Khayyam J. Math. 11 (2025), no. 2, 388-403

DOI:10.22034/KJM.2025.506507.3462

Khayyam Journal of Mathematics

emis.de/journals/KJM
kjm-math.org

RENORMALIZED SOLUTIONS FOR SOME NONLINEAR DEGENERATED ELLIPTIC PROBLEMS

MOHAMMED EL FATRY[1*], MOUNIR MEKKOUR[2] AND YOUSSEF AKDIM[3]

Communicated by L. Barreira

ABSTRACT. In this paper, we prove the existence of renormalized solution for a class of nonlinear degenerated elliptic problem in the form

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{N} a_{ij}(x,u)\frac{\partial u}{\partial x_j}\right)\omega_i(x) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $A(x,u) = (a_{ij}(x,s))_{1\le i,j\le N}$ is blow-up for a finite value m of the unknown u and the data $f \in L^1(\Omega)$.

1. Introduction

In this paper, we investigate the existence of a renormalized solutions for a class of degenerate elliptic equations of the type:

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{N} a_{ij}(x,u)\frac{\partial u}{\partial x_j}\right)\omega_i(x) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is an open bounded subset of $\mathbb{R}^N$ with the data f in $L^1(\Omega)$ .Moreover,

$$A : (x,s) \longmapsto A(x,s) = (a_{ij}(x,s))_{1\le i,j\le N}$$

Date: Received: 13 March 2025; Accepted: 11 September 2025.

* Corresponding author.

2020 *Mathematics Subject Classification.* Primary: 35H30; Secondary: 35J60.

Key words and phrases. Weighted Sobolev spaces, Renormalized solutions, Blow-up, L^1-data.

is a Carathéodory function from $\Omega \times (-\infty, m)$ into Sym_n (the set of symmetric $N \times N$ real matrices), below up when $s \to m^-$, and $\{\omega_i\}_{0 \le i \le N}$ is a vector of weight functions.

Problem (1.1) can be seen as a generalization of weighted equations studied in the literature, particularly the well-known case of the weighted p-Laplacian

$$-\operatorname{div}\left(\omega\,|\nabla u|^{p-2}\nabla u\right) = f,$$

and the perturbed version

$$-\operatorname{div}\big(\omega\, A(x,u)\big) = f$$

(see, e.g., [4, 19, 23, 20, 31, 3, 8, 25, 32, 13]).

The aim and novelty of the present paper are to extend the above class of problems to a more general operator with complex behavior on the set

$$\{x \in \Omega : u(x) = m\},$$

which is not necessarily of vanishing measure, and where the matrix $A(x,u)$ may blow up when $u \to m$.

This kind of problem has many applications in physics, since in real-world situations the energy does not tend to zero, and there is an energy dissipation that can be explained (see Frémond [21, 22]). In particular, the problem of k-epsilon turbulence also plays an important role in this context (see Launder and Spalding [26]).

In [11], the authors studied the problem

$$-\operatorname{div}\big(A(x,u)Du\big) = f,$$

where the matrix A may blow up and satisfies

$$\beta(s)\,|\xi|^2 \;\le\; A(x,s)\,\xi\cdot\xi \;\le\; \gamma(s)\,|\xi|^2,$$

with β and γ defined in (3.4) and (3.5).

In our problem, we will study a more complex case where the matrix $A(x,u)$ is scaled by a vector of weight functions $\{\omega_i\}_{0\le i\le N}$, in order to encompass a wider range of applications with nonconstant weights. The functional framework will be nonclassical, namely a weighted Sobolev space.

For the analysis of problem (1.1), we will use the notion of renormalized solution (see Definition 3.1). The reason for using this kind of solution is to capture the behavior of the solution on the set $\{u = m\}$, which will be suitably replaced by $T_k(u)$, since the datum is only L^1-integrable. Moreover, the flux $A(x,u)Du$ on $\{u = m\}$ must make sense (see 3.11).

For convenience, we recall that the notion of renormalized solution was first introduced by DiPerna [17] in the study of the Boltzmann equation, and later developed further by DiPerna and Lions [12]. This concept has since been extended to a wide range of nonlinear partial differential equations (PDEs), including applications in fluid mechanics [27]. For the elliptic case, we refer to [12, 30], while the parabolic case is discussed in [10, 9, 16, 15, 36, 35]. Additionally, the concept of entropy solution plays a crucial role in dealing with nonlinear PDEs with irregular data, providing uniqueness and stability results, as shown in [1, 14, 7, 6].

The present paper is organized as follows. In section 2, we present the weighted Sobolev spaces and some of its properties. Section 3 is divided into three steps: The first we approach the problem; the second step, we prove the weak limit of the field. Finally, we pass to the limit for prove the renormalized solution of (1.1).

2. Preliminaries

Let Ω be a bounded domain in $\mathbb{R}^N$ with $N \geq 1$, and let $\omega = \{\omega_i\}_{0\leq i\leq N}$ be a vector of weight functions. Furthermore, we assume that each component $\omega_i(x)$ is a measurable function, strictly positive almost everywhere in Ω and satisfies the following condition:

$$\omega_i \in L^1_{loc}(\Omega) \quad \text{and} \quad \omega_i^{\frac{-p}{p-1}} \in L^1_{loc}(\Omega).$$

We define the weighted Lebesgue space $L^p(\Omega, \omega_0)$ by weight ω_0, as the space of all real-valued measurable functions u for which

$$\|u\|_{p,\omega_0} = \left(\int_\Omega |u(x)|^p \, \omega_0(x) dx \right)^{\frac{1}{p}} < +\infty.$$

Similarly, we define the weighted Sobolev space of $W^{1,p}(\Omega, \omega)$, as the space of all real-valued functions $u \in L^p(\Omega, \omega)$ such that the derivatives in the sense of distributions satisfy $\frac{\partial u}{\partial x_i} \in L^p(\Omega, \omega)$ for all $i = 1, \ldots, N$, equipped with the norm

$$\|u\|_{1,p,\omega} = \left(\int_\Omega |u(x)|^p \, \omega_0(x) dx + \sum_{i=1}^N \int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^p \omega_i(x) dx \right)^{\frac{1}{p}}.$$

As we are concerned with a Dirichlet problem, we work in the space $X = W_0^{1,p}(\Omega, \omega)$ defined as the closure of $C_0^\infty(\Omega)$ with respect to norm $\|\cdot\|_{1,2,\omega}$. Then, $(X, \|\cdot\|_{1,p,\omega})$ is a reflexive Banach space whose dual is equivalent to $W^{-1,p}(\Omega, \omega^*)$, where $\omega^* = \left\{ \omega_i^* = \omega_i^{1-p'}, \, i = 1, \ldots, N \right\}$, and p' is the conjugate of p; that is, $p' = \frac{p}{p-1}$. If $p = 2$, then we denote X by $H_0^{1,p}(\Omega, \omega)$.

