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(C, α)-SUMMABILITY WITH VARYING-PARAMETERS OF QUADRATICAL PARTIAL SUMS OF DOUBLE FOURIER SERIES ON THE GROUP OF 2-ADIC INTEGERS

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Abstract. We prove that the maximal operator of the (C, α_n)-means of the quadratical partial sums of double Fourier series on the group of 2-adic integers is quasi-local, weak type (L^1, L^1) and of strong type (L^∞, L^∞). Moreover, we prove almost everywhere convergence of (C, α_n) means of integrable functions (i.e., $\sigma_n^{\alpha_n} f \longrightarrow f \quad a.e.$), for $f \in L^1(\mathbb{I}^2)$, $\mathbb{I}^2$ is two-dimensional group on the 2-adic integers in which $\alpha_n \in (0,1)$ and α_n is varying parameter.

1. Introduction and preliminaries

1.1. **Introduction.** We introduce basics by following the standard notions of dyadic analysis introduced by Schipp, Simon, and Wade (see, e.g., [26]) and others. Denote by $\mathbb{N} := \{0, 1, \ldots\}, \mathbb{P} := \mathbb{N} \setminus \{0\}$, the set of natural numbers, the set of positive integers and $\mathbb{I} := [0, 1)$ the unit interval. Denote by $|B|$ the Lebesgue measure of a set $B (B \subset \mathbb{I})$. Denote by $L^p(\mathbb{I})$ the usual Lebesgue spaces and $\|\cdot\|_p$ the corresponding norms $(1 \leq p \leq \infty)$. Set

$$\tau := \left\{ \left[\frac{p}{2^n}, \frac{p+1}{2^n} \right) : p,\, n \in \mathbb{N} \right\},$$

the set of 2-adic intervals, and for given $x \in \mathbb{I}$ let $I_n(x)$ denote the interval $I_n(x) \in \tau$ of length 2^{-n} that contains $x (n \in \mathbb{N})$. Also, use the notation $I_n :=$

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$I_n(0)$ $(n \in \mathbb{N})$. Let

$$x = \sum_{n=0}^{\infty} x_n 2^{-(n+1)}$$

be the 2-adic expansion of $x \in \mathbb{I}$, where $x_n = 0$ or 1. If x is a dyadic rational number $(x \in \left\{\frac{p}{2^n} : p, n \in \mathbb{N}\right\})$, then we choose the expansion that terminates in $0's$. The notion of the Hardy space $H(\mathbb{I})$ is introduced in the following way [26]. A function $a \in L^{\infty}(\mathbb{I})$ is called an atom if either $a = 1$ or a satisfies the following properties:

$$\begin{aligned} &i. \quad \operatorname{supp} a \subset I_a, \\ &ii. \quad \|a\|_{\infty} \leq |I_a|^{-1}, \\ &iii. \quad \int_{\mathbb{I}} a = 0, \end{aligned}$$

for some $I_a \in \mathbb{I}$. We say that the function f belongs to H if f can be represented as $f = \sum_{i=0}^{\infty} \lambda_i a_i$, where $a_i' s$ are atoms and for the coefficients (λ_i) the inequality $\sum_{i=0}^{\infty} |\lambda_i| < \infty$ is true. It is known that H is a Banach space with respect to the norm

$$\|f\|_H := \inf \sum_{i=0}^{\infty} |\lambda_i|,$$

where the infimum is taken over all decompositions $f = \sum_{i=0}^{\infty} \lambda_i a_i \in H$. The 2-adic (or arithmetic) sum $a + b := \sum_{n=0}^{\infty} r_n 2^{-(n+1)} (a, b \in \mathbb{I})$, where bits $q_n, r_n \in \{0,1\}$ $(n \in \mathbb{N})$ are defined recursively by $q_{-1} := 0, a_n + b_n + q_{n-1} = 2q_n + r_n$ for $n \in \mathbb{N}$. Since q_n, r_n take only the values 0 and 1, these equations uniquely determine the coefficients q_n and r_n. The group $(\mathbb{I}, +)$ is called the group of 2-adic integers.

Set

$$\epsilon(t) := \exp(2\pi i t), \qquad t \in \mathbb{R},$$

where $i = \sqrt{-1}$.

Set

$$\psi_{2^n} := \epsilon\left(\frac{x_n}{2} + \cdots + \frac{x_o}{2^{n+1}}\right) \qquad (x \in \mathbb{I}, n \in \mathbb{N})$$

and

$$\psi_n := \prod_{i=0}^{\infty} \psi_{2^i}^{n_i},$$

where $n = \sum_{i=0}^{\infty} n_i 2^i \quad (n_i \in \{0,1\} \, (i \in \mathbb{N})), \, n \in \mathbb{N}$. It is known [12] that the system $(\psi_n, n \in \mathbb{N})$ is the character system of $(\mathbb{I}, +)$. For more details on the Fourier theory with respect to the character system of the group of 2-adic integers see, for instance, [12]. Denote by

$$\hat{f}(n) := \int_{\mathbb{I}} f \bar{\psi}_n d\lambda, \quad D_n := \sum_{k=0}^{n-1} \psi_k, \quad K_n^1 := \frac{1}{n+1} \sum_{k=0}^{n} D_k,$$

the Fourier coefficients, the Dirichlet, and the Fejér or $(C,1)$ kernels, respectively. It is also known that the Fejér or $(C,1)$ mean of f is

$$\begin{aligned}\sigma_n f(y) :&= \frac{1}{n+1}\sum_{k=0}^{n} S_k f(y) = \int_{\mathbb{I}} f(x) K_n^1(y-x)d\lambda(x)\\ &= \frac{1}{n+1}\int_{\mathbb{I}} f(x) D_k^1(y-x)d\lambda(x) \qquad (n\in\mathbb{N},\, y\in\mathbb{I}).\end{aligned}$$

