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# STRONGLY SINGULAR CONVOLUTION OPERATORS AND THEIR COMMUTATORS ON VARIABLE HERZ-TYPE HARDY SPACES

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Abstract. The aim of this paper is to prove that strongly singular convolution operators are bounded from $H\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$ to $\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$ when $\alpha(0) = n - n/p(0)$ and $\alpha_\infty = n - n/p_\infty$. Also, the boundedness of their commutators from the inhomogeneous Herz-type Hardy spaces to the inhomogeneous Herz spaces has been obtained.

## 1. Introduction and preliminaries

As usual, we denote by $\mathbb{R}^n$ the $n$-dimensional real Euclidean space. The Euclidean scalar product of $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ is given by $x \cdot y = x_1y_1 + \cdots + x_ny_n$. By $\mathcal{S}(\mathbb{R}^n)$ we denote the Schwartz space of all complex-valued, in finitely differentiable and rapidly decreasing functions on $\mathbb{R}^n$. Given a function $f \in \mathcal{S}(\mathbb{R}^n)$, its Fourier transform will be denoted by

$$\mathcal{F}(f)(z) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-ix\cdot z} f(x)dx.$$

Moreover, $\rho$ is a smooth radial cut-off function such that

$$\rho(z) = \begin{cases} 1, & \text{if} \quad |z| \geqslant 1 \\ 0, & \text{if} \quad |z| \leqslant \frac{1}{2}. \end{cases}$$

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Define the multipliers

$$\mathbb{T}_{\gamma,d} : \mathcal{F}(\mathbb{T}_{\gamma,d}(f))(z) = \rho(z)\frac{e^{i|z|^{\gamma}}}{|z|^{d}}\mathcal{F}(f)(z),$$

where $0 < \gamma < 1$ and $0 < d \leqslant n\gamma/2$. When $d = n\gamma/2$ the resulting operator will be denoted by $\mathbb{T}_{\gamma}f$. In this paper, we study the multipliers

$$\mathbb{T}_{\gamma} : \mathcal{F}(\mathbb{T}_{\gamma}(f))(z) = \rho(z)\frac{e^{i|z|^{\gamma}}}{|z|^{n\gamma/2}}\mathcal{F}(f)(z).$$

The kernel for $\mathbb{T}_{\gamma}$ is very singular. Roughly speaking, it looks like

$$\mathbb{K}_{\gamma'}(x) = \frac{e^{i|x|^{-\gamma'}}}{|x|^{n}},$$

where $\frac{1}{\gamma'} + \frac{1}{\gamma} = 1$. Indeed the cancellation is minimal and if one makes a quick computation for $|x| \geqslant 2\,|y|$, we have

$$|\mathbb{K}_{\gamma'}(x-y) - \mathbb{K}_{\gamma'}(x)| \leqslant \frac{|y|}{|x|^{n+\gamma'+1}}. \tag{1.1}$$

Strongly singular convolution operators (SSCOs) are not the usual Calderón–Zygmund operators; their kernels are more singular near the diagonal, and they often exhibit oscillatory behavior or fractional-type singularities.

It is well known that SSCOs have been a central topic in modern analysis and are now of increasing applications in areas such as harmonic analysis and partial differential equations. In harmonic analysis, SSCOs are studied for boundedness properties in function spaces like Lebesgue, weighted Morrey, Hardy, and Herz-type spaces. Recently, the boundedness of SSCOs $\mathbb{T}_{\gamma}$ on function spaces has attracted great attention (see [4, 3, 17, 22]). Li and Lu [14] proved that the operators $\mathbb{T}_{\gamma}$ are bounded from the homogeneous weighted Herz-type Hardy spaces $H\dot{K}_{p}^{\alpha,q}(\varrho_1,\varrho_2)$ to the homogeneous weighted Herz spaces $\dot{K}_{p}^{\alpha,q}(\varrho_1,\varrho_2)$ and also obtained the boundedness of these operators on the inhomogeneous weighted Herz-type Hardy spaces. Wang and Yan [20] generalized this results to Herz-type Hardy spaces with variable exponent $p$ but $\alpha$ and $q$ are constants.

In partial differential equations, meanwhile, whereas classical convolution kernels (e.g., Gaussian, Poisson, Laplace) provide regularization for heat and Laplace equations, more strongly singular (Cauchy-type) kernels are used to construct solutions to integro-differential and boundary-value problems in fields such as elasticity, electrostatics, and heat conduction. For many applications, see [12].

The purpose of this paper is to generalize the boundedness theorems of SSCOs $\mathbb{T}_{\gamma}$ and their commutators on variable Herz-type Hardy spaces, where all parameters defining the spaces are variables.

We define the set of variable exponents by

$$\mathcal{P}_0(\mathbb{R}^n) := \{p \text{ measurable: } p(\cdot) : \mathbb{R}^n \to [c,\infty) \text{ for some } c > 0\}.$$

The subset of variable exponents with range $[1,\infty)$ is denoted by $\mathcal{P}(\mathbb{R}^n)$. For $p \in \mathcal{P}_0(\mathbb{R}^n)$, we use the notation $p^- = ess\ \inf_{x\in\mathbb{R}^n} p(x)$, $\quad p^+ = ess\ \sup\limits_{x\in\mathbb{R}^n} p(x)$.

**Definition 1.1.** Let $p \in \mathcal{P}_0(\mathbb{R}^n)$. The variable exponent Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ is the class of all measurable functions $f$ on $\mathbb{R}^n$ such that the modular

$$\eta_{p(\cdot)}(f) := \int_{\mathbb{R}^n} |f(x)|^{p(x)}\, dx$$

is finite. This space is a quasi-Banach function space equipped with the norm

$$\|f\|_{p(\cdot)} := \inf \Big\{ \mu > 0 : \eta_{p(\cdot)}(\frac{1}{\mu} f) \leqslant 1 \Big\}.$$

If $p(x) \equiv p$ is constant, then $L^{p(\cdot)}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ is the classical Lebesgue space.

**Definition 1.2.** We say that a function $\beta : \mathbb{R}^n \to \mathbb{R}$ is locally log-Hölder continuous if there exists a constant $c_{\log} > 0$ such that

$$|\beta(x) - \beta(y)| \leqslant \frac{c_{\log}}{\log(e + 1/|x-y|)}$$

for all $x, y \in \mathbb{R}^n$. If

$$|\beta(x) - \beta(0)| \leqslant \frac{c_{\log}}{\log(e + 1/|x|)}$$

for all $x \in \mathbb{R}^n$, then we say that $\beta$ is log-Hölder continuous at the origin (or has a log decay at the origin). If, for some $\beta_\infty \in \mathbb{R}$ and $c_{\log} > 0$, there holds

$$|\beta(x) - \beta_\infty| \leqslant \frac{c_{\log}}{\log(e + |x|)}$$

for all $x \in \mathbb{R}^n$, then we say that $\beta$ is log-Hölder continuous at infinity (or has a log decay at infinity).

As an example of a function locally log-Hölder continuous, see Nakai and Sawano [18, Example 1.3].

The sets $\mathcal{P}_0^{\log}(\mathbb{R}^n)$ and $\mathcal{P}_\infty^{\log}(\mathbb{R}^n)$ consist of all exponents $r \in \mathcal{P}(\mathbb{R}^n)$ that have a log decay at the origin and at infinity, respectively. The set $\mathcal{P}^{\log}(\mathbb{R}^n)$ is used for all those exponents $r \in \mathcal{P}(\mathbb{R}^n)$ that are locally log-Hölder continuous and have a log decay at infinity, with $r_\infty := \lim_{|x|\to\infty} r(x)$.

For variable exponents, Hölder's inequality takes the form

$$\|f \cdot g\|_{\tau(\cdot)} \leqslant c\, \|f\|_{r(\cdot)}\, \|g\|_{t(\cdot)}\,, \quad r, t, \tau \in \mathcal{P}(\mathbb{R}^n),$$

where $\tau$ is defined pointwise by $\frac{1}{\tau(x)} := \frac{1}{r(x)} + \frac{1}{t(x)}$ and $c > 0$. Often we use the particular case $\tau(x) := 1$ corresponding to the situation when $t = r'$ is the conjugate exponent of $r$.

**Definition 1.3.** Let $p, q \in \mathcal{P}_0(\mathbb{R}^n)$. The mixed Lebesgue-sequence space $\ell^{q(\cdot)}(L^{p(\cdot)})$ is defined on sequences of $L^{p(\cdot)}$-functions by the modular

$$\eta_{\ell^{q(\cdot)}(L^{p(\cdot)})}((f_v)_v) = \sum_v \inf \Big\{ \lambda_v > 0 : \eta_{p(\cdot)}(\frac{f_v}{\lambda_v^{1/q(\cdot)}}) \leqslant 1 \Big\}.$$

The (quasi)-norm is defined from this as usual:

$$\|(f_v)_v\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} = \inf \Big\{ \gamma > 0 : \eta_{\ell^{q(\cdot)}(L^{p(\cdot)})}(\frac{1}{\gamma}(f_v)_v) \leqslant 1 \Big\}.$$

Since $q^+ < \infty$, then we can replace by the simpler expression

$$\eta_{\ell^{q(\cdot)}(L^{p(\cdot)})}((f_v)_v) = \sum_v \left\| |f_v|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}}.$$

If $E \subset \mathbb{R}^n$ is a measurable set, then $|E|$ stands for the (Lebesgue) measure of $E$ and $\chi_E$ denotes its characteristic function.

Before giving the definition of variable Herz spaces, let us introduce the following notations:

$$B_k := B(0, 2^k), \quad R_k := B_k \setminus B_{k-1} \text{ and } \chi_k = \chi_{R_k}, \quad k \in \mathbb{Z}.$$

**Definition 1.4.** Let $p, q \in \mathcal{P}_0(\mathbb{R}^n)$ and $\alpha : \mathbb{R}^n \to \mathbb{R}$ with $\alpha \in L^\infty(\mathbb{R}^n)$. The inhomogeneous Herz space $K_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$ consists of all $f \in L_{Loc}^{p(\cdot)}(\mathbb{R}^n)$ such that

$$\|f\|_{K_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}} := \|f\,\chi_{B_0}\|_{p(\cdot)} + \left\| \left(2^{k\alpha(\cdot)} f\,\chi_k\right)_{k\geqslant 1} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} < \infty.$$

Similarly, the homogeneous Herz space $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$ is defined as the set of all $f \in L_{Loc}^{p(\cdot)}(\mathbb{R}^n \setminus \{0\})$ such that

$$\|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)} := \left\| \left(2^{k\alpha(\cdot)} f\,\chi_k\right)_{k\in\mathbb{Z}} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} < \infty.$$

The variable Herz spaces $K_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$ and $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$, were first introduced by Izuki and Noi [13]. In [11], the authors obtained a new equivalent norm of these function spaces. We refer the reader to the recent paper [9] for further results for these function spaces. If $\alpha(\cdot)$, $p(\cdot)$ and $q(\cdot)$ are constants, then $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$ is the classical Herz space $\dot{K}_p^{\alpha,q}(\mathbb{R}^n)$.