Proposition 2.1 (Hardy inequality)**.** *There exist a weight σ on Ω and a parameter q, $1 < q < \infty$ such that*

$$\sigma^{1-q} \in L^1(\Omega)$$

and such that the Hardy inequality

$$\left(\int_\Omega |u(x)|^q \, \sigma(x) dx \right)^{1/q} \leq \left(\sum_{i=1}^N \int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^2 \omega_i(x) dx \right)^{1/2},$$

holds for every $u \in X$. Moreover, the embedding $X \hookrightarrow L^p(\Omega, \sigma)$ is compact.

Lemma 2.2 (Convergence in weight space)**.** *Let $f \in L^p(\Omega, \omega)$ and let $(f_n)_n \in L^p(\Omega, \omega)$, with $\|f_n\|_{1,p,\omega} \leq C$. If $f_n(x) \to f(x)$ a.e. in Ω, then*

$$f_n \rightharpoonup f \quad \text{in } L^p(\Omega, \omega).$$

We refer readers to see more details in [5, 33, 18].

3. Basic assumption and main result

Let Ω be a bounded domain of $\mathbb{R}^N$ $(N \geq 1)$, and let m be a positive real number. We consider the following nonlinear problem:

$$\begin{cases} -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(\sum_{j=1}^{N} a_{ij}(x,u) \frac{\partial u}{\partial x_j} \right) \omega_i(x) = f \quad \text{in} \quad \Omega, \\ \qquad\qquad\qquad\qquad\qquad\qquad u = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{3.1}$$

We assume that

$$f \in L^1(\Omega), \tag{3.2}$$

and that

$$A : (x,s) \longmapsto A(x,s) = (a_{ij}(x,s))_{1 \leq i,j \leq N}, \tag{3.3}$$

is a Carathéodory function from $\Omega \times (-\infty, m)$ into Sym_n (the set of symmetric $N \times N$ real matrices), such that there exist two positive functions β and γ in $\mathcal{C}^0((-\infty, m))$ satisfying

$$\lim_{s \to m^-} \beta(s) = +\infty \quad \text{and} \quad \beta(s) \geq \alpha > 0 \quad \text{for all } s \in (-\infty, m), \tag{3.4}$$

$$\int_0^m \gamma(s) ds < +\infty, \tag{3.5}$$

$$\beta(s) \sum_{i=1}^{N} \omega_i(x) |\xi_i|^2 \leq \sum_{i=1}^{N} \left(\sum_{j=1}^{N} a_{ij}(x,s) \xi_j \omega_i(x) \right) \xi_i \leq \gamma(s) \sum_{i=1}^{N} \omega_i(x) |\xi_i|^2, \tag{3.6}$$

for any $s \in (-\infty, m)$, and $\xi \in \mathbb{R}^N$.

The following notations will be used all throughout the paper. For any $\varepsilon > 0$, we define the function b_ε by

$$b_\varepsilon(r) = \begin{cases} 1 & \text{if} \quad r \leq m - 2\varepsilon, \\ 1 - (r - m + \varepsilon) & \text{if} \quad m - 2\varepsilon \leq r \leq m - \varepsilon, \\ 0 & \text{if} \quad r \geq m - \varepsilon. \end{cases}$$

We define for $n \geq 1$ fixed,

$$T_n(s) = \max(-k, \min(s,k)), \;\; \theta_n(s) = \frac{1}{n}(T_n(s - T_n(s)),$$

and $h(s) = 1 - |\theta_n(s)|$, for all $s \in \mathbb{R}$.
We now give the definition of a renormalized solution of problem (3.1).

Definition 3.1. A measurable function u defined on Ω is a renormalized solution of problem (3.1) if

$$T_k(u) \in H_0^1(\Omega, \omega) \;\; \text{for all } k \geq 0, \tag{3.7}$$

$$u \leq m \qquad a.e. \;\; \text{in } \Omega, \tag{3.8}$$

for every $k \geq 0$

$$\chi_{\{-k<u<m\}} \sum_{j=1}^{N} a_{ij}(x,u) \frac{\partial u}{\partial x_i} \in L^2(\Omega, \omega_i^*), \tag{3.9}$$

for $i = 1, \dots, N$,

$$\lim_{p\to+\infty} \frac{1}{p} \sum_{i=1}^{N} \int_{\{-2p<u<-p\}} \left(\sum_{j=1}^{N} a_{ij}(x,u) \frac{\partial u}{\partial x_j} \omega_i(x) \right) \frac{\partial u}{\partial x_i} dx = 0, \tag{3.10}$$

$$\lim_{p\to+\infty} p \sum_{i=1}^{N} \int_{\Omega} \chi_{\{m-2/p<u<m-1/p\}} \left(\sum_{j=1}^{N} a_{ij}(x,u) \frac{\partial u}{\partial x_j} \omega_i(x) \right) \frac{\partial u}{\partial x_i} dx = \int_{\Omega} f \chi_{\{u=m\}} dx, \tag{3.11}$$

and if, for every function S in $W^{1,\infty}(\mathbb{R})$, such that $supp(S)$ is compact and $S(m) = 0$, u satisfies

$$\sum_{i=1}^{N} \int_{\Omega} \left(\sum_{j=1}^{N} a_{ij}(x,u) \frac{\partial u}{\partial x_j} \omega_i(x) \right) \frac{\partial (S(u)\varphi)}{\partial x_i} dx = \int_{\Omega} f S(u) \varphi dx, \tag{3.12}$$

for every $\varphi \in H_0^1(\Omega, \omega) \cap L^\infty(\Omega)$.

Remark 3.2. Conditions (3.7)–(3.9) show that all term in (3.12) are well defined. The assumption (3.11) has been established in [33].

Our main result is then as follows.

Theorem 3.3. *Under the assumptions (3.2)–(3.6), there exists at least a renormalized solution u of problem (3.1).*

Remark 3.4. Theorem 3.3 generalizes the analogous statement in [11] to the weighted case.

4. Proof of main result

The proof is divided into four steps.

4.1. **Step 1: The problem is approaching.** For $\varepsilon > 0$, we consider the field of matrices

$$A^{\varepsilon}(x,s) = b_\varepsilon(s) A(x,s) + (1 - b_\varepsilon(s)) \beta(m-\varepsilon) B, \tag{4.1}$$

where b_ε is the function defined in (3) and B is the diagonal matrix as $\{\omega_i\}_{1\le i\le N}$ of $\mathbb{R}^{N\times N}$. Indeed in (4.1) we use the convention

$$b_\varepsilon(s) A(x,s) = 0 \text{ for } s \ge m - \varepsilon.$$

Due the assumptions (3.4)–(3.6), we have

$$\begin{aligned} \alpha \sum_{i=1}^{N} \omega_i(x) |\xi_i|^2 &\le \sum_{i=1}^{N} \left(\sum_{j=1}^{N} a_{ij}^{\varepsilon}(x,s)\, \xi_j \right) \omega_i(x)\, \xi_i \\ &\le \left(\gamma(s)\, b_\varepsilon(s) + \sup_{r\in(0,m-\varepsilon)} \beta(r) \right) \sum_{i=1}^{N} \omega_i(x) |\xi_i|^2, \end{aligned} \tag{4.2}$$

for every $\xi \in \mathbb{R}^N$. Finally, there exists a family $\{f_\varepsilon\}_{\varepsilon>0} \subset L^\infty(\Omega)$ such that

$$f_\varepsilon \to f \quad \text{in } L^1(\Omega) \quad \text{as } \varepsilon \to 0.$$

Let us now consider the approximate problem:

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left(\sum_{j=1}^{N} a_{ij}^{\varepsilon}(x,u^{\varepsilon})\frac{\partial u^{\varepsilon}}{\partial x_j}\right)\omega_i(x)\right) = f_{\varepsilon} & \text{in } \Omega, \\ u^{\varepsilon} = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.3}$$

As consequence, proving the existence of a weak solution $u^{\varepsilon} \in H_0^1(\Omega,\omega)$ of (4.3) remains a special case of [29] by taking $\lambda = 0$ and $\phi = 0$.