It is known from [12] that for $n\in\mathbb{N}, x\in\mathbb{I}$,

$$D_{2^n}(x) = \begin{cases} 0, & \text{if } x\in I_n,\\ 2^n, & \text{if } x\notin I_n,\end{cases}$$

and also that

$$D_n(x) = \psi_n(x)\sum_{k=0}^{\infty} D_{2^k}(x) n_k (-1)^{x_k}.$$

Denote by K_n^α the kernel of the summability method (C,α), called the (C,α) kernel or Cesáro kernel, for $\alpha\in\mathbb{R}\setminus\{\dots,-3-2,-1\}$

$$K_n^\alpha = \frac{1}{A_n^\alpha}\sum_{v=0}^{n} A_{n-v}^{\alpha-1} D_v,$$

where

$$A_k^\alpha = \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+k)}{k!}.$$

The (C,α) means of an integrable function f are

$$\sigma_n^\alpha f(y) = \frac{1}{A_n^\alpha}\sum_{k=0}^{n} A_{n-k}^{\alpha-1} S_k f(y) = \int_{\mathbb{I}} f(x) K_n^\alpha(y-x)d\lambda(x),$$

$$\sigma_n^\alpha f(y) = \frac{1}{A_n^\alpha}\sum_{k=0}^{n} A_{n-k}^{\alpha-1}\int_{\mathbb{I}} f(x) D_k(y-x)d\lambda(x).$$

It is well known from [33] that

$$A_n^\alpha = \sum_{k=0}^{n} A_{n-k}^{\alpha-1}, \quad A_n^\alpha - A_{n-1}^\alpha = A_n^{\alpha-1}, \quad A_n^\alpha \approx n^\alpha.$$

Basically, in order to prove Theorem 2.5, we verify that the maximal operator $\sigma_{*,q}^\alpha$ is of weak type (L^1, L^1). In order to do this, we investigate the kernel functions and their maximal function on the unit interval $\mathbb{I}$ by making a hole around zero. To have the proof of Theorem 2.6, we follow the standard way by following the fact that $\sigma_{*,q}^\alpha$ is of weak type (L^1, L^1). We need several lemmas.

The following notations as well as definitions of functions and operators are used through the proofs of this paper.

For $a,\, s,\, n\in\mathbb{N}$ let $n_{(s)} := \sum_{j=0}^{s-1} n_i 2^i$, that is, $n_{(0)} = 0$, $n_{(1)} = n_0$ and for $2^B \le n < 2^{B+1}$, $|n| := B$, $n_{(B+1)} = n$.
Define two-variable functions $P(n,\alpha) := \sum_{i=0}^{\infty} n_i (2^i)^\alpha$ for $n\in\mathbb{N}$, $\alpha\in\mathbb{R}$ [4]. For

example, $P(n,1) = n$. Besides, set for sequences $\alpha = (\alpha_n)$ and positive reals q, define the subset of natural numbers

$$\mathbb{N}_{\alpha,q} := \left\{ n \in \mathbb{N} : \frac{P(n,\alpha_n)}{n^{\alpha_n}} \le q \right\}. \tag{1.1}$$

For a sequence α such that $0 < \alpha_0 \le \alpha_n < 1$ we have $\mathbb{N}_{\alpha,q} = \mathbb{N}$ for some q depending only on α_0. We remark that $2^n \in \mathbb{N}_{\alpha,q}$ for every $\alpha = (\alpha_n)$, $0 < \alpha_n < 1$ and $q \ge 1$. In this paper, C denotes an absolute constant and C_q denotes another one, which may depend only on q. Define the following kernel function and operator:

$$\begin{aligned} T_n^{\alpha_a} &:= \frac{1}{A_n^{\alpha_a}} \sum_{k=0}^{2^B-1} A_{n-k}^{\alpha_a - 1} D_k, \\ t_n^{\alpha_a} f(y) &:= \int_{\mathbb{I}} f(x) T_n^{\alpha_a}(y+x) d\mu(x), \qquad a \in \mathbb{N},\ \alpha_a \in [0,1). \end{aligned} \tag{1.2}$$

For $f \in L^1(\mathbb{I})$ and for all real number $\alpha_n \ne -1, -2, -3, \ldots$, define the kernel of (C, α_n) summability method as follows:

$$K_n^{\alpha_n} := \frac{1}{A_n^{\alpha_n}} \sum_{t=0}^{n} A_{n-t}^{\alpha_n - 1} D_t, \tag{1.3}$$

and where $A_k^{\alpha_n}$ is defined in [6] for the case where $\alpha = (\alpha_n)$.

Let $\mathbb{I} \times \mathbb{I}$ be a direct product of 2-adic group $\mathbb{I}$. Let (x^1, x^2), $(y^1, y^2) \in \mathbb{I} \times \mathbb{I}$, $f \in L^1(\mathbb{I} \times \mathbb{I})$. The two-dimensional Fourier coefficients, the rectangular partial sums, Dirichlet Kernel, (C, α_n) kernel, (C, α_n) means for $\alpha_n \in \mathbb{R} \setminus \{\ldots, -3, -2, -1\}$, Marcinkiewicz means and the Marcinkiewicz kernels with respect to the character systems of two-dimensional 2-adic group are defined as follows, respectively:

$$\begin{aligned} \hat{f}(n_1, n_2) &:= \int_{\mathbb{I} \times \mathbb{I}} f(x^1, x^2) \bar{\psi}_{n_1}(x^1) \bar{\psi}_{n_2}(x^2) d\mu(x^1, x^2), \\ S_{n_1,n_2} f(y^1, y^2) &:= \sum_{k_1=0}^{n_1-1} \sum_{k_2=0}^{n_2-1} \hat{f}(k_1, k_2) \psi_{k_1}(y^1) \psi_{k_2}(y^2), \\ D_{n_1,n_2}(y, x) &:= \sum_{k_1=0}^{n_1-1} \sum_{k_2=0}^{n_2-1} \psi_{k_1}(y^1) \psi_{k_2}(y^2) \bar{\psi}_{k_1}(x^1) \bar{\psi}_{k_2}(x^2), \\ \Omega_n^{\alpha_n}(x^1, x^2) &= \frac{1}{A_{n-k}^{\alpha_n - 1}} \sum_{j=0}^{n-1} D_{j,j}(x^1, x^2), \\ \sigma_n^{\alpha_n} f(y^1, y^2) &= \frac{1}{A_n^{\alpha_n}} \sum_{k=0}^{n} A_{n-k}^{\alpha_n - 1} S_{k,k} f(y^1, y^2), \\ \Omega_n(x^1, x^2) &:= \frac{1}{n} \sum_{j=1}^{n} D_j(x^1) D_j(x^2), \end{aligned}$$

$$\sigma_n(x^1,x^2) := \frac{1}{n}\sum_{j=1}^{n} S_{j,j}f(x^1,x^2), \tag{1.4}$$

where

$$A_k^{\alpha_n} = \frac{(\alpha_n+1)(\alpha_n+2)\cdots(\alpha_n+k)}{k!}.$$

Each natural number n can be uniquely expressed as $n = \sum_{i=0}^{\infty} n_i 2^i \quad (n_i \in \{0,1\}\,,\ i \in \mathbb{N})$, where only a finite number of n_i 's differ from zero. It is already defined in [4] that

$$P(n,\alpha) := \sum_{i=0}^{\infty} n_i 2^{\alpha i} \tag{1.5}$$

for $n \in \mathbb{N}, \alpha \in \mathbb{R}$. For example, $P(n,1) = n$. Besides, the set of sequences $\alpha = (\alpha_n)$ and positive real q, the subset of natural numbers

$$\mathbb{N}_{\alpha,\,q} := \left\{ n \in \mathbb{N} : \frac{P(n,\alpha)}{n^{\alpha_n}} \le q \right\}. \tag{1.6}$$

