, we have

$$s_5(y) \star (f_2(y)q(y), g_2(y)q(y), u^2(b_2(y) + ub_3(y))q(y))q_1(y) \in span(E_6)$$

else ways, by the division algorithm, there exist polynomials $r_6(y)$, $s_6(y) \in \mathbb{Z}_2[y]$ such that $s_5(y) = s_6(y)q_2(y) + r_6(y)$ with $r_6(y) = 0$ or $\deg(r_6(y)) < \deg(q_2(y))$. Thus, we obtain

$$\begin{aligned} s_5(y) \star (f_2(y)q(y), g_2(y)q(y), u^2(b_2(y) + ub_3(y))q(y))q_1(y) \\ =& s_6(y) \star (f_2(y)q(y), g_2(y)q(y), u^3b_3(y)q(y))q_1(y)q_2(y) \\ &+ r_6(y) \star (f_2(y)q(y), g_2(y)q(y), u^2(b_2(y) \\ &+ ub_3(y))q(y))q_1(y), \end{aligned}$$

where $r_6(y) \star (f_2(y)q(y), g_2(y)q(y), u^2(b_2(y) + ub_3(y))q(y))q_1(y) \in span(E_6)$. In the case when $\deg(s_6(y)) < \deg(q_3(y))$, we have

$$s_6(y) \star (f_2(y)q(y), g_2(y)q(y), u^3b_3(y)q(y))q_1(y)q_2(y) \in span(E_7);$$

otherwise, by the division algorithm there exist polynomials $r_7(y)$, $s_7(y) \in \mathbb{Z}_2[y]$ such that $s_6(y) = s_7(y)q_3(y) + r_7(y)$, where $r_7(y) = 0$ or $\deg(r_7(y)) < \deg(q_3(y))$. Thus, we have

$$\begin{aligned} s_6(y) \star (f_2(y)q(y), g_2(y)q(y), u^3b_3(y)q(y))q_1(y)q_2(y) \\ =& s_7(y) \star (f_2(y)q(y), g_2(y)q(y), u^3b_3(y)q(y))q_1(y)q_2(y)q_3(y) \\ &+ r_7(y) \star (f_2(y)q(y), g_2(y)q(y), u^3b_3(y)q(y))q_1(y)q_2(y), \end{aligned}$$

where $r_7(y) \star (f_2(y)q(y), g_2(y)q(y), u^3b_3(y)q(y))q_1(y)q_2(y) \in span(E_7)$. Furthermore, we have $q(y)q_1(y)q_2(y)q_3(y) = \frac{y^n-1}{b_3(y)}$. Also, from previous lemma, we have $f(y)|\mu(y)f_1(y) + \left(\frac{y^n-1}{b_3(y)}\right)f_2(y)$ implies that $\left(\frac{y^n-1}{b_3(y)}\right)f_2(y) = \lambda(y)f(y) - \mu(y)f_1(y)$ and $\left(\frac{y^n-1}{b_3(y)}\right)g_2(y) = (g_1(y) + ua_1(y))\mu(y)$ for some $\mu(y) \in \mathbb{Z}_2[y]$. Thus, we get

$$\begin{aligned} s_7(y) \star (f_2(y)q(y), g_2(y)q(y), u^3b_3(y)q(y))q_1(y)q_2(y)q_3(y) \\ =& s_7(y) \star (\lambda(y)f(y) - \mu(y)f_1(y), (g_1(y) + ua_1(y))\mu(y), 0) \\ =& s_7(y) \star (f(y), 0, 0)\lambda(y) + (f_1(y), (g_1(y) + ua_1(y)), 0)\mu(y) \\ \in& span(E_1 \cup E_2). \end{aligned}$$

