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SUPERCENTRALIZING MAPS ON GENERALIZED QUATERNION RINGS

AJDA FOŠNER [1] AND LEILA HEIDARI ZADEH [2*]

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Abstract. Let $\mathcal{A} = H_{\alpha,\beta}(\mathcal{S})$ be a generalized quaternion ring over an arbitrary unital ring $\mathcal{S}$ with supercenter $Z_s(\mathcal{A})$. In this article, we provide the structure of a supercentralizing additive map F on a generalized quaternion superring $\mathcal{A}$ and demonstrate that F on $\mathcal{A}$ is proper. Furthermore, we establish that any supercentralizing derivation on the quaternion ring is zero.

1. Introduction

Let $\mathcal{A}$ be an associative ring with unity and center $Z(\mathcal{A})$. For each $x, y \in \mathcal{A}$, denote the commutator of x and y by $[x, y] = xy - yx$. An additive map $F\colon \mathcal{A} \to \mathcal{A}$ is said to be centralizing if $[F(x), x] \in Z(\mathcal{A})$ for all $x \in \mathcal{A}$. In the special case where $[F(x), x] = 0$ for all $x \in \mathcal{A}$, F is called commuting. Posner [16] initiated the study of centralizing maps, stating that a noncommutative prime ring cannot exhibit a nonzero commuting derivation. Mayne [15] proved the analogous result for centralizing automorphisms. Later, Brešar [2] showed that a centralizing additive map F on a prime ring $\mathcal{A}$ of characteristic not 2 is of the form $F(x) = \lambda x + \mu(x)$ for all $x \in \mathcal{A}$, where $\lambda \in \mathcal{C}$, the extended centroid of $\mathcal{A}$, and μ is an additive map of $\mathcal{A}$ into $\mathcal{C}$. Several authors have since obtained various results on centralizing maps. (see, for example, [2, 3, 1, 13, 14]).

In recent years, a significant portion of associative ring theory has been extended to superrings and superalgebras by various authors. This naturally leads to the consideration of Posner's theorem for superderivations on superalgebras

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* Corresponding author.
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(see [7, 8, 9, 10]). In this article, we explore supercentralizing maps on generalized quaternion rings. For more information on quaternion rings, refer to [6, 9, 11] and the cited references

In the following section, we define and state the lemmas needed to prove the theorems. Subsequently, we will prove the theorem in Section 3.

2. Preliminaries

A *superring* $\mathcal{A}$, is a $\mathbb{Z}_2$-graded ring. This means there exist additive subgroups $\mathcal{A}_0$ and $\mathcal{A}_1$ of $\mathcal{A}$ such that $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ and $\mathcal{A}_i\mathcal{A}_j \subseteq \mathcal{A}_{i+j}$, where indices are computed modulo 2. A superring is called trivial if $\mathcal{A}_1 = 0$. Elements in $\mathcal{A}_0 \cup \mathcal{A}_1$, are said to be *homogeneous* of *degree* i, and we write $|x| = i$ to mean $x \in \mathcal{A}_i$. We refer to $\mathcal{A}_0$ as the *even* and $\mathcal{A}_1$ as the *odd* part of $\mathcal{A}$.

Define a product in $\mathcal{A}_0 \cup \mathcal{A}_1$, called the *supercommutator*, by

$$[x, y]_s = xy - (-1)^{|x||y|}yx$$

for $x, y \in \mathcal{A}_0 \cup \mathcal{A}_1$ and extend this product on $\mathcal{A}$ additively. Thus,

$$[x, y]_s = [x_0, y_0]_s + [x_1, y_0]_s + [x_0, y_1]_s + [x_1, y_1]_s,$$

where $x = x_0 + x_1$ and $y = y_0 + y_1$. The supercenter of $\mathcal{A}$ is the set

$$Z_s(\mathcal{A}) = \{a \in \mathcal{A} \mid [a, x]_s = 0 \quad \text{for all } x \in \mathcal{A}\}.$$

We set

$$Z_s(\mathcal{A})_0 = Z_s(\mathcal{A}) \cap \mathcal{A}_0 \ \text{ and } \ Z_s(\mathcal{A})_1 = Z_s(\mathcal{A}) \cap \mathcal{A}_1.$$

One can easily check that $Z_s(\mathcal{A})$ is a graded subring of $\mathcal{A}$. Certainly, in a trivial superring, a supercommutator is just an ordinary commutator, and the supercenter coincides with the center.