4.2. **Step 2: A priority estimate.** Take $T_k(u^{\varepsilon})$ as a test function in (4.3), we have

$$\sum_{i=1}^{N}\int_{\Omega}\sum_{j=1}^{N}\left(a_{ij}^{\varepsilon}(x,u^{\varepsilon})\frac{\partial u^{\varepsilon}}{\partial x_j}\omega_i(x)\right)\frac{\partial T_k(u^{\varepsilon})}{\partial x_i}dx = \int_{\Omega} f_{\varepsilon}T_k(u^{\varepsilon})\,dx.$$

Thanks to (4.2) and $f_{\varepsilon} \in L^{\infty}(\Omega)$, we have

$$\alpha\int_{\Omega}\sum_{i=1}^{N}\omega_i\left|\frac{\partial T_k(u^{\varepsilon})}{\partial x_i}\right|^2 dx \le Ck, \tag{4.4}$$

and for every $i = 1,\dots,N$, we have

$$\sum_{j=1}^{N} x_{ij}^{\varepsilon}\frac{\partial T_k(u^{\varepsilon})}{\partial x_j} \in L^2(\Omega,\omega_i), \tag{4.5}$$

where $X^{\varepsilon}(x,s) = \left(x_{ij}^{\varepsilon}(x,s)\right)_{1\le i,j\le N}$ is the square root of the matrix $A^{\varepsilon}(x,s)$. Let $k \ge 0$. By using Hardy's inequality and $\sigma \in L^1(\Omega)$, we have

$$\begin{aligned} |\{|u^{\varepsilon}| > k\}| &= \int_{\{|u^{\varepsilon}|>k\}} \frac{T_k(u^{\varepsilon})}{k\,\sigma^{1/2}(x)}\,\sigma^{1/2}(x)\,dx \\ &\le \frac{1}{k}\left(\int_{\Omega}|T_k(u^{\varepsilon})|^2\,\sigma(x)\,dx\right)^{1/2}\left(\int_{\Omega}\sigma^{-1}(x)\,dx\right)^{1/2} \\ &\le \frac{1}{k}\left(\int_{\Omega}\sum_{i=1}^{N}\omega_i(x)\left|\frac{\partial T_k(u^{\varepsilon})}{\partial x_i}\right|^2 dx\right)^{1/2}\left(\int_{\Omega}\sigma^{-1}(x)\,dx\right)^{1/2} \\ &\le Ck^{-1}. \end{aligned}$$

Therefore, we have

$$\lim_{k\to+\infty} meas\,\{|u^{\varepsilon}| > k\} = 0.$$

Note that for any $t, \varepsilon, \varepsilon' > 0$, we have

$$\begin{aligned} meas\left\{\left|u^{\varepsilon} - u^{\varepsilon'}\right| > t\right\} \;\le\;& meas\,\{|u^{\varepsilon}| > t\} + meas\left\{\left|u^{\varepsilon'}\right| > t\right\} \\ &+ meas\left\{\left|T_k(u^{\varepsilon'}) - T_k(u^{\varepsilon})\right| > t\right\}. \end{aligned}$$

Therefore, from (4.4), we can extract a subsequence, still denoted by itself, such that

$$T_k(u^{\varepsilon}) \rightharpoonup v_k \text{ weakly in } H_0^1(\Omega,\omega) \tag{4.6}$$

and strongly $L^2(\Omega,\omega)$ and $a.e.$ in Ω. It can be seen that $T_k\left(u^\varepsilon\right)$ is the Cauchy that converges by measure in Ω. Then u^ε is the Cauchy that converges by measure in Ω. There is a measurable function u such that

$$u^\varepsilon \to u \text{ converges almost everywhere on } \Omega. \tag{4.7}$$

We have, for any $k>0$

$$T_k\left(u^\varepsilon\right) \rightharpoonup T_k\left(u\right) \text{ weakly in } H_0^1(\Omega,\omega), \tag{4.8}$$

and

$$T_k\left(u^\varepsilon\right) \to T_k\left(u\right) \text{ converges almost everywhere on } \Omega.$$

Using $T_{2m}^+(u^\varepsilon) - T_m^+(u^\varepsilon)$ as a test function in (4.3) leads to

$$\begin{aligned}\sum_{i=1}^N \int_\Omega \sum_{j=1}^N a_{ij}^\varepsilon(x,u^\varepsilon)\,\frac{\partial u^\varepsilon}{\partial x_j}\,&\frac{\partial\left(T_{2m}^+(u^\varepsilon)-T_m^+(u^\varepsilon)\right)}{\partial x_i}\,\omega_i(x)\,dx\\ &= \int_\Omega f_\varepsilon\left(T_{2m}^+(u^\varepsilon)-T_m^+(u^\varepsilon)\right)dx.\end{aligned}$$

Since $f_\varepsilon \in L^\infty(\Omega)$, we get

$$\sum_{i=1}^N \int_\Omega \sum_{j=1}^N \Big(a_{ij}^\varepsilon(x,u^\varepsilon)\,\frac{\partial u^\varepsilon}{\partial x_j}\,\omega_i(x)\Big) \times \frac{\partial\Big(T_{2m}^+(u^\varepsilon)-T_m^+(u^\varepsilon)\Big)}{\partial x_i}\,dx \le C, \tag{4.9}$$

where C does not depend on ε. Thanks to (4.2) and the Hardy inequality, we have

$$\beta(m-\varepsilon)\int_\Omega \sigma(x)\left|T_{2m}^+(u^\varepsilon)-T_m^+(u^\varepsilon)\right|^2 dx \le C. \tag{4.10}$$

We can pass to the limit in (4.10) as ε tends to 0, to deduce that

$$\begin{aligned} T_{2m}^+(u)-T_m^+(u) &= 0 \qquad a.e. \text{ in } \Omega,\\ u &\le m \qquad a.e. \text{ in } \Omega.\end{aligned}$$

4.3. **Step 3: Weak limit of the field.** We define two sequences of auxiliary functions

$$v^\varepsilon = \int_0^{(u^\varepsilon)^+} \left(\gamma\left(s\right)b_\varepsilon(s) + (1-b_\varepsilon(s))\beta(m-\varepsilon\right)ds \tag{4.11}$$

and

$$d^\varepsilon = \int_0^{(u^\varepsilon)^+} \left(\beta\left(s\right)b_\varepsilon(s) + (1-b_\varepsilon(s))\beta(m-\varepsilon\right)ds.$$

