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ON DESIGNS FROM THE CHARACTER TABLES OF FINITE GROUPS

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ABSTRACT. The aim of this paper is to construct certain designs using the maximal subgroups and conjugacy classes of finite primitive groups with known character tables. Our method is based on a result of Key and Moori, commonly referred to in the literature as the second Key–Moori method. We analyze the character table to identify permutation characters corresponding to the group's primitive actions. Our focus is on the symplectic group $PSp_4(q)$, where q is a prime power. However, the approach is general and can be applied to any primitive group.

1. INTRODUCTION

In [12], two methods for constructing 1-designs and codes from finite simple groups have been discussed. These methods, developed by Key and Moori, are sometimes referred to as the Key–Moori methods. In the first method, the designs are symmetric and constructed using the primitive permutation representations of simple groups. In the second method, the 1-designs are mainly nonsymmetric and are constructed using the conjugacy classes of a primitive group. These methods have been applied to several families of simple groups [8, 17, 18, 20, 21] as well as some sporadic simple groups [5, 14, 15, 13, 16, 19]. Recently, in [11], Moori developed the third method where the designs can be constructed using the fixed points of a primitive group (see [10, 22, 23]). The aim of this paper is to apply the second Key–Moori method to another family of simple groups, namely $PSp_4(q)$. However, our approach differs from previous works, as we use

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the character table of the group to determine the permutation characters of G corresponding to its primitive actions. Throughout this paper, we assume that q is a prime power which can be either odd or even, with $q > 2$. We use the notation from $\mathbb{A}$TLAS [2], for groups. Recall that the symplectic group $Sp_4(q)$ is the set of all 4×4 matrices X that satisfy $XJX^T = J$, where

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

The center of the symplectic group is given by $Z(Sp_4(q)) = \{I, -I\}$. Here I denotes the 4×4 identity matrix. The quotient group $Sp_4(q)/Z(Sp_4(q))$ is the projective symplectic group $PSp_4(q)$, which is a simple group for $q > 2$. The order of $PSp_4(q)$ is given by $|PSp_4(q)| = \frac{q^4(q^4-1)(q^2-1)}{gcd(2,q-1)}$. The full set of maximal subgroups of $PSp_4(q)$ is provided in [6]. We aim to construct designs that remain invariant under the following maximal subgroups of $PSp_4(q)$.

(1) Maximal subgroups of index $q^3 + q^2 + q + 1$ (two classes for all q, say M_1 and M_2).
(2) Maximal subgroups of index $\frac{q^2(q^2+1)}{2}$ (one class for q odd, say M_3 and two classes for q even, say M_3' and M_3'').
(3) Maximal subgroups of index $\frac{q^2(q^2-1)}{2}$ (one class for q odd, say M_4 and two classes for q even, say M_4' and M_4'').

We use the above notation for maximal subgroups of G throughout this paper.

The conjugacy classes of $PSp_4(q)$ are derived from those of $Sp_4(q)$. For more details on the conjugacy classes and their representatives (see [3, 24, 25, 26]), particularly regarding the class representatives of each conjugacy class. It is important to note that the conjugacy classes differ for odd and even values of q. Therefore, we approach the problem by considering each case separately.

In section 2, we focus on the terminology and notation used throughout this paper, including basic results on design theory and the permutation characters of groups. In sections 3 and 4, we present Method 2 and derive the parameters of the resulting designs.

2. Terminology, notation and preliminaries

Our notation for designs follows that of [1]. Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ be an incidence structure, where

- $\mathcal{P}$ is the set of points,
- $\mathcal{B}$ is the set of blocks, disjoint from $\mathcal{P}$, and
- $\mathcal{I} \subseteq \mathcal{P} \times \mathcal{B}$ is the incidence relation.

If the ordered pair $(p, B) \in \mathcal{I}$, we say that p is incident to B. It is often convenient to assume that the blocks in $\mathcal{B}$ are subsets of $\mathcal{P}$, so that $(p, B) \in \mathcal{I}$ if and only if $p \in B$. For a positive integer t, we say that $\mathcal{D}$ is a t-design if every block $B \in \mathcal{B}$ is incident with exactly k points, and every t distinct points are together incident with λ blocks. In this case, we say that $\mathcal{D}$ is a t-(v, k, λ) design, where $v = |\mathcal{P}|$. We say that $\mathcal{D}$ is symmetric if it has the same number of points and blocks; otherwise, $\mathcal{D}$ is nonsymmetric.

Definition 2.1. [9] If a group G acts on a set Ω, then the function $\chi_\Omega : G \to \mathbb{C}$ defined by

$$\chi_\Omega(g) = |Fix(g)|,$$

for all $g \in G$, is called the permutation character of G with respect to the action.

Definition 2.2. Let G act transitively on a set Ω, and let G_α be the stabilizer of a point α on Ω. Consider the action of G_α on Ω. Each orbit of this action is called a suborbit of G. The rank of G (with respect to this action) is defined as the number of distinct suborbits of G and is denoted by $Rank(G)$.

Theorem 2.3. *Let G act transitively on the set Ω. Then the following properties hold:*

(1) *The function χ_Ω is an ordinary character of G with degree $|\Omega|$.*
(2) *χ_Ω can be expressed as a sum of irreducible characters:*

$$\chi_\Omega = \sum_{i=1}^{k} r_i \psi_i,$$

where r_i are positive integers and each ψ_i is an irreducible character of G.
(3) *For some index j, the character ψ_j is the trivial character of G, and we have $r_j = 1$.*
(4) *The rank of G is given by*

$$Rank(G) = \sum_{i=1}^{k} r_i^2.$$

Proof. See [4, Chapter 5]. □

Remark 2.4. Let M be a maximal subgroup of G. We denote by χ_M the permutation character afforded by the action of G on the set of conjugates of M in G. Note that χ_M vanishes on all elements of G that are not conjugate to any element of M.

Our main results for constructing designs are based on the following result.