An additive map $F\colon \mathcal{A} \to \mathcal{A}$ is said to be *supercentralizing* if $[F(x), x]_s \in Z_s(\mathcal{A})$ for all $x \in \mathcal{A}$.

Definition 2.1. Let $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ be a superring. We call an additive map $F\colon \mathcal{A} \to \mathcal{A}$ a *proper supercentralizing* map if

$$F(x) = \lambda x + \mu(x)$$

for all $x \in \mathcal{A}$, where $\lambda \in Z(\mathcal{A})_0$ with $2\lambda{\mathcal{A}_1}^2 \subseteq Z(\mathcal{A})_0$, and $\mu\colon \mathcal{A} \to Z_s(\mathcal{A})$ is an additive map. In particular, if $F(x) \in Z_s(A)$ for all $x \in \mathcal{A}$, then we call F a supercentral map.

The following result shows that a proper supercentralizing map is a supercentralizing map.

Lemma 2.2. *Let $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ be a superring. Then $F(x) = \lambda x + \mu(x)$ for all $x \in \mathcal{A}$, is a supercentralizing map of $\mathcal{A}$, where $\lambda \in Z(\mathcal{A})_0$ with $2\lambda{\mathcal{A}_1}^2 \subseteq Z(\mathcal{A})_0$, and $\mu\colon \mathcal{A} \to Z_s(\mathcal{A})$ is a supercentral map.*

Proof. For any $x = x_0 + x_1 \in \mathcal{A}$, we have

$$\begin{aligned}[F(x), x]_s &= [\lambda x + \mu(x), x]_s = \lambda[x, x]_s = \lambda[x_0 + x_1, x_0 + x_1]_s \\ &= \lambda[x_0, x_0] + \lambda[x_0, x_1] + \lambda[x_1, x_0] + \lambda[x_1, x_1]_s \\ &= \lambda[x_1, x_1]_s = 2\lambda x_1^2 \\ &\in Z(\mathcal{A})_0.\end{aligned}$$

We conclude that F is a supercentralizing map. □

A supercentralizing map of $\mathcal{A}$ is said to be of degree i if $F_i(\mathcal{A}_j) \subseteq \mathcal{A}_{i+j}$ (index modulo 2).

A *superderivation* of degree i is an additive map $D_i \colon \mathcal{A} \to \mathcal{A}$ such that $D_i(\mathcal{A}_j) \subseteq \mathcal{A}_{i+j}$ (index modulo 2) and

$$D_i(xy) = D_i(x)y + (-1)^{i|x|} x D_i(y)$$

for all $x, y \in \mathcal{A}_0 \cup \mathcal{A}_1$.

A superderivation of $\mathcal{A}$ is the sum of a superderivation of degree 0 and a superderivation of degree 1. Note that every superderivation of degree 0 is actually a derivation from $\mathcal{A}$ to $\mathcal{A}$. For example, for any $a_0 \in \mathcal{A}_0$ and $a_1 \in \mathcal{A}_1$, the maps $I_{a_0}(x) = [a_0, x]_s$ and $I_{a_1}(x) = [a_1, x]_s$ are superderivations of degree 0 and 1, respectively. Hence, for any $a = a_0 + a_1 \in \mathcal{A}$, the linear mapping $I_a(x) = [a, x]_s$ is a superderivation since $[a, x]_s = [a_0, x]_s + [a_1, x]_s$. The superderivation $I_a(x) = [a, x]_s$ will be called the *inner superderivation.*