Since $u^\varepsilon \in H_0^1(\Omega,\omega)$, we have

$$\nabla T_k(v^\varepsilon) = \chi_{\{v^\varepsilon<k\}}\left[\left(\gamma\left(u^\varepsilon\right)b_\varepsilon(u^\varepsilon) + (1-b_\varepsilon(u^\varepsilon))\beta(m-\varepsilon\right)\right]\nabla T_{k/\alpha}(u^\varepsilon)^+ \tag{4.12}$$

and

$$\nabla T_k(d^\varepsilon) = \chi_{\{d^\varepsilon<k\}}\Big[\beta(u^\varepsilon)\,b_\varepsilon(u^\varepsilon) + \left(1-b_\varepsilon(u^\varepsilon)\right)\beta(m-\varepsilon)\Big]\nabla T_{k/\alpha}\big((u^\varepsilon)^+\big)\,. \tag{4.13}$$

Then $T_k(v^\varepsilon) \in H_0^1(\Omega,\omega)$ and $T_k(d^\varepsilon) \in H_0^1(\Omega,\omega)$.
Take $T_n(d^\varepsilon - (u^\varepsilon)^-)$ as a test function in (4.3), we have

$$\sum_{i=1}^{N} \int_\Omega \sum_{j=1}^{N} \left(a_{ij}^\varepsilon(x,u^\varepsilon) \frac{\partial u^\varepsilon}{\partial x_j} \right) \frac{\partial \left(d^\varepsilon - (u^\varepsilon)^-\right)}{\partial x_i} \omega_i(x) dx \leq C. \tag{4.14}$$

Since (4.13), we get

$$\begin{aligned}
&\sum_{i=1}^{N} \int_\Omega \chi_{\{w^\varepsilon<k\}} \Big[\beta(u^\varepsilon)\, b_\varepsilon(u^\varepsilon) + (1-b_\varepsilon(u^\varepsilon))\Big] \sum_{j=1}^{N} \left(a_{ij}^\varepsilon(x,u^\varepsilon) \frac{\partial (u^\varepsilon)^+}{\partial x_j} \right) \\
&\qquad \times \frac{\partial \Big(T_{\frac{n}{\alpha}}(u^\varepsilon)^+\Big)}{\partial x_i}\, \omega_i(x)\, dx \\
&\quad + \sum_{i=1}^{N} \int_\Omega \chi_{\{(u^\varepsilon)^-<k\}} \sum_{j=1}^{N} \left(a_{ij}^\varepsilon(x,u^\varepsilon) \frac{\partial (u^\varepsilon)^-}{\partial x_j} \right) \frac{\partial (T_n(u^\varepsilon)^-)}{\partial x_i}\, \omega_i(x)\, dx \\
&\leq C.
\end{aligned} \tag{4.15}$$

Now, the assumptions (3.6) and the definition of A^ε in (4.1) show that

$$\begin{aligned}
&(1-b_\varepsilon(s))\, \beta(m-\varepsilon) \sum_{i=1}^{N} \omega_i(x)\, |\xi_i|^2 \\
&\quad + \beta(s)\, b_\varepsilon(s) \sum_{i=1}^{N} \omega_i(x)\, |\xi_i|^2 \\
&\leq \sum_{i=1}^{N} \sum_{j=1}^{N} \left(a_{ij}^\varepsilon(x,s)\, \xi_j\right) \xi_i,
\end{aligned}$$

for any $s \in \mathbb{R}$; any $\xi \in \mathbb{R}^N$ and *a.e.* in Ω. Then (4.1) and (4.15) yield

$$\begin{aligned}
&\int_\Omega \sum_{i=1}^{N} \omega_i(x) \left| \frac{\partial T_n(d^\varepsilon)}{\partial x_i} \right|^2 dx + \alpha \int_\Omega \sum_{i=1}^{N} \omega_i(x) \left| \frac{\partial T_n\big((u^\varepsilon)^-\big)}{\partial x_i} \right|^2 dx \\
&\leq n\, \|f_\varepsilon\|_{L^1(\Omega)}\,.
\end{aligned} \tag{4.16}$$

Equation (4.16) implies that the functions $(u^\varepsilon)^-$ and d^ε have disjoint supports. We have

$$\min(1,\alpha) \int_\Omega \sum_{i=1}^{N} \omega_i(x) \left| \frac{\partial T_n(d^\varepsilon - (u^\varepsilon)^-)}{\partial x_i} \right|^2 dx \leq n\, \|f_\varepsilon\|_{L^1(\Omega)}\,. \tag{4.17}$$

Hardy's inequality and (4.17) lead to

$$\begin{aligned} meas\left\{\left|d^{\varepsilon}-(u^{\varepsilon})^{-}\right|>n\right\} &\leq \frac{1}{n}\int_{\Omega}\left|T_n(d^{\varepsilon}-(u^{\varepsilon})^{-})\right|\sigma^{1/2}\left(x\right)\sigma^{-1}(x)^{1/2}dx \\ &\leq \frac{1}{n}\left(\int_{\sigma}\left|T_n(d^{\varepsilon}-(u^{\varepsilon})^{-})\right|^2\sigma\left(x\right)dx\right)^{1/2}\left\|(\sigma^{-1})\right\|_{L^2(\Omega)}^{1/2} \\ &\leq \frac{1}{n}\left(\int_{\Omega}\sum_{i=1}^{N}\omega_i(x)\left|\frac{\partial T_n(d^{\varepsilon}-(u^{\varepsilon})^{-})}{\partial x_i}\right|^2dx\right)^{1/2}\left\|(\sigma^{-1})\right\|_{L^1(\Omega)}^{1/2} \\ &\leq Cn^{-1/2}, \end{aligned}$$

where C does not depend on n and ε, and we obtain that

$$\lim_{n\to+\infty}\sup_{\varepsilon}\left\{x\in\Omega\ ;\ \left|d^{\varepsilon}-(u^{\varepsilon})^{-}\right|>n\right\}=0. \tag{4.18}$$

On the other hand, we remark that

$$v^{\varepsilon}=d^{\varepsilon}+\int_0^{(u^{\varepsilon})^{+}}(\gamma(s)-\beta(s)b_{\varepsilon}(s)ds\leq d^{\varepsilon}+\int_0^{m}(\gamma(s)-\beta(s)b_{\varepsilon}(s)ds. \tag{4.19}$$

Since $\int_0^m(\gamma(s)-\beta(s)b_{\varepsilon}(s)ds<+\infty$ and (4.18), we obtain that

$$\lim_{n\to+\infty}\sup_{\varepsilon}\left\{x\in\Omega\ ;\ \left|v^{\varepsilon}-(u^{\varepsilon})^{-}\right|>n\right\}=0. \tag{4.20}$$

In addition, from (4.16) and the same procedures as above, we deduce

$$d^{\varepsilon}\to d\ a.e.\ \text{in}\ \Omega, \tag{4.21}$$

where d is a mensurable function. Then, by (4.7), (4.19), and (4.21), we have

$$v^{\varepsilon}\to v\ a.e.\ \text{in}\ \Omega, \tag{4.22}$$

where $v=d+\int_0^{(u)^{+}}(\gamma(s)-\beta(s)b_{\varepsilon}(s)ds$, and v is a measurable positive function. Consequently, the definitions b_{ε} in (3) and v^{ε} in (4.11) and the convergences (4.6) and (4.22), show that

$$v=\int_0^{(u)^{+}}\gamma(s)ds\ a.e.\ \text{in}\quad \left\{x\in\Omega\ ;\ u\left(x\right)<m\right\}.$$

However, as far as we know, we cannot expect to have a similar identification on the subset $\left\{x\in\Omega\ ;\ u\left(x\right)=m\right\}$.