Theorem 2.5 (Method 2)**.** [12] *Let G be a finite primitive group, let M be a maximal subgroup of G, and let x^G be a conjugacy class of elements of order n in G such that $M \cap x^G \neq \emptyset$. Define*

$$\mathcal{B} = \{(M \cap x^G)^y |\ y \in G\} \text{ and } \mathcal{P} = x^G.$$

Then, we obtain a $1-\big(|x^G|, |M \cap x^G|, \chi_M(g)\big)$ design $\mathcal{D}$, where $g \in x^G$.

Remark 2.6. The group G acts as an automorphism group on $\mathcal{D}$, acting primitively on the blocks and transitively (but not necessarily primitively) on the points of $\mathcal{D}$.

Lemma 2.7. *Let $\mathcal{D}$ be a $1-(v,k,\lambda)$ design constructed using Method 2. Then*

$$k = \frac{|M| \cdot (\chi_M(g) \cdot |x^G|)}{|G|}.$$

Proof. According to [12], we have $kb = \lambda v$. Then

$$k = |M \cap x^G| = \frac{\chi_M(g) \cdot |x^G|}{[G{:}M]} = \frac{|M| \cdot (\chi_M(g) \cdot |x^G|)}{|G|}.$$

□

Theorem 2.8. [3, 25] *Let $G = PSp(4,q)$. If q is even, then the set $cd(G)$ of character degrees of G is given by*

$$\{1, \frac{q(q^2+1)}{2}, \frac{q(q+1)^2}{2}, q^4, \frac{q(q^2-1)}{2}, q^4-1, (q-1)^2(q^2+1), (q^2-1)^2,$$
$$(q+1)(q^2+1), (q-1)(q^2+1), q(q+1)(q^2+1), q(q-1)(q^2+1), (q+1)^2(q^2+1)\},$$

where the last degree appears only for $q > 4$. If q is odd and $q > 7$, then the following additional degrees appear:

$$\{\frac{(q^4-1)}{2}, q(q^2+1), \frac{(q^2+1)}{2}, \frac{q^2(q^2+1)}{2}, \frac{(q-1)^2(q^2+1)}{2},$$
$$(q-1)^2(q^2+1), \frac{(q+1)^2(q^2+1)}{2}, \frac{(q+\epsilon)(q^2+1)}{2}, \frac{q(q+\epsilon)(q^2+1)}{2}\},$$

where $\epsilon = (-1)^{\frac{q-1}{2}}$.

Remark 2.9. Let $G = PSp(4,q)$. For the special cases where $q = 3, 5,$ or 7, the sets of character degrees $cd(G)$ are given as follows:

(1) For $G = PSp(4,3)$, we have
$$cd(G) = \{1, 5, 6, 10, 15, 20, 24, 30, 40, 45, 60, 64, 81\}.$$

(2) For $G = PSp(4,5)$, we have
$$cd(G) = \{1, 13, 40, 65, 78, 90, 104, 130, 156, 208,$$
$$312, 325, 390, 520, 576, 624, 625, 780\}.$$

(3) For $G = PSp(4,7)$, we have
$$cd(G) = \{1, 25, 126, 150, 175, 224, 300, 350, 400, 900, 1050,$$
$$1200, 1225, 1600, 1800, 2100, 2304, 2400, 2401, 2800\}.$$

3. Main results

3.1. **Designs from subgroups of index $1+q+q^2+q^3$.** In this section, we provide the parameters of designs from $G = PSp(4,q)$ using Method 2 with respect to maximal subgroups of index q^3+q^2+q+1, which form two conjugacy classes, denoted by M_1 and M_2, in G for both even and odd q.

Theorem 3.1. *The parameters of designs from $G = PSp_4(q)$ using Method 2 with respect to M_1 and M_2 are fully given in the following tables:*

(1) *For q even, the parameters are given in full in Tables 2 and 3,*
(2) *For q odd, the parameters are given in full in Tables 4 and 5, respectively.*

Proof. As in [6], let M_1 and M_2 be the stabilizers of rank-3 actions of G. By Theorem 2.3, we have $\chi_{M_i} = 1 + \psi_i + \chi_i$ (for $i = 1, 2$), where ψ_i and χ_i are irreducible characters of G and the degrees satisfy

$$\psi_1(1) + \psi_2(1) = q + q^2 + q^3.$$

The characters ψ_i, χ_i and θ_i are defined in [25]. These decompositions play a crucial role in understanding the structure and character theory of the corresponding permutation representations. According to the character table of G, the group has a unique irreducible character θ_9 of degree $\frac{q(q+1)^2}{2}$ and exactly two irreducible characters θ_{11} and θ_{12}, both of degree $\frac{q(q^2+1)}{2}$. It is clear that

$$\theta_9 + \theta_{11} = \theta_9 + \theta_{12} = q + q^2 + q^3.$$

We claim that $1 + \theta_9 + \theta_{11}$ and $1 + \theta_9 + \theta_{12}$ are permutation characters of the rank-3 actions of G. In fact, a computation of the character degrees of G (Theorem 2.8) shows that there is no other pair of irreducible character degrees of G whose sum equals $q+q^2+q^3$. Now, using the values of these two permutation characters given in Table 1, we can determine the third parameters of each design. If $G = PSp(4, q)$, then in the special cases where $q = 3, 5, \text{or } 7$, the sets of irreducible character degrees $cd(G)$ are given explicitly in 2.9. For $q = 3$ we have $1+q+q^2+q^3 = 1+3+3^2+3^3 = 40$; $q = 5$, $1+q+q^2+q^3 = 1+5+5^2+5^3 = 156$ and for $q = 7$, $1+q+q^2+q^3 = 1+7+7^2+7^3 = 400$. These cases are considered exceptional because the general formulas for character degrees of symplectic groups often simplify or behave irregularly for small q. Note that if $\chi_{M_i}(g) = 0$, then the conjugacy class of G does not intersect M_i (for $i = 1, 2$), so we exclude the conjugacy classes on which the permutation characters vanish. To find the second parameter of the design, we use Lemma 2.7. This completes the proof. □

In [3, 24, 25], the notation used in the tables of conjugacy classes and their representatives is given in detail.