For a sequence α such that $0 < \alpha_0 \le \alpha_n < 1$ we have $\mathbb{N}_{\alpha,q} = \mathbb{N}$ for some q depending only on α_0. We remark that $2^n \in \mathbb{N}_{\alpha,\,q}$ for every $\alpha = (\alpha_n)$, $0 < \alpha_n < 1$ and $q \ge 1$. In this case C denotes a constant and C_q another one, which may depend only on q.

Let $1 \le p \le +\infty$. We say that an operator $T : L^1 \to L^0$ ($L^0(\mathbb{I})$ is the space of measurable functions on $\mathbb{I}$) of type (p,p) if there exists an absolute constant $C > 0$ for which $\|Tf\|_p \le C\|f\|_p$ for all $f \in L^p$. Then T is said to be of weak type $(1,1)$ if there exists an absolute constant $C > 0$ for which $\mu(\{x : |(Tf)(x)| > \lambda\}) \le C\|f\|_1/\lambda$ for all $\lambda > 0$ and $f \in L^1(\mathbb{I})$. Throughout this paper, consider μ as measure.

For constant parameter setting, the almost everywhere convergence of $(C,1)$ means of two-dimensional trigonometric Fourier series to f for all f in the space $L^1 \log L^1([0,2\pi]^2)$ was proved by Marcinkiewicz and Zygmund in 1939 [23]. In 1968, for the two-dimensional trigonometric Fourier series, Zhizhiashvili [32] studied that for all $f \in L^1([0,2\pi]^2)$ the (C,α) means converges to f a.e. for any $\alpha > 0$. Dyachenko [7] proved this for dimensions greater than 2. The dyadic analog of Marcinkiewicz and Zygmund for the double Walsh–Fourier series was proved by Weisz [30]. Goginava and Weisz [21] investigated that the maximal operator of the triangular $(C,1)$ means of a two-dimensional Walsh–Fourier series is bounded from the dyadic Hardy space H^p to L^p for all $1/2 < p \le \infty$ and, consequently, is of weak type $(1,1)$. As a consequence, they obtained that (2^n)th of the triangular$(C,1)$ means of a function $f \in L^1([0,1]^2)$ converge a.e. to f. Almost everywhere convergence of (C,α) means of cubical partial sums of d-dimensional Walsh–Fourier series was proved by Goginava [20].

Gát [13] proved the almost everywhere convergence of Cesáro means of one-dimensional Fourier series on the group of 2-adic integers. Gát and Gábor proved the almost everywhere convergence of the (C,α) Marczinkiewicz-means of integrable functions of two-dimensional Fourier series on the group of 2-adic integers [17].

The almost everywhere convergence of this summability method for a constant parameter in the quadratic partial sums of double Vilenkin–Fourier series was proved by Gát and Goginava in 2006 [18]. On the point-wise convergence of Cesàro means of two-variable functions with respect to unbounded Vilenkin systems, see the paper of Gát [15]. With respect to more general orthonormal systems, two-dimensional Vilenkin-like systems, the author studied almost everywhere convergence of generalized Marcinkiewicz means [10].

To add more, a lot of authors studied the one- and two-dimensional cases of convergence and summability of Cesáro means of in Lebesgue and martingale Hardy spaces. We mention as Blahota, Persson and Tephnadze [1], Persson, Tephnadze and Weisz [27], Blahota, Tephnadze and Toledo [3], Blahota and Tephnadze [2], Fridli [8], Gát [16], Nagy [24, 25], Weisz [31].

In 1960, Kaplan [22] introduced the idea of Cesàro means with varying parameters of numerical sequences. In 2007, Akhobadez [6] introduced the notion of Cesàro means of Trigonometric Fourier series with varying parameter setting. Anas and Gát [4] proved for the varying parameter settings (C,α) means for one-dimensional Walsh-Paley system converges a.e. Maximal operators of Cesàro means with varying parameters of Walsh–Fourier series were investigated by Gát and Goginava [19].

For the one- and two-dimensional Walsh-Paley system, these were proved by Anas and Gát (see [4, 5]). Cesàro and Riesz summability with varying parameters of the multidimensional Walsh–Fourier series was proved by Weisz [30]. Cesàro means in a varying parameter setting with respect to the one-dimensional character systems of the group of 2-adic integers and with respect to Walsh–Kaczmarz systems were proved by A. Tilahun (see[28, 29]). The one-dimensional case with respect to one-dimensional Vilenkin systems was proved by Gát and A. Tilahun [9]. The two-dimensional case with respect to character systems of Fourier series on the group of 2-adic integers in a varying parameter setting has not been proved yet.

Thus, in this paper the almost everywhere convergence of varying parameter setting Cesàro means in the case of two-dimensional Fourier series on the group of 2-adic integers is studied. We used Cesàro and (C,α) nomenclatures alternatively.

Define the following functions and operators, which are very important through out the proof of main results of the paper as follows: Let $n^{(j)} := \sum_{i=0}^{j-1} n_i 2^{\alpha i}$, that is, $n^{(0)} = 0$, $n^{(1)} = n_0$ and for $2^B \leq n < 2^{B+1}$, let $|n| := B$, $n = n^{(B+1)}$ for a, n, $j \in \mathbb{N}$.