The following proposition is very important for the proof of the main results in this paper; it is from Drihem and seghiri [11]. Recall that the expression $A \lesssim B$ means that $A \leqslant cB$ for some independent constant $c$ (and nonnegative functions $A$ and $B$), and $A \approx B$ means $A \lesssim B \lesssim A$.

**Proposition 1.5.** *Let $\alpha \in L^\infty(\mathbb{R}^n)$ and $p, q \in \mathcal{P}_0(\mathbb{R}^n)$. If $\alpha$ and $q$ are* log*-Hölder continuous at infinity, then*

$$K_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n) = K_{p(\cdot)}^{\alpha_\infty,q_\infty}(\mathbb{R}^n).$$

*Additionally, if $\alpha(\cdot)$ and $q(\cdot)$ have a* log *decay at the origin, then*

$$\|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)} \approx \Big(\sum_{k=-\infty}^{-1} \|2^{k\alpha(0)} f\,\chi_k\|_{p(\cdot)}^{q(0)}\Big)^{1/q(0)} + \Big(\sum_{k=0}^{\infty} \|2^{k\alpha_\infty} f\,\chi_k\|_{p(\cdot)}^{q_\infty}\Big)^{1/q_\infty}.$$

The Hardy–Littlewood maximal operator $\mathcal{M}$ is defined on $L^1_{\mathrm{loc}}$ by

$$\mathcal{M}(f)(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|dy,$$

where $B(x,r)$ is the open ball in $\mathbb{R}^n$ centered at $x \in \mathbb{R}^n$ and radius $r > 0$. It was shown in [7, Theorem 4.3.8] that $\mathcal{M} : L^{p(\cdot)} \to L^{p(\cdot)}$ is bounded if $p \in \mathcal{P}^{\log}$ and $p^- > 1$; see also [6, Theorem 1.2].

Let $\psi \in C_0^\infty(\mathbb{R}^n)$ with $\operatorname{supp}\psi \subseteq B(0,1)$, $\int_{\mathbb{R}^n} \psi(x)dx \neq 0$ and $\psi_t(\cdot) = t^{-n}\psi\left(\frac{\cdot}{t}\right)$ for any $t > 0$. Let $M_\psi(f)$ be the grand maximal function of $f$ defined by

$$\mathcal{M}_\psi(f)(x) := \sup_{t>0} |\psi_t * f(x)|.$$

Here we give the definition of the homogeneous Herz-type Hardy spaces $H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}$.

**Definition 1.6.** Let $p, q \in \mathcal{P}_0(\mathbb{R}^n)$ and $\alpha : \mathbb{R}^n \to \mathbb{R}$ with $\alpha \in L^\infty(\mathbb{R}^n)$. The homogeneous Herz-type Hardy space $H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$ is defined as the set of all $f \in S'(\mathbb{R}^n)$ such that $\mathcal{M}_\psi(f) \in \dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$, and we define

$$\|f\|_{H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}} := \|\mathcal{M}_\psi(f)\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}}.$$

It can be shown that if $\alpha(\cdot)$, $p(\cdot)$ and $q(\cdot)$ satisfy the conditions of Definition 1.6, then the quasi-norm $\|f\|_{H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}}$ does not depend, up to the equivalence of quasi-norms, on the choice of the function $\psi$ and, hence, the space $H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$ is defined independently of the choice of $\psi$. If $p \in \mathcal{P}_0^{\log}(\mathbb{R}^n) \cap \mathcal{P}_\infty^{\log}(\mathbb{R}^n)$ with $-\frac{n}{p^+} < \alpha^- \leqslant \alpha^+ < n - \frac{n}{p^-}$ and $q \in \mathcal{P}_0(\mathbb{R}^n)$, then $H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n) = \dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$. If $\alpha(\cdot) = 0, p(\cdot) = q(\cdot)$, then $H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$ and $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$ coincide with $L^{p(\cdot)}(\mathbb{R}^n)$.

One recognizes immediately that if $\alpha(\cdot)$, $p(\cdot)$ and $q(\cdot)$ are constants, then the spaces $H\dot{K}_p^{\alpha,q}$ are just the usual Herz-type Hardy spaces, which were recently studied in [15, 16]. Wang and Liu [19] studied homogeneous Herz-type Hardy spaces $H\dot{K}_{p(\cdot)}^{\alpha,q}(\mathbb{R}^n)$ with variable $p$, but fixed $\alpha$ and $q$, where the characterization of these function spaces by atomic and molecular decompositions.

We refer the reader to the recent monograph [7, section 4.5] for further details, historical remarks and more references on variable exponent spaces.

## 2. Some technical lemmas

In this section, we present some lemmas used to prove our main theorems in sections 3 and 4.

The following lemma plays an important role in this paper, which was established in [1, Lemma 2.2].

**Lemma 2.1.** *Let $p \in \mathcal{P}_\infty^{\log}(\mathbb{R}^n)$ and $S = B(0,v) \setminus B(0,\frac{v}{2})$. If $|S| \geqslant 2^{-n}$, then*

$$\|\chi_S\|_{p(\cdot)} \approx |S|^{\frac{1}{p(x)}} \approx |S|^{\frac{1}{p_\infty}}$$

*with the implicit constants independent of $v$ and $x \in S$.*

*The left-hand side equivalence remains true for every $|S| > 0$ if we assume, additionally, $p \in \mathcal{P}_0^{\log}(\mathbb{R}^n) \cap \mathcal{P}_\infty^{\log}(\mathbb{R}^n)$.*

The next lemma is a Hardy-type inequality, which is easy; see [8, Lemma 3.4].

**Lemma 2.2.** *Let $0 < t < 1$ and $0 < s \leqslant \infty$. Let $\{\kappa_k\}_{k\in\mathbb{Z}}$ be a sequence of positive real numbers, such that*

$$\left\|\{\kappa_k\}_{k\in\mathbb{Z}}\right\|_{\ell^s} = I < \infty.$$

*Then the sequences* $\left\{\vartheta_k : \vartheta_k = \sum_{j\leqslant k} t^{k-j}\kappa_j\right\}_{k\in\mathbb{Z}}$ *and* $\left\{\mu_k : \mu_k = \sum_{j\geqslant k} t^{j-k}\kappa_j\right\}_{k\in\mathbb{Z}}$ *belong to* $\ell^s$, *and*

$$\left\|\{\vartheta_k\}_{k\in\mathbb{Z}}\right\|_{\ell^s} + \left\|\{\mu_k\}_{k\in\mathbb{Z}}\right\|_{\ell^s} \leqslant c\, I,$$

*with* $c > 0$ *only depending on* $t$ *and* $s$.

A nonnegative locally integrable function $\varrho$ on $\mathbb{R}^n$ is said to belong to $A_s$ $(1 \leqslant s \leqslant \infty)$ if there is a constant $c > 0$ such that

$$\sup_Q \left(\frac{1}{|Q|}\int_Q \varrho(x)dx\right)\left(\frac{1}{|Q|}\int_Q \varrho^{1-s'}(x)dx\right)^{s-1} \leqslant c < \infty,$$

or

$$\sup_Q \left(\frac{1}{|Q|}\int_Q \varrho(x)dx\right) \leqslant c\, ess\inf_Q \varrho(x) < \infty,\ \text{for } s = 1,$$

where $\frac{1}{s}+\frac{1}{s'} = 1$. $Q$ denotes a cube in $\mathbb{R}^n$ with its sides parallel to the coordinate axes. We let $A_\infty$ denote the union of all the $A_s$ classes, $1 \leqslant s < \infty$.

The next lemma treats the $(L^s, L^s)$-boundedness of $\mathbb{T}_\gamma$; see [2, Theorem A].

**Lemma 2.3.** *Let* $1 < s < \infty$ *and* $\varrho \in A_s$. *Then, there exists a constant* $c$ *such that*

$$\int_{\mathbb{R}^n} |\mathbb{T}_\gamma(f)(x)|^s \varrho(x)dx \leqslant c\int_{\mathbb{R}^n} |f(x)|^s \varrho(x)dx.$$

Hereafter $\mathcal{B}$ denotes a family of ordered pairs of non-negative, measurable functions $(f, g)$.

The next lemma is Lemma 13 in [20]; see also [5, Theorem 1.3 ].

**Lemma 2.4.** *Given a family* $\mathcal{B}$ *and an open set* $E \subset \mathbb{R}^n$, *assume that for some* $s_0$, $0 < s_0 < \infty$ *and for every* $\varrho \in A_\infty$,

$$\int_E f(x)^{s_0}\varrho(x)dx \leqslant c_0\int_E g(x)^{s_0}\varrho(x)dx, \quad (f,g)\in\mathcal{B}.$$

*Let* $s$ *be* log-*Hölder continuous, both at the origin and at infinity. Then for all* $(f, g) \in \mathcal{B}$ *such that* $f \in L^{s(\cdot)}(E)$, *we have*

$$\|f\|_{L^{s(\cdot)}(E)} \leqslant c\,\|g\|_{L^{s(\cdot)}(E)}.$$

*Remark* 2.5. Since $A_s \subset A_\infty$, by Lemmas 2.3 and 2.4 it is easy to get the $(L^{s(\cdot)}, L^{s(\cdot)})$ -boundedness of $\mathbb{T}_\gamma$.

To prove our main results, we also need the following two lemmas; see [20, 22].

**Lemma 2.6.** *The kernel for the multiplier operator* $\mathbb{T}_\gamma(f)(x)$ *is given by*

$$c\frac{e^{i\alpha_\gamma|x|^{-\gamma'}}}{|x|^n}\chi(|x|\leqslant 1) + g(x),\quad \frac{1}{\gamma'}+\frac{1}{\gamma} = 1,$$

*with* $|g(x)| \leqslant \frac{c}{(1+|x|)^{n+1}} + \frac{c}{|x|^{n-\varepsilon}}\chi(|x|\leqslant 1)$, $\varepsilon > 0$. *Here* $\alpha_\gamma = \gamma^{\frac{\gamma}{1-\gamma}} - \gamma^{\frac{1}{1-\gamma}}$ *and* $\varepsilon$ *depend only on* $\gamma$.

**Lemma 2.7.** *Let* $\widetilde{\mathbb{K}}_{\gamma',q}(x) = e^{i\alpha_\gamma|x|^{-\gamma'}}/|x|^{n(\gamma'+2)/q}$ *and* $n(\gamma'+2)/q < 1$. *Then*

$$\left\|\widetilde{\mathbb{K}}_{\gamma',q} * f\right\|_q \leqslant c\,\|f\|_{q'},\qquad \frac{1}{q}+\frac{1}{q'} = 1.$$

## 3. Variable Herz estimate of SSCOs

In this section, we present two results concerning the SSCOs $\mathbb{T}_\gamma$. In the first, we show that $\mathbb{T}_\gamma$ can be extended to a bounded operator from $H\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$ to $\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$ for $\alpha(\cdot), p(\cdot)$ and $q(\cdot)$ satisfies some conditions.