TABLE 1. Permutation characters of the rank-3 actions of G

Character	A_1	A_{21}	A_{31}	A_{32}	A_{41}	$B_1(i)$	$B_2(i)$	$B_3(i,j)$
θ_9	$\frac{q(q+1)^2}{2}$	$\frac{q(q+1)}{2}$	q			-1		2
θ_{11}	$\frac{q(q^2+1)}{2}$	$\frac{q(q-1)}{2}$	q				1	1
θ_{12}	$\frac{q(q^2+1)}{2}$	$\frac{q(q-1)}{2}$		q			-1	1
Character	$B_4(i,j)$	$B_5(i,j)$	$B_6(i)$	$B_7(i)$	$B_8(i)$	$B_9(i)$	$C_1(i)$	$C_{21}(i)$
θ_9					$q+1$	1		
θ_{11}	-1	-1	q		1	1	-1	-1
θ_{12}	-1	1	-1	-1	q		q	
Character	$C_3(i)$	$C_{41}(i)$	D_1	D_{21}	D_{31}	D_{32}		
θ_9	$q+1$	1	$\frac{(q+1)^2}{2}$	$\frac{q+1}{2}$	$\frac{q-1}{2}$	$\frac{1-q}{2}$		
θ_{11}	q		$\frac{(q^2-1)}{2}+q$	$\frac{q-1}{2}$	$-\frac{q+1}{2}$	$\frac{q-1}{2}$		
θ_{12}	1	1	$\frac{1+2q-q^2}{2}$	$\frac{q+1}{2}$	$\frac{1-q}{2}$	$\frac{q+1}{2}$		

TABLE 2. Designs from G under M_1 (q even).

Representative	$v = \|x^G\|$	$k = \|M \cap x^G\|$	$\lambda = \chi_{M_1}$
$\overline{A}_{11}$	$(q^2-1)(q^4-1)$	$(q^2-1)^2$	$q+1$
$\overline{A}_{21}$	q^4-1	$(q-1)(q^2+q+1)$	q^2+q+1
$\overline{A}_{22}$	q^4-1	q^2-1	$q+1$
$\overline{A}_{31}$ and $\overline{A}_{32}$	$q(q^4-1)(q+1)$	$q(q^2-1)(q\pm 1)$	$q+1$
$\overline{A}_{41}$ and $\overline{A}_{42}$	$q^2(q^4-1)(q^2-1)$	$q^2(q^2-1)(q-1)$	1
$\overline{B}_2(i)$	$q^4(q^4-1)$	$2q^4(q-1)$	2
$\overline{B}_3(i,j)$ and $\overline{B}_3(j,i)$	$q^4(q^2+1)(q+1)^2$	$4q^4(q+1)$	4
$\overline{B}_5(i,j)$	$q^4(q^4-1)$	$2q^4(q-1)$	2
$\overline{B}_6(i)$ and $\overline{C}_1(i)$	$q^3(q^2+1)(q-1)$	$q^3(q-1)$	$q+1$
$\overline{B}_8(i)$	$q^3(q^2+1)(q+1)$	$2q^3(q+1)$	$2(q+1)$
$\overline{B}_9(i)$	$q^3(q^4-1)(q+1)$	$2q^3(q^2-1)$	2
	$q^3(q^2+1)(q-1)$	$q^3(q-1)$	$q+1$
$\overline{C}_{21}(i)$ and $\overline{C}_{22}(i)$	$q^3(q^4-1)(q-1)$	$q^3(q-1)^2$	1
$\overline{C}_3(i)$	$q^3(q^2+1)(q+1)$	$q^3(q+3)$	$q+3$
$\overline{C}_{41}(i)$	$q^3(q^4-1)(q+1)$	$3q^3(q^2-1)$	3
$\overline{D}_1$	$q^2(q^2+1)$	$2q^2$	$2(q+1)$
$\overline{D}_{21}$ and $\overline{D}_{22}$	$q^2(q^2+1)(q^2-1)$	$q^2(q-1)(q+2)$	$q+2$
$\overline{D}_{31}$ and $\overline{D}_{32}$	$\frac{q^2(q^2-1)(q^4-1)}{2}$	$\frac{q^2(q-1)(q^2-1)}{2}$	1
$\overline{D}_{34}$	$\frac{q^2(q^4-1)(q^2-1)}{4}$	$\frac{q^2(q-1)(q^2-1)}{2}$	2

Table 3. Designs from G under M_2 (q even).

Representative	$v = \|x^G\|$	$k = \|M \cap x^G\|$	$\lambda = \chi_{M_2}$
$\overline{A}_{11}$	$(q^2-1)(q^4-1)$	$(q^2-1)^2$	$q+1$
$\overline{A}_{21}$ and $\overline{A}_{22}$	q^4-1	q^2-1	$q+1$
$\overline{A}_{31}$	$q(q^4-1)(q+1)$	$q(q^2-1)(2q+1)$	$2q+1$
$\overline{A}_{32}$	$q(q^4-1)(q-1)$	$q(q-1)^2$	1
$\overline{A}_{41}$	$q^2(q^4-1)(q^2-1)$	$q^2(q^2-1)(q-1)$	1
$\overline{B}_2(i)$	$q^4(q^4-1)$	$2q^4(q-1)$	2
$\overline{B}_3(i,j)$ and $\overline{B}_3(j,i)$	$q^4(q^2+1)(q+1)^2$	$4q^4(q+1)$	4
$\overline{B}_5(i,j)$	$q^4(q^4-1)$	$2q^4(q-1)$	2
$\overline{B}_6(i)$	$q^3(q^2+1)(q-1)$	$q^3(q-1)$	$q+1$
$\overline{B}_7(i)$	$q^3(q^4-1)(q-1)$	$q^3(q-1)^2$	1
$\overline{B}_8(i)$ and $\overline{B}_8(i)$	$q^3(q^2+1)(q+1)$	$q^3(q+3)$	$q+3$
$\overline{B}_8(i)$	$q^3(q^2+1)(q+1)$	$2q^3(q+1)$	$2(q+1)$
$\overline{B}_9(i)$ and $\overline{B}_9(i)$	$q^3(q^4-1)(q+1)$	$2q^3(q^2-1)$	2
$\overline{B}_9(i)$	$q^3(q^4-1)(q+1)$	$3q^3(q^2-1)$	3
$\overline{C}_{21}(i)$	$q^3(q^4-1)(q-1)$	$q^3(q-1)^2$	1
$\overline{C}_3(i)$	$q^3(q^2+1)(q+1)$	$2q^3(q+1)$	$2(q+1)$
$\overline{C}_{41}(i)$	$q^3(q^4-1)(q+1)$	$2q^3(q^2-1)$	2
$\overline{D}_1$	$q^2(q^2+1)$	$q^2(q+1)$	$(q+1)^2$
$\overline{D}_{21}$ and $\overline{D}_{22}$	$q^2(q^2+1)(q^2-1)$	$q^2(q^2-1)$	$q+1$
$\overline{D}_{31}$ and $\overline{D}_{32}$	$\frac{q^2(q^2-1)(q^4-1)}{2}$	$\frac{q^2(q-1)(q^2-1)}{2}$	1
$\overline{D}_{34}$	$\frac{q^2(q^4-1)(q^2-1)}{4}$	$\frac{q^2(q-1)(q^2-1)}{4}$	1