Definition 2.3. Let $\mathcal{S}$ be a ring with identity, and let α be a unit in $\mathcal{S}$. Set

$$H_{\alpha,\beta}(\mathcal{S}) = \{x + yi + zj + wk : x, y, z, w \in \mathcal{S}\} = \mathcal{S} \oplus \mathcal{S}i \oplus \mathcal{S}j \oplus \mathcal{S}k,$$

where $i^2 = -\alpha$, $j^2 = -\beta$, $k^2 = -\alpha\beta$, $ij = -ji = k$, $jk = -kj = \beta i$ and $ki = -ik = \alpha j$. Then, with componentwise addition and multiplication subject to the given relations and the convention that i, j, k commute with $\mathcal{S}$ elementwise, $H_{\alpha,\beta}(\mathcal{S})$ is a ring called *generalized quaternion ring* over $\mathcal{S}$. This ring is a generalization of Hamilton's division ring of real quaternions $\mathbb{H} = H(\mathbb{R})$. Specifically, when $\alpha = \beta = 1$, we denote the quaternion ring by $H(\mathcal{S})$ in reference to Hamilton's quaternions. In general, $H_{\alpha,\beta}(\mathcal{S})$ is not commutative unless $\mathcal{S}$ is commutative with $char(\mathcal{S}) = 2$.

Remark 2.4. If $\alpha = -\beta = 1$, then $H_{sp}(\mathcal{S})$ represents the *split quaternion ring* over $\mathcal{S}$. When $\alpha = 1$ and $\beta = 0$, we refer to $H_{\alpha,\beta}(\mathcal{S})$ as the *semiquaternion ring*, denoted by $H_s(\mathcal{S})$. Additionally, if $\alpha = -1$ and $\beta = 0$, then it becomes the *split semiquaternion ring* denoted as $H_{sps}(\mathcal{S})$.

From now on, we assume that $\mathcal{S}$ is a ring with identity such that $\dfrac{1}{2} \in \mathcal{S}$ and $\mathcal{A} = H_{\alpha,\beta}(\mathcal{S})$.

Lemma 2.5. *Let $\mathcal{S}$ and $\mathcal{A}$ be as described above. Then $Z(\mathcal{A}) = Z(\mathcal{S})$.*

Proof. Let $q = a + bi + cj + dk \in Z(\mathcal{A})$. Since q commutes with i and j, we have $2b = 2c = 2d\alpha = 0$. Given that 2 and α are invertible in $\mathcal{S}$, it follows that $b = c = d = 0$. Furthermore, since $sq = qs$ for every $s \in \mathcal{S}$, we conclude that $a \in Z(\mathcal{S})$. Therefore, $Z(\mathcal{A}) \subseteq Z(\mathcal{S})$. The reverse inclusion is straightforward. □

Let

$$\mathcal{A}_0 = \mathcal{S} \oplus \mathcal{S}i \quad \text{and} \qquad \mathcal{A}_1 = \mathcal{S}j \oplus \mathcal{S}k.$$

It is easily verified that $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ becomes a superring. In the following, we consider the above-mentioned superring structure on a generalized quaternion ring.

Remark 2.6. Using an argument similar to that of Lemma 2.5, and since $[x, y]_s = [x, y]$ for all $x \in \mathcal{A}_0, y \in \mathcal{A}$, then it can be easily verified that

$$Z_s(\mathcal{A})_0 = Z(\mathcal{A})_0 = Z(\mathcal{S}) = Z_s(\mathcal{A}).$$

Lemma 2.7. *Let F be a supercentralizing additive map on $\mathcal{A} = H_{\alpha,\beta}(\mathcal{S})$. Then $F = F_0 + F_1$ where F_0 is a supercentralizing map of degree 0 on $\mathcal{A}$ and F_1 is a supercentralizing map of degree 1 on $\mathcal{A}$.*