Next, we present the boundedness of $\mathbb{T}_\gamma$ on variable Herz-type Hardy spaces into itself. Motivated by [20, 22], we generalize the boundedness for SSCOs $\mathbb{T}_\gamma$ to the case of variable Herz and Herz-type Hardy spaces (all exponents are variables). One of our main results can be stated as follows.

**Theorem 3.1.** *Let $q \in \mathcal{P}_0(\mathbb{R}^n)$ with $0 < q^- \leqslant q^+ \leqslant 1$, $p \in \mathcal{P}(\mathbb{R}^n)$ with $1 < p^- \leqslant p^+ < \infty$ and $\alpha \in L^\infty(\mathbb{R}^n)$. If $\alpha, p$ and $q$ are* log*-Hölder continuous, both at the origin and at infinity such that*

$$\alpha^- > 0,\ \alpha(0) = n - n/p(0) \ \text{ and } \alpha_\infty = n - n/p_\infty,$$

*then $\mathbb{T}_\gamma$ can be extended to a bounded operator from $H\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$ (or $HK^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$) to $\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$ (or $K^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}$).*

*Remark* 3.2. We would like to mention if $\alpha(\cdot)$ and $q(\cdot)$ are constants, then the statements corresponding to Theorem 3.1 can be found in [20, Theorem 6].

To prove Theorem 3.1, we need the notation of atomic decomposition.

**Definition 3.3.** Let $\alpha \in L^\infty(\mathbb{R}^n)$, $p \in \mathcal{P}(\mathbb{R}^n)$, $q \in \mathcal{P}_0(\mathbb{R}^n)$ and $m \in \mathbb{N}_0$. A function $a$ is said to be a central $(\alpha(\cdot), p(\cdot))$-atom if

(i) $\operatorname{supp} a \subset \overline{B(0,r)} = \{x \in \mathbb{R}^n : |x| \leqslant r\}, r > 0$.
(ii) $\|a\|_{p(\cdot)} \leqslant |\overline{B(0,r)}|^{-\alpha(0)/n}, \qquad 0 < r < 1$.
(iii) $\|a\|_{p(\cdot)} \leqslant |\overline{B(0,r)}|^{-\alpha_\infty/n}, \qquad r \geqslant 1$.
(iv) $\int_{\mathbb{R}^n} x^\beta a(x)dx = 0, \quad |\beta| \leqslant m$.

A function $a$ on $\mathbb{R}^n$ is said to be a central $(\alpha(\cdot), p(\cdot))$-atom of restricted type if it satisfies the conditions $(iii)$, $(iv)$ above and $\operatorname{supp} a \subset B(0,r), r \geqslant 1$.

If $r = 2^k$ for some $k \in \mathbb{Z}$ in Definition 3.3, then the corresponding central $(\alpha(\cdot), p(\cdot))$-atom is called a dyadic central $(\alpha(\cdot), p(\cdot))$-atom.

The following theorem presents the atomic decomposition characterization of variable Herz-type Hardy spaces; see [11].

**Theorem 3.4.** *Let $\alpha$ and $q$ be* log*-Hölder continuous, both at the origin and at infinity and $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $1 < p^- \leqslant p^+ < \infty$. For any $f \in H\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$, we have*

$$f = \sum_{k=-\infty}^{\infty} \phi_k a_k, \tag{3.1}$$

*where the series converges in the sense of distributions, $\phi_k \geqslant 0$, each $a_k$ is a central $(\alpha(\cdot), p(\cdot))$-atom with* $\operatorname{supp} a \subset B_k$ *and*

$$\Big(\sum_{k=-\infty}^{-1} \phi_k^{q(0)}\Big)^{1/q(0)} + \Big(\sum_{k=0}^{\infty} \phi_k^{q_\infty}\Big)^{1/q_\infty} \leqslant c\|f\|_{H\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}},$$

*where the constant $c$ depends on the parameters.*

*Conversely, if $\alpha(\cdot) \geqslant n(1-\frac{1}{p^-})$ and $m \geqslant [\alpha^+ + n(\frac{1}{p^-}-1)]$ and if (3.1) holds, then $f \in H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$, and*

$$\|f\|_{H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}} \approx \inf\Big\{\Big(\sum_{k=-\infty}^{-1}\phi_k^{q(0)}\Big)^{1/q(0)} + \Big(\sum_{k=0}^{\infty}\phi_k^{q_\infty}\Big)^{1/q_\infty}\Big\},$$

*where the infimum is taken over all the decomposition of $f$ as above.*

In the following, we use $c$ as a generic positive constant, that is, a constant whose value may change from appearance to appearance.

*Proof of Theorem 3.1.* It suffices to prove that $\mathbb{T}_\gamma$ is bounded from $H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$ to $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$. The nonhomogeneous case can be proved similarly.

We show that

$$\|\mathbb{T}_\gamma(f)\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q\ (\cdot)}(\mathbb{R}^n)} \leqslant c\,\|f\|_{H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)}$$

for all $f \in H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$. Using Proposition 1.5, we have $\mathbb{T}_\gamma(f)$ in $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$-norm is equivalent to

$$\Big\{\sum_{k=0}^{\infty}2^{k\alpha(0)q(0)}\,\|\mathbb{T}_\gamma(f).\chi_k\|_{p(\cdot)}^{q(0)}\Big\}^{1/q(0)} + \Big\{\sum_{k=0}^{\infty}2^{k\alpha_\infty q_\infty}\,\|\mathbb{T}_\gamma(f)\cdot\chi_k\|_{p(\cdot)}^{q_\infty}\Big\}^{1/q_\infty}.$$

By the atomic decomposition characterization of variable Herz spaces (see [10, Theorem 3.8]), we can write

$$f = \sum_{j=-\infty}^{\infty}\phi_j a_j,$$

where $\phi_j \geqslant 0$, $\operatorname{supp} a_j \subseteq B_j$. By the Minkowski inequality, we have

$$\begin{aligned}
\|\mathbb{T}_\gamma(f)\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)} \lesssim\ & \Big\{\sum_{k=-\infty}^{-1}2^{k\alpha(0)q(0)}\Big(\sum_{j=-\infty}^{k-2}\phi_j\,\|\mathbb{T}_\gamma(a_j)\cdot\chi_k\|_{p(\cdot)}\Big)^{q(0)}\Big\}^{1/q(0)}\\
&+\Big\{\sum_{k=-\infty}^{-1}2^{k\alpha(0)q(0)}\Big(\sum_{j=k-1}^{\infty}\phi_j\,\|\mathbb{T}_\gamma(a_j)\cdot\chi_k\|_{p(\cdot)}\Big)^{q(0)}\Big\}^{1/q(0)}\\
&+\Big\{\sum_{k=0}^{\infty}2^{k\alpha_\infty q_\infty}\Big(\sum_{j=-\infty}^{k-2}\phi_j\,\|\mathbb{T}_\gamma(a_j)\cdot\chi_k\|_{p(\cdot)}\Big)^{q_\infty}\Big\}^{1/q_\infty}\\
&+\Big\{\sum_{k=0}^{\infty}2^{k\alpha_\infty q_\infty}\Big(\sum_{j=k-1}^{\infty}\phi_j\,\|\mathbb{T}_\gamma(a_j)\cdot\chi_k\|_{p(\cdot)}\Big)^{q_\infty}\Big\}^{1/q_\infty}\\
:=\ & H_1+H_2+H_3+H_4.
\end{aligned}$$

Let us estimate $H_2+H_4$. By $\left(L^{p(\cdot)}\left(\mathbb{R}^n\right), L^{p(\cdot)}\left(\mathbb{R}^n\right)\right)$ of $\mathbb{T}_\gamma$. Then

$$\begin{aligned} H_2+H_4 \lesssim\ & \left\{\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q(0)}\Big(\sum_{j=k-1}^{\infty} \phi_j \left\|a_j\right\|_{p(\cdot)}\Big)^{q(0)}\right\}^{1/q(0)} \\ & +\left\{\sum_{k=0}^{\infty} 2^{k\alpha_\infty q_\infty}\Big(\sum_{j=k-1}^{\infty} \phi_j \left\|a_j\right\|_{p(\cdot)}\Big)^{q_\infty}\right\}^{1/q_\infty}. \end{aligned} \tag{3.2}$$

For $k\leqslant -1$, we split

$$\sum_{j=k-1}^{\infty} \phi_j \left\|a_j\right\|_{p(\cdot)} = \sum_{j=k-1}^{-1} \cdots + \sum_{j=0}^{\infty} \cdots .$$

Since $a_i$'s are $(\alpha(\cdot), p(\cdot))$-atoms, (3.2) is bounded by

$$\begin{aligned} & c\left\{\sum_{k=-\infty}^{-1}\Big(\sum_{j=k-1}^{-1} \phi_j 2^{(k-j)\alpha(0)}\Big)^{q(0)}\right\}^{1/q(0)} + c\left\{\sum_{k=-\infty}^{-1}\Big(\sum_{j=0}^{\infty} \phi_j 2^{(k-j)\alpha^-}\Big)^{q(0)}\right\}^{1/q(0)} \\ & + c\left\{\sum_{k=0}^{\infty}\Big(\sum_{j=k-1}^{\infty} \phi_j 2^{(k-j)\alpha_\infty}\Big)^{q_\infty}\right\}^{1/q_\infty}. \end{aligned}$$

Since $\alpha(0)$, $\alpha_\infty$ and $\alpha^- > 0$, by Lemma 2.2 (with $t = 2^{-\alpha(0)}, 2^{-\alpha^-}$ and $2^{-\alpha_\infty}$ respectively), $H_2+H_4$ is estimated by

$$c\left\{\sum_{k=-\infty}^{-1} \phi_k^{q(0)}\right\}^{1/q(0)} + c\left\{\sum_{k=0}^{\infty} \phi_k^{q_\infty}\right\}^{1/q_\infty} \leqslant c\left\|f\right\|_{H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)}. \tag{3.3}$$