TABLE 4. Designs from G under M_1 (q odd).

Representative	$v = \lvert x^G \rvert$	$k = \lvert M \cap x^G \rvert$	$\lambda = \chi_{M_1}$
$\overline{A}_{21}$ and $\overline{A}_{22}$	$\frac{q^4-1}{2}$	$\frac{(q-1)(q^2+q+1)}{2}$	q^2+q+1
$\overline{A}_{31}$ and $\overline{A}_{32}$	$\frac{q(q^4-1)(q\pm1)}{2}$	$\frac{q(q^2-1)(q\pm1)}{2}$	$q+1$
$\overline{A}_{41}$ and $\overline{A}_{42}$	$\frac{q^2(q^4-1)(q^2-1)}{2}$	$\frac{q^2(q^2-1)(q-1)}{2}$	1
$\overline{B}_2(i)\ i \in\ R_1''$	$q^4(q^4-1)$	$2q^4(q-1)$	2
$\overline{B}_3(i,j)\ j = \frac{q-1}{2} - i$	$\frac{q^4(q^2+1)(q+1)^2}{2}$	$2q^4(q+1)$	4
$\overline{B}_3(i,j)\ i < j \leq \frac{q-3}{2} - i$	$q^4(q^2+1)(q+1)^2$	$4q^4(q+1)$	4
$\overline{B}_5(i,j)$	$q^4(q^4-1)$	$2q^4(q-1)$	2
$\overline{B}_7(i)$	$q^3(q^4-1)(q-1)$	$q^3(q-1)^2$	1
$\overline{B}_8(i)\ i = \frac{q-1}{2}$	$\frac{q^3(q^2+1)(q+1)}{2}$	$q^3(q+1)$	$2(q+1)$
$\overline{B}_9(i)\ i = \frac{q-1}{4}$	$\frac{q^3(q^4-1)(q+1)}{2}$	$q^3(q^2-1)$	2
$\overline{B}_9(i)\ 1 \leq i \leq \frac{q-3}{4}$	$\frac{q^3(q^4-1)(q+1)}{2}$	$\frac{3q^3(q^2-1)}{2}$	3
$\overline{C}_1(i)$	$q^3(q^2+1)(q-1)$	$q^3(q-1)$	$q+1$
$\overline{C}_{21}(i)$ and $\overline{C}_{22}(i)$	$\frac{q^3(q^4-1)(q-1)}{2}$	$\frac{q^3(q-1)^2}{2}$	1
$\overline{C}_3(i)$	$q^3(q^2+1)(q+1)$	$q^3(q+3)$	$q+3$
$\overline{C}_{41}(i)$	$\frac{q^3(q^4-1)(q+1)}{2}$	$\frac{3q^3(q^2-1)}{2}$	3
$\overline{D}_1$	$\frac{q^2(q^2+1)}{2}$	q^2	$2(q+1)$
$\overline{D}_{21}$ and $\overline{D}_{22}$	$\frac{q^2(q^2+1)(q^2-1)}{2}$	$\frac{q^2(q-1)(q+2)}{2}$	$q+2$
$\overline{D}_{31}$	$\frac{q^2(q^2-1)(q^4-1)}{4}$	$\frac{q^2(q-1)(q^2-1)}{2}$	2
$\overline{D}_{32}$ and $\overline{D}_{34}$	$\frac{q^2(q^4-1)(q^2-1)}{8}$	$\frac{q^2(q-1)(q^2-1)}{4}$	2

TABLE 5. Designs from G under M_2 (q odd).