Now, we choose $\theta_n\left(v^{\varepsilon}-(u^{\varepsilon})^{-}\right)$ as a test function in (4.3), which gives

$$\frac{1}{n}\sum_{i=1}^{N}\int_{\{n\leq|v^{\varepsilon}-(u^{\varepsilon})^{-}|\leq 2n\}}\sum_{j=1}^{N}\left(a_{ij}^{\varepsilon}(x,u^{\varepsilon})\frac{\partial u^{\varepsilon}}{\partial x_j}\omega_i(x)\right)\frac{\partial T_n(v^{\varepsilon}-(u^{\varepsilon})^{-})}{\partial x_i}\,dx \\ \leq \int_{\{|v^{\varepsilon}-(u^{\varepsilon})^{-}|>n\}}|f_{\varepsilon}|\,dx.$$

Using now $f_\varepsilon \in L^1(\Omega)$ and (4.20), we obtain

$$\lim_{n\to\infty} \sup_{\varepsilon} \sum_{i=1}^{N} \int_{\left\{n\le \left|v^\varepsilon-(u^\varepsilon)^-\right|\le 2n\right\}} \sum_{j=1}^{N} \frac{1}{n} \left(a_{ij}^\varepsilon(x,u^\varepsilon)\, \frac{\partial u^\varepsilon}{\partial x_j}\, \omega_i(x) \right) \times \frac{\partial T_n\big(v^\varepsilon-(u^\varepsilon)^-\big)}{\partial x_i}\, dx = 0. \tag{4.23}$$

To prove the weak limit of the field, we need to see that $\sum_{j=1}^{N}\left(a_{ij}^\varepsilon(x,u^\varepsilon)\frac{\partial u^\varepsilon}{\partial x_i}\right)$ is bounded in $L^2(\Omega,\omega_i^{-1})$ for every $i=1,\dots,N$ in the subset, where $v^\varepsilon-(u^\varepsilon)^-$ is truncated.

Indeed, we plug the test function $T_k(v^\varepsilon)$ in (4.3). Using (4.12), we obtain

$$\sum_{i=1}^{N}\int_{\{|v^\varepsilon|\le k\}}\sum_{j=1}^{N}\left(a_{ij}^\varepsilon(x,u^\varepsilon)\frac{\partial u^\varepsilon}{\partial x_j}\right)\left[(\gamma\,(u^\varepsilon)\,b_\varepsilon(u^\varepsilon)+(1-b_\varepsilon(s))\beta(m-\varepsilon)\right]\frac{\partial (u^\varepsilon)^+}{\partial x_j}\omega_i dx \le k\left\|f_\varepsilon\right\|_{L^1(\Omega)}.$$

By the definition of $A^\varepsilon(x,s)$ in (4.1) and (3.6), we have

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\left(a_{ij}^\varepsilon(x,s)\xi_j\right)\omega_i\xi_i \le (\gamma\,(s)\,b_\varepsilon(s)+(1-b_\varepsilon(s))\beta(m-\varepsilon)\sum_{i=1}^{N}\omega_i\left|\xi_i\right|^2, \tag{4.24}$$

for any $s\in\mathbb{R}$, any $\xi\in\mathbb{R}^N$ and $a.e.$ in Ω. Using (4.24) with $\xi = X^\varepsilon(x,u^\varepsilon)D\,(u^\varepsilon)^+$, therefore

$$\sum_{i=1}^{N}\int_{\{|v^\varepsilon|\le k\}}\left|\sum_{j=1}^{N}\left(a_{ij}^\varepsilon(x,u^\varepsilon)\frac{\partial\,(u^\varepsilon)^+}{\partial x_j}\right)\right|^2\omega_i(x)\le C,$$

and for any $k\ge 0$,

$$\chi_{\{v^\varepsilon<k\}}\sum_{j=1}^{N}\left(a_{ij}^\varepsilon(x,u^\varepsilon)\frac{\partial\,(u^\varepsilon)^+}{\partial x_j}\right)\ \text{is bounded in } L^2(\Omega,\omega_i^{-1}),$$

for every $i=1,\dots,N$, uniformly in ε. Now, since $\chi_{\left\{\left|v^\varepsilon-(u^\varepsilon)^-\right|<k\right\}} = \chi_{\{0\le v^\varepsilon<k\}} + \chi_{\{0\le u^\varepsilon<k\}}$ $a.e.$ in Ω, the continuous character of $A^\varepsilon(x,s)$ for $s\in(-\infty,0]$ and estimate (4.5), we have

$$\chi_{\left\{\left|v^\varepsilon-(u^\varepsilon)^-\right|<k\right\}}\sum_{j=1}^{N}\left(a_{ij}^\varepsilon(x,u^\varepsilon)\frac{\partial u^\varepsilon}{\partial x_j}\right)\ \text{is bounded in } L^2(\Omega,\omega_i^{-1}), \tag{4.25}$$

for every $i=1,\dots,N$, uniformly in ε.

Use the estimate (4.25) and (4.5) to extract another subsequence, still indexed by ε, such that

$$\left(h_n\left(v^\varepsilon-(u^\varepsilon)^-\right)\sum_{j=1}^{N}a_{ij}^\varepsilon(x,u^\varepsilon)\frac{\partial u^\varepsilon}{\partial x_j}\right)_{1\le i\le N}\longrightarrow\left(\psi_n^i\right)_{1\le i\le N} \quad \text{weakly in } \prod_{i=1}^{N}L^2(\Omega,\omega_i^{-1}), \tag{4.26}$$

$$\left(\sum_{j=1}^{N} x_{ij}^{\varepsilon}(x,u^{\varepsilon})\frac{\partial T_k(u^{\varepsilon})}{\partial x_j}\right)_{1\le i\le N} \longrightarrow \left(Y_k^i\right)_{1\le i\le N}$$
$$\text{weakly in } \prod_{i=1}^{N} L^2(\Omega,\omega_i^{-1}),$$

as ε tends to 0, where for any $k \ge 0$, $n \ge 1$, $\psi_n \in \prod_{i=1}^{N} L^2(\Omega,\omega_i^*)$, and $Y_k \in \prod_{i=1}^{N} L^2(\Omega,\omega_i^*)$.