Proof. Let π_i be the canonical projection of $\mathcal{A}$ onto $\mathcal{A}_i$ for $i = 0, 1$ and let $F_0 = \pi_0 F \pi_0 + \pi_1 F \pi_1$ and $F_1 = \pi_0 F \pi_1 + \pi_1 F \pi_0$. Then each F_i is an additive map of $\mathcal{A}$ to $\mathcal{A}$ and $F = F_0 + F_1$. Moreover, $F_i(\mathcal{A}_j) \subseteq \mathcal{A}_{i+j}$ for $i = 0, 1$. Since $Z_s(\mathcal{A}) = Z(\mathcal{S}) = Z(\mathcal{A})_0$, for $i = 0$ or 1 we have

$$[F(x_i), x_i]_s = [F_0(x_i), x_i]_s + [F_1(x_i), x_i]_s \in Z(\mathcal{S}).$$

Since $[F_0(x_i), x_i]_s$ is even and $[F_1(x_i), x_i]_s$ is odd, we have

$$[F_1(x_i), x_i]_s = 0 \tag{2.1}$$

and

$$[F_0(x_i), x_i]_s \in Z(\mathcal{S}) \tag{2.2}$$

for all $x_i \in \mathcal{A}_i$. On the other hand, we see that

$$[F(x_0), x_1]_s + [F(x_1), x_0]_s \in Z(\mathcal{S}). \tag{2.3}$$

Replacing F with $F_0+F_1 = F$ in (2.3), and noting that the odd part $[F_0(x_0), x_1]_s + [F_0(x_1), x_0]_s = 0$, we see that the even part

$$[F_1(x_0), x_1]_s + [F_1(x_1), x_0]_s \tag{2.4}$$

belongs to $Z(\mathcal{S})$. Hence, by using (2.1), (2.2), and (2.4), it becomes straightforward to see that each F_i satisfies the condition that $[F_i(x), x]_s \in Z_s(\mathcal{A})$ for $i = 0$ or 1. □

The following results will be used to prove our main results.

Lemma 2.8. [10, Theorem 2.2] *If $\alpha = \beta = 1$, then $\mathcal{A} = H(\mathcal{S})$ is the quaternion ring, and $D\colon \mathcal{A} \to \mathcal{A}$ is a superderivation if and only if there exists a derivation $\delta\colon \mathcal{S} \to \mathcal{S}$ such that $D(q) = \delta(x) + \delta(y)i + \delta(z)j + \delta(w)k + [t, q]_s$, where $q = x + yi + zj + wk \in H(\mathcal{S})$ and $t = \frac{1}{2}ai - \frac{1}{2}bj - \frac{1}{2}ck$ for some $a, b, c \in \mathcal{S}$.*

We conclude this section with an example that illustrates how a centralizing map is generally not a supercentralizing map.

Example 2.9. Let $\mathcal{S}$ be a noncommutative ring and let $H(\mathcal{S})$ be the quaternion ring. We define a linear map $F\colon H(\mathcal{S}) \to H(\mathcal{S})$ by $F(x) = x$ for all $x \in H(\mathcal{S})$. Then F is a centralizing map on $H(\mathcal{S})$, but F is not a supercentralizing map on $H(\mathcal{S})$.

Proof. It is clear that F is a centralizing map. Assume that F is a supercentralizing map. Given that $\mathcal{S}$ is noncommutative, there exist elements $s, s' \in \mathcal{S}$ such that $ss' \neq s's$. Consider $x = sj + s'k$. We get that $[x, x]_s \in Z(H(\mathcal{S}))$. Specifically, we have that

$$-2(s^2 + s'^2) + 2(ss' - s's)i \in Z(\mathcal{S}),$$

which is a contradiction. Hence, F is not a supercentralizing map. □

3. Main results

Theorem 3.1. *Let $\mathcal{A} = H_{\alpha,\beta}(\mathcal{S})$ be a generalized quaternion ring. Then every supercentralizing additive map F on $\mathcal{A}$ is proper, that is, $F(q) = \lambda q + \mu(q)$ for all $q \in \mathcal{A}$, where $\lambda \in Z(\mathcal{A})_0$ with $2\lambda {\mathcal{A}_1}^2 \subseteq Z(\mathcal{A})_0$, and $\mu \colon \mathcal{A} \to Z_s(\mathcal{A})$ is an additive map.*