Representative	$v = \lvert x^G \rvert$	$k = \lvert M \cap x^G \rvert$	$\lambda = \chi_{M_2}$
$\overline{A}_{21}$ and $\overline{A}_{22}$	$\frac{q^4-1}{2}$	$\frac{q^2-1}{2}$	$q+1$
$\overline{A}_{31}$	$\frac{q(q^4-1)(q+1)}{2}$	$\frac{q(q^2-1)(2q+1)}{2}$	$2q+1$
$\overline{A}_{32}$	$\frac{q(q^4-1)(q-1)}{2}$	$\frac{q(q-1)^2}{2}$	1
$\overline{A}_{41}$ and $\overline{A}_{42}$	$\frac{q^2(q^4-1)(q^2-1)}{2}$	$\frac{q^2(q^2-1)(q-1)}{2}$	1
$\overline{B}_2(i)\ i \in\ R_1''$ and $\overline{B}_2(i)\ i \in\ R_2''$	$\frac{q^4(q^4-1)}{2}$	$q^4(q-1)$	2
$\overline{B}_3(i,j)\ j = \frac{q-1}{2} - i$	$\frac{q^4(q^2+1)(q+1)^2}{2}$	$2q^4(q+1)$	4
$\overline{B}_3(i,j)\ i < j \leq \frac{q-3}{2} - i$	$q^4(q^2+1)(q+1)^2$	$4q^4(q+1)$	4
$\overline{B}_5(i,j)$	$q^4(q^4-1)$	$2q^4(q-1)$	2
$\overline{B}_6(i)$	$q^3(q^2+1)(q-1)$	$q^3(q-1)$	$q+1$
$\overline{B}_6(i)$	$\frac{q^3(q^2+1)(q-1)}{2}$	$\frac{q^3(q-1)}{2}$	$q+1$
$\overline{B}_7(i)$	$q^3(q^4-1)(q-1)$	$q^3(q-1)^2$	1
$\overline{B}_7(i)$	$\frac{q^3(q^4-1)(q-1)}{2}$	$\frac{q^3(q-1)^2}{2}$	1
$\overline{B}_8(i)\ i = \frac{q-1}{2}$	$\frac{q^3(q^2+1)(q+1)}{2}$	$\frac{q^3(q+3)}{2}$	$q+3$
$\overline{B}_8(i)\ 1 \leq i \leq \frac{q-3}{4}$	$q^3(q^2+1)(q+1)$	$q^3(q+3)$	$q+3$
$\overline{B}_8(i)\ 1 \leq i \leq \frac{q-5}{4}$	$q^3(q^2+1)(q+1)$	$2q^3(q+1)$	$2(q+1)$
$\overline{B}_9(i)\ i = \frac{q-1}{4}$ and $\overline{C}_{41}(i)$	$\frac{q^3(q^4-1)(q+1)}{2}$	$q^3(q^2-1)$	2
$\overline{B}_9(i)\ 1 \leq i \leq \frac{q-5}{4}$	$q^3(q^4-1)(q+1)$	$2q^3(q^2-1)$	2
$\overline{B}_9(i)\ 1 \leq i \leq \frac{q-3}{4}$	$\frac{q^3(q^4-1)(q+1)}{2}$	$\frac{3q^3(q^2-1)}{2}$	3
$\overline{C}_3(i)$	$q^3(q^2+1)(q+1)$	$2q^3(q+1)$	$2(q+1)$
$\overline{D}_1$	$\frac{q^2(q^2+1)}{2}$	$\frac{q^2(q+1)}{2}$	$(q+1)^2$
$\overline{D}_{21}$ and $\overline{D}_{22}$	$\frac{q^2(q^2+1)(q^2-1)}{2}$	$\frac{q^2(q^2-1)}{2}$	$q+1$
$\overline{D}_{31}$	$\frac{q^2(q^2-1)(q^4-1)}{4}$	$\frac{q^2(q-1)(q^2-1)}{4}$	1
$\overline{D}_{32}$ and $\overline{D}_{34}$	$\frac{q^2(q^4-1)(q^2-1)}{8}$	$\frac{q^2(q-1)(q^2-1)}{8}$	1

3.2. **Designs from subgroups of indices $\frac{q^2(q^2\pm 1)}{2}$.** In this section, we provide the parameters of designs from $G = PSp_4(q)$ using Method 2, with respect to maximal subgroups of index $\frac{q^2(q^2+1)}{2}$, which appear as one class for q odd and two classes for q even, denoted by M_3, M_3' and M_3''. We also consider maximal subgroups of index $\frac{q^2(q^2-1)}{2}$, which appear as one class for q odd and two classes for q even, denoted M_4, M_4' and M_4''.

Theorem 3.2. *The parameters of designs from $G = PSp_4(q)$ ($q > 2$, even) using Method 2 are fully given in Table 6 for two classes of subgroups of index $\frac{q^2(q^2+1)}{2}$, and in Table 7 for two classes of subgroups of index $\frac{q^2(q^2-1)}{2}$.*

Proof. According to Theorem 2.8, the degree of the permutation representation of M_3 is $\frac{q^2(q^2+1)}{2}$, which is a linear combination of some irreducible characters of G.

We claim that $1 + \theta_9 + \chi_3$ and $1 + \theta_9' + \chi_3$ are permutation characters corresponding to specific actions of G. A computation of the character degrees of G (Theorem 2.8) suggests that these are the only combinations of irreducible character degrees whose sum to $\frac{q^2(q^2+1)}{2}$.

Similarly, we claim that $1+\theta_{11}+\chi_8$ and $1+\theta_{12}+\chi_8$ are permutation characters corresponding to specific actions of G. A computation of the character degrees of G (Theorem 2.8) implies that there is no other combination of irreducible character degrees that sums to $\frac{q^2(q^2-1)}{2}$.

We need to find the permutation characters of the action in each case:

$$\frac{q^2(q^2+1)}{2} = 1 + \theta_9(1) + \left(\frac{q-2}{2}\right)\chi_3(1) = 1 + \theta_9'(1) + \left(\frac{q-2}{2}\right)\chi_3(1).$$

and

$$\frac{q^2(q^2-1)}{2} = 1 + \theta_{11}(1) + \left(\frac{q-2}{2}\right)\chi_8(1) = 1 + \theta_{12}(1) + \left(\frac{q-2}{2}\right)\chi_8(1).$$

We claim that the permutation characters for these two actions are

$$1 + \theta_9 + \left(\frac{q-2}{2}\right)\chi_3, 1 + \theta_9' + \left(\frac{q-2}{2}\right)\chi_3,$$

and

$$1 + \theta_{11} + \left(\frac{q-2}{2}\right)\chi_8, 1 + \theta_{12} + \left(\frac{q-2}{2}\right)\chi_8.$$

Now, using Maxima [7], we can verify that no other linear combinations of the character degrees of G sum to $\frac{q^2(q^2+1)}{2}$. If $G = PSp(4, q)$, then in the special cases where $q = 3, 5,$ or 7, the sets of irreducible character degrees $cd(G)$ are given explicitly in 2.9. A similar argument shows that the permutation characters of degree $\frac{q^2(q^2-1)}{2}$ are as follows:

$$1 + \theta_9(1) + \left(\frac{q-2}{2}\right)\chi_3(1), 1 + \theta_9'(1) + \left(\frac{q-2}{2}\right)\chi_3(1),$$

and

$$1+\theta_{11}(1)+\left(\frac{q-2}{2}\right)\chi_8(1), 1+\theta_{12}(1)+\left(\frac{q-2}{2}\right)\chi_8(1).$$

The proof is now complete. $\square$

TABLE 6. Designs under M_3' and M_3'' (q even).