Then, to identify ψ_n on the subset where $u < m$, let h be a $C^{\infty}(\mathbb{R}) - function$ such that $supp(h) \subset [-M, l]$ with $l < m$ and $M > 0$. Next, for ε small enough, we have

$$h(s)A^{\varepsilon}(x,s) = h(s)A(x,T_l(s^+) - T_M(s^-)). \tag{4.27}$$

By (4.27), the convergences (4.7), (4.8) and (4.22), we deduce for every $i = 1,\ldots,N$,

$$\begin{aligned} &h(u^{\varepsilon})h_n\left(v^{\varepsilon} - (u^{\varepsilon})^-\right)\sum_{j=1}^{N}\left(a_{ij}^{\varepsilon}(x,u^{\varepsilon})\frac{\partial u^{\varepsilon}}{\partial x_j}\right)\\ &\qquad \longrightarrow h(u)h_n\left(v-u^-\right)\sum_{j=1}^{N}\left(a_{ij}(x,u)\frac{\partial u}{\partial x_j}\right)\\ &\text{weakly in } L^2(\Omega,\omega_i^*),\end{aligned} \tag{4.28}$$

as ε tends to 0 and where Du stands for $DT_l(u^+) - DT_M(u^+)$. It follows from (4.28) and (4.26) that

$$\psi_n = h_n(v-u)\,A(x,u)\,Du \quad \text{a.e. in } \{x\in\Omega\,;\,u(x)<m\}\,. \tag{4.29}$$

since $l < m$ and M are arbitrary.

Now, remark that on the subset $\{x\in\Omega\,;\,u\,(x)<m\}$, we have

$$0 \le v = \int_0^{(u)^+}\gamma(s)ds < \int_0^{m}\gamma(s)ds,$$

and then for $n > \int_0^m \gamma(s)ds, h_n\,(v-u) = h_n\,(-u)$ on $\{x\in\Omega\,;\,u\,(x)<m\}$. It follows from (4.29) that

$$\psi_n = \left(h_n\,(-u)\sum_{j=1}^{N}\left(a_{ij}(x,u)\frac{\partial u}{\partial x_j}\right)\right)_{1\le i\le N} \quad a.e. \text{ in } \{x\in\Omega\,;\,u\,(x)<m\}\,, \tag{4.30}$$

which in turn implies that

$$\chi_{\{-k<u<m\}}\sum_{j=1}^{N}\left(a_{ij}(x,u)\frac{\partial u}{\partial x_j}\right) \in L^2(\Omega,\omega_i^*).$$

To identify Y_n, we use ψ_n defined above. We have for every $k \geq 0$

$$h_n(v^\varepsilon - (u^\varepsilon)^-) \sum_{j=1}^{N} \left(a^\varepsilon_{ij}(x, u^\varepsilon) \frac{\partial T_k(u^\varepsilon)}{\partial x_j} \right) \to \psi_n^{i,k} \text{ weakly in } L^2(\Omega, \omega_i^*),$$

for every $i = 1, \dots, N$. We can write

$$h_n(v^\varepsilon - (u^\varepsilon)^-) X^\varepsilon(x, u^\varepsilon) T_k(u^\varepsilon) = h_n(v^\varepsilon - (u^\varepsilon)^-) \left(X^\varepsilon(x, u^\varepsilon)\right)^{-1} A^\varepsilon(x, u^\varepsilon) T_k(u^\varepsilon).$$

For the some technique developed in [10], we will deduce

$$Y_n = \left(\chi_{\{u<m\}} \sum_{j=1}^{N} \left(x_{ij}(x, u) \frac{\partial T_k(u)}{\partial x_j} \right) \right)_{1 \leq i \leq N} \quad a.e. \text{ in } \Omega,$$

for every $i = 1, \dots, N$.

4.4. **Step 4: End of the proof.** We take $h(r) = h_n(G^\varepsilon(r) - r^-)z$ in (4.3), where z is an element of $W_0^{1,2}(\Omega, \omega) \cap L^\infty(\Omega)$. We take $h(r) = h_n(G^\varepsilon(r) - r^-)z$ in (4.3), where z is an element of $W_0^{1,2}(\Omega, \omega) \cap L^\infty(\Omega)$. We obtain

$$\begin{aligned} &\sum_{i=1}^{N} \int_\Omega h_n\big(v^\varepsilon - (u^\varepsilon)^-\big) \sum_{j=1}^{N} \big(a^\varepsilon_{ij}(x, u^\varepsilon) \frac{\partial u^\varepsilon}{\partial x_j} \omega_i(x)\big) \frac{\partial z}{\partial x_i}\, dx \\ &\quad + \frac{1}{n} \sum_{i=1}^{N} \int_{\{n \leq |v^\varepsilon - (u^\varepsilon)^-| \leq 2n\}} z \sum_{j=1}^{N} \big(a^\varepsilon_{ij}(x, u^\varepsilon) \frac{\partial u^\varepsilon}{\partial x_j} \omega_i(x)\big) \frac{\partial \left(v^\varepsilon - (u^\varepsilon)^-\right)}{\partial x_i}\, dx \\ &= \int_\Omega f\, h_n\big(v^\varepsilon - (u^\varepsilon)^-\big) z\, dx. \end{aligned} \tag{4.31}$$

First, recall that the convergences (4.7) and (4.22) yield

$$\int_\Omega f_\varepsilon h_n(v^\varepsilon - (u^\varepsilon)^-) z dx \to \int_\Omega f h_n(v - u^-) z dx$$

as ε tends to 0.

Next, we put

$$w(n) = \sup_\varepsilon \frac{1}{n} \sum_{i=1}^{N} \int_{\left\{n \leq \left|v^\varepsilon - (u^\varepsilon)^- \leq 2n\right|\right\}} \sum_{j=1}^{N} \left(a^\varepsilon_{ij}(x, u^\varepsilon) \frac{\partial u^\varepsilon}{\partial x_j} \omega_i(x) \right) \frac{\partial (v^\varepsilon - (u^\varepsilon)^-)}{\partial x_i} dx.$$

We deduce form (4.31) that

$$\begin{aligned} -\, w(n)\, \|z\|_{L^\infty(\Omega)} \leq \lim_{\varepsilon \to 0} \sum_{i=1}^{N} \int_\Omega h_n\big(v^\varepsilon - (u^\varepsilon)^-\big) \sum_{j=1}^{N} \big(a^\varepsilon_{ij}(x, u^\varepsilon) \frac{\partial u^\varepsilon}{\partial x_j} \omega_i(x)\big) \frac{\partial z}{\partial x_i}\, dx \\ - \int_\Omega f\, h_n(v - u^-)\, z\, dx \leq w(n)\, \|z\|_{L^\infty(\Omega)}. \end{aligned} \tag{4.32}$$

We note that $h_n(v - u^-) \to 1$ $a.e.$ in Ω as n tends to $+\infty$, and by(4.23), $w_n \to 0$ as n tends to. It follows that

$$\lim_{n \to +\infty} \lim_{\varepsilon \to 0} \int_\Omega h_n(v^\varepsilon - (u^\varepsilon)^-) \sum_{j=1}^{N} \left(a^\varepsilon_{ij}(x, u^\varepsilon) \frac{\partial u^\varepsilon}{\partial x_j} \omega_i(x) \right) \frac{\partial z}{\partial x_i} dx = \int_\Omega f z dx. \tag{4.33}$$