Thus, we find that $v_3(y) \star (f_2(y), g_2(y), h(y) + ub_1(y) + u^2b_2(y) + u^3b_3(y)) \in span(E_1 \cup E_2 \cup E_4 \cup E_5 \cup E_6 \cup E_7)$. Finally, we conclude that $G_M = \bigcup_{i=1}^{7} E_i$ forms a minimal generating set for C, and it contains

$$2^{\deg(p_f(y))}4^{\deg(p_1(y))}2^{\deg(p_2(y))}16^{\deg(q(y))}8^{\deg(q_1(y))}4^{\deg(q_2(y))}2^{\deg(q_3(y))}$$

codewords. □

Corollary 3.7. *Let $C = \langle (f(y), 0), (f_1(y), g_1(y) + ua_1(y)) \rangle$ be a $\mathbb{Z}_2R_1$-additive cyclic code of block length (l, m) with $f(y)p_f(y) \equiv (y^l - 1)$, $g_1(y)p_1(y) \equiv (y^m - 1)$*

$\bmod u^2$, $g_1(y) = a_1(y)p_2(y) \mod u^2$, *where* $p_f(y), p_i(y)| \in \mathbb{Z}_2[y]$, *for* $i = 1, 2$. *Define the sets*

$$E_1 = \bigcup_{i=0}^{\deg(p_f(y))-1} y^i \star (f(y), 0),$$

$$E_2 = \bigcup_{i=0}^{\deg(p_1(y))-1} y^i \star (f_1(y), g_1(y) + ua_1(y)),$$

$$E_3 = \bigcup_{i=0}^{\deg(p_2(y))-1} y^i \star (f_1(y)p_1(y), ua_1(y)p_1(y)).$$

Then $G_M = E_1 \cup E_2 \cup E_3$ *forms a minimal generating set for* C, *and the code* C *has* $2^{\deg(p_f(y))}4^{\deg(p_1(y))}2^{\deg(p_2(y))}$ *count of codewords.*

Corollary 3.8. *Let* $C = \langle (f(y), 0), (f_2(y), h(y) + ub_1(y) + u^2b_2(y) + u^3b_3(y))\rangle$ *be a* $\mathbb{Z}_2R_2$*-additive cyclic code of length* (l, n) *with* $f(y)p_f(y) \equiv (y^l - 1)$, $h(y)q(y) \equiv (y^n - 1) \mod u^4$, $h(y) = b_1(y)q_1(y)$, $b_1(y) = b_2(y)q_2(y)$ *and* $b_2(y) = b_3(y)q_3(y)$, *where* $p_f(y), q_j(y)| \in \mathbb{Z}_2[y]$, $j = 1, 2, 3, 4$. *Define the sets*

$$E_1 = \bigcup_{i=0}^{\deg(p_f(y))-1} y^i \star (f(y), 0),$$

$$E_2 = \bigcup_{i=0}^{\deg(q(y))-1} y^i \star (f_2(y), h(y) + ub_1(y) + u^2b_2(y) + u^3b_3(y)),$$

$$E_3 = \bigcup_{i=0}^{\deg(q_1(y))-1} y^i \star (f_2(y)q(y), u(b_1(y) + ub_2(y) + u^2b_3(y))q(y)),$$

$$E_4 = \bigcup_{i=0}^{\deg(q_2(y))-1} y^i \star (f_2(y)q(y)q_1(y), u^2(b_2(y) + ub_3(y))q(y)q_1(y)),$$

$$E_5 = \bigcup_{i=0}^{\deg(q_3(y))-1} y^i \star (f_2(y)q(y)q_1(y)q_2(y), u^3b_3(y)q(y)q_1(y)q_2(y)).$$

Then $G_M = \bigcup_{i=1}^{5} E_i$ *forms a minimal generating set for* C, *and it contains* $2^{\deg(p_f(y))}16^{\deg(q(y))}8^{\deg(q_1(y))}4^{\deg(q_2(y))}2^{\deg(q_3(y))}$ *number of codewords.*

Example 3.9. Let C be a $\mathbb{Z}_2R_1R_2$-additive cyclic code with a block length $(7, 9, 9)$ in $R_{7,9,9}[y] = \frac{\mathbb{Z}_2[y]}{<y^7-1>} \times \frac{R_1[y]}{<y^9-1>} \times \frac{R_2[y]}{<y^9-1>}$ generated by

$$\langle (f(y), 0, 0), (f_1(y), g_1(y)+ua_1(y), 0)(f_2(y), g_2(y), h(y)+ub_1(y)+u^2b_2(y)+u^3b_3(y))\rangle.$$

