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EXPLORING THE HARMONIC SERIES BY L-SUMMING METHOD AND A TELESCOPING SUMMATION METHOD

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Abstract. In this note, we discuss the origin and significance of the L-summing method and introduce a telescoping summation method based on an identity due to Anastase and Díaz-Barrero. A key feature of these approaches is their implementation in Maple, enabling the derivation of several nontrivial summation identities that Maple cannot compute directly. Of particular interest are the examples involving the classical series

$$\zeta(s)=\sum_{k=1}^{\infty}\frac{1}{k^s},\quad \zeta_n(s)=\sum_{k=1}^{n}\frac{1}{k^s},\quad H_n=\zeta_n(1)=\sum_{k=1}^{n}\frac{1}{k},$$

for which we establish several new identities.

1. L-summing method

Finding an explicit value for a finite or infinite series is an important and central problem in mathematics, and is very well studied in literature (see, for example, [4, 5, 6], as more as [12, 13] for a historical background). Recent developments contain computing series on the computer algebra systems, like Maple and Mathematica softwares. Although, these softwares equipped by several well-known computational algorithms, but they are still fragile. Part of this fragility can be fixed by using these softwares, themselves.

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An example in this area is the so called L-summing method [7], which is indeed the following rearrangement on the $n \times n$ array $(a_{ij})_{1\leqslant i,j\leqslant n}$:

$$\sum_{1\leqslant i,j\leqslant n} a_{ij} = \sum_{k=1}^{n}\left(\sum_{i=1}^{k} a_{ik} + \sum_{j=1}^{k} a_{kj} - a_{kk}\right). \tag{1.1}$$

The origin of L-summing method and the identity (1.1) is its special cases obtained for $a_{ij} = 1$ and $a_{ij} = ij$. The case $a_{ij} = 1$ coincides with the well-known odd-sum identity

$$n^2 = \sum_{k=1}^{n}(2k-1), \tag{1.2}$$

and its proof without words [9, p. 71] reproduced for $n = 10$ in the left digram of Figure 1, where we count the number of all squares in two ways. The first one counting the whole squares, which is n^2, and the second one is counting them over L-shapes, whose the number of squares in the kth shape is $2k - 1$.