Representative	$v = \lvert x^G\rvert$	$k = \lvert M\cap x^G\rvert$	$\lambda = \chi'_{M_3}$ or χ''_{M_3}
$\overline{A}_{11}$	$(q^2-1)(q^4-1)$	$\frac{4(q^4-1)}{q}$	$2q$
$\overline{A}_{21}$	q^4-1	$2(q^2-1)$	q^2
$\overline{A}_{22}$	q^4-1	$4(q^2-1)$	$2q^2$
$\overline{B}_2(i)$	$q^4(q^4-1)$	$2q^2(q^2-1)$	1
$\overline{B}_4(i,j)$	$q^4(q^2+1)(q-1)^2$	$2q^2(q-1)^2$	1
$\overline{B}_5(i,j)$	$q^4(q^4-1)$	$2q^2(q^2-1)$	1
$\overline{B}_6(i)$	$q^3(q^2+1)(q-1)$	$2q(q-1)(q+2)$	$q+2$
$\overline{B}_7(i)$ and $\overline{B}_7'(i)$	$q^3(q^4-1)(q-1)$	$4q(q-1)(q^2-1)$	2
$\overline{B}_8(i)$ and $\overline{C}_1(i)$ and $\overline{C}_1'(i)$	$q^3(q^2+1)(q\pm1)$	$2q(q\pm1)$	1
$\overline{B}_9(i)$	$q^3(q^4-1)(q+1)$	$2q(q^2-1)(q+1)$	1
$\overline{C}_{21}(i)$ and $\overline{C}_{22}(i)$	$q^3(q^4-1)(q-1)$	$2q(q^2-1)(q-1)$	1
$\overline{C}_3(i)$	$q^3(q^2+1)(q+1)$	$4q(q+1)^2$	$2(q+1)$
$\overline{C}_{41}(i)$	$q^3(q^4-1)(q+1)$	$4q(q^2-1)(q+1)$	2
$\overline{D}_{31}$	$\frac{q^2(q^2-1)(q^4-1)}{2}$	$q(q^2-1)^2$	q

Remark 3.3. The representative $\overline{C}_1(i)'$ appears only on M_3'.

TABLE 7. Designs under M_4' and M_4'' (q even).

Representative	$v = \lvert x^G\rvert$	$k = \lvert M\cap x^G\rvert$	$\lambda = \chi'_{M_4}$ or χ''_{M_4}
$\overline{A}_{11}$	$(q^2-1)(q^4-1)$	$\frac{4(q^4-1)}{q}$	$2q$
$\overline{A}_{22}$	q^4-1	$4(q^2+1)$	$2q^2$
$\overline{B}_1(i)$, $\overline{B}_1'(i)$, $\overline{B}_1''(i)$ and $\overline{B}_1'''(i)$	$q^4(q^2-1)^2$	$2q^2(q^2-1)$	1
$\overline{B}_2(i)$ and $\overline{B}_5(i,j)$	$q^4(q^4-1)$	$2q^2(q^2+1)$	1
$\overline{B}_6(i)$	$q^3(q^2+1)(q-1)$	$4q(q^2+1)$	$2(q+1)$
$\overline{B}_7(i)$ and $\overline{B}_7(i)'$	$q^3(q^4-1)(q-1)$	$4q(q^2+1)(q-1)$	2
$\overline{B}_8(i)$	$q^3(q^2+1)(q+1)$	$\frac{2q(q^2+1)(q+2)}{q-1}$	$q+2$
$\overline{B}_9(i)$	$q^3(q^4-1)(q+1)$	$4q(q^2+1)(q+1)$	2
$\overline{C}_1(i)$	$q^3(q^2+1)(q-1)$	$4q(q^2+1)$	$2(q+1)$
$\overline{C}_{21}(i)$ and $\overline{C}_{22}(i)$	$q^3(q^4-1)(q-1)$	$4q(q^2+1)(q-1)$	2
$\overline{C}_3(i)$	$q^3(q^2+1)(q+1)$	$\frac{2q(q^2+1)(q+2)}{q-1}$	$q+2$
$\overline{C}_{41}(i)$ and $\overline{C}_{42}(i)$	$q^3(q^4-1)(q+1)$	$4q(q^2+1)(q+1)$	2
$\overline{D}_{31}$	$\frac{q^2(q^2-1)(q^4-1)}{2}$	$q(q^4-1)$	q

Remark 3.4. (i) The representative $\overline{B}_7(i)'$ appears only on M_4'.
(ii) The representative $\overline{B}_9$ appears only on M_4''.

Theorem 3.5. *The parameters of designs from $G = PSp_4(q)$ (q odd) using Method 2 are given fully in Tables 8 and 9 for two classes of subgroups of index $\frac{q^2(q^2+1)}{2}$, and in Tables 10 and 11 for two classes of subgroups of index $\frac{q^2(q^2-1)}{2}$.*

Proof. Based on Theorem 2.8, the degree of the permutation representation of $\frac{q^2(q^2+1)}{2}$, and it can be written as a linear combination of specific irreducible characters of G.

We assert that the expressions $1 + \theta_9 + \chi_3$ and $1 + \theta_9' + \chi_3$ serve as permutation characters for specific actions of the group G. A complete evaluation of the character degrees of G, as outlined in Theorem 2.8, indicates that no other combination of irreducible character degrees sums to $\frac{q^2(q^2+1)}{2}$. This suggests that no other combinations can achieve this total.