Let us estimate $H_1+H_3$. Since the kernel for the multiplier operator $\mathbb{T}_\gamma(f)$ is given by (see Lemma 2.6)

$$c\frac{e^{i\alpha_\gamma |x|^{-\gamma'}}}{|x|^n}\chi\left(|x|\leqslant 1\right) + g(x), \quad \frac{1}{\gamma}+\frac{1}{\gamma'}=1,$$

by the Minkowski inequality, we have

$$\begin{aligned} H_1+H_3 \lesssim\ & \left\{\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q(0)}\Big(\sum_{j=-\infty}^{k-2} \phi_j \left\|\left(\mathbb{K}_{\gamma'} * a_j\right)\chi_k\right\|_{p(\cdot)}\Big)^{q(0)}\right\}^{1/q(0)} \\ & +\left\{\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q(0)}\Big(\sum_{j=-\infty}^{k-2} \phi_j \left\|\left(g * a_j\right)\chi_k\right\|_{p(\cdot)}\Big)^{q(0)}\right\}^{1/q(0)} \\ & +\left\{\sum_{k=0}^{+\infty} 2^{k\alpha_\infty q_\infty}\Big(\sum_{j=-\infty}^{k-2} \phi_j \left\|\left(\mathbb{K}_{\gamma'} * a_j\right)\chi_k\right\|_{p(\cdot)}\Big)^{q_\infty}\right\}^{1/q_\infty} \\ & +\left\{\sum_{k=0}^{+\infty} 2^{k\alpha_\infty q_\infty}\Big(\sum_{j=-\infty}^{k-2} \phi_j \left\|\left(g * a_j\right)\chi_k\right\|_{p(\cdot)}\Big)^{q_\infty}\right\}^{1/q_\infty} \\ := \ & H_1^a + H_1^b + H_3^a + H_3^b. \end{aligned} \tag{3.4}$$

For $|x| \geqslant 2^{j+1}$ and by Lemma 2.6, we have

$$\begin{aligned} |g * a_j(x)| &\lesssim \int_{B_j} |g(x-t)|\,|a_j(t)|\,dt \\ &\lesssim \int_{B_j} \Big(\frac{1}{(1+|x-t|)^{n+1}} + \frac{\chi(|x-t| \leqslant 1)}{|x-t|^{n+\varepsilon}}\Big) |a_j(t)|\,dt \\ &\lesssim \Big(\frac{1}{(1+|x|)^{n+1}} + \frac{\chi(|x| \leqslant 2)}{|x|^{n-\varepsilon}}\Big) \int_{B_j} |a_j(t)|\,dt. \end{aligned}$$

On other hand, by generalized Hölder's inequality in $L^1(\mathbb{R}^n)$, with $\frac{1}{p\prime(\cdot)} + \frac{1}{p(\cdot)} = 1$, we obtain

$$\begin{aligned} \int_{B_j} |a_j(t)|\,dt &= \int_{\mathbb{R}^n} |a_j(t)|\,\chi_{B_j} dt \\ &\leqslant \|a_j\|_{p(\cdot)} \left\|\chi_{B_j}\right\|_{p\prime(\cdot)}, \end{aligned}$$

it is known that for $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$, we have

$$\|\chi_{B_j}\|_{p(\cdot)} \|\chi_{B_j}\|_{p'(\cdot)} \approx |B_j|. \tag{3.5}$$

Also,

$$\|\chi_B\|_{p(\cdot)} \approx |B|^{\frac{1}{p(x)}}, \quad x \in B_j$$

for small balls $B_j$ and

$$\|\chi_{B_j}\|_{p(\cdot)} \approx |B_j|^{\frac{1}{p_\infty}}$$

for large balls, with constants only depending on the log-Hölder constant of $p$ (see, for example, [10]).

Note that (3.5) is true. If we replace the ball $B$ by an arbitrary (dyadic) annulus $S = B(0,v)\backslash B(0,v/2)$ (see, [1, Lemma 2.2]), then we can obtain

$$\left\|\chi_{B_j}\right\|_{p\prime(\cdot)} \approx \left\|\chi_j\right\|_{p\prime(\cdot)} \approx \begin{cases} 2^{jn/p\prime(0)} & \text{for } j \leqslant -1, \\ 2^{jn/p'_\infty} & \text{for } j \geqslant 0, \end{cases} \tag{3.6}$$

where $\frac{1}{p\prime(0)} + \frac{1}{p(0)} = 1$ and $\frac{1}{p'_\infty} + \frac{1}{p_\infty} = 1$.

Therefore, we obtain

$$|g * a_j(x)| \lesssim \|a_j\|_{p(\cdot)} \|\chi_j\|_{p'(\cdot)} \Big[\frac{1}{(1+|x|)^{n+1}} + \frac{\chi(|x| \leqslant 2)}{|x|^{n-\varepsilon}}\Big].$$

Therefore, $H_1^b$ is bounded by

$$c\Big\{\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q(0)}\Big(\sum_{j=-\infty}^{k-2} \phi_j \left\|a_j\right\|_{p(\cdot)} \left\|\chi_j\right\|_{p'(\cdot)} \Big\|\frac{1}{(1+|\cdot|)^{n+1}}\chi_k\Big\|_{p(\cdot)}\Big)^{q(0)}\Big\}^{1/q(0)}$$

$$+c\Big\{\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q(0)}\Big(\sum_{j=-\infty}^{k-2} \phi_j \left\|a_j\right\|_{p(\cdot)} \left\|\chi_j\right\|_{p'(\cdot)} \Big\|\frac{\chi(|\cdot|\leqslant 2)}{|\cdot|^{n-\varepsilon}}\chi_k\Big\|_{p(\cdot)}\Big)^{q(0)}\Big\}^{1/q(0)}$$

$$\lesssim c\Big\{\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q(0)}\Big(\sum_{j=-\infty}^{k-2} \frac{\phi_j}{(1+2^k)^{n+1}} \left\|a_j\right\|_{p(\cdot)} \left\|\chi_j\right\|_{p'(\cdot)} \left\|\chi_k\right\|_{p(\cdot)}\Big)^{q(0)}\Big\}^{1/q(0)}$$

$$+c\Big\{\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q(0)}\Big(\sum_{j=-\infty}^{k-2} \frac{\phi_j}{2^{(n-\varepsilon)k}} \left\|a_j\right\|_{p(\cdot)} \left\|\chi_j\right\|_{p'(\cdot)} \left\|\chi_k\right\|_{p(\cdot)}\Big)^{q(0)}\Big\}^{1/q(0)}.$$

Since $k$ and $j$ are negatives numbers, by Lemma 2.1 (see (3.6)), the last expression is dominated by

$$c\Big\{\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q(0)}\Big(\sum_{j=-\infty}^{k-2} \frac{\phi_j}{(1+2^k)^{n+1}} 2^{-j\alpha(0)}\cdot 2^{jn/p'(0)}\cdot 2^{kn/p(0)}\Big)^{q(0)}\Big\}^{1/q(0)}$$

$$+c\Big\{\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q(0)}\Big(\sum_{j=-\infty}^{k-2} \frac{\phi_j}{2^{(n-\varepsilon)k}} 2^{-j\alpha(0)}\cdot 2^{jn/p'(0)}\cdot 2^{kn/p(0)}\Big)^{q(0)}\Big\}^{1/q(0)}$$

$$\lesssim c\Big\{\sum_{k=-\infty}^{-1}\Big(\sum_{j=-\infty}^{k-2} \phi_j 2^{(k-j)(\alpha(0)+n/p(0)-n-1)}\Big)^{q(0)}\Big\}^{1/q(0)}$$

$$+c\Big\{\sum_{k=-\infty}^{-1} 2^{\varepsilon kq(0)}\Big(\sum_{j=-\infty}^{k-2} \phi_j 2^{(k-j)(\alpha(0)+n/p(0)-n)}\Big)^{q(0)}\Big\}^{1/q(0)}. \tag{3.7}$$

Since $\alpha(0)=n-n/p(0)$, by Lemma 2.2 (with $t=2^{-1}$), we obtain that the first term of (3.7) is bounded by

$$c\Big\{\sum_{k=-\infty}^{-1} \phi_k^{q(0)}\Big\}^{1/q(0)} \leqslant c\left\|f\right\|_{H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)}. \tag{3.8}$$

Since $\varepsilon>0$ and $q(0)\leqslant 1$, by the injection $\ell^{q(0)}\hookrightarrow \ell^1$ the second term of (3.7) is bounded by

$$\Big\{\sum_{k=-\infty}^{-1}\phi_k\Big\}\Big\{\sum_{k=-\infty}^{-1} 2^{\varepsilon kq(0)}\Big\}^{1/q(0)} \lesssim \Big\{\sum_{k=-\infty}^{-1}\phi_k^{q(0)}\Big\}^{1/q(0)} \leqslant c\left\|f\right\|_{H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)}. \tag{3.9}$$

Let us estimate $H_3^b$. We write

$$\sum_{j=-\infty}^{k-2} \phi_j \left\|(g*a_j)\,\chi_k\right\|_{p(\cdot)} = \sum_{j=-\infty}^{-1}\cdots + \sum_{j=0}^{k-2}\cdots.$$

Repeating the same arguments used in the estimation of $H_1^b$, the term $H_3^b$ in the right hand side of (3.4) is dominated by

$$\begin{aligned} &c\Big\{\sum_{k=0}^{\infty}2^{k\alpha_\infty q_\infty}\Big(\sum_{j=-\infty}^{-1}\frac{\phi_j}{(1+2^k)^{n+1}}\left\|a_j\right\|_{p(\cdot)}\left\|\chi_j\right\|_{p'(\cdot)}\left\|\chi_k\right\|_{p(\cdot)}\Big)^{q(0)}\Big\}^{1/q(0)} \\ &+c\Big\{\sum_{k=0}^{\infty}2^{k\alpha_\infty q_\infty}\Big(\sum_{j=0}^{k-2}\frac{\phi_j}{(1+2^k)^{n+1}}\left\|a_j\right\|_{p(\cdot)}\left\|\chi_j\right\|_{p'(\cdot)}\left\|\chi_k\right\|_{p(\cdot)}\Big)^{q(0)}\Big\}^{1/q(0)} \\ &+c\Big\{\sum_{k=0}^{1}2^{k\alpha_\infty q_\infty}\Big(\sum_{j=-\infty}^{k-2}\frac{\phi_j}{2^{(n-\varepsilon)k}}\left\|a_j\right\|_{p(\cdot)}\left\|\chi_j\right\|_{p'(\cdot)}\left\|\chi_k\right\|_{p(\cdot)}\Big)^{q(0)}\Big\}^{1/q(0)}. \end{aligned} \tag{3.10}$$

By Lemma 2.1, the second term of (3.10) can be estimated by

$$c\Big\{\sum_{k=0}^{\infty}2^{-kq_\infty}\Big(\sum_{j=-\infty}^{k-2}\phi_j 2^{(k-j)(\alpha_\infty+n/p_\infty-n)}\Big)^{q_\infty}\Big\}^{1/q_\infty}.$$

Since $\alpha_\infty = n - n/p_\infty$ and $q_\infty \leqslant 1$, by the injection $\ell^{q_\infty} \hookrightarrow \ell^1$, the second term of (3.10) is bounded by

$$c\Big\{\sum_{k=0}^{\infty}\phi_k^{q_\infty}\Big\}^{1/q_\infty} \leqslant c\left\|f\right\|_{H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)}. \tag{3.11}$$

By Lemma 2.1 and the injection $\ell^{q(0)} \hookrightarrow \ell^1$, the first term of (3.10) can be estimated by

$$c\Big(\sum_{j=-\infty}^{-1}\phi_j\Big)\Big\{\sum_{k=0}^{\infty}2^{-kq_\infty}\Big\}^{1/q_\infty} \lesssim \Big\{\sum_{j=-\infty}^{-1}\phi_j^{q(0)}\Big\}^{1/q(0)} \leqslant c\left\|f\right\|_{H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)}. \tag{3.12}$$