Proof. According to Lemma 2.7, $F = F_0 + F_1$, where $F_i(\mathcal{A}_j) \subseteq \mathcal{A}_{i+j}$ and $[F_i(x), x]_s \in Z_s(\mathcal{A})$ for $i, j = 0$ or 1. By replacing x with $x+y$ in $[F_i(x), x]_s \in Z_s(\mathcal{A})$ and using $Z_s(\mathcal{A}) = Z(\mathcal{S})$, we get

$$[F_i(x), y]_s + [F_i(y), x]_s \in Z(\mathcal{S}) \tag{3.1}$$

for all $x, y \in \mathcal{A}$.

We divide the proof into two steps:

Step 1: $F_0(q) = \lambda q + \mu_0(q)$ for all $q \in \mathcal{A}$, where $\lambda \in Z(\mathcal{A})_0$ and $\mu_0 \colon \mathcal{A} \to Z(\mathcal{A})_0$ is an additive map.

Let $F_0(i) = a + bi$ and $F_0(j) = cj + dk$ for some $a, b, c, d \in \mathcal{S}$. By taking $x = i$ and $y = j$ into (3.1) we get $2(b + c)k - 2d\alpha j \in Z(\mathcal{S})$. From the invertibility of 2 and α, we obtain $c = b$ and $d = 0$. Similarly, by taking $F_0(k) = ej + fk$, $x = i$, and $y = k$ into (3.1), we get $2(f - b)\alpha j - 2ek \in Z(\mathcal{S})$. Hence, $f = b$ and $e = 0$. Therefore, we have

$$F_0(i) = a + bi, \quad F_0(j) = bj, \quad F_0(k) = bk.$$

Now, let $s \in \mathcal{S}$ be arbitrary, and let $F_0(s) = g + hi$ for some $g, h \in \mathcal{S}$. Considering $[F_o(s), i]_s - [s, F_0(i)]_s \in Z(\mathcal{S})$, we compute that $(as - sa) + (bs - sb)i \in Z(\mathcal{S})$. Thus, $b \in Z(\mathcal{S})$. Taking $x = s$ and $y = j$ in (3.1), we obtain $h = 0$. Let $g = \delta_1(s)$, where $\delta_1 \colon \mathcal{S} \to \mathcal{S}$ is an additive map uniquely determined by F_0. Note that δ_1 is centralizing on $\mathcal{S}$ since $[F_0(s), s]_s \in Z_s(\mathcal{A})$.

On the other hand, by taking $F_0(si) = g'j + h'k$, and using $x = si$ and $y = i$ in (3.1), we get $a \in Z(\mathcal{S})$. Also, in view of $[F_0(si), j]_s + [F_0(j), si]_s \in Z(\mathcal{S})$, we have $h' = bs$. By defining $g' = \delta_2(s)$, where $\delta_2 \colon \mathcal{S} \to \mathcal{S}$ is an additive map uniquely determined by F_0, we see $F_0(si) = \delta_2(s) + bsi$.

If $F_0(sj) = mj + nk$ for some $m, n \in \mathcal{S}$, and using $x = sjandy = i$ in (3.1), then we arrive at $n = 0$ and $F_0(sj) = bsj$. Similarly, by taking $F_0(sk) = m'j + n'k$, and using $x = sk$ and $y = i$ in (3.1), we get $m' = 0$ and $n' = bs$, so $F_0(sk) = bsk$.