Similarly, we claim that $1 + \theta_{11} + \chi_8$ and $1 + \theta_{12} + \chi_8$ are permutation characters of the actions of G. An explicit computation of the character degrees of G, as presented in Theorem 2.8, indicates that this is the only combination of irreducible character degrees that sums to $\frac{q^2(q^2-1)}{2}$. This indicates that no alternative combination of irreducible character degrees can produce this sum.

Hence, we have

$$1 + \theta_9(1) + \left(\frac{q+2}{2}\right)\chi_3(1), 1 + \theta_9'(1) + \left(\frac{q+2}{2}\right)\chi_3(1),$$

and

$$1 + \theta_{11}(1) + \left(\frac{q+2}{2}\right)\chi_8(1), 1 + \theta_{12}(1) + \left(\frac{q+2}{2}\right)\chi_8(1).$$

Now, using a similar argument to the proof of the previous theorem, we can find the corresponding permutation characters of G and the result follows. $\square$

TABLE 8. Designs under M_3 for $q = 4n - 1$, where $n \in \mathbb{N}$.

Representative	$v = \|x^G\|$	$k = \|M \cap x^G\|$	$\lambda = \chi_{M_3}$
$\overline{A}_{22}$	$\frac{q^4-1}{2}$	$q^2 - 1$	q^2
$\overline{A}_{31}$	$\frac{q(q^4-1)(q+1)}{2}$	$n(q+1)(q^2-1)$	nq
$\overline{A}_{32}$	$\frac{q(q^4-1)(q-1)}{2}$	$2n(q-1)(q^2-1)$	$2nq$
$\overline{B}_2(i)$ $i \in R_1''$	$\frac{q^4(q^4-1)}{2}$	$q^2(q^2-1)$	1
$\overline{B}_6(i)$ $i = \frac{q+1}{4}$	$q^3(q^2+1)(q-1)$	$2nq(q-1)$	n
$\overline{B}_6(i)$ $i = \frac{q+1}{4}$	$\frac{q^3(q^2+1)(q-1)}{2}$	$q(q-1)(q+2)$	$q+2$
$\overline{B}_7(i)$	$\frac{q^3(q^4-1)(q-1)}{2}$	$q(q-1)^2(q+1)$	1
$\overline{B}_8(i)$ $i = \frac{q-1}{2}$	$\frac{q^3(q^2+1)(q+1)}{2}$	$nq(q+1)$	n
$\overline{C}_{21}(i)$	$\frac{q^3(q^4-1)(q-1)}{2}$	$2q(q-1)^2(q+1)$	2
$\overline{C}_{22}(i)$	$\frac{q^3(q^4-1)(q-1)}{2}$	$q(q^2-1)(q-1)$	1
$\overline{D}_1$	$\frac{q^2(q^2+1)}{2}$	$\frac{q^3-2q+(q+2n)}{2}$	$\frac{q^3-2q+(q+2n)}{2}$
$\overline{D}_{21}$	$\frac{q^2(q^4-1)}{2}$	$(q^2-1)n$	n
$\overline{D}_{22}$	$\frac{q^2(q^2+1)(q^2-1)}{2}$	$\frac{q^3-2q+(q+2n)}{2}$	$\frac{q^3-2q+(q+2n)}{2}$
$\overline{D}_{31}$	$\frac{q^2(q^2-1)(q^4-1)}{4}$	$\frac{(q-1)(q^2-1)^2}{4}$	$\frac{q-1}{2}$
$\overline{D}_{32}$ and $\overline{D}_{34}$	$\frac{q^2(q^4-1)(q^2-1)}{8}$	$\frac{(q+n)(q^2-1)^2}{4}$	$q+n$

Table 9. Designs under M_3 for $q = 4n + 1$, where $n \in \mathbb{N}$.

Representative	$v = \|x^G\|$	$k = \|M \cap x^G\|$	$\lambda = \chi_{M_3}$
$\overline{A}_{21}$ and $\overline{A}_{22}$	$\frac{q^4-1}{2}$	$q^2 - 1$	q^2
$\overline{A}_{31}$	$\frac{q(q^4-1)(q+1)}{2}$	$2n(q+1)(q^2-1)$	$2nq$
$\overline{A}_{32}$	$\frac{q(q^4-1)(q-1)}{2}$	$(2n+1)(q-1)(q^2-1)$	$(2n+1)q$
$\overline{B}_2(i)\ i \in\ R_1''$	$\frac{q^4(q^4-1)}{2}$	$2q^2(q^2-1)$	2
$\overline{B}_2(i)\ i \in\ R_1''$	$\frac{q^4(q^4-1)}{2}$	$3q^2(q^2-1)$	3
$\overline{B}_4(i,j)\ i < j \leq \frac{q-3}{2} - i$	$\frac{q^4(q^2+1)(q-1)^2}{2}$	$q^2(q-n)(q-1)^2$	$q - n$
$\overline{B}_5(i,j)$	$q^4(q^4-1)$	$2q^2(q^2-1)$	1
$\overline{B}_6(i)\ i = \frac{q+1}{4}$	$q^3(q^2+1)(q-1)$	$2q(q-1)$	1
$\overline{B}_6'(i)\ i = \frac{q+1}{4}$	$q^3(q^2+1)(q-1)$	$4q^2(q-1)$	$2q$
$\overline{B}_7(i)$	$\frac{q^3(q^4-1)(q-1)}{2}$	$q(q-1)^2(q+1)$	1
$\overline{B}_8(i)\ i = \frac{q-1}{2}$	$\frac{q^3(q^2+1)(q+1)}{2}$	$q[(4n-1)q+2](q+1)$	$(4n-1)q+2$
$\overline{B}_9(i)\ i = \frac{q-1}{4}$	$\frac{q^3(q^4-1)(q+1)}{2}$	$2q(q^2-1)(q+1)$	2
$\overline{C}_1(i)$	$q^3(q^2+1)(q-1)$	$2q(q-1)$	1
$\overline{C}_{21}(i)$ and $\overline{C}_{22}(i)$	$\frac{q^3(q^4-1)(q-1)}{2}$	$q(q^2-1)(q-1)$	1
$\overline{C}_3(i)$	$q^3(q^2+1)(q+1)$	$2q(q+1)$	1
$\overline{C}_{41}(i)$ and $\overline{C}_{42}(i)$	$\frac{q^3(q^4-1)(q+1)}{2}$	$q(q^2-1)(q+1)$	1
$\overline{D}_1$	$\frac{q^2(q^2+1)}{2}$	$\frac{q^3-2q+(q+2n)}{2}$	$\frac{q^3-2q+(q+2n)}{2}$
$\overline{D}_{21}$ and $\overline{D}_{22}$	$\frac{q^2(q^4-1)}{2}$	$n(q^2-1)$	n
$\overline{D}_{31}$	$\frac{q^2(q^2-1)(q^4-1)}{4}$	$\frac{n(q^2-1)^2}{2}$	n
$\overline{D}_{32}$ and $\overline{D}_{34}$	$\frac{q^2(q^4-1)(q^2-1)}{8}$	$\frac{(q+n)(q^2-1)^2}{4}$	$q + n$