Then, we have $y^7 - 1 = (y+1)(y^3+y+1)(y^3+y^2+1)$ and $y^9 - 1 = (y+1)(y^2+y+1)(y^6+y^3+1)$ over $\mathbb{Z}_2$. Consider, $f(y) = (y+1)(y^3+y+1) \in R_{1,7}[y]$ and $g_1(y) = (y+1)(y^6+y^3+1)$, $a_1(y) = (y^6+y^3+1) \in R_{1,9}[y]$. Moreover, assume that $h(y) = y^9 - 1$, $b_1(y) = (y^2+y+1)(y^6+y^3+1)$, $b_2(y) = y^2+y+1$, $b_3(y) = 1 \in R_{1,9}[y]$. Then, by applying Theorem 3.3, we find that $f_1(y) = f_2(y) = y^3+y+1$.

Let $g_2(y) = (y + 1 + u)$. Then, the relation $(g_1(y) + ua_1(y))q(y) = (\frac{y^n-1}{b_3(y)})g_2(y)$ implies that $q(y) = y^3 - 1$. Now, by using Theorem 3.6, we have

$$\begin{aligned} f(y)p_f(y) &= y^7 - 1 \Rightarrow p_f(y) = y^3 + y^2 + 1, \\ g_1(y)p_1(y) &= y^9 - 1 \Rightarrow p_1(y) = y^2 + y + 1, \\ g_1(y) &= a_1(y)p_2(y) \Rightarrow p_2(y) = y + 1, \\ h(y)q(y) &= y^9 - 1 \Rightarrow q(y) = 1, \\ h(y) &= b_1(y)q_1(y) \Rightarrow q_1(y) = y + 1, \\ b_1(y) &= b_2(y)q_2(y) \Rightarrow q_2(y) = y^6 + y^3 + 1, \\ b_2(y) &= b_3(y)q_3(y) \Rightarrow q_3(y) = y^2 + y + 1. \end{aligned}$$

Now, by using Theorem 3.6, the minimum generating set for this code is given by $G_M = \bigcup_{i=1}^{6} E_i$, where

$$\begin{aligned} E_1 &= \bigcup_{i=0}^{2} y^i \star ((y + 1)(y^3 + y + 1), 0, 0), \\ E_2 &= \bigcup_{i=0}^{1} y^i \star ((y^3 + y + 1), (y^6 + y^3 + 1)(y + 1 + u), 0), \\ E_3 &= ((y^2 + y + 1)(y^3 + y + 1), u(y^2 + y + 1)(y^6 + y^3 + 1), 0), \\ E_4 &= ((y^3 + y + 1), (y + 1 + u), u(y^2 + y + 1)(y^6 + y^3 + 1) + u^2(y^2 + y + 1) + u^3), \\ E_5 &= \bigcup_{i=0}^{5} y^i \star ((y^3 + y + 1)(y + 1), (y + 1 + u)(y + 1), (u^2(y^2 + y + 1) + u^3)(y + 1)), \\ E_6 &= \bigcup_{i=0}^{1} y^i \star ((y^3 + y + 1), (y + 1 + u), u^3)(y + 1)(y^6 + y^3 + 1). \end{aligned}$$

4. Dual of an additive cyclic codes

In this module, the structure of dual code of additive cyclic codes over $\mathbb{Z}_2R_1R_2$ and their properties are studied. For this, first we define the inner product over the mixed alphabet $\mathbb{Z}_2^l R_1^m R_2^n$. Suppose that $A_1 = ((a_{1,i})|(b_{1,j})|(c_{1,k}))_{i,j,k=1}^{l,m,n}$ and $A_2 = ((a_{2,i})|(b_{2,j})|(c_{2,k}))_{i,j,k=1}^{l,m,n}$ are two elements. Then, we define a new inner product over $\mathbb{Z}_2^l R_1^m R_2^n$ as follows:

$$\langle A_1, A_2 \rangle = u^3 \sum_{i=1}^{l} a_{1,i}.a_{2,i} + u^2 \sum_{j=1}^{m} b_{1,j}.b_{2,j} + \sum_{k=1}^{n} c_{1,k}.c_{2,k}.$$

The dual code $C^\perp$ of the $\mathbb{Z}_2R_1R_2$-linear code C with a block length (l, m, n) is defined as

$$C^\perp = \{w \in \mathbb{Z}_2^l R_1^m R_2^n | \ \langle v, w \rangle = 0, \forall v \in C\}.$$

It can be proven that $C^\perp$ is linear, provided C is linear. Now, we check out the cyclic behavior of $C^\perp$ in the following result.