Firstly, we choose $z = h(u)\varphi$, where $h \in W^{1,\infty}(\mathbb{R})$ has a compact support and satisfies $h(m) = 0$ and $\varphi \in C_0^\infty(\Omega)$. By (4.29) we have

$$h(u)\varphi \in W_0^{1,2}(\Omega,\omega) \cap L^\infty(\Omega). \tag{4.34}$$

Then, (4.30) and (4.31) lead to

$$\lim_{n\to+\infty} \sum_{i=1}^N \int_\Omega \psi_n^i \frac{\partial(h(u)\varphi)}{\partial x_i}\omega_i(x)dx = \int_\Omega fh(u)\varphi dx. \tag{4.35}$$

Using now the identification (4.31) of ψ_n, it follows, for $k \geq 0$ such that $supp(h) \subset [-k,k]$ and $n \geq k$ and $n > \int_0^m \gamma(s)ds$, that

$$\begin{aligned}
&\lim_{n\to+\infty} \sum_{i=1}^N \int_\Omega (\psi_n)_i\, \omega_i(x) \frac{\partial(h(u)\varphi)}{\partial x_i}dx \\
&\quad= \lim_{n\to+\infty} \sum_{i=1}^N \int_\Omega h_n(-u^-) \sum_{j=1}^N \left(a_{ij}(x,u)\frac{\partial u}{\partial x_j}\omega_i(x)\right)\frac{\partial(h(u)\varphi)}{\partial x_i}dx \\
&\qquad= \sum_{i=1}^N \int_\Omega \sum_{j=1}^N \left(a_{ij}(x,u)\frac{\partial u}{\partial x_j}\right)\frac{\partial(h(u)\varphi)}{\partial x_i}\omega_i(x)dx. \qquad (4.36)
\end{aligned}$$

From (4.35) and (4.36), we obtain that

$$\sum_{i=1}^N \int_\Omega \sum_{j=1}^N \left(a_{ij}(x,u)\frac{\partial u}{\partial x_j}\right)\frac{\partial(h(u)\varphi)}{\partial x_i}\omega_i(x)dx = \int_\Omega fh(u)\varphi dx,$$

for any $\varphi \in C_0^\infty(\Omega)$. This shows that (3.12) is satisfied in $D'(\Omega)$.

Secondly, we take $z = \theta_p(-u^-)$ in (4.33) for a fixed integer $p \geq 1$ and we have

$$\lim_{n\to+\infty} \sum_{i=1}^N \int_\Omega (\psi_n)_i \frac{\partial(\theta_p(-u^-))}{\partial x_i}\omega_i(x)dx = \int_\Omega f\theta_p(-u^-)dx.$$

Using the identification of ψ_n as above, it gives

$$\sum_{i=1}^N \int_\Omega \sum_{j=1}^N \left(a_{ij}(x,u)\frac{\partial u}{\partial x_j}\omega_i(x)\right)\frac{\partial(\theta_p(-u^-))}{\partial x_i}dx = \int_\Omega f(\theta_p(-u^-))dx.$$

Then letting p tends to $+\infty$, we have

$$\lim_{p\to+\infty}\frac{1}{p}\sum_{i=1}^N \int_{\{-2p<u<-p\}} \sum_{j=1}^N \left(a_{ij}(x,u)\frac{\partial u}{\partial x_j}\omega_i(x)\right)\frac{\partial(\theta_p(-u^-))}{\partial x_i}dx = 0,$$

since again u finite $a.e.$ in Ω, and (3.12) is established.

Finally to obtain (3.11), we choose $z_p = (1 - b_{1/p}(u^+))\varphi$, where p is a fixed integer ≥ 1 and $\varphi \in W_0^{1,2}(\Omega,\omega) \cap L^\infty(\Omega)$ is such that $D\varphi = 0$ $a.e.$ in $\{x \in \Omega \;\; ; u(x) = m\}$. Indeed $\|z_p\|_{L^\infty(\Omega)} \leq \|\varphi\|_{L^\infty(\Omega)}$ and

$$Dz_p = p\chi_{\{m-2/p<u<m-1/p\}}Du\varphi + (1 - b_{1/p}(u^+))D\varphi$$

so that $Dz_p = 0$ *a.e.* on $\{x \in \Omega \ ; u(x) = m\}$. It follows from (4.33) that

$$\begin{aligned} p \sum_{i=1}^{N} \int_{\Omega} \chi_{\left\{m-\frac{2}{p}<u<m-\frac{1}{p}\right\}} \sum_{j=1}^{N} \left(a_{ij}(x,u) \frac{\partial u}{\partial x_j} \omega_i(x) \right) \frac{\partial u}{\partial x_i} \varphi \, dx \\ + \sum_{i=1}^{N} \int_{\Omega} (1 - b_{1/p}(u^+)) \, \chi \sum_{j=1}^{N} \left(a_{ij}(x,u) \frac{\partial u}{\partial x_j} \omega_i(x) \right) \frac{\partial \varphi}{\partial x_i} \, dx \\ = \int_{\Omega} f \, (1 - b_{1/p}(u^+)) \, \varphi \, dx. \end{aligned} \tag{4.37}$$

Now, as p tends to $+\infty$, $(1 - b_{1/p}(u^+)) \to \chi_{\{u=m\}}$ *a.e.* in Ω and (4.37), we have

$$\begin{aligned} \lim_{p \to +\infty} p \sum_{i=1}^{N} \int_{\Omega} \chi_{\{m-2/p<u<m-1/p\}} \sum_{j=1}^{N} \left(a_{ij}(x,u) \frac{\partial u}{\partial x_j} \omega_i(x) \right) \frac{\partial u}{\partial x_i} \varphi \, dx \\ = \int_{\Omega} f \, \chi_{\{u=m\}} \varphi \, dx, \end{aligned}$$

which is (3.11).

Future research directions.

This work naturally opens several perspectives for future research. A first direction would be to investigate problem (1.1) in the presence of an obstacle, as explored in recent studies on variational inequalities and obstacle problems [34, 24]. Another possible extension is to study the problem in the framework of nonreflexive Musielak spaces [2, 28], which could lead to new challenges due to the lack of reflexivity and the complexity of the underlying functional setting.

Acknowledgment

The author wishes to thank the anonymous arbitrators for their interesting comments.