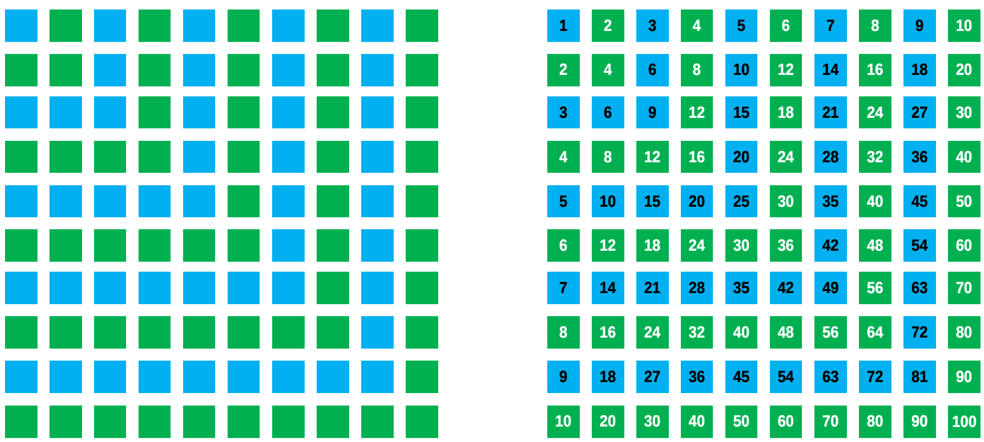


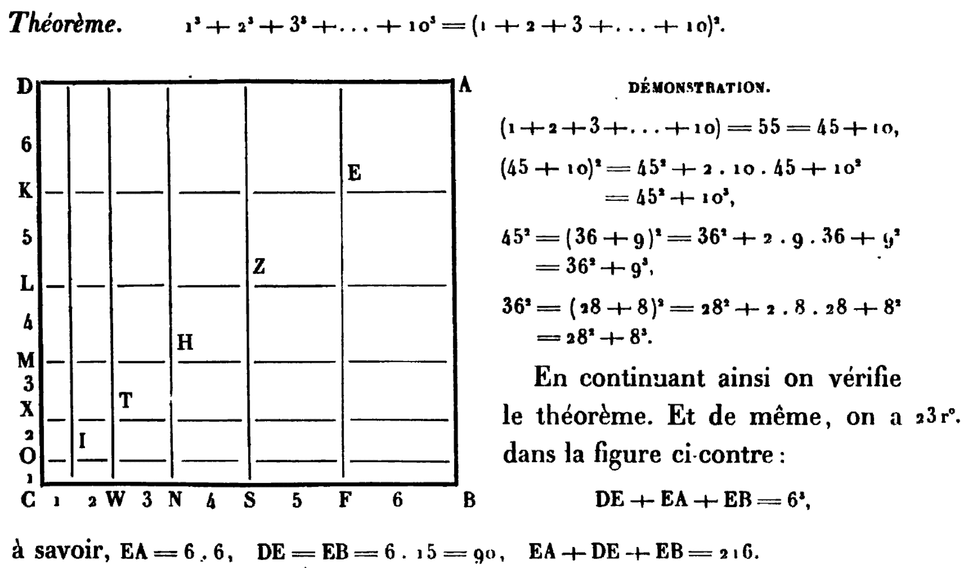
Figure 1. Left: A reproduction of the proof without words of the odd-sum identity $n^2 = \sum_{k=1}^{n}(2k-1)$ for $n = 10$. Right: 10×10 multiplication table and computing summation of all its entries over L-shapes

The case $a_{ij} = ij$ is related by the classical multiplication table and the problem of computing the sum of all numbers in an $n \times n$ multiplication table, as pictured in the right digram of Figure 1 for $n = 10$. Accordingly, (1.1) ends in the following well-known identity:

$$\left(\sum_{k=1}^{n} k\right)^2 = \sum_{k=1}^{n} k^3. \tag{1.3}$$

Generalizing identities (1.2) and (1.3), and summing over L shapes lead the author [7] to introduce the so called L-summing method (1.1).

The identity (1.3) was considered for $n = 10$ by Al-Karaji, as documented by the German historian Woepcke on page 61 of his essay [16] on Al-Karaji's works. For further information on Al-Karaji, we refer the reader to [10] and the references cited therein. The diagram in Figure 2 corresponds to the right-hand side diagram of Figure 1 with $n = 6$. Al-Karaji's argument is based on evaluating the areas of rectangles, where the area of the (i, j)th rectangle (counted from the lower-left corner) equals ij.


Théorème. $1^3 + 2^3 + 3^3 + \ldots + 10^3 = (1 + 2 + 3 + \ldots + 10)^2$.

DÉMONSTRATION.

$(1+2+3+\ldots+10) = 55 = 45+10,$

$(45+10)^2 = 45^2 + 2 \,.\, 10 \,.\, 45 + 10^2 = 45^2 + 10^3,$

$45^2 = (36+9)^2 = 36^2 + 2 \,.\, 9 \,.\, 36 + 9^2 = 36^2 + 9^3,$

$36^2 = (28+8)^2 = 28^2 + 2 \,.\, 8 \,.\, 28 + 8^2 = 28^2 + 8^3.$

En continuant ainsi on vérifie le théorème. Et de même, on a 23r°. dans la figure ci-contre :

$\mathrm{DE} + \mathrm{EA} + \mathrm{EB} = 6^3,$

à savoir, $\mathrm{EA} = 6 \,.\, 6$, $\mathrm{DE} = \mathrm{EB} = 6 \,.\, 15 = 90$, $\mathrm{EA} + \mathrm{DE} + \mathrm{EB} = 216$.