Theorem 4.1. *Let C be a $\mathbb{Z}_2R_1R_2$-additive cyclic code with a block length (l,m,n). Then, the dual code $C^{\perp}$ is also a $\mathbb{Z}_2R_1R_2$-additive cyclic code.*

Proof. Suppose that C is a $\mathbb{Z}_2R_1R_2$-additive cyclic code. Then it is obvious that C is a linear code. It needs to be demonstrated that C forms a cyclic code over $\mathbb{Z}_2R_1R_2$. For this, assume that $A_1 = ((a_{1,i})|(b_{1,j})|(c_{1,k}))_{i,j,k=0}^{l-1,m-1,n-1} \in C$ and that $A_2 = ((a_{2,i})|(b_{2,j})|(c_{2,k}))_{i,j,k=0}^{l-1,m-1,n-1}$ is an element in $C^{\perp}$. Let $\nu = lcm(l,m,n)$. It is given that C is cyclic; therefore $A_1^{\nu-1} \in C$, where

$$A_1^{\nu-1} = ((a_{1,i+1 \mod l})|(b_{1,j+1 \mod m})|(c_{1,k+1 \mod n}))_{i,j,k=0}^{l-1,m-1,n-1}.$$

Consider

$$\begin{aligned}\langle A_1^{\nu-1}, A_2\rangle =& u^3\sum_{i=0}^{l-1} a_{1,i+1 \mod l}a_{2,i} + u^2\sum_{j=0}^{m-1} b_{1,j+1 \mod m}b_{2,j} + \sum_{k=0}^{n-1} c_{1,k+1 \mod n}c_{2,k}\\ =& u^3(a_{1,0}a_{2,l-1} + \sum_{i=1}^{l-1} a_{1,i}a_{2,i-1}) + u^2(b_{1,0}b_{2,m-1}\sum_{j=1}^{m-1} b_{1,j}b_{2,j-1})\\ &+ (c_{1,0}c_{2,n-1} + \sum_{k=1}^{n-1} c_{1,k}c_{2,k-1})\\ =& \langle A_1, A_2^1\rangle.\end{aligned}$$

Since we are given that A_1 is an arbitrary element, $A_1^{\nu-1} \in C$ and $A_2 \in C^{\perp}$. Therefore, $\langle A_1, A_2^1\rangle = 0$ employs that $A_2^1 \in C^{\perp}$. □

Consider $\nu = lcm(l,m,n)$ and $\theta_\nu(y) = \sum_{i=0}^{\nu-1} y^i$. Then it can be shown that

$$y^\nu - 1 = (y^l-1)\theta_{\frac{\nu}{l}}(y^l) = (y^m-1)\theta_{\frac{\nu}{m}}(y^m) = (y^n-1)\theta_{\frac{\nu}{n}}(y^n).$$

Let us introduce the bilinear map φ and examine the orthogonal properties of polynomials in $R_{l,m,n,}[y]$ regarding φ. Consider two arbitrary elements $A_1(y) = (a_1(y)|b_1(y)|c_1(y))$ and $A_2(y) = (a_2(y)|b_2(y)|c_2(y))$ in $R_{l,m,n,}[y]$. Then, we define a bilinear map φ as follows:

$$\varphi : R_{l,m,n,}[y] \times R_{l,m,n,}[y] \longrightarrow \frac{R_2[y]}{(y^\nu - 1)},$$

$$\begin{aligned}&\varphi(A_1(y), A_2(y))\\ &= u^3a_1(y)\theta_{\frac{\nu}{l}}(y^l).y^{\nu-1-\deg(a_2(y))}a_2^*(y) + u^2b_1(y)\theta_{\frac{\nu}{m}}(y^m).y^{\nu-1-\deg(b_2(y))}b_2^*(y)\\ &\quad + c_1(y)\theta_{\frac{\nu}{n}}(y^n).y^{\nu-1-\deg(c_2(y))}c_2^*(y) \mod (y^\nu - 1).\end{aligned}$$