Table 10. Designs under M_4 for $q = 4n - 1$, where $n \in \mathbb{N}$.

Representative	$v = \|x^G\|$	$k = \|M \cap x^G\|$	$\lambda = \chi_{M_4}$
$\overline{A}_{31}$	$\frac{q(q^4-1)(q+1)}{2}$	$n(q+1)(q^2+1)$	nq
$\overline{A}_{32}$	$\frac{q(q^4-1)(q-1)}{2}$	$2n(q-1)(q^2+1)$	$2nq$
$\overline{B}_1(i)\ i \in\ R_1''$	$q^4(q^2-1)^2$	$2q^2(q^2-1)$	1
$\overline{B}_2(i)\ i \in\ R_1''$	$\frac{q^4(q^4-1)}{2}$	$q^2(q^2+1)(q-1)$	$q - 1$
$\overline{B}_5(i,j)$	$q^4(q^4-1)$	$2q^2(q^2+1)$	1
$\overline{B}_6(i)\ i = \frac{q+1}{4}$	$\frac{q^3(q^2+1)(q-1)}{2}$	$2q(q^2+1)$	$2(q+1)$
$\overline{B}_7(i)$	$\frac{q^3(q^4-1)(q-1)}{2}$	$q(q-1)(q^2+1)$	1
$\overline{C}_{21}(i)$	$\frac{q^3(q^4-1)(q-1)}{2}$	$2q(q^2+1)(q-1)$	2
$\overline{D}_1$	$\frac{q^2(q^2+1)}{2}$	$\frac{q(q^2+1)}{q-1}$	$q(q+1)$
$\overline{D}_{21}$	$\frac{q^2(q^4-1)}{2}$	$(q^2+1)n$	n
$\overline{D}_{22}$	$\frac{q^2(q^2+1)(q^2-1)}{2}$	$\frac{q^3-2q+(q+2n)}{2}$	$\frac{q^3-2q+(q+2n)}{2}$
$\overline{D}_{31}$	$\frac{q^2(q^2-1)(q^4-1)}{4}$	$\frac{q(q^4-1)}{2}$	q

Table 11. Designs under M_4 for $q = 4n + 1$, where $n \in \mathbb{N}$.

Representative	$v = \|x^G\|$	$k = \|M \cap x^G\|$	$\lambda = \chi_{M_4}$
$\overline{A}_{31}$	$\frac{q(q^4-1)(q+1)}{2}$	$2n(q+1)(q^2+1)$	$2nq$
$\overline{A}_{32}$	$\frac{q(q^4-1)(q-1)}{2}$	$(q^2+1)(q-1)(q-2n)$	$q(q-2n)$
$\overline{B}_1(i)\ i \in\ R_1''$, $\overline{B}_1'(i)\ i \in\ R_1''$ and $\overline{B}_1''(i)\ i \in\ R_1''$	$q^4(q^2-1)^2$	$2q^2(q^2-1)$	1
$\overline{B}_2(i)\ i \in\ R_1''$	$\frac{q^4(q^4-1)}{2}$	$q^2(q^2+1)(q-1)$	$q - 1$
$\overline{B}_4(i,j)\ i < j \leq \frac{q-3}{2} - i$	$\frac{q^4(q^2+1)(q-1)^2}{2}$	$\frac{q^2(q^2+1)(q-1)(q-2)}{q+1}$	$q - 2$
$\overline{B}_5(i,j)$	$q^4(q^4-1)$	$2q^2(q^2+1)$	1
$\overline{B}_8(i)\ i = \frac{q-1}{2}$	$\frac{q^3(q^2+1)(q+1)}{2}$	$\frac{2q(q^2+1)(q+1)}{q-1}$	$2(q+1)$
$\overline{C}_1(i)$	$q^3(q^2+1)(q-1)$	$\frac{2q^2(q^2+1)(q-2)}{q+1}$	$q(q-2)$
$\overline{C}_{21}(i)$	$\frac{q^3(q^4-1)(q-1)}{2}$	$q(q^2+1)(q-1)$	1
$\overline{C}_3(i)$	$q^3(q^2+1)(q+1)$	$\frac{2q(q^2+1)(q-2n)}{q-1}$	$q - 2n$
$\overline{C}_{41}(i)$	$\frac{q^3(q^4-1)(q+1)}{2}$	$2q(q^2+1)(q+1)$	2
$\overline{D}_1$	$\frac{q^2(q^2+1)}{2}$	$\frac{2q(q^2+1)}{q-1}$	$2q(q+1)$
$\overline{D}_{21}$	$\frac{q^2(q^4-1)}{2}$	$n(q^2-1)$	n
$\overline{D}_{31}$	$\frac{q^2(q^2-1)(q^4-1)}{4}$	$\frac{(4n+1)(q^4-1)}{2}$	$4n+1$
$\overline{D}_{32}$	$\frac{q^2(q^4-1)(q^2-1)}{8}$	$\frac{(q+n)(q^4-1)}{4}$	$q + n$

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