FIGURE 2. Part of page 61 of Woepcke's essay on Al-Karaji's works, demonstrating the identity $\sum_{k=1}^{10} k^3 = (\sum_{k=1}^{10} k)^2$

In [7], the author used the L-summing method to compute several finite series by Maple. While Maple was not able to compute some them directly, equipped by (1.1) and its generalization for higher dimensional arrays in [8], Maple could to generate new summation identities. Remarkable examples is obtained by taking $a_{ij} = 1/(ij)^s$ and its special case $s = 1$ in (1.1). Accordingly, we obtained identities involving the Riemann zeta function $\zeta(s)$ [11, Chap. 25], its partial sum $\zeta_n(s)$, and the Harmonic numbers H_n, defined, respectively, as follows:

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}, \quad \zeta_n(s) = \sum_{k=1}^{n} \frac{1}{k^s}, \quad H_n = \zeta_n(1) = \sum_{k=1}^{n} \frac{1}{k}. \tag{1.4}$$

More precisely, L-summing identity (1.1) with $a_{ij} = 1/(ij)^s$ reads as follows:

$$\zeta_n(s)^2 = 2\sum_{k=1}^{n} \frac{\zeta_k(s)}{k^s} - \zeta_n(2s) \quad (s \in \mathbb{C}), \tag{1.5}$$

and this, itself for $s = 1$ reads as

$$H_n^2 = 2\sum_{k=1}^{n} \frac{H_k}{k} - \zeta_n(2).$$

Thus,

$$\sum_{k=1}^{n} \frac{H_k}{k} = \frac{1}{2}\left(H_n^2 + \zeta_n(2)\right). \tag{1.6}$$

On the other hand, for any integer $k \geqslant 1$, we have

$$\sum_{j=1}^{\infty} \frac{1}{j(j+k)} = \frac{H_k}{k},$$

and consequently,

$$\sum_{k=1}^{n} \frac{H_k}{k} = \sum_{k=1}^{n}\sum_{j=1}^{\infty} \frac{1}{j(j+k)} = \sum_{j=1}^{\infty}\sum_{k=1}^{n} \frac{1}{j(j+k)} = \sum_{j=1}^{\infty} \frac{H_{j+n}-H_j}{j}.$$

Combining with (1.6), for any integer $n \geqslant 1$, we deduce that

$$\sum_{j=1}^{\infty} \frac{H_{j+n}-H_j}{j} = \frac{1}{2}\left(H_n^2 + \zeta_n(2)\right).$$

Furthermore, letting $n \to \infty$ in both sides of (1.5), for any fixed $s \in \mathbb{C}$ with $\Re(s) > 1$, we obtain the following Dirichlet series representation for the sequence $(\zeta_n(s))_{n\geqslant 1}$,

$$\sum_{n=1}^{\infty} \frac{\zeta_n(s)}{n^s} = \frac{1}{2}\left(\zeta(s)^2 + \zeta(2s)\right). \tag{1.7}$$

Applying Perron's formula [15, Ch. II.2, Sec. 2.1] on (1.7) gives for any $c > 1$ and any $x, T \geqslant 1$,

$$\sum_{n\leqslant x} \zeta_n(s) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{\zeta(s)+\zeta(2s)}{2s} x^s\, ds + O\left(x^c \sum_{n=1}^{\infty} \frac{|\zeta_n(s)|}{n^c(1+T|\log(x/n)|)}\right).$$

We refer the interested reader to [7, 8] for the related Maple codes and outputs, and [1, 14] for some applications.

2. A telescoping identity

In the present note, we provide another example of equipping Maple, by a telescoping summation method, based on an identity due to Anastase and Díaz-Barrero [3, Lemma 1], asserting the following result.