Theorem 4.2. *Let A_1 and A_2 be two elements in $\mathbb{Z}_2^l R_1^m R_2^n$. Then, A_1 is orthogonal to A_2, as well as all its cyclic shifts if and only if $\varphi(A_1(y), A_2(y)) = 0$, where $A_1(y)$, $A_2(y)$ are polynomial representation of $A_1, A_2 \in \mathbb{Z}_2^l R_1^m R_2^n$.*

Proof. Suppose that $A_1 = ((a_{1,i})|(b_{1,j})|(c_{1,k}))_{i,j,k=0}^{l-1,m-1,n-1}$ and

$$A_2 = ((a_{2,i})|(b_{2,j})|(c_{2,k}))_{i,j,k=0}^{l-1,m-1,n-1}$$

are two elements in $\mathbb{Z}_2^l R_1^m R_2^n$. For any positive integer e, consider

$$A_2^e = ((a_{2,i-e})|(b_{2,j-e})|(c_{2,k-e}))_{i,j,k=0}^{l-1,m-1,n-1},$$

be the eth cyclic shift of A_2. Then, A_1 is orthogonal to A_2^e if and only if $\langle A_1, A_2^e\rangle = 0$. Applying the inner product defined above, we obtain

$$u^3 \sum_{i=0}^{l-1} a_{1,i}a_{2,i-e} + u^2 \sum_{j=0}^{m-1} b_{1,j}b_{2,j-e} + \sum_{k=0}^{n-1} c_{1,k}c_{2,k-e} = 0.$$

Let us consider $\nu = lcm(l, m, n)$. Then, by the definition of the bilinear map φ, we have

$$\begin{aligned}
&\varphi(A_1(y), A_2(y))\\
&=u^3 a_1(y)\theta_{\frac{\nu}{l}}(y^l).y^{\nu-1-\deg(a_2(y))}a_2^*(y) + u^2 b_1(y)\theta_{\frac{\nu}{m}}(y^m).y^{\nu-1-\deg(b_2(y))}b_2^*(y)\\
&\quad + c_1(y)\theta_{\frac{\nu}{n}}(y^n).y^{\nu-1-\deg(c_2(y))}c_2^*(y) \mod (y^\nu - 1)\\
&=u^3 \sum_{r=0}^{l-1}\theta_{\frac{\nu}{l}}(y^l)\sum_{i=0}^{l-1} a_{1,i}a_{2,i-r}y^{\nu-1-r} + u^2\sum_{s=0}^{m-1}\theta_{\frac{\nu}{m}}(y^m)\sum_{j=0}^{m-1} b_{1,j}b_{2,j-s}y^{\nu-1-s}\\
&\quad + \sum_{y=0}^{n-1}\theta_{\frac{\nu}{n}}(y^n)\sum_{k=0}^{n-1} c_{1,k}c_{2,k-y}y^{\nu-1-y}\\
&=\theta_{\frac{\nu}{l}}(y^l).u^3\sum_{r=0}^{l-1}\sum_{i=0}^{l-1} a_{1,i}a_{2,i-r}y^{\nu-1-r} + \theta_{\frac{\nu}{m}}(y^m).u^2\sum_{s=0}^{m-1}\sum_{j=0}^{m-1} b_{1,j}b_{2,j-s}y^{\nu-1-s}\\
&\quad + \theta_{\frac{\nu}{n}}(y^n)\sum_{y=0}^{n-1}\sum_{k=0}^{n-1} c_{1,k}c_{2,k-y}y^{\nu-1-y}.
\end{aligned}$$