Since $k \in \{0,1\}$, by Lemma 2.1 and the injection $\ell^{q(0)} \hookrightarrow \ell^1$, the third term of (3.10) is estimated by

$$c\sum_{j=-\infty}^{-1}\phi_j\left\|a_j\right\|_{p(\cdot)}\left\|\chi_j\right\|_{p'(\cdot)} \lesssim \Big\{\sum_{j=-\infty}^{-1}\phi_j^{q(0)}\Big\}^{1/q(0)}.$$

Let us estimate $H_3^a$. Let $|x| \geqslant 2^{j+1}$. By the vanishing moment condition on $a_j(x)$, we have

$$|\mathbb{T}_{\gamma'} * a_j(x)| \leqslant \int_{B_j}|\mathbb{K}_{\gamma'}(x-y)-\mathbb{K}_{\gamma'}(x)|\,|a_j(y)|\,dy.$$

By the condition (1.1) of $\mathbb{K}_{\gamma'}(x)$,

$$|\mathbb{K}_{\gamma'}(x-y)-\mathbb{K}_{\gamma'}(x)| \leqslant c\frac{|y|}{|x|^{n+\gamma'+1}}.$$

If $|x| \geqslant 2\,|y|$, then

$$|\mathbb{K}_{\gamma'} * a_j(x)| \lesssim \frac{2^j}{|x|^{n+\gamma'+1}}\int_{B_j}|a_j(y)|\,dy \lesssim \frac{2^j}{|x|^{n+\gamma'+1}}\left\|a_j\right\|_{p(\cdot)}\left\|\chi_j\right\|_{p'(\cdot)}.$$

By the Minkowski inequality, we get that

$$\begin{aligned} H_3^a &\lesssim \Big\{\sum_{k=0}^{\infty} 2^{k(\alpha_\infty - n + n/p_\infty - \gamma' - 1)q_\infty}\Big(\sum_{j=-\infty}^{-1} 2^j \phi_j \left\|a_j\right\|_{p(\cdot)} \left\|\chi_j\right\|_{p'(\cdot)}\Big)^{q_\infty}\Big\}^{1/q_\infty} \\ &\quad + \Big\{\sum_{k=0}^{\infty} 2^{k(\alpha_\infty - n + n/p_\infty - \gamma' - 1)q_\infty}\Big(\sum_{j=0}^{k-2} 2^j \phi_j \left\|a_j\right\|_{p(\cdot)} \left\|\chi_j\right\|_{p'(\cdot)}\Big)^{q_\infty}\Big\}^{1/q_\infty} \\ &:= H_3^{a1} + H_3^{a2}. \end{aligned}$$

Since $\alpha_\infty = n - n/p_\infty$ and $\alpha(0) = n - n/p(0)$, by Lemma 2.1 and the injection $\ell^{q(0)} \hookrightarrow \ell^\infty$, $H_3^{a1}$ is dominated by

$$\begin{aligned} & c\Big\{\sum_{k=0}^{\infty} 2^{-k(\gamma'+1)q_\infty}\Big(\sum_{j=-\infty}^{-1} 2^j \phi_j\Big)^{q_\infty}\Big\}^{1/q_\infty} \\ & \lesssim \sup_j (\phi_j) \Big\{\sum_{k=0}^{\infty} 2^{-k(\gamma'+1)q_\infty}\Big\}^{1/q_\infty} \sum_{j=-\infty}^{-1} 2^j \\ & \lesssim \Big\{\sum_{k=0}^{\infty} \phi_k^{q(0)}\Big\}^{1/q(0)} \leqslant c \left\|f\right\|_{H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)}. \end{aligned} \tag{3.13}$$

For $H_3^{a2}$, we have

$$\begin{aligned} H_3^{a2} &\lesssim \Big\{\sum_{k=0}^{\infty} 2^{-k(\gamma'+1)q_\infty}\Big(\sum_{j=0}^{k-2} 2^j \phi_j \left\|a_j\right\|_{p(\cdot)} \left\|\chi_j\right\|_{p'(\cdot)}\Big)^{q_\infty}\Big\}^{1/q_\infty} \\ &\lesssim \sup_{j\geqslant 0} (\phi_j) \Big\{\sum_{k=0}^{\infty} 2^{-k\gamma' q_\infty}\Big\}^{1/q_\infty} \\ &\lesssim \Big\{\sum_{k=0}^{\infty} \phi_k^{q_\infty}\Big\}^{1/q_\infty} \\ &\leqslant c \left\|f\right\|_{H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)}. \end{aligned} \tag{3.14}$$

Let us estimate $H_1^a$. We split $\mathbb{K}_{\gamma'} * a_j(x)$ as follows:

$$\begin{aligned} & \mathbb{K}_{\gamma'} * a_j(x) \\ &= c \int_{B_j} \frac{e^{i\alpha_p |x-y|^{-\gamma'}}}{|x-y|^{n(\gamma'+2)/s}} \Big[\frac{1}{|x-y|^{n(1-(\gamma'+2)/s)}} - \frac{1}{|x|^{n(1-(\gamma'+2)/s)}}\Big] a_j(y) dy \\ &\quad + c(\widetilde{\mathbb{K}}_{\gamma',s} * a_j(x)) \frac{1}{|x|^{n(1-(\gamma'+2)/s)}} \\ &:= E(x) + F(x), \end{aligned} \tag{3.15}$$

where $\widetilde{\mathbb{K}}_{\gamma',s}$ is define in Lemma 2.7 and let $s > \max(p^+, 2)$ satisfy $n(\gamma'+2)/s < 1$.

For $|x| \geqslant 2^{j+1}$, we have the pointwise estimate for $E(x)$

$$|E(x)| \lesssim \int_{B_j} \frac{1}{|x-y|^{n(\gamma'+2)/s}} \Big[\frac{1}{|x-y|^{n(1-(\gamma'+2)/s)}} - \frac{1}{|x|^{n(1-(\gamma'+2)/s)}}\Big] |a_j(y)| \, dy.$$

By mean value theorem, we have

$$\Big|\frac{1}{|x-y|^{n(1-(\gamma'+2)/s)}}-\frac{1}{|x|^{n(1-(\gamma'+2)/s)}}\Big|\lesssim\frac{|y|}{|x-y|^{1+n(1-(\gamma'+2)/s)}}.$$

Therefore, we obtain $|E(x)|$ is dominated by

$$|E(x)|\lesssim\frac{1}{|x|^{n+1}}\int_{B_j}|y|\,|a_j(y)|\,dy\lesssim\frac{2^j}{|x|^{n+1}}\left\|a_j\right\|_{p(\cdot)}\left\|\chi_j\right\|_{p'(\cdot)}. \tag{3.16}$$

By the Minkowski inequality, Lemmas 2.1 and 2.2 (with $t=2^{-1}$), we get

$$\begin{aligned}
&\Big\{\sum_{k=-\infty}^{-1}2^{k\alpha(0)q(0)}\Big(\sum_{j=-\infty}^{k-2}\phi_j\left\|E(\cdot).\chi_k\right\|_{p(\cdot)}\Big)^{q(0)}\Big\}^{1/q(0)}\\
&\lesssim\Big\{\sum_{k=-\infty}^{-1}\Big(\sum_{j=-\infty}^{k-2}2^{(k-j)(\alpha(0)-n+n/p(0)-1)}\phi_j\Big)^{q(0)}\Big\}^{1/q(0)}\\
&\lesssim\Big\{\sum_{k=-\infty}^{-1}\phi_k^{q(0)}\Big\}^{1/q(0)}\\
&\leqslant c\left\|f\right\|_{H\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)}.
\end{aligned}\tag{3.17}$$

Noting that for $x\in R_k$, we have

$$\begin{aligned}
&\Big\{\sum_{k=-\infty}^{-1}2^{k\alpha(0)q(0)}\Big(\sum_{j=-\infty}^{k-2}\phi_j\left\|F(\cdot).\chi_k\right\|_{p(\cdot)}\Big)^{q(0)}\Big\}^{1/q(0)}\\
&\lesssim\Big\{\sum_{k=-\infty}^{-1}2^{k\alpha(0)q(0)}\Big(\sum_{j=-\infty}^{k-2}\phi_j\Big\|(\widetilde{\mathbb{K}}_{\gamma',s}*a_j(\cdot))\frac{1}{|\cdot|^{n(1-(\gamma'+2)/s)}}\chi_k\left(\cdot\right)\Big\|_{p(\cdot)}\Big)^{q(0)}\Big\}^{1/q(0)}\\
&\lesssim\Big\{\sum_{k=-\infty}^{-1}2^{k(\alpha(0)-n(1-(\gamma'+2)/s))q(0)}\Big(\sum_{j=-\infty}^{k-2}\phi_j\left\|(\widetilde{\mathbb{K}}_{\gamma',s}*a_j(\cdot))\chi_k\left(\cdot\right)\right\|_{p(\cdot)}\Big)^{q(0)}\Big\}^{1/q(0)}.
\end{aligned}$$

Applying Hölder's inequality with $\frac{1}{p(\cdot)}=\frac{1}{s}+\frac{1}{r(\cdot)}$ and the $L^s\left(\mathbb{R}^n\right)$- boundedness of $\widetilde{\mathbb{K}}_{\gamma',s}*a_j(x)$, we get

$$\Big\{\sum_{k=-\infty}^{-1}2^{k(\alpha(0)-n(1-(\gamma'+2)/s))q(0)}\Big(\sum_{j=-\infty}^{k-2}\phi_j\left\|\widetilde{\mathbb{K}}_{\gamma',s}*a_j\right\|_{s}\left\|\chi_k\right\|_{r(\cdot)}\Big)^{q(0)}\Big\}^{1/q(0)}.$$

By the $L^s\left(\mathbb{R}^n\right)$- boundedness of $\widetilde{\mathbb{K}}_{\gamma',s}*a_j(x)$ (see Lemma 2.7), we get

$$\begin{aligned}
&\Big\{\sum_{k=-\infty}^{-1}2^{k(\alpha(0)+n/p(0)-n/s-n(1-(\gamma'+2)/s))q(0)}\Big(\sum_{j=-\infty}^{k-2}\phi_j\left\|a_j\right\|_{s'}\Big)^{q(0)}\Big\}^{1/q(0)}\\
&\lesssim\Big\{\sum_{k=-\infty}^{-1}2^{k(\alpha(0)+n/p(0)-n/s-n(1-(\gamma'+2)/s))q(0)}\Big(\sum_{j=-\infty}^{k-2}\phi_j\left\|a_j\right\|_{s'}\Big)^{q(0)}\Big\}^{1/q(0)}.
\end{aligned}\tag{3.18}$$