In particular,

$$[F_0(si), s']_s + [F_0(s'), si]_s \in Z(\mathcal{S})$$

for all $s, s' \in \mathcal{S}$. Therefore,

$$\delta_2(s)s' - s'\delta_2(s) + ((\delta_1(s') - bs')s - s(\delta_1(s') - bs'))i \in Z(\mathcal{S}).$$

Hence, $\delta_1(s) - bs \in Z(\mathcal{S})$ for all $s \in \mathcal{S}$. On the other hand, by choosing $x = sj$ and $y = s'i$ in (3.1), we see that $\delta_2(s) \in Z(\mathcal{S})$.

By a straightforward computation, we see that $F_0(q) = \delta_1(x) + \delta_2(y) + byi + bzj + bwk$ for all $q = x + yi + zj + wk \in H_{\alpha,\beta}(\mathcal{S})$. Set $\lambda = b$ and $\mu_0(q) = -bx + \delta_1(x) + \delta_2(y)$, as desired. The proof of $2\lambda{\mathcal{A}_1}^2 \subseteq Z(\mathcal{S})$ can be seen in Lemma 2.2.

Step 2: $F_1(\mathcal{A}_0) = 0$ and $F_1(\mathcal{A}_1) \subseteq Z(\mathcal{A})_0$.

Assume that $F_1(i) = aj + bk$, $F_1(j) = c + di$ and $F(k) = e + fi$ for some $a, b, c, d, e, f \in \mathcal{S}$. In view of $[F_1(i), i]_s \in Z(\mathcal{S})$, $[F_1(j), j]_s \in Z(\mathcal{S})$ and $[F_1(k), k]_s \in Z(\mathcal{S})$, we routinely compute that $F_1(i) = 0$, $F_1(j) = c$ and $F_1(k) = e$. Now, let $s \in \mathcal{S}$ be arbitrary, and let $F_1(s) = gj + hk$ for some $g, h \in \mathcal{S}$. Repeating the same computational process and taking $x = s$ and $y = i$ in (3.1), we get $-2gk + 2h\alpha j \in Z(\mathcal{S})$. So $g = h = 0$, and hence, $F_1(s) = 0$. Similarly, suppose $F_1(si) = mj + nk$ for some $m, n \in \mathcal{S}$ and using $x = si$ and $y = i$ in (3.1), we get $m = n = 0$. Hence, $F_1(\mathcal{A}_0) = 0$.

Next, we show that $F_1(\mathcal{A}_1) \subseteq Z(\mathcal{A})_0$. If we take $F_1(sj) = g' + h'i$, and use $x = sj$ and $y = j$ in (3.1), then $2h'k - (sc - cs)j \in Z(\mathcal{S})$. Thus, $h' = 0$ and $F_1(sj) = g'$. Therefore, $F_1(sj) = \delta_1(s)$ for some additive mapping $\delta_1\colon \mathcal{S} \to \mathcal{S}$ uniquely determined by F_1. Likewise, by taking $x = sk$ and $y = k$ in (3.1), we have $F_1(sk) = \delta_2(s)$ for all $s \in \mathcal{S}$ and for some mapping $\delta_2\colon \mathcal{S} \to \mathcal{S}$ uniquely determined by F_1.

Finally, let us take $x = sj + sk$ and $y = s'i$ in (3.1). Then the identity $[F_1(x), y]_s \in Z_s(\mathcal{A})$ gives $(\delta_1(s) + \delta_2(s))s' = s'(\delta_1(s) + \delta_2(s))$ for all $s, s' \in \mathcal{S}$, which implies that $\delta_1 + \delta_2 \in Z(\mathcal{S})$. Thus, the statement of Step 2 is proved.

Setting $\mu_1 = F_1$ and $\mu = \mu_0 + \mu_1$, completes the proof. $\square$

If $\mathcal{S}$ is commutative or $\beta = 0$, then, the condition $2\lambda{\mathcal{A}_1}^2 \subseteq Z(\mathcal{S})$ in the above theorem can be omitted.