After permuting the terms, we assume that

$$T_e = u^3 \sum_{i=0}^{l-1} a_{1,i}a_{2,i-e} + u^2 \sum_{j=0}^{m-1} b_{1,j}b_{2,j-e} + \sum_{k=0}^{n-1} c_{1,k}c_{2,k-e}.$$

Then, we have

$$\varphi(A_1(y), A_2(y)) = \sum_{e=0}^{\nu-1} T_e x^{\nu-1-e} \mod (y^\nu - 1).$$

Henceforth, $\varphi(A_1(y), A_2(y)) = 0$ if and only if $T_e = \langle A_1, A_2^e\rangle = 0$ for all $0 \le e \le \nu - 1$. □

Corollary 4.3. *Let $A_1(y) = (a_1(y)|b_1(y)|c_1(y))$ and $A_2(y) = (a_2(y)|b_2(y)|c_2(y))$ be two polynomials in $R_{l,m,n,}[y]$, as defined above such that $\varphi(A_1(y), A_2(y)) = 0$.*

1. *If $a_1(y) = 0$ or $a_2(y) = 0$, then $u^2\sum_{j=0}^{m-1} b_{1,j}b_{2,j-s} + \sum_{k=0}^{n-1} c_{1,k}c_{2,k-s} = 0$.*
2. *If $b_1(y) = 0$ or $b_2(y) = 0$, then $u^3\sum_{i=0}^{l-1} a_{1,i}a_{2,i-r} + \sum_{k=0}^{n-1} c_{1,k}c_{2,k-y} = 0$.*
3. *If $c_1(y) = 0$ or $c_2(y) = 0$, then $u^3\sum_{i=0}^{l-1} a_{1,i}a_{2,i-r} + u^2\sum_{j=0}^{m-1} b_{1,j}b_{2,j-s} = 0$.*

Proof. Suppose that $A_1(y) = (a_1(y)|b_1(y)|c_1(y))$ and $A_2(y) = (a_2(y)|b_2(y)|c_2(y))$ are two polynomials in $R_{l,m,n,}[y]$ corresponding to the codewords

$$A_1 = ((a_{1,i})|(b_{1,j})|(c_{1,k}))_{i,j,k=0}^{l-1,m-1,n-1}$$

and

$$A_2 = ((a_{2,i})|(b_{2,j})|(c_{2,k}))_{i,j,k=0}^{l-1,m-1,n-1}$$

in $\mathbb{Z}_2^l R_1^m R_2^n$, respectively. Furthermore, suppose that $a_1(y) = 0$ or $a_2(y) = 0$ and $\nu_1 = lcm(m, n)$. Then, by the definition of the bilinear map φ, we have

$$\begin{aligned}
&\varphi(A_1(y), A_2(y)) \\
&= u^2 b_1(y)\theta_{\frac{\nu_1}{m}}(y^m).y^{\nu_1-1-\deg(b_2(y))}b_2^*(y) + c_1(y)\theta_{\frac{\nu_1}{n}}(y^n).y^{\nu_1-1-\deg(c_2(y))}c_2^*(y) \\
&= u^2 \sum_{s=0}^{m-1} \theta_{\frac{\nu_1}{m}}(y^m) \sum_{j=0}^{m-1} b_{1,j}b_{2,j-s}y^{\nu_1-1-s} + \sum_{y=0}^{n-1} \theta_{\frac{\nu_1}{n}}(y^n) \sum_{k=0}^{n-1} c_{1,k}c_{2,k-y}y^{\nu_1-1-y} \\
&= \theta_{\frac{\nu_1}{m}}(y^m).u^2 \sum_{s=0}^{m-1}\sum_{j=0}^{m-1} b_{1,j}b_{2,j-s}y^{\nu_1-1-s} + \theta_{\frac{\nu_1}{n}}(y^n) \sum_{y=0}^{n-1}\sum_{k=0}^{n-1} c_{1,k}c_{2,k-y}y^{\nu_1-1-y}.
\end{aligned}$$