Moreover, let $n \in \mathbb{N}$ and $1 > \alpha_a > 0 \quad (a \in \mathbb{N})$, where (x^1,x^2), $(y^1,y^2) \in \mathbb{I}\times\mathbb{I}$. Then,

$$\begin{aligned}
&\Omega_n^{\alpha_a,1}(x^1,x^2) := \frac{1}{A_n^{\alpha_n}} \sum_{j=0}^{2^B-1} A_{n-j}^{\alpha_a-1} D_{j,j}(x^1,x^2),\\
&t_n^{\alpha_a} f(y^1,y^2) := \int_{\mathbb{I}\times\mathbb{I}} f(x^1,x^2)\Omega_n^{\alpha_a,1}(y^1+x^1,y^2+x^2)d\lambda(x^1,x^2).
\end{aligned} \tag{1.7}$$

1.2. **Preliminary lemmas.**

Lemma 1.1. [13] *For $j,\ n \in \mathbb{N}, j < 2^n$, we have*

$$D_{2^n-j} = D_{2^n} - \psi_{2^n-1}\bar{D}_j.$$

Lemma 1.2. [14]*For $k,\ n \in \mathbb{N}$, we have*

$$\int_{\overline{I_k \times I_k}} \sup_{|n| \geq k} |\Omega_n| \leq C.$$

2. Main results

In the following Lemma, we prove the operator $t_n^{\alpha_a}$ is quasi-local (for the notion of quasi-locality see [26]).

Lemma 2.1. *Let $1 > \alpha_a > 0,\ B := |n|\ ,\ f \in L_1(\mathbb{I} \times \mathbb{I})$ such that*

$$\operatorname{supp} f \subset I_k(u^1 \times I_k(u^2)),$$

$$\int_{I_k(u^1) \times I_k(u^2)} f d\lambda = 0$$

for some 2-adic rectangle $I_k(u^1) \times I_k(u^2) \subset \mathbb{I} \times \mathbb{I}$.

Then,

$$\int_{\overline{I_k(u^1) \times I_k(u^2)}} \sup_{n,a \in \mathbb{N}} |t_n^{\alpha_a} f|\, d\lambda \leq C \left\| f \right\|_1.$$

Proof. By the equality (see Lemma 1.1)

$$D_{2^B-j} = D_{2^B} - \psi_{2^B-1}\bar{D}_j, \tag{2.1}$$

we have

$$\begin{aligned}
\Omega_n^{\alpha_a,1}(x^1,x^2) &= \frac{1}{A_n^{\alpha_a}} \sum_{j=0}^{2^B-1} A_{2^B+n^{(B)}-j}^{\alpha_a-1} D_{j,j}(x^1,x^2) \\
&= \frac{1}{A_n^{\alpha_a}} \sum_{j=0}^{2^B-1} A_{n^{(B)}+j}^{\alpha_a-1} D_{2^B-j,\,2^B-j}(x^1,x^2) \\
&= \frac{D_{2^B}(x^1)D_{2^B}(x^2)}{A_n^{\alpha_a}} \sum_{j=0}^{2^B-1} A_{n^{(B)}+j}^{\alpha_a-1} \\
&\quad - \frac{\psi_{2^B-1}(x^1)D_{2^B}(x^2)}{A_n^{\alpha_a}} \sum_{j=0}^{2^B-1} A_{n^{(B)}+j}^{\alpha_a-1} D_j(x^1) \\
&\quad - \frac{\psi_{2^B-1}(x^2)D_{2^B}(x^1)}{A_n^{\alpha_a}} \sum_{j=0}^{2^B-1} A_{n^{(B)}+j}^{\alpha_a-1} D_j(x^2) \\
&\quad + \frac{\psi_{2^B-1}(x^1)\psi_{2^B-1}(x^2)}{A_n^{\alpha_a}} \sum_{j=0}^{2^B-1} A_{n^{(B)}+j}^{\alpha_a-1} D_{j,j}(x^1,x^2) \\
&=: (1) + (2) + (3) + (4).
\end{aligned} \tag{2.2}$$

Therefore, with the help of Abel's transform, we get

$$\begin{aligned}
|(4)| =&\frac{1}{A_n^{\alpha_a}}\left|\sum_{j=0}^{2^B-1} A_{n^{(B)}+j}^{\alpha_a-1} D_{j,j}(x^1,x^2)\right| \\
=&\frac{1}{A_n^{\alpha_a}}\left|(\sum_{j=0}^{2^B-2} A_{n^{(B)}+j}^{\alpha_a-1} - A_{n^{(B)}+j+1}^{\alpha_a-1}) \sum_{i=0}^{j} D_{i,i}(x^1,x^2) + \frac{A_{n^{(B)}+2^B}^{\alpha_a-1}}{A_n^{\alpha_a}} \sum_{i=0}^{2^B-1} D_{i,i}(x^1,x^2)\right| \\
=&\left|(1-\alpha_a)\sum_{j=0}^{2^B-2} \frac{A_{n^{(B)}+j}^{\alpha_a-1}}{A_n^{\alpha_a}} \frac{j+1}{n^{(B)}+j+1}\Omega_j(x^1,x^2) + \frac{A_n^{\alpha_a-1}}{A_n^{\alpha_a}}(2^B)\Omega_{2^B-1}(x^1,x^2)\right| \\
\leq&(1-\alpha_a)\sum_{j=0}^{2^k-1} \frac{A_{n^{(B)}+j}^{\alpha_a-1}}{A_n^{\alpha_a}} \frac{j+1}{n^{(B)}+j+1}\left|\Omega_j(x^1,x^2)\right| \\
&+(1-\alpha_a)\sum_{j=2^k}^{2^B-2} \frac{A_{n^{(B)}+j}^{\alpha_a-1}}{A_n^{\alpha_a}} \times \frac{j+1}{n^{(B)}+j+1}\left|\Omega_j(x^1,x^2)\right| \\
&+\frac{A_n^{\alpha_a-1}}{A_n^{\alpha_a}}(2^B)\left|\Omega_{2^B-1}(x^1,x^2)\right| =: I+II+III.
\end{aligned} \tag{2.3}$$

We know from [9, Lemma 1.1],

$$\frac{A_{n^{(B)}+j}^{\alpha_a-1}}{A_n^{\alpha_a}} \leq \frac{\alpha_a(n^{(B)}+j)^{\alpha_a-1}}{(n)^{\alpha_a}}. \tag{2.4}$$