Corollary 3.2. *Suppose that the ring $\mathcal{S}$ is commutative. Then any supercentralizing additive map F on $\mathcal{A}$ is of the form $F(q) = \lambda q + \mu(q)$ for all $q \in \mathcal{A}$, where $\lambda \in Z(\mathcal{A})_0$ and $\mu\colon \mathcal{A} \to Z_s(\mathcal{A})$ is an additive map.*

Proof. Let F be a supercentralizing map on $\mathcal{A}$. By Theorem 3.1, there exist $\lambda \in Z(\mathcal{S})$ and a supercentral map $\mu\colon \mathcal{A} \to Z_(\mathcal{A})_0$ such that $F(q) = \lambda q + \mu(q)$ for all $q \in \mathcal{A}$. It suffices to prove that $2\lambda{\mathcal{A}_1}^2 \subseteq Z(\mathcal{S})$. It is clear that for $x_1 = sj + s'k \in \mathcal{A}_1$ $(s, s' \in \mathcal{S})$, $2\lambda{x_1}^2 = -2\lambda\beta[(s^2 + s'^2\alpha) + (s's - ss')i]$. Since $\mathcal{S}$ is commutative, then $s's = ss'$ and $2\lambda{x_1}^2 \in \mathcal{S} = Z(\mathcal{A})_0$ for all $x_1 \in \mathcal{A}_1$. $\square$

Remark 3.3. Based on the proof of Corollary 3.2, it is particularly significant that when $\beta = 0$, the condition $2\lambda{\mathcal{A}_1}^2 \subseteq Z(\mathcal{S})$ is redundant in Theorem 3.1 for both the semiquaternion ring $H_s(\mathcal{S})$ and the split semiquaternion ring $H_{sps}(\mathcal{S})$.

Chen [4] proved a version of Posner's theorem for superalgebras. He demonstrated that the existence of a nonzero supercentralizing derivation on a prime superring forces the ring to be commutative. In the next theorem, we extend this result to a noncommutative quaternion superring $H(\mathcal{S})$ that is not necessarily prime as a $\mathbb{Z}_2$-graded ring. In fact, by [4, Lemma 2.2] if the quaternion ring $H(\mathcal{S})$

is a prime as $\mathbb{Z}_2$-graded ring, then $H(\mathcal{S})$ is a semiprime (as a ring) is true if and only if $\mathcal{S}$ is semiprime (see [6, Corollary 5.3]).

Theorem 3.4. *Let $H(\mathcal{S})$ be the quaternion ring. Then any supercentralizing superderivation on $H(\mathcal{S})$ is zero.*

Proof. Let D be a superderivation of $H(\mathcal{S})$ such that $[D(q), q]_s \in Z_s(H(\mathcal{S}))$ for all $q \in H(\mathcal{S})$. By Lemma 2.8, we have

$$D(q) = \delta(x) + \delta(y)i + \delta(z)j + \delta(w)k + [t, q]_s,$$

where δ is a derivation on $\mathcal{S}$ and $t = \frac{1}{2}ai - \frac{1}{2}bj - \frac{1}{2}ck$ for some $a, b, c \in \mathcal{S}$. Applying D to $q = i$, we get $D(i) = [t, i]_s$. Since $[D(i), i]_s \in Z_s(H(\mathcal{S}))$, we have $2bj - 2ck \in Z_s(H(\mathcal{S}))$. Since $H(\mathcal{S})$ contains $\dfrac{1}{2}$, we have $b = c = 0$. Similarly, by applying $[D(j), j]_s \in Z_s(H(\mathcal{S}))$ we obtain $a = 0$. Considering $[D(sk), j] - [sk, D(j)] \in Z_s(H(\mathcal{S}))$, $s \in \mathcal{S}$, we see that $\delta = 0$. Hence, $D = 0$, as desired. ☐

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[1] Faculty of Management, University of Primorska, Slovenia.
Email address: fajda.fosner@upr.si

[2] Department of Mathematics and Statistics, Sho.C., Islamic Azad University, Shoushtar, Iran
Email address: leilahz@iau.ac.ir