Consider $T_{e_1} = u^2 \sum_{j=0}^{m-1} b_{1,j}b_{2,j-e_1} + \sum_{k=0}^{n-1} c_{1,k}c_{2,k-e_1}$. Then, we have

$$\varphi(A_1(y), A_2(y)) = \sum_{e_1=0}^{\nu_1-1} T_{e_1} y^{\nu_1-1-e_1} \mod (y_1^{\nu} - 1).$$

Hence, $\varphi(A_1(y), A_2(y)) = 0$ if and only if $T_{e_1} = u^2 \sum_{j=0}^{m-1} b_{1,j}b_{2,j-e_1} + \sum_{k=0}^{n-1} c_{1,k}c_{2,k-e_1} = 0$. With the similar arguments as in (1), one can prove (2) and (3). □

Corollary 4.4. *Let $A_1(y) = (a_1(y)|b_1(y)|c_1(y))$ and $A_2(y) = (a_2(y)|b_2(y)|c_2(y))$ be two polynomials of $R_{l,m,n,}[y]$ such that $\varphi(A_1(y), A_2(y)) = 0$.*

1. *If $a_1(y) = 0$ or $a_2(y) = 0$ and $b_1(y) = 0$ or $b_2(y) = 0$, then $c_1(y)c_2^*(y) \equiv 0 \mod (y^n - 1)$.*
2. *If $a_1(y) = 0$ or $a_2(y) = 0$ and $c_1(y) = 0$ or $c_2(y) = 0$, then $b_1(y)b_2^*(y) \equiv 0 \mod (y^m - 1)$.*
3. *If $b_1(y) = 0$ or $b_2(y) = 0$ and $c_1(y) = 0$ or $c_2(y) = 0$, then $a_1(y)a_2^*(y) \equiv 0 \mod (y^l - 1)$.*

Proof. Suppose that $A_1(y) = (a_1(y)|b_1(y)|c_1(y))$ and $A_2(y) = (a_2(y)|b_2(y)|c_2(y))$ are two polynomials in $R_{l,m,n,}[y]$ with $a_1(y) = 0$ or $a_2(y) = 0$ and $b_1(y) = 0$ or $b_2(y) = 0$. Then, using the definition of the bilinear map defined above, we obtain

$$\varphi(A_1(y), A_2(y)) = c_1(y)\theta_{\frac{\nu}{n}}(y^n).y^{\nu-1-\deg(c_2(y))}c_2^*(y) \mod (y^{\nu} - 1).$$

We are given that $\varphi(A_1(y), A_2(y)) = 0$. Then, we have

$$\begin{aligned}
c_1(y)\theta_{\frac{\nu}{n}}(y^n).y^{\nu-1-\deg(c_2(y))}c_2^*(y) &= 0 \mod (y^{\nu} - 1), \\
c_1(y)\theta_{\frac{\nu}{n}}(y^n).y^{\nu-1-\deg(c_2(y))}c_2^*(y) &= (y^{\nu} - 1)h(y) \text{ for some } h(y) \in Z_2[y], \\
c_1(y)\theta_{\frac{\nu}{n}}(y^n).y^{\nu}c_2^*(y) &= (y^{\nu} - 1)h(y)y^{1+\deg(c_2(y))} \\
&= (y^{\nu} - 1)h_1(y),
\end{aligned}$$

for some $h_1(y) = h(y)y^{1+\deg(c_2(y))} \in Z_2[y]$. Since we have $y^\nu - 1 = (y^n-1)\theta_{\frac{\nu}{n}}(y^n)$. Applying this value in the above expression, we have

$$\begin{aligned} c_1(y)(y^\nu - 1).y^\nu c_2^*(y) =& (y^\nu - 1)(y^n - 1)h_1(y) \\ \Rightarrow c_1(y)c_2^*(y) =& \frac{(y^n - 1)h_1(y)}{y^\nu}. \end{aligned}$$

Furthermore, we have $\gcd((y^n-1), y^\nu) = 1$. Hence, we conclude that $c_1(y)c_2^*(y) = 0 \mod (y^n - 1)$. By employing similar arguments as in (1), one can prove (2), and (3). □

Acknowledgement. The authors are greatly indebted to the referee for his/her constructive comments and suggestion, which improves the quality of the paper a lot.

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