Thus,

$$\begin{aligned}
I &\leq \alpha_a(1-\alpha_a)\sum_{j=0}^{2^k-1} \frac{(n^{(B)}+j)^{\alpha_a-1}}{(n)^{\alpha_a}} \frac{j+1}{n^{(B)}+j+1}\left|\Omega_j(x^1,x^2)\right| \\
&\leq \alpha_a(1-\alpha_a)\sum_{j=0}^{2^k-1} \frac{(n^{(B)}+j)^{\alpha_a-2}}{(n)^{\alpha_a}}(j+1)\left|\Omega_j(x^1,x^2)\right| \\
&= \frac{\alpha_a(1-\alpha_a)}{(n)^{\alpha_a}}\sum_{j=0}^{2^k-1}\gamma(j)\left|\Omega_j(x^1,x^2)\right| =: W,
\end{aligned} \tag{2.5}$$

$$\text{where} \quad \gamma(j) := \frac{j+1}{(n^{(B)}+j)^{2-\alpha}}. \tag{2.6}$$

Similarly,

$$\begin{aligned}
II &= (1-\alpha_a)\sum_{j=2^k}^{2^B-2} \frac{A_{n^{(B)}+j}^{\alpha_a-1}}{A_n^{\alpha_a}} \frac{j+1}{n^{(B)}+j+1}\left|\Omega_j(x^1,x^2)\right| \\
&\leq \frac{\alpha_a(1-\alpha_a)}{(n)^{\alpha_a}}\sum_{j=2^k}^{2^B-2}\gamma(j)\left|\Omega_j(x^1,x^2)\right| =: M.
\end{aligned} \tag{2.7}$$

Consider III as it is. Using Abel's transform and the same procedure as above, we have the following results for (2) and (3).
The situation for (2),

$$\begin{aligned}|(2)| \leq & \frac{\alpha_a(1-\alpha_a)D_{2^B}(x^2)}{(n)^{\alpha_a}} \sum_{j=2^k}^{2^B-2} \gamma(j) \Big| K_j(x^1) \Big| \\ & + \frac{\alpha_a(1-\alpha_a)D_{2^B}(x^2)}{(n)^{\alpha_a}} \sum_{j=0}^{2^k-1} \gamma(j) \Big| K_j(x^1) \Big| + \frac{A_n^{\alpha_a-1}}{A_n^{\alpha_a}} D_{2^B}(x^2) \Big| K_{2^B-1}(x^1) \Big|.\end{aligned} \tag{2.8}$$

The situation for (3),

$$\begin{aligned}|(3)| \leq & \frac{\alpha_a(1-\alpha_a)D_{2^B}(x^1)}{(n)^{\alpha_a}} \sum_{j=2^k}^{2^B-2} \gamma(j) \Big| K_j(x^2) \Big| \\ & + \frac{\alpha_a(1-\alpha_a)D_{2^B}(x^1)}{(n)^{\alpha_a}} \sum_{j=0}^{2^k-1} \gamma(j) \Big| K_j(x^2) \Big| + \frac{A_n^{\alpha_a-1}}{A_n^{\alpha_a}} D_{2^B}(x^1) \Big| K_{2^B-1}(x^2) \Big|.\end{aligned} \tag{2.9}$$

By the above written, we get

$$\begin{aligned}\Big| \Omega_n^{\alpha_a,1}(x^1,\, x^2) \Big| \leq & \frac{D_{2^B}(x^1)D_{2^B}(x^2)}{A_n^{\alpha_a}} \sum_{j=0}^{2^B-1} A_{n^{(B)}+j}^{\alpha_a-1} \\ & + \frac{\alpha_a(1-\alpha_a)}{n^{\alpha_a}} \sum_{j=0}^{2^k-1} \gamma(j) \Bigg[\Big| D_{2^B}(x^1) K_j(x^2) \Big| \\ & + \Big| D_{2^B}(x^2) K_j(x^1) \Big| + \Big| \Omega_j(x^1,\, x^2) \Big| \Bigg] \\ & + \frac{\alpha_a(1-\alpha_a)}{n^{\alpha_a}} \sum_{j=2^k}^{2^B-2} \gamma(j) \Bigg[\Big| D_{2^B}(x^1) K_j(x^2) \Big| + \Big| D_{2^B}(x^2) K_j(x^1) \Big| \\ & + \Big| \Omega_j(x^1,\, x^2) \Big| \Bigg] + \frac{A_n^{\alpha_a-1}}{A_n^{\alpha_a}} \Bigg[D_{2^B}(x^1) \Big| K_{2^B-1}(x^2) \Big| \\ & + 2^B D_{2^B}(x^2) \Big| K_{2^B-1}(x^1) \Big| + \Big| 2^B \Omega_{2^B-1}(x^1,x^2) \Big| \Bigg].\end{aligned} \tag{2.10}$$

It can be supposed that $u = 0$, that since Haar measure is shift invariant. That is, $I_k(u) = I_k(0) = I_k$. Besides, since for $n < 2^k$, $(x^1, x^2) \in I_k(u^1) \times I_k(u^2)$ and $(y^1, y^2) \in \overline{I_k(u^1) \times I_k(u^2)}$ we have $\Omega_n^{\alpha_a,1}(y^1+x^1,\, y^2+x^2) = \Omega_n^{\alpha_a,1}(y^1+u^1,\, y^2+u^2)$. Thus,

$$\begin{aligned}& \int_{I_k(u^1)\times I_k(u^2)} f(x^1,x^2)\Omega_j^{\alpha_a,1}(y^1+x^1, y^2+x^2) d\lambda(x^1,x^2) \\ & = \Omega_n^{\alpha_a,1}(y^1+u^1,\, y^2+u^2) \int_{I_k(u^1)\times I_k(u^2)} f(x^1,x^2) d\lambda(x^1,x^2) = 0.\end{aligned} \tag{2.11}$$

As a result,

$$\int_{\overline{I_k(u^1)\times I_k(u^2)}} \sup_{n,a\in\mathbb{N}} |t_n^{\alpha_a} f|\, d\lambda = \int_{\overline{I_k(u^1)\times I_k(u^2)}} \sup_{n\geq 2^k,\, a\in\mathbb{N}} |t_n^{\alpha_a} f|\, d\lambda.$$