By Hölder's inequality with $\frac{1}{p(\cdot)/s'}+\frac{1}{(p(\cdot)-s')/p(\cdot)}=1$, we have

$$\|a_j\|_{s'} \leqslant \left\|a_j^{s'}\right\|_{p(\cdot)/s'}^{1/s'} \|\chi_{B_j}\|_{\frac{p(\cdot)}{p(\cdot)-s'}}^{1/s'} \leqslant \|a_j\|_{p(\cdot)} \|\chi_{B_j}\|_{\frac{p(\cdot)}{p(\cdot)-s'}}^{1/s'}. \tag{3.19}$$

Since $\alpha(0)=n-n/p(0)$, by Lemma 2.1 and the estimation (3.19), the term in the right-hand side of (3.18) is dominated by

$$\begin{aligned}
&c\Big\{\sum_{k=-\infty}^{-1} 2^{k(-n/s+n(\gamma'+2)/s)q(0)}\Big(\sum_{j=-\infty}^{k-2} 2^{jn/s}\phi_j\Big)^{q(0)}\Big\}^{1/q(0)}\\
&\lesssim \Big\{\sum_{k=-\infty}^{-1} 2^{kn(\gamma'+2)q(0)/s}\Big(\sum_{j=-\infty}^{k-2} 2^{(j-k)n/s}\phi_j\Big)^{q(0)}\Big\}^{1/q(0)}\\
&\lesssim \Big\{\sum_{k=-\infty}^{-1} \Big(\sum_{j=-\infty}^{k-2} 2^{-(k-j)n/s}\phi_j\Big)^{q(0)}\Big\}^{1/q(0)}.
\end{aligned}$$

By Lemma 2.2 (with $t=2^{-n/s}$), we get

$$c\Big\{\sum_{k=-\infty}^{-1}\phi_k^{q(0)}\Big\}^{1/q(0)} \leqslant c\,\|f\|_{H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)}. \tag{3.20}$$

Therefore, by (3.3), (3.8), (3.9), (3.11), (3.12), (3.13), (3.14), (3.17) and (3.20) we complete the proof of Theorem 3.1. □

In the next result, we treat the boundedness of SSCOs $\mathbb{T}_\gamma$ on variable inhomogeneous Herz-type Hardy spaces.

**Theorem 3.5.** *Let $q\in\mathcal{P}_0(\mathbb{R}^n)$ with $0<q^-\leqslant q^+\leqslant 1$, let $p\in\mathcal{P}(\mathbb{R}^n)$ with $1<p^-\leqslant p^+<\infty$, and let $\alpha\in L^\infty(\mathbb{R}^n)$. If $\alpha,p$ and $q$ are* log*-Hölder continuous at infinity such that*

$$n-n/p_\infty \leqslant \alpha_\infty \leqslant n-n/p_\infty+1,$$

*then $\mathbb{T}_\gamma$ can be extended to a bounded operator from $HK_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$ to $HK_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$.*

*Remark* 3.6. We would like to mention if $\alpha(\cdot)$ and $q(\cdot)$ are constants, then the statements corresponding to Theorem 3.5 in $HK_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$ can be found in [20, Theorem 7]. Since $K_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)=K_{p(\cdot)}^{\alpha_\infty,q_\infty}(\mathbb{R}^n)$, the proof of Theorem 3.5 is an immediate consequence of [20, Theorem 7].

## 4. Commutators of SSCOs on variable Herz-type Hardy spaces

In this section, we present the boundedness of the commutators of SSCOs on variable Herz spaces into variable Herz-type Hardy.

**Definition 4.1.** The space $BMO(\mathbb{R}^n)$ consists of all locally integrable functions $f$ such that

$$\|f\|_{BMO(\mathbb{R}^n)} := \sup_Q \frac{1}{|Q|}\int_Q |f(x)-f_Q|\,dx<\infty,$$

where $f_Q = \frac{1}{|Q|}\int_Q f(y)dy$, the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ with sides parallel to the coordinate axes.

It was presented in [21, Lemma 0.5] that if $p \in \mathcal{P}^{\log}$, then for all $h \in BMO(\mathbb{R}^n)$, we have

$$\frac{1}{c}\|h\|^k_{BMO(\mathbb{R}^n)} \leqslant \sup_B \frac{1}{\|\chi_B\|_{p(\cdot)}}\|(h-h_B)^k\chi_B\|_{p(\cdot)} \leqslant c\|h\|^k_{BMO}$$

and all $i, j \in \mathbb{Z}$ with $j > i$,

$$\|(h-h_{B_i})^k\chi_{B_j}\|_{p(\cdot)} \leqslant c(j-i)^k\|h\|^k_{BMO}\|\chi_{B_j}\|_{p(\cdot)}.$$

We introduce the commutators $[h;\mathbb{T}_\gamma]$ as follows.

**Definition 4.2.** Let $h$ be a $BMO(\mathbb{R}^n)$ function and let $\mathbb{T}_\gamma$ be a strongly singular operator. The commutator $[h;\mathbb{T}_\gamma]$ is defined by

$$[h;\mathbb{T}_\gamma]f(x) = h(x)\mathbb{T}_\gamma f(x) - \mathbb{T}_\gamma(hf)(x).$$

The following lemma is [22, Lemma 2.6] and treats the weighted $(L^p, L^p)$-boundedness of $[h;\mathbb{T}_\gamma]$.

**Lemma 4.3.** *Suppose $h \in BMO(\mathbb{R}^n)$, $1 < p < \infty$, $\varrho \in A_1$. Then there exists a constant $c$ such that for any $f$ in weighted $L^p(\mathbb{R}^n)$*

$$\int_{\mathbb{R}^n}\left|[h;\mathbb{T}_\gamma]f(x)\right|^s\varrho(x)dx \leqslant c\int_{\mathbb{R}^n}|f(x)|^s\varrho(x)dx.$$

*Remark* 4.4. Since $A_1 \subset A_\infty$, by Lemmas 2.4 and 4.3 it is easy to get the $(L^{p(\cdot)}, L^{p(\cdot)})$ -boundedness of $[h;\mathbb{T}_\gamma]$.

Here, we present the boundedness of $[h;\mathbb{T}_\gamma]$ on variable Herz-type Hardy spaces.

**Theorem 4.5.** *Let $\alpha \in L^\infty(\mathbb{R}^n), q \in \mathcal{P}_0(\mathbb{R}^n)$ with $0 < q^- \leqslant q^+ \leqslant 1$, $p \in \mathcal{P}(\mathbb{R}^n)$ with $1 < p^- \leqslant p^+ < \infty$, and let $h \in BMO$. If $\alpha, p$ and $q$ be* log*-Hölder continuous at infinity such that*

$$\alpha_\infty = n - n/p_\infty.$$

*Then $[h;\mathbb{T}_\gamma]$ can be extended to a bounded operator from $HK^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$ to $K^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$.*

*Remark* 4.6. We would like to mention if $\alpha(\cdot)$ $p(\cdot)$ and $q(\cdot)$ are constants, then the statements corresponding to Theorem 4.5 in homogeneous spaces can be found in [22, Theorem 2.1].

*Proof of Theorem 4.5.* Our proofs use partially some decomposition techniques already used in [22] where the constant exponent case was studied. We show that

$$\|[h;\mathbb{T}_\gamma]f\|_{K^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)} \leqslant c\|f\|_{HK^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)}$$

for all $f \in HK^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$. Using Proposition 1.5, we have $[h;\mathbb{T}_\gamma]f$ in $K^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$-norm is equivalent to

$$\|[h;\mathbb{T}_\gamma]f\cdot\chi_{B_0}\|_{p(\cdot)} + \left\{\sum_{k=1}^\infty 2^{k\alpha_\infty q_\infty}\|[h;\mathbb{T}_\gamma]f\cdot\chi_k\|^{q_\infty}_{p(\cdot)}\right\}^{1/q_\infty}. \tag{4.1}$$

By the atomic decomposition Theorem in variable Herz spaces, we can write $f = \sum_{j=0}^{\infty} \phi_j a_j$, where $\phi_j \geqslant 0$, $\operatorname{supp} a_j \subseteq B_j$. Then, by the Jensen inequality, we have (4.1) is bounded by

$$\Big(\sum_{j=0}^{\infty} \|[h;\mathbb{T}_\gamma]a_j \cdot \chi_{B_0}\|_{p(\cdot)}\Big)^{q_\infty} + \sum_{j=0}^{\infty} \phi_j^{q_\infty} \Big(\sum_{k=1}^{\infty} 2^{k\alpha_\infty q_\infty} \|[h;\mathbb{T}_\gamma]a_j \cdot \chi_k\|_{p(\cdot)}^{q_\infty}\Big).$$

Since

$$\sum_{j=0}^{\infty} \|[h;\mathbb{T}_\gamma]a_j \cdot \chi_{B_0}\|_{p(\cdot)} \lesssim \sum_{j=0}^{\infty} \|a_j\|_{p(\cdot)} \lesssim \sum_{j=0}^{\infty} 2^{-j\alpha_\infty} \leqslant c, \tag{4.2}$$

then, we only need to prove that, there exists a constant $c > 0$ such that

$$\sum_{k=1}^{\infty} 2^{k\alpha_\infty q_\infty} \|[h;\mathbb{T}_\gamma]a_j \cdot \chi_k\|_{p(\cdot)}^{q_\infty} \leqslant c,$$

where $c$ is independent of the atom $a$.

We split

$$\sum_{k=1}^{\infty} 2^{k\alpha_\infty q_\infty} \|[h;\mathbb{T}_\gamma]a_j \cdot \chi_k\|_{p(\cdot)}^{q_\infty} = \sum_{k=1}^{j+1} \cdots + \sum_{k=j+2}^{\infty} \cdots := I_1 + I_2.$$

Let us estimate $I_1$. By the $(L^{p(\cdot)}(\mathbb{R}^n), L^{p(\cdot)}(\mathbb{R}^n))$-boundedness of $[h;\mathbb{T}_\gamma]$, $I_1$ is bounded by

$$I_1 \lesssim 2^{-j\alpha_\infty q_\infty} \sum_{k=1}^{j+1} 2^{k\alpha_\infty q_\infty} \leqslant c. \tag{4.3}$$

Let us estimate $I_2$. By Lemma 2.6, we have

$$\begin{aligned} I_2 &\lesssim \sum_{k=j+2}^{\infty} 2^{k\alpha_\infty q_\infty} \Big\| \int_{B_j} \mathbb{K}_{\gamma'}(\cdot - y)(h(\cdot) - h(y)) a_j(y) dy \cdot \chi_k \Big\|_{p(\cdot)}^{q_\infty} \\ &\quad + \sum_{k=j+2}^{\infty} 2^{k\alpha_\infty q_\infty} \Big\| \int_{B_j} g(\cdot - y)(h(\cdot) - h(y)) a_j(y) dy \cdot \chi_k \Big\|_{p(\cdot)}^{q_\infty} \\ &:= I_2^a + I_2^b. \end{aligned}$$