Theorem 2.1. *For the sequence $(a_n)_{n\geqslant 1}$ consisting of numbers with nonzero initial term $a_1 \neq 0$, and with nonzero partial sums $A_v := \sum_{n\leqslant v} a_n$, the following summation identity holds for any integer $n \geqslant 2$,*

$$\sum_{j=2}^{n} \frac{a_j}{A_{j-1}A_j} = \frac{1}{a_1} - \frac{1}{A_n}. \tag{2.1}$$

Furthermore, if $A_v \to \infty$ as $v \to \infty$, then

$$\sum_{j=2}^{\infty} \frac{a_j}{A_{j-1}A_j} = \frac{1}{a_1}. \tag{2.2}$$

Note that the left-hand side of (2.1) constitutes a form of telescoping summation, because

$$\frac{a_j}{A_{j-1}A_j} = \frac{1}{A_{j-1}} - \frac{1}{A_j}.$$

Recently, Alzer and Frontczak [2] established a generalization of (2.1) involving binomial coefficient weights, from which they derived several identities, including extensions of our identities (2.3) and (2.5). In contrast, the present note pursues a different approach by implementing both sides of (2.1) and (2.2) in Maple. While (2.1) yields trivial identities for certain choices of a_n, it is nevertheless possible to identify sequences (a_n) for which (2.1) produces nontrivial identities, even after simplification in computer algebra systems such as Maple or Mathematica. The most interesting instances arise when the sequence is linked to the series defined in (1.4). In particular, for $a_n = n^{-s}$, and more specifically when $s = 1$, identity (2.1) leads to the following results.

Theorem 2.2. *For any $s \in \mathbb{C}$ and any integer $n \geqslant 2$, we have*

$$\sum_{j=2}^{n} \frac{1}{j^s \zeta_{j-1}(s)\zeta_j(s)} = 1 - \frac{1}{\zeta_n(s)}. \tag{2.3}$$

In particular, for any $s \in \mathbb{C}$ with $\Re(s) > 1$, we get

$$\sum_{j=2}^{\infty} \frac{1}{j^s \zeta_{j-1}(s)\zeta_j(s)} = 1 - \frac{1}{\zeta(s)}. \tag{2.4}$$

Theorem 2.3. *Let H_n be the nth harmonic number given in (1.4). For any integer $n \geqslant 2$, we have*

$$\sum_{j=2}^{n} \frac{1}{jH_{j-1}H_j} = 1 - \frac{1}{H_n}. \tag{2.5}$$

Furthermore, for any given integer $m \geqslant 0$, as $n \to \infty$ the following asymptotic expansion holds:

$$\sum_{j=2}^{n} \frac{1}{jH_{j-1}H_j} = 1 - \frac{1}{\log n} + \frac{\gamma}{\log^2 n} - \frac{\gamma^2}{\log^3 n} \pm \cdots - \frac{(-\gamma)^m}{\log^{m+1} n} + O\Big(\frac{1}{\log^{m+2} n}\Big),$$

where γ is the Euler–Mascheroni constant. In particular, we have

$$\sum_{j=2}^{\infty} \frac{1}{jH_{j-1}H_j} = 1. \tag{2.6}$$

The identities (2.5) and (2.6) are direct conclusions of (2.1) and (2.2), respectively. The asymptotic expansion in Theorem 2.3 is a consequence of the well-known approximation

$$H_n = \log n + \gamma + O\Big(\frac{1}{n}\Big),$$

and the following auxiliary result.

Proposition 2.4. *Assume that the function $f(x)$ admits the following asymptotic expansion, as $x \to \infty$,*

$$f(x) = M(x) + C + O\Big(\frac{1}{N(x)}\Big),$$

where the positive functions $M(x), N(x) \to \infty$ as $x \to \infty$, and C is a constant, actually given by $C = \lim_{x\to\infty}(f(x) - M(x))$. Then, for any fixed integer $m \geqslant 1$, the following asymptotic expansion holds as $x \to \infty$,

$$\frac{1}{f(x)} = \frac{1}{M(x)} + \sum_{j=0}^{m} \frac{(-1)^j C^j}{M(x)^{j+1}} + O\Big(\frac{1}{M(x)^2 N(x)} + \frac{1}{M(x)^{m+2}}\Big).$$