Thus,

$$\begin{aligned}
&\int_{\overline{I_k\times I_k}} \sup_{n\geq 2^k, a\in\mathbb{N}} \left| \int_{I_k\times I_k} \Omega_n^{\alpha_a,1}(y^1+x^1,\, y^2+x^2) f(x^1,\, x^2) d\lambda(x^1,\, x^2) \right| d\lambda(y^1,\, y^2) \\
&\leq \int_{\overline{I_k\times I_k}} \sup_{n\geq 2^k, a\in\mathbb{N}} \int_{I_k\times I_k} \left|\Omega_n^{\alpha_a,1}(y^1+x^1,\, y^2+x^2)\right| \left|f(x^1,\, x^2)\right| d\lambda(x^1,\, x^2) d\lambda(y^1,\, y^2) \\
&\leq \int_{\overline{I_k\times I_k}} \sup_{n\geq 2^k, a\in\mathbb{N}} \left| \int_{I_k\times I_k} \left|f(x^1,\, x^2)\right| \left[\frac{D_{2^B}(y^1+x^1) D_{2^B}(y^2+x^2)}{A_n^{\alpha_a}} \right.\right. \\
&\quad \times \sum_{j=0}^{2^B-1} A_{n_{(B)}+j}^{\alpha_a-1} d\lambda(x^1,\, x^2) \Bigg| d\lambda(y^1,\, y^2) + \int_{\overline{I_k\times I_k}} \sup_{n\geq 2^k, a\in\mathbb{N}} \left| \int_{I_k\times I_k} \left|f(x^1,\, x^2)\right| \right] \\
&\quad \times \left[\frac{\alpha_a(1-\alpha_a)}{n^{\alpha_a}} \sum_{j=0}^{2^k-1} \gamma(j) \right] \left| D_{2^B}(y^1+x^1) K_j(y^2+x^2) \right| d\lambda(x^1,\, x^2) \Bigg| d\lambda(y^1,\, y^2) \\
&\quad + \int_{\overline{I_k\times I_k}} \sup_{n\geq 2^k, a\in\mathbb{N}} \left| \int_{I_k\times I_k} f(x^1,\, x^2) \frac{\alpha_a(1-\alpha_a)}{n^{\alpha_a}} \sum_{j=0}^{2^k-1} \gamma(j) \right| D_{2^B}(y^2+x^2) \\
&\quad \times K_j(y^1+x^1) \Bigg\| d\lambda(x^1,\, x^2) \Bigg| d\lambda(y^1,\, y^2) + \int_{\overline{I_k\times I_k}} \sup_{n\geq 2^k, a\in\mathbb{N}} \left| \int_{I_k\times I_k} f(x^1,\, x^2) \right. \\
&\quad \times \left[\frac{\alpha_a(1-\alpha_a)}{n^{\alpha_a}} \sum_{j=0}^{2^k-1} \gamma(j) \left| \Omega_j(y^1+x^1,\, y^2+x^2) \right| \right] d\lambda(x^1,\, x^2) \Bigg| d\lambda(y^1,\, y^2) \\
&\quad + \int_{\overline{I_k\times I_k}} \sup_{n\geq 2^k, a\in\mathbb{N}} \left| \int_{I_k\times I_k} f(x^1,\, x^2) \left[\frac{\alpha_a(1-\alpha_a)}{n^{\alpha_a}} \sum_{j=2^k}^{2^B-2} \gamma(j) \right| D_{2^B}(y^1+x^1) \right. \\
&\quad \times K_j(y^2+x^2) \Bigg| \Bigg] d\lambda(x^1,\, x^2) \Bigg| d\lambda(y^1,\, y^2) + \int_{\overline{I_k\times I_k}} \sup_{n\geq 2^k, a\in\mathbb{N}} \left| \int_{I_k\times I_k} f(x^1,\, x^2) \right. \\
&\quad \times \left[\frac{\alpha_a(1-\alpha_a)}{n^{\alpha_a}} \sum_{j=2^k}^{2^B-2} \gamma(j) \left| D_{2^B}(y^2+x^2) K_j(y^1+x^1) \right| \right] d\lambda(x^1,\, x^2) \Bigg| d\lambda(y^1,\, y^2) \\
&\quad + \int_{\overline{I_k\times I_k}} \sup_{n\geq 2^K, a\in\mathbb{N}} \left| \int_{I_k\times I_k} f(x^1,\, x^2) \left[\frac{\alpha_a(1-\alpha_a)}{n^{\alpha_a}} \sum_{j=2^K}^{2^B-2} \gamma(j) \right.\right. \\
&\quad \times \left| \Omega_j(y^1+x^1,\, y^2+x^2) \right| \Bigg] d\lambda(x^1,\, x^2) \Bigg| d\lambda(y^1,\, y^2)
\end{aligned}$$

$$
\begin{aligned}
&+\int_{\overline{I_k\times I_k}} \sup_{n\geq 2^K, a\in\mathbb{N}} \left| \int_{I_k\times I_k} f(x^1,\, x^2) \left[\frac{A_n^{\alpha_a-1}}{A_n^{\alpha_a}} (2^B) D_{2^B}(y^1+x^1) \right.\right. \\
&\times \left.\left. \left| K_{2^B-1}(y^2+x^2) \right| \right] d\lambda(x^1,\, x^2) \right| d\lambda(y^1,\, y^2) \\
&+\int_{\overline{I_k\times I_k}} \sup_{n\geq 2^K, a\in\mathbb{N}} \left| \int_{I_k\times I_k} f(x^1,\, x^2) \left[\frac{A_n^{\alpha_a-1}}{A_n^{\alpha_a}} (2^B) D_{2^B}(y^2+x^2) \right.\right. \\
&\times \left.\left. \left| K_{2^B-1}(y^1+x^1) \right| \right] d\lambda(x^1,\, x^2) \right| d\lambda(y^1,\, y^2) \\
&+\int_{\overline{I_k\times I_k}} \sup_{n\geq 2^K, a\in\mathbb{N}} \left| \int_{I_k\times I_k} f(x^1,\, x^2) \right. \\
&\times \left. \left[\frac{A_n^{\alpha_a-1}}{A_n^{\alpha_a}} 2^B \left| \Omega_{2^B-1}(y^1+x^1, y^2+x^2) \right| \right] d\lambda(x^1,\, x^2) \right| d\lambda(y^1,\, y^2) \\
&=: \sum_{i=0}^{10} \Theta_i.
\end{aligned}
\tag{2.12}
$$