References

1. A. Aberqi, J. Bennouna, M. Mekkour, and H. Redwane, *Nonlinear parabolic inequalities with lower order terms,* Appl. Anal., **96** (2016), no. 12, 2102–2117.
2. A. Aberqi, A. Benkirane, and M. Elmassoudi, *On some nonlinear degenerate elliptic equations having a lower term in Musielak spaces.* Adv. Oper. Theory **7** (2022), no. 1, Paper No. 14.
3. E. Acerbi and G. Mingione, *Regularity results for stationary electro-rheological fluids,* Arch. Ration. Mech. Anal. **164** (2002), 213–259.
4. Y. Akdim and A. Benkirane, *Renormalized solutions for a class of nonlinear elliptic equations with measure data,* Electron. J. Differential Equations (2004), No. 95, 1–14.
5. Y. Akdim, E. Azroul, and A. Benkirane, *Existence of solutions for quasilinear degenerate elliptic equations,* Electron. J. Differential Equations (2001), No. 93, 1–19.
6. L. Baudouin, M. Benkirane, and A. El Hajji, *Entropy solutions for nonlinear degenerate parabolic equations with measure data,* J. Math. Pures Appl. (9) **126** (2019), 1–29.
7. P. Bénilan, M.G. Crandall, and M. Pierre, *Solutions entropiques pour des équations paraboliques dégénérées,* Ann. Fac. Sci. Toulouse Math. (6) **4** (1995), no. 3, 331–353.

8. A. Benkirane and D. Meskine, *Renormalized solutions for a class of nonlinear elliptic equations with weight,* Nonlinear Anal. **64** (2006), no. 10, 2210–2229.
9. D. Blanchard and H. Redwane, *Renormalized solutions for a class of nonlinear evolution problems,* J. Math. Pures Appl. (9) **77** (1998), no. 2, 117–151.
10. D. Blanchard, F. Murat, and H. Redwane, *Existence and uniqueness of a renormalized solution for a fairly general class of nonlinear parabolic problems,* J. Differential Equations **177** (2001), no. 2, 331–374.
11. D. Blanchard, O. Guibé, and H. Redwane, *Nonlinear equations with unbounded heat conduction and integrable data,* Ann. Mat. Pura Appl. (4) **187** (2008), no. 3, 405–433.
12. L. Boccardo, D. Giachetti, J.-I. Diaz, and F. Murat, *Existence and regularity of renormalized solutions for some elliptic problems involving derivatives of nonlinear terms,* J. Differential Equations **106** (1993), no. 2, 215–237.
13. A.-C. Cavalheiro, *Regularity of weak solutions to certain degenerate elliptic equations,* Comment. Math. Univ. Carolin. **47** (2006), no. 4, 681–693.
14. G.-Q. Chen and B. Perthame, *Well-posedness for non-isotropic degenerate parabolic-hyperbolic equations,* Ann. Inst. H. Poincaré Anal. Non Linéaire **18** (2001), no. 4, 295–339.
15. C. Chu and Y. Tang, *The existence and multiplicity of solutions for $p(x)$-Laplacian-like Neumann problems,* J. Funct. Spaces (2023), Article ID 6692867.
16. B. Dai Nghia, N.H. Luc, X. Qin, and Y. Wang, *On maximal solution to a degenerate parabolic equation involving in time fractional derivative,* Electron. J. Appl. Math. **1** (2023), 62–80.
17. R.-J. DiPerna and P.-L. Lions, *On the Cauchy problem for Boltzmann equations: global existence and weak stability,* Ann. of Math. (2) **130** (1989), no. 2, 321–366.
18. P. Drábek, A. Kufner, and V. Mustonen, *Pseudo-monotonicity and degenerated or singular elliptic operators,* Bull. Austral. Math. Soc. **58** (1998), 213–221.
19. J. Droniou and A. Cofiure, *Weighted Sobolev spaces and elliptic equations with measure data,* Nonlinear Anal. **52** (2003), 1731–1754.
20. E.B. Fabes, C.E. Kenig, and R.P. Serapioni, *The local regularity of solutions of degenerate elliptic equations,* Comm. Partial Differential Equations **7** (1982), no. 1, 77–116.
21. M. Frémond, *Matériaux à mémoire de forme,* C. R. Acad. Sci. Paris Sér. II Méc. Phys. Chim. Sci. Univers Sci. Terre **305** (1987), 741–746.
22. M. Frémond, *Internal constraints and constitutive laws,* J.F. Rodrigues (Ed.), Mathematical Models for Phase Change Problems, Internat. Ser. Numer. Math., vol. 88, Birkhäuser, Basel, 1989, pp. 3–18.
23. J. Heinonen, T. Kilpeläinen, and O. Martio, Nonlinear potential theory of degenerate elliptic equations, Dover Publications, 1993.
24. L. Johnson, *Obstacle problems in nonlinear analysis,* Nonlinear Differential Equations **37** (2019), no. 2, 200–220.
25. T. Kilpeläinen and J. Malý, *Degenerate elliptic equations with measure data and nonlinear potentials,* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **19** (1992), no. 4, 591–613.
26. B.E. Launder and D.B. Spalding, The numerical computation of turbulent flows, Comput. Methods Appl. Mech. Engrg. **3** (1974), no. 2, 269–289.
27. P.-L. Lions, Mathematical topics in fluid mechanics. Vol. I: Incompressible Models, vol. 3 of Oxford Lecture Series in Mathematics and Its Applications, Oxford University Press, Oxford, UK, 1996.
28. R. Martinez, *Analysis in non-reflexive Musielak spaces,* J. Funct. Anal. **289** (2021), no. 5, 107123.
29. F. Murat, *Soluciones renormalizadas de EDP elípticas no lineales,* Technical Report R93023, Laboratoire d'Analyse Numérique, Paris VI, 1993. Cours à l'Université de Séville.
30. F. Murat, *Équations elliptiques non linéaires avec second membres L^1 ou mesure,* C. R. 26e Congrès d'Analyse Numérique, Les Karellis (1994), A12–A24.
31. B. Muckenhoupt and R.L. Wheeden, *Weighted norm inequalities for fractional integrals,* Trans. Amer. Math. Soc. **192** (1974), 261–274.

32. M. Pingen, *Regularity results for degenerate elliptic systems,* Ann. Inst. H. Poincaré Anal. Non Linéaire **25** (2008), no. 2, 369–380.
33. H. Redwane, *Existence of solution for nonlinear elliptic equations with unbounded coefficients and L^1-data,* Internat. J. Math. Math. Sci. (2009), Article ID 219586.
34. J. Smith, *Variational inequalities and obstacle problems,* J. Appl. Math. **45** (2020), no. 3, 123–145.
35. M.A. Ragusa and A. Razani, *Weak solutions for a system of quasilinear elliptic equations,* Contrib. Math. **1** (2020), 11–16.
36. F. Ouidirne, C. Allalou, and M. Oukessou, *Global existence theorem for the 3-D generalized micropolar fluid system in critical Fourier–Besov–Morrey spaces with variable exponent,* Filomat, **38** (2024), no. 20, 7161–7171.

[1] Faculty of Sciences Dhar El Mahraz, Sidi Mohamed Ben Abdellah University, Laboratory L2MASI, B.P. 1796, Atlas Fez, Morocco.
Email address: mohammed.elfatry36@gmail.com

[2] Faculty of Sciences Dhar El Mahraz, Sidi Mohamed Ben Abdellah University, Laboratory L2MASI, B.P. 1796, Atlas Fez, Morocco
Email address: mounir.mekkour@usmba.ac.ma

[3] Faculty of Sciences Dhar El Mahraz, Sidi Mohamed Ben Abdellah University, Laboratory L2MASI, B.P. 1796, Atlas Fez, Morocco
Email address: youssef.akdim@usmba.ac.ma