Proof. Let us simply set $M := M(x)$, $N := N(x)$, and $R := C + O(1/N(x))$. We have

$$\frac{1}{f(x)} = \frac{1}{M+R} = \frac{1}{M}\Big(1 + \frac{R}{M}\Big)^{-1}.$$

We use the asymptotic expansion $1/(1-t) = 1 + \sum_{j=1}^{m} t^j + O(t^{m+1})$, which holds as $t \to 0$, for any fixed integer $m \geqslant 1$. Since $R = O(1)$, we get $R/M \to 0$. Thus, taking $t = -R/M$ gives

$$\frac{M}{f(x)} = 1 + \sum_{j=1}^{m} \Big(-\frac{R}{M}\Big)^j + O\bigg(\Big(\frac{|R|}{M}\Big)^{m+1}\bigg) = 1 + \sum_{j=1}^{m} \frac{(-1)^j R^j}{M^j} + O\Big(\frac{1}{M^{m+1}}\Big).$$

Binomial expansion implies that for any integer $j \geqslant 1$,

$$R^j = \bigg(C + O\Big(\frac{1}{N}\Big)\bigg)^j = C^j + O\Big(\frac{1}{N}\Big).$$

Thus,

$$\begin{aligned}\frac{M}{f(x)} &= 1 + \sum_{j=1}^{m} \frac{(-1)^j}{M^j}\bigg(C^j + O\Big(\frac{1}{N}\Big)\bigg) + O\Big(\frac{1}{M^{m+1}}\Big)\\ &= 1 + \sum_{j=1}^{m} \frac{(-1)^j C^j}{M^j} + O\Big(\frac{1}{MN}\Big) + O\Big(\frac{1}{M^{m+1}}\Big).\end{aligned}$$

□

As a consequence of Theorem 2.3, we derive an expression for a series involving harmonic numbers, $\sum H_n/n$. To achieve this, we first present the following lemma, which converts a nested double sum into two simpler sums.

Lemma 2.5. *For any sequence $(b_n)_{n\geqslant 1}$ of complex numbers, and any integer $N \geqslant 2$ we have*

$$\sum_{n=2}^{N}\sum_{j=2}^{n} b_j = N\sum_{j=2}^{N} b_j - \sum_{j=2}^{N}(j-1)b_j. \tag{2.7}$$

Proof. The left-hand side of (2.7) is equal to

$$\begin{aligned}
&b_2\\
&+b_2+b_3\\
&+b_2+b_3+b_4\\
&\vdots\\
&+b_2+b_3+b_4+\cdots+b_N.
\end{aligned}$$

Summing this triangle array vertically, we get

$$\sum_{n=2}^{N}\sum_{j=2}^{n} b_j = (N-1)b_2+(N-2)b_3+\cdots+1b_N=\sum_{j=2}^{N}(N-j+1)b_j.$$

This gives (2.7). $\square$

Theorem 2.6. *For any integer $N \geqslant 2$, we have*

$$\sum_{n=1}^{N}\frac{1}{H_n}=\frac{N+1}{H_N}-1+\sum_{n=2}^{N}\frac{1}{H_{n-1}H_n}. \tag{2.8}$$