It is simple to find out that

$$
\frac{D_{2^B}(y^1+x^1)D_{2^B}(y^2+x^2)}{A_n^{\alpha_a}} \sum_{j=0}^{2^B-1} A_{n^{(B)}+j}^{\alpha_a-1} = 0 \tag{2.13}
$$

for any $(y^1+x^1,\, y^2+x^2) \in \overline{I_k \times I_k} = \overline{I_k} \times I_k \cup I_k \times \overline{I_k} \cup \overline{I_k} \times \overline{I_k}$. From this we can easily see that there is a possibility in which either $y^1+x^1 \in \overline{I_k}$, $y^2+x^2 \in \overline{I_k}$ or both y^1+x^1 and $y^2+x^2 \in \overline{I_k}$. This implies that either $D_{2^B}(y^1+x^1) = 0$ or $D_{2^B}(y^2+x^2) = 0$ must hold true. Thus (2.13) becomes true because $D_{2^B}(y^1+x^1)D_{2^B}(y^2+x^2) = 0$ for $B = |n| \geq k$ and $(y^1+x^1,\, y^2+x^2) \in \overline{I_k \times I_k}$. Thus, $\Theta_1 = 0$. Moreover, since for any $j < 2^k$ we have that the Marcinkiewicz kernel $\Omega_j(y^1+x^1,\, y^2+x^2)$ depends with respect to (x^1, x^2) only on coordinates $x_0^1 = 0, \ldots, x_{k-1}^1 = 0,\; x_0^2 = 0, \ldots, x_{k-1}^2 = 0$, then

$$
\begin{aligned}
&\int_{I_k\times I_k} f(x^1,\, x^2) \left| \Omega_j(y^1+x^1,\, y^2+x^2) \right| d\lambda(x^1,\, x^2) \\
&= \left| \Omega_j(y^1,\, y^2) \right| \int_{I_k\times I_k} f(x^1,\, x^2) d\lambda(x^1,\, x^2) = 0.
\end{aligned}
\tag{2.14}
$$

Thus, (2.14) implies $\Theta_4 = 0$.

Since for any $j < 2^K$ we have that the kernel $K_j(y+x)$ depends (with respect to x) only on coordinates $x_0^1 = 0,\, \ldots,\, x_{k-1}^1 = 0$, then

$$
\int_{I_k\times I_k} f(x) \left| K_j(y+x) \right| d\lambda(x) = \left| K_j(y) \right| \int_{I_k\times I_k} f(x) d\lambda(x) = 0. \tag{2.15}
$$

Thus, (2.15) implies $\Theta_b = 0$ for $b = 2,\, 3$.

On the other hand,

$$\begin{aligned}
M &= \frac{\alpha_a(1-\alpha_a)}{(n)^{\alpha_a}} \sum_{j=2^K}^{2^B-2} \gamma(j) \left|\Omega_j(x^1,x^2)\right| \\
&\le \sup_{j\ge 2^K} \left|\Omega_j(x^1,x^2)\right| \frac{\alpha_a(1-\alpha_a)}{(n)^{\alpha_a}} \sum_{j=1}^{n} \gamma(j) \\
&\le \sup_{j\ge 2^k} \left|\Omega_j(x^1,x^2)\right| \frac{\alpha_a(1-\alpha_a)}{(n)^{\alpha_a}} \sum_{j=1}^{n} \frac{j+1}{j^{2-\alpha_a}} \\
&\le C \sup_{j\ge 2^k} \left|\Omega_j(x^1,x^2)\right| \frac{\alpha_a(1-\alpha_a)}{(n)^{\alpha_a}} \sum_{j=1}^{n} j^{\alpha_a-1} \\
&\le C \sup_{j\ge 2^k} \left|\Omega_j(x^1,x^2)\right| \frac{\alpha_a(1-\alpha_a)}{(n)^{\alpha_a}} \int_{j=1}^{n} j^{\alpha_a-1} \\
&\le C \sup_{j\ge 2^k} \left|\Omega_j(x^1,x^2)\right| \frac{\alpha_a(1-\alpha_a)}{(n)^{\alpha_a}} \frac{n^{\alpha_a}}{\alpha_a} \\
&= C(1-\alpha_a) \sup_{j\ge 2^k} \left|\Omega_j(x^1,x^2)\right| \qquad (2.16)
\end{aligned}$$

for some constant C.

Thus, using Lemma 1.2, we can find that

$$\begin{aligned}
&\int_{\overline{I_k\times I_k}} \sup_{n\ge 2^k, a\in\mathbb{N}} M d\lambda(x^1,x^2) \\
&\le C(1-\alpha_a) \int_{\overline{I_k\times I_k}} \sup_{n\ge 2^k, a\in\mathbb{N}} \sup_{j\ge 2^B} \left|\Omega_j(x^1,x^2)\right| d\lambda(x^1,x^2) \\
&\le C.
\end{aligned}$$

Thus, $\Theta_7 \le C\|f\|_1$. The situation for Θ_5 becomes

$$\begin{aligned}
\theta_5 \le &\int_{\overline{I}_k\times I_k} \sup_{n\ge 2^k} \left| \int_{I_k\times I_k} f(x^1,\, x^2) \left[\frac{\alpha_a(1-\alpha_a)}{n^{\alpha_a}} \sum_{j=2^k}^{2^B-2} \gamma(j) \right.\right. \\
&\times \left.\left.\left| D_{2^B}(y^1+x^1) K_j(y^2+x^2) \right| \right] d\lambda(x^1,\, x^2) \right| d\lambda(y^1,\, y^2) \\
&+ \int_{I_k\times\overline{I}_k} \sup_{n\ge 2^k} \left| \int_{I_k\times I_k} f(x^1,\, x^2) \left[\frac{\alpha_a(1-\alpha_a)}{n^{\alpha_a}} \sum_{j=2^k}^{2^B-2} \gamma(j) \right.\right. \\
&\times \left.\left.\left| D_{2^B}(y^1+x^1) K_j(y^2+x^2) \right| \right] d\lambda(x^1,\, x^2) \right| d\lambda(y^1,\, y^2) \\
&+ \int_{\overline{I}_k\times\overline{I}_k} \sup_{n\ge 2^k} \left| \int_{I_k\times I_k} f(x^1,\, x^2) \left[\frac{\alpha_a(1-\alpha_a)}{n^{\alpha_a}} \sum_{j=2^B}^{2^B-2} \gamma(j) \right.\right.
\end{aligned}$$

$$\times \left| D_{2^B}(y^1+x^1)K_j(y^2+x^2) \right| \Bigg] d\lambda(x^1,\, x^2) \Bigg| d\lambda(y^1,\, y^2)$$