Let us estimate $I_2^b$. By the Minkowski inequality, we get that

$$\begin{aligned} I_2^b &\lesssim \sum_{k=j+2}^{\infty} 2^{k\alpha_\infty q_\infty} \Big\| \int_{B_j} g(\cdot - y)(h(\cdot) - h_{B_j}) a_j(y) dy \cdot \chi_k \Big\|_{p(\cdot)}^{q_\infty} \\ &\quad + \sum_{k=j+2}^{\infty} 2^{k\alpha_\infty q_\infty} \Big\| \int_{B_j} g(\cdot - y)(h_{B_j} - h(y)) a_j(y) dy \cdot \chi_k \Big\|_{p(\cdot)}^{q_\infty} \\ &:= I_2^{b1} + I_2^{b2}. \end{aligned}$$

Let us estimate $I_2^{b1}$. For $|x| \geqslant 2^{j+1}$, we have

$$\begin{aligned} |g * a_j(x)| &\lesssim \int_{B_j} |g(x-y)|\,|a_j(y)|\,dy \\ &\lesssim \int_{B_j} \left| \frac{1}{(1+|x-y|)^{n+1}} + \frac{1}{|x-y|^{n-\varepsilon}} \chi(|x-y| \leqslant 1) \right| |a_j(y)|\,dy \\ &\lesssim \|a_j\|_{p(\cdot)} \|\chi_j\|_{p'(\cdot)} \Big( \frac{1}{(1+|x|)^{n+1}} + \frac{1}{|x|^{n-\varepsilon}} \chi(|x| \leqslant 2) \Big). \end{aligned}$$

By Lemma 2.1, we obtain that $I_2^{b1}$ is bounded by

$$\begin{aligned} & c \sum_{k=j+2}^{\infty} 2^{k\alpha_\infty q_\infty} \|a_j\|_{p(\cdot)}^{q_\infty} \|\chi_j\|_{p'(\cdot)}^{q_\infty} \frac{1}{(1+2^k)^{(n+1)q_\infty}} \left\| (h(\cdot) - h_{B_j}) \cdot \chi_k \right\|_{p(\cdot)}^{q_\infty} \\ & + c \sum_{k=j+2}^{1} 2^{k\alpha_\infty q_\infty} \|a_j\|_{p(\cdot)}^{q_\infty} \|\chi_j\|_{p'(\cdot)}^{q_\infty} \frac{1}{2^{k(n-\varepsilon)q_\infty}} \left\| (h(\cdot) - h_{B_j}) \cdot \chi_k \right\|_{p(\cdot)}^{q_\infty} \\ & \lesssim \sum_{k=j+2}^{\infty} 2^{k(\alpha_\infty - n - 1)q_\infty} 2^{-j(\alpha_\infty - n + n/p_\infty)q_\infty} \left\| (h(\cdot) - h_{B_j}) \cdot \chi_k \right\|_{p(\cdot)}^{q_\infty} \\ & \quad + \sum_{k=j+2}^{\infty} 2^{k(\alpha_\infty - n + \varepsilon)q_\infty} 2^{-j(\alpha_\infty - n + n/p_\infty)q_\infty} \left\| (h(\cdot) - h_{B_j}) \cdot \chi_k \right\|_{p(\cdot)}^{q_\infty}. \end{aligned}$$

Since $\alpha_\infty = n - n/p_\infty$ and $h \in BMO$, the term $I_2^{b1}$ is estimated by

$$c \|h\|_{BMO}^{q_\infty} \left\{ \sum_{k=j+2}^{\infty} \frac{(k-j)^{q_\infty}}{2^{kq_\infty}} + \sum_{k=j+2}^{1} \frac{(k-j)^{q_\infty}}{2^{k\varepsilon q_\infty}} \right\} \leqslant c.$$

*Remark* 4.7. Since $x \in R_k$ and $j \geqslant 0$, then we put $\sum_{k=j+2}^{1} \cdots = 0$, and in the following proof, we can avoid the term $\frac{\chi(|x| \leqslant 2)}{|x|^{n-\varepsilon}}$ in inhomogeneous spaces.

Let us estimate $I_2^{b2}$. By Lemma 2.6, we have

$$\begin{aligned} & \left\| \int_{B_j} g(\cdot - y)(h_{B_j} - h(y)) a_j(y) dy . \chi_k \right\|_{p(\cdot)} \\ & \lesssim \left\| \int_{B_j} \Big( \frac{1}{(1+|\cdot - y|)^{n+1}} + \frac{1}{|\cdot - y|^{n-\varepsilon}} \chi(|\cdot - y| \leqslant 2) \Big) (h_{B_j} - h(y)) a_j(y) dy \cdot \chi_k \right\|_{p(\cdot)}. \end{aligned}$$

By the Hölder's inequality, with $\frac{1}{p'(\cdot)} + \frac{1}{p(\cdot)} = 1$, we have

$$\begin{aligned} & \left\| \int_{B_j} \frac{1}{(1+|\cdot - y|)^{n+1}} \left( h_{B_j} - h(y) \right) a_j(y) dy \cdot \chi_k \right\|_{p(\cdot)} \\ & \lesssim \frac{1}{2^{k(n+1)}} \left\| h_{B_j} - h(\cdot) \cdot \chi_{B_j} \right\|_{p'(\cdot)} \|a_j\|_{p(\cdot)} \|\chi_k\|_{p(\cdot)} \\ & \lesssim \frac{1}{2^{k(n+1)}} \|h\|_{BMO} \left\| \chi_{B_j} \right\|_{p'(\cdot)} \|a_j\|_{p(\cdot)} \|\chi_k\|_{p(\cdot)}, \end{aligned}$$

which yields that $I_2^{b2}$ is bounded by

$$c\sum_{k=j+2}^{\infty}2^{k\alpha_\infty q_\infty}\frac{1}{2^{k(n+1)q_\infty}}\left\|h\right\|_{BMO}^{q_\infty}\left\|a_j\right\|_{p(\cdot)}^{q_\infty}\left\|\chi_{B_j}\right\|_{p'(\cdot)}^{q_\infty}\left\|\chi_k\right\|_{p(\cdot)}^{q_\infty}.$$

By Lemma 2.1, the last term is bounded by

$$c\left\|h\right\|_{BMO}^{q_\infty}\sum_{k=j+2}^{\infty}2^{k(\alpha_\infty-n-1+n/p_\infty)q_\infty}2^{-j(\alpha_\infty+n-n/p_\infty)q_\infty}.$$

Since $\alpha_\infty=n-n/p_\infty$ and $h\in BMO$, the term $I_2^{b2}$ is estimated by

$$I_2^{b2}\lesssim c\sum_{k=j+2}^{\infty}\frac{1}{2^{kq_\infty}}\leqslant c. \tag{4.4}$$

Let us estimate $I_2^{a}$, by the Minkowski inequality, we get that

$$\begin{aligned}
I_2^{a} &\lesssim \sum_{k=j+2}^{j_0}2^{k\alpha_\infty q_\infty}\left\|\int_{B_j}\mathbb{K}_{\gamma'}\left(\cdot-y\right)\left(h(\cdot)-h\left(y\right)\right)a_j(y)dy\cdot\chi_k\right\|_{p(\cdot)}^{q_\infty}\\
&\quad+\sum_{k=j_0}^{\infty}2^{k\alpha_\infty q_\infty}\left\|\int_{B_j}\mathbb{K}_{\gamma'}\left(\cdot-y\right)\left(h(\cdot)-h\left(y\right)\right)a_j(y)dy\cdot\chi_k\right\|_{p(\cdot)}^{q_\infty}\\
&:= I_2^{a1}+I_2^{a2},
\end{aligned}$$

where $j_0\in\mathbb{N}$ satisfies $2^{j_0-1}<2^{j(1-\gamma)}\leqslant 2^{j_0}$ if we fixed $j$.

For $I_2^{a2}$, by the Minkowski inequality, we get that

$$\begin{aligned}
I_2^{a2} &\lesssim \sum_{k=j_0}^{\infty}2^{k\alpha_\infty q_\infty}\left\|\int_{B_j}\mathbb{K}_{\gamma'}\left(\cdot-y\right)\left(h(\cdot)-h_{B_j}\right)a_j(y)dy\cdot\chi_k\right\|_{p(\cdot)}^{q_\infty}\\
&\quad+\sum_{k=j_0}^{\infty}2^{k\alpha_\infty q_\infty}\left\|\int_{B_j}\mathbb{K}_{\gamma'}\left(\cdot-y\right)\left(h_{B_j}-h(y)\right)a_j(y)dy\cdot\chi_k\right\|_{p(\cdot)}^{q_\infty}\\
&:= I_2^{a21}+I_2^{a22}.
\end{aligned}$$

Let $|x|\geqslant 2^{j+1}$. Then, by the vanishing moment condition on $a_j(x)$, we have

$$\left|\mathbb{T}_{\gamma'}*a_j(x)\right|\lesssim\int_{B_j}\left|\mathbb{K}_{\gamma'}(x-y)-\mathbb{K}_{\gamma'}(x)\right|\left|a_j(y)\right|dy.$$

From the condition of $\mathbb{K}_{\gamma'}(x)$, we have

$$\left|\mathbb{K}_{\gamma'}(x-y)-\mathbb{K}_{\gamma'}(x)\right|\lesssim\frac{|y|}{|x|^{n+\gamma'+1}}.$$

If $|x| \geqslant 2|y|$, by Hölder's inequality and Lemma 2.1 it follows that

$$\begin{aligned} |\mathbb{K}_{\gamma'} * a_j(x)| &\lesssim \frac{2^j}{|x|^{n+\gamma'+1}} \int_{B_j} |a_j(y)|\, dy \\ &\lesssim \frac{2^j}{|x|^{n+\gamma'+1}} \|a_j\|_{p(\cdot)} \|\chi_j\|_{p'(\cdot)} \\ &\lesssim \frac{2^{j(1-\alpha_\infty+n-n/p_\infty)}}{|x|^{n+\gamma'+1}}. \end{aligned} \tag{4.5}$$

Therefore

$$\begin{aligned} I_2^{a21} &\lesssim \sum_{k=j_0}^{\infty} 2^{k\alpha_\infty q_\infty} 2^{j(1-\alpha_\infty+n-n/p_\infty)q_\infty} \left\| \frac{1}{|\cdot|^{n+\gamma'+1}} \cdot (h(\cdot) - h_{B_j}) \cdot \chi_k \right\|_{p(\cdot)}^{q_\infty} \\ &\lesssim \sum_{k=j_0}^{\infty} 2^{k(\alpha_\infty-n-\gamma'-1)q_\infty} 2^{j(1-\alpha_\infty+n-n/p_\infty)q_\infty} \left\| (h(\cdot) - h_{B_j}) \cdot \chi_k \right\|_{p(\cdot)}^{q_\infty} \\ &\lesssim \|h\|_{BMO}^{q_\infty} \sum_{k=j_0}^{\infty} (k-j)^{q_\infty} 2^{k(\alpha_\infty-n+n/p_\infty-\gamma'-1)q_\infty} 2^{j(1-\alpha_\infty+n-n/p_\infty)q_\infty}. \end{aligned}$$