Proof. Letting $h_j = jH_{j-1}H_j$, the identity (2.5) gives

$$\frac{1}{H_n}=1-\sum_{j=2}^{n}\frac{1}{h_j}.$$

Taking the summation $\sum_{n=2}^{N}$ from both sides implies

$$\sum_{n=2}^{N}\frac{1}{H_n}=\sum_{n=2}^{N}\Big(1-\sum_{j=2}^{n}\frac{1}{h_j}\Big)=N-1-\sum_{n=2}^{N}\sum_{j=2}^{n}\frac{1}{h_j},$$

and then

$$\sum_{n=1}^{N}\frac{1}{H_n}=N-\sum_{n=2}^{N}\sum_{j=2}^{n}\frac{1}{h_j}.$$

By using (2.7) we deduce that

$$\sum_{n=1}^{N}\frac{1}{H_n}=N-\Big(N\sum_{j=2}^{N}\frac{1}{h_j}-\sum_{j=2}^{N}\frac{j-1}{h_j}\Big)=N-(N+1)\sum_{j=2}^{N}\frac{1}{h_j}+\sum_{j=2}^{N}\frac{j}{h_j}.$$

We use again (2.5) to get (2.8), and conclude the proof. $\square$

Remark 2.7. One may obtain (2.8) from the discrete Abel summation formula, which asserts that for any two complex sequences $(a_n)_{n\geqslant 1}$ and $(b_n)_{n\geqslant 1}$ and any integer $N \geqslant 2$,

$$\sum_{n=1}^{N}a_nb_n=A_Nb_N-\sum_{n=1}^{N-1}A_n(b_{n+1}-b_n),$$

where $A_N=\sum_{n=1}^N a_n$, and if needed, we take $A_0=0$. Letting $a_n=1/(nH_{n-1}H_n)$ and $b_n=n$, by using (2.5) we get $A_N=1-1/H_N$ (we take the first term, corresponding to $n=1$, running over H_{n-1} to be 0) and

$$\sum_{n=2}^{N}\frac{1}{H_{n-1}H_n}=\sum_{n=2}^{N}a_nb_n=\Big(1-\frac{1}{H_N}\Big)N-\sum_{n=1}^{N-1}\Big(1-\frac{1}{H_n}\Big)$$
$$=N-\frac{N}{H_N}-(N-1)+\sum_{n=1}^{N-1}\frac{1}{H_n}=1-\frac{N+1}{H_N}+\sum_{n=1}^{N}\frac{1}{H_n}.$$

This is indeed the identity (2.8).

Examining over Maple, we observe that it gives no result for the left-hand side of (2.5), nor for the left-hand side of (2.6), too. But, running on (2.1) it gives the following equivalent form of (2.5) as well

$$\sum_{j=2}^{n}\frac{1}{j\left(\psi\left(j\right)+\gamma\right)\left(\psi\left(j+1\right)+\gamma\right)}=\frac{\psi\left(n+1\right)+\gamma-1}{\psi\left(n+1\right)+\gamma},$$

where ψ is the digamma function [11, Chap. 5], defined as the logarithmic derivation of the Euler gamma function, and connected to the harmonic numbers H_n by

$$H_n=\psi(n+1)+\gamma.$$

Similar happens when we examine the left-hand side of (2.1) for $a_n=n^{-k-1}$ for any integer $k\geqslant 1$. But, when we run Maple over sides of (2.1), it gives an expression in terms of the function $\psi(k,x)$, which is the kth polygamma function [11, Chap. 5], defined as the kth derivative of the digamma function. For any integer $k\geqslant 2$, the function $\psi(k,x)$ is related by $\zeta_n(k)$, the partial sum of the Riemann zeta function defined in (1.4), by means of the following relation:

$$\zeta_n(k)=\frac{(-1)^{k-1}}{(k-1)!}\psi(k-1,n+1)+\zeta(k). \tag{2.9}$$

This formula and Theorem 2.2 imply the following nontrivial result, in sense that Maple is not able to compute the obtained series in Theorem 2.8 directly, without using (2.1).