Since $\alpha_\infty = n - n/p_\infty$, the term $I_2^{a21}$ is estimated by

$$\begin{aligned} c2^{jq_\infty} \|h\|_{BMO}^{q_\infty} \sum_{k=j_0}^{+\infty} \frac{(k-j)^{q_\infty}}{2^{k(\gamma'+1)q_\infty}} &\lesssim \|h\|_{BMO} \frac{2^{jq_\infty}}{2^{j_0(\gamma'+1)q_\infty}} \\ &\lesssim \|h\|_{BMO}\, 2^{-\gamma' j q_\infty} \\ &\leqslant c. \end{aligned} \tag{4.6}$$

Let us estimate $I_2^{a22}$. By the vanishing moment condition on $a_j$, we have

$$I_2^{a22} \lesssim \sum_{k=j_0}^{\infty} 2^{k\alpha_\infty q_\infty} \left\| \int_{B_j} \left(\mathbb{K}_{\gamma'}(x-y) - \mathbb{K}_{\gamma'}(x)\right) (h_{B_j} - h(y)) a_j(y) dy \cdot \chi_k \right\|_{p(\cdot)}^{q_\infty},$$

by (4.5) and the Hölder's inequality with $\frac{1}{p(\cdot)} = \frac{1}{p'(\cdot)} = 1$, we obtain

$$I_2^{a22} \lesssim \sum_{k=j_0}^{\infty} 2^{k(\alpha_\infty-n-\gamma'-1)q_\infty} 2^{j(1-\alpha_\infty+n-n/p_\infty)} \|h\|_{BMO}^{q_\infty} \|\chi_j\|_{p'(\cdot)}^{q_\infty} \|a_j\|_{p(\cdot)}^{q_\infty} \|\chi_k\|_{p(\cdot)}^{q_\infty}.$$

By Lemma 2.1 and since $\alpha_\infty = n - n/p_\infty$, we have

$$I_2^{a22} \lesssim \|h\|_{BMO}^{q_\infty} \frac{2^{jq_\infty}}{2^{j_0(\gamma'+1)q_\infty}} \lesssim \|h\|_{BMO}^{q_\infty} 2^{-\gamma' j q_\infty} \leqslant c. \tag{4.7}$$

Now, we estimate $I_2^{a1}$.

$$
\begin{aligned}
I_2^{a1} &\leqslant \sum_{k=j+2}^{j_0} 2^{k\alpha_\infty q_\infty} \left\| \int_{B_j} \mathbb{K}_{\gamma'}(\cdot - y)\,(h(\cdot) - h_{B_j}) a_j(y) dy \cdot \chi_k \right\|_{p(\cdot)}^{q_\infty} \\
&\quad + \sum_{k=j+2}^{j_0} 2^{k\alpha_\infty q_\infty} \left\| \int_{B_j} \mathbb{K}_{\gamma'}(\cdot - y)\,(h_{B_j} - h(y)) a_j(y) dy \cdot \chi_k \right\|_{p(\cdot)}^{q_\infty} \\
&:= I_2^{a11} + I_2^{a12}.
\end{aligned}
$$

Let us estimate $I_2^{a11}$. As in (3.15), we decompose $\mathbb{K}_{\gamma'} * a_j(x)$ into $E(x) + F(x)$. We have

$$
\begin{aligned}
I_2^{a11} &\lesssim \sum_{k=j+2}^{j_0} 2^{k\alpha_\infty q_\infty} \left\| (E(\cdot) + F(\cdot)) \cdot (h(\cdot) - h_{B_j}) \cdot \chi_k \right\|_{p(\cdot)}^{q_\infty} \\
&\lesssim \sum_{k=j+2}^{j_0} 2^{k\alpha_\infty q_\infty} \left\| E(\cdot) \cdot (h(\cdot) - h_{B_j}) \cdot \chi_k \right\|_{p(\cdot)}^{q_\infty} \\
&\quad + \sum_{k=j+2}^{j_0} 2^{k\alpha_\infty q_\infty} \left\| F(\cdot) \cdot (h(\cdot) - h_{B_j}) \cdot \chi_k \right\|_{p(\cdot)}^{q_\infty} \\
&:= I_2^{a111} + I_2^{a112}.
\end{aligned}
$$

For $I_2^{a111}$, by (3.16), we have

$$
\begin{aligned}
I_2^{a111} &\lesssim \sum_{k=j+2}^{j_0} 2^{k\alpha_\infty q_\infty} 2^{jq_\infty} \|a_j\|_{p(\cdot)} \|\chi_j\|_{p'(\cdot)} \left\| \frac{1}{|\cdot|^{n+1}} \cdot (h(\cdot) - h_{B_j}) \cdot \chi_k \right\|_{p(\cdot)}^{q_\infty} \\
&\lesssim \sum_{k=j+2}^{j_0} 2^{k(\alpha_\infty - n - 1)q_\infty} 2^{jq_\infty} \|a_j\|_{p(\cdot)} \|\chi_j\|_{p'(\cdot)} \left\| (h(\cdot) - h_{B_j}) \cdot \chi_k \right\|_{p(\cdot)}^{q_\infty}.
\end{aligned}
$$

By Lemma 2.1 and since $\alpha_\infty = n - n/p_\infty$, we have

$$
\begin{aligned}
I_2^{a111} &\lesssim \sum_{k=j+2}^{j_0} 2^{k(\alpha_\infty - n - 1)q_\infty} 2^{j(1-\alpha_\infty + n - n/p_\infty)q_\infty} \left\| (h(\cdot) - h_{B_j}) \cdot \chi_k \right\|_{p(\cdot)}^{q_\infty} \\
&\lesssim \|h\|_{BMO}^{q_\infty} \sum_{k=j+2}^{j_0} (k-j)^{q_\infty}\, 2^{k(\alpha_\infty - n + n/p_\infty - 1)q_\infty} 2^{j(-\alpha_\infty + n - n/p_\infty)q_\infty} \\
&\lesssim \|h\|_{BMO}^{q_\infty} 2^{jq_\infty} \sum_{k=j+2}^{j_0} \frac{(k-j)^{q_\infty}}{2^{kq_\infty}} \leqslant c. \qquad (4.8)
\end{aligned}
$$

For $I_2^{a112}$,

$$\begin{aligned} I_2^{a112} &= \sum_{k=j+2}^{j_0} 2^{k\alpha_\infty q_\infty} \left\| (\widetilde{\mathbb{K}}_{\gamma',s} * a_j(\cdot)) \frac{1}{|\cdot|^{n(1-(\gamma'+2)/s)}} \cdot (h(\cdot) - h_{B_j}) \cdot \chi_k \right\|_{p(\cdot)}^{q_\infty} \\ &\lesssim \sum_{k=j+2}^{j_0} 2^{k(\alpha_\infty - n + n(\gamma'+2)/s)q_\infty} \left\| \widetilde{\mathbb{K}}_{\gamma',s} * a_j(\cdot) \cdot (h(\cdot) - h_{B_j}) \cdot \chi_k \right\|_{p(\cdot)}^{q_\infty}. \end{aligned}$$

In other hand, by Lemma 2.6 and the Hölder's inequality with $\frac{1}{p(\cdot)} = \frac{1}{s} + \frac{1}{\theta(\cdot)}$, we have

$$\begin{aligned} \left\| \widetilde{\mathbb{K}}_{\gamma',s} * a_j(\cdot).(h(\cdot) - h_{B_j}).\chi_k \right\|_{p(\cdot)} &\lesssim \left\| \widetilde{\mathbb{K}}_{\gamma',s} * a_j(\cdot) \right\|_s^{q_\infty} \left\| (h(\cdot) - h_{B_j}) \cdot \chi_k \right\|_{\theta(\cdot)}^{q_\infty} \\ &\lesssim (k-j)^{q_\infty} \left\| a_j(\cdot) \right\|_{s'}^{q_\infty} \left\| h \right\|_{BMO}^{q_\infty} \left\| \chi_k \right\|_{\theta(\cdot)}^{q_\infty}, \end{aligned}$$

which gives

$$\begin{aligned} I_2^{a112} &\lesssim \|h\|_{BMO}^{q_\infty} \sum_{k=j+2}^{j_0} (k-j)^{q_\infty} 2^{k(\alpha_\infty - n + n(\gamma'+2)/s)q_\infty} \left\| a_j(\cdot) \right\|_{s'}^{q_\infty} \left\| \chi_k \right\|_{\theta(\cdot)}^{q_\infty} \\ &\lesssim \|h\|_{BMO}^{q_\infty} \sum_{k=j+2}^{j_0} (k-j)^{q_\infty} 2^{k(\alpha_\infty + n/p_\infty - n - n/s + n(\gamma'+2)/s)q_\infty} \left\| a_j(\cdot) \right\|_{s'}^{q_\infty}. \end{aligned}$$

By (3.19), we have

$$\begin{aligned} I_2^{a112} &\lesssim \|h\|_{BMO}^{q_\infty} \sum_{k=j+2}^{j_0} (k-j)^{q_\infty} 2^{k(\sigma - 1/s + n(\gamma'+2)/s)q_\infty} \left\| a_j(\cdot) \right\|_{p(\cdot)}^{q_\infty} \left\| \chi_{B_j} \right\|_{\frac{p(\cdot)}{p(\cdot)-s'}}^{q_\infty/s'} \\ &\lesssim \|h\|_{BMO}^{q_\infty} \sum_{k=j+2}^{j_0} (k-j)^{q_\infty} 2^{k(\sigma - n/s + n(\gamma'+2)/s)q_\infty} 2^{j(-\alpha_\infty + n - n/p_\infty - n/s)q_\infty}. \end{aligned}$$

Since $\sigma = \alpha_\infty - n + n/p_\infty = 0$, $I_2^{a112}$ is bounded by

$$\begin{aligned} I_2^{a112} &\lesssim \|h\|_{BMO}^{q_\infty} \sum_{k=j+2}^{j_0} (k-j)^{q_\infty} 2^{nkq_\infty(\gamma'+1)/s} 2^{-jnq_\infty/s} \\ &\lesssim \left( \frac{2^{j_0(1-\gamma)(\gamma'+1)}}{2^j} \right)^{nq_\infty/s} \\ &\lesssim \left( \frac{2^{j(1-\gamma)(\gamma'+1)}}{2^j} \right)^{nq_\infty/s} \\ &\leqslant c. \end{aligned} \tag{4.9}$$

We can estimate $I_2^{a12}$ by the same arguments used in the estimation of $I_2^{a22}$.

Therefore, by (4.4), (4.2), (4.3), (4.6), (4.7), (4.8), and (4.9), we complete the proof of Theorem 4.5. □

*Remark* 4.8. A homogeneous counterpart of Theorem 4.5 is available under the conditions $\alpha(0) = n - n/p(0)$ and $\alpha_\infty = n - n/p_\infty$, to be addressed in future work.

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