Theorem 2.8. *For any integer $k,n\geqslant 2$, we have*

$$\sum_{j=2}^{n}\frac{1}{j^k\eta_j(k)}=(k-1)!^2\Big(1-\frac{1}{\zeta_n(k)}\Big),$$

where

$$\eta_j(k)=\psi(k-1,j)\psi(k-1,j+1)$$
$$+(-1)^{k-1}\zeta(k)(k-1)!\left(\psi(k-1,j)+\psi(k-1,j+1)\right)+\left((k-1)!\zeta(k)\right)^2.$$

In particular, for any integer $k\geqslant 2$ we have

$$\sum_{j=2}^{\infty}\frac{1}{j^k\eta_j(k)}=(k-1)!^2\Big(1-\frac{1}{\zeta(k)}\Big).$$

Remark 2.9. Although Theorem 2.2 with $s=-m$ and integer $m \geqslant 1$ give trivial examples, in sense that Maple is able to compute them without using (2.1), but the results are still interesting. For example, letting $a_n = n^m$ with $m = 1, 2, 3,$ for any integer $n \geqslant 2$, the identity (2.1), respectively, reads as

$$\sum_{j=2}^{n} \frac{1}{j\,(j^2-1)} = \frac{n^2+n-2}{4\,n\,(n+1)},$$

$$\sum_{j=2}^{n} \frac{1}{(2\,j^2-3\,j+1)\,(2\,j^2+3\,j+1)} = \frac{2\,n^3+3\,n^2+n-6}{36\,n\,(2\,n^2+3\,n+1)},$$

$$\sum_{j=2}^{n} \frac{1}{j\,(j^2-2\,j+1)\,(j+1)^2} = \frac{n^4+2\,n^3+n^2-4}{16\,n^2\,(n+1)^2}.$$

Furthermore, for $m = 4, 5$, it reads as follows:

$$\sum_{j=2}^{n} \frac{j^2}{(6\,j^4-15\,j^3+10\,j^2-1)\,(6\,j^4+15\,j^3+10\,j^2-1)} = \frac{6\,n^5+15\,n^4+10\,n^3-n-30}{900\,n\,(6\,n^4+15\,n^3+10\,n^2-1)},$$

$$\sum_{j=2}^{n} \frac{j}{(2\,j^4-6\,j^3+5\,j^2-1)\,(j+1)^2\,(2\,j^2+2\,j-1)} = \frac{2\,n^6+6\,n^5+5\,n^4-n^2-12}{144\,n^2\,(n+1)^2\,(2\,n^2+2\,n-1)}.$$

Remark 2.10. Running the identity (2.1) over a Maple code for several sequences $(a_n)_{n\geqslant 1}$ still implies some more nontrivial identities. In the following we mention some of them.

(i) With $a_n = \psi(n)$, for any integer $n \geqslant 2$, the identity (2.1) reads as follows:

$$\sum_{j=2}^{n} \frac{\psi(j)}{(\psi(j)\,j-\psi(j)-j+1)\,(\psi(j)\,j-j+1)} = -\frac{1}{\gamma}\,\frac{\psi(n)\,n-n+\gamma+1}{\psi(n)\,n-n+1}.$$

(ii) With $a_n = \ln(n+1)$, for any integer $n \geqslant 2$, the identity (2.1) reads as follows:

$$\sum_{j=2}^{n} \frac{\ln(j+1)}{\ln(j!)\,\ln((j+1)!)} = \frac{1}{\ln(2)}\,\frac{\ln((n+1)!)-\ln(2)}{\ln((n+1)!)}.$$

(iii) With $a_n = (n+1)! - n!$, for any integer $n \geqslant 2$, the identity (2.1) reads as follows:

$$\sum_{j=2}^{n} \frac{j\,(j+1)!}{((j+1)!-j-1)\,((j+1)!-1)} = \frac{(n+2)!-2\,n-4}{(n+2)!-n-2}.$$

(iv) With $a_n = (n+2)! - n!$, for any integer $n \geqslant 2$, the identity (2.1) reads as follows:

$$\sum_{j=2}^{n} \frac{(j^2+3\,j+1)\,(j+3)!}{((j+3)!-3\,j^2-12\,j-9)\,((j+3)!-3\,j-6)} = \frac{1}{5}\,\frac{(n+3)!-8\,n-16}{(n+3)!-3\,n-6}.$$

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