$$\begin{aligned}
&\leq \int_{I_k\times I_k} \left| f(x^1,\, x^2) \right| \sup_{n\geq 2^k} \int_{\overline{I}_k\times I_k} \Bigg| \Bigg[\frac{\alpha_a(1-\alpha_a)}{n^{\alpha_a}} \sum_{j=1}^{n} \gamma(j) \\
&\quad \times \left| D_{2^B}(y^1+x^1) \sup_{j\geq 2^k} \left| K_j(y^2+x^2) \right| \right| \Bigg] d\lambda(y^1,\, y^2) \Bigg| d\lambda(x^1,\, x^2) \\
&\quad + \int_{I_k\times I_k} \left| f(x^1,\, x^2) \right| \sup_{n\geq 2^k} \int_{I_k\times \overline{I}_k} \Bigg| \Bigg[\frac{\alpha_a(1-\alpha_a)}{n^{\alpha_a}} \sum_{j=1}^{n} \gamma(j) \\
&\quad \times \left| D_{2^B}(y^1+x^1) \sup_{j\geq 2^k} \left| K_j(y^2+x^2) \right| \right| \Bigg] d\lambda(y^1,\, y^2) \Bigg| d\lambda(x^1,\, x^2) \\
&\quad + \int_{I_k\times I_k} \left| f(x^1,\, x^2) \right| \sup_{n\geq 2^k} \int_{\overline{I}_k\times \overline{I}_k} \Bigg| \Bigg[\frac{\alpha_a(1-\alpha_a)}{n^{\alpha_a}} \sum_{j=1}^{n} \gamma(j) \\
&\quad \times \left| D_{2^B}(y^1+x^1) \sup_{j\geq 2^k} \left| K_j(y^2+x^2) \right| \right| \Bigg] d\lambda(y^1,\, y^2) \Bigg| d\lambda(x^1,\, x^2) \\
&=: N_1 + N_2 + N_3.
\end{aligned}$$

By using [12, Lemma 3] we have

$$\int_{\overline{I}_k} \sup_{j\geq 2^k} \left| K_j \right| \leq C.$$

Hence N_2 becomes

$$\begin{aligned}
N_2 &\leq \int_{I_k\times I_k} \left| f(x^1,\, x^2) \right| \sup_{n\geq 2^k} \int_{\overline{I}_k\times \overline{I}_k} \Bigg| \Bigg[\frac{\alpha_a(1-\alpha_a)}{n^{\alpha_a}} \sum_{j=1}^{n} \gamma(j) \\
&\quad \times \left| D_{2^B}(y^1+x^1) \sup_{j\geq 2^k} \left| K_j(y^2+x^2) \right| \right| \Bigg] d\lambda(y^1,\, y^2) \Bigg| d\lambda(x^1,\, x^2) \\
&\leq C \int_{I_k\times I_k} \left| f(x^1,\, x^2) \right| d\lambda(x^1,\, x^2) \sup_{n\geq 2^k} \left(\int_{I_k} \left| D_{2^B}(y^1+x^1) \right| d\lambda(y^1) \right) \\
&\leq C \int_{I_k\times I_k} \left| f(x^1,\, x^2) \right| d\lambda(x^1,\, x^2) \\
&\leq C \left\| f \right\|_1 . \qquad (2.17)
\end{aligned}$$

Analogous with the proof of N_2, we have $N_3 \leq C \left\| f \right\|_1$. The situation for N_1 becomes

$$N_1 \leq \int_{I_k\times I_k} \left| f(x^1,\, x^2) \right| d\lambda(x^1, x^2) \sup_{n\geq 2^k} \int_{\overline{I}_k} \frac{\alpha_a(1-\alpha_a)}{n^{\alpha_a}}$$

$$\times \sum_{j=1}^{n} \gamma(j) \left| D_{2^B}(y^1 + x^1) \right| d\lambda(y^1) \sup_{j \geq 2^k} \int_{I_k} \left| K_j(y^2 + x^2) \right| d\lambda(y^2) \Big|$$

$$\leq C \int_{I_k \times I_k} \left| f(x^1,\, x^2) \right| d\lambda(x^1, x^2) \sup_{n \geq 2^k} \int_{\overline{I}_k} \frac{\alpha_a(1 - \alpha_a)}{n^{\alpha_a}}$$

$$\times \sum_{j=1}^{n} \gamma(j) \left| D_{2^B}(y^1 + x^1) \right| d\lambda(y^1) \leq C \int_{I_k \times I_k} \left| f(x^1,\, x^2) d\lambda(x^1, x^2) \right|$$

$$= C \left\| f \right\|_1 . \tag{2.18}$$

Therefore,

$$\int_{\overline{I_k \times I_k}} \sup_{n \geq 2^k, a \in \mathbb{N}} \left| t_n^{\alpha_a} f \right| d\lambda$$

$$= \int_{\overline{I_k \times I_k}} \sup_{n \geq 2^k, a \in \mathbb{N}} \left| \int_{I_k \times I_k} f(x^1, x^2) \Omega_n^{\alpha_a, 1}(y^1 + x^1, y^2 + x^2) d\lambda(x^1, x^2) \right| d\lambda(y^1, y^2)$$

$$\leq \int_{I_k \times I_k} \left| f(x^1, x^2) \right| d\lambda(x^1, x^2) \int_{\overline{I_k \times I_k}} \sup_{n \geq 2^k, a \in \mathbb{N}} \left| \Omega_n^{\alpha_a, 1}(y^1 + x^1, y^2 + x^2) \right| d\lambda(y^1, y^2)$$

$$\leq C \int_{I_k \times I_k} \left| f(x^1, x^2) \right| d\lambda(x^1, x^2)$$

$$\leq C \left\| f \right\|_1 .$$

□

Define the maximal operator

$$t_*^{\alpha_a} f := \sup_{n, a \in \mathbb{N}} \left| t_n^{\alpha_a, 1} f \right| = \sup_{n, a \in \mathbb{N}} \left| f * \Omega_n^{\alpha_a, 1}(x^1,\, x^2) \right| .$$

Corollary 2.2. *Let $\lambda > 0$ and $f \in L^1(\mathbb{I}^2)$, $g \in L^\infty(\mathbb{I}^2)$. Then,*

$$\left\| t_n^{\alpha_a, 1} f \right\|_1 \leq C \left\| f \right\|_1$$

and

$$\left\| t_n^{\alpha_a, 1} g \right\|_\infty \leq C \left\| g \right\|_\infty .$$

Proof. The proof is a direct consequence of Lemma 2.1. Besides, from the fact that L^1-norms of the Fejér kernels and also the Dirichlet kernels with indices of the form 2^B are uniformly bounded, we get

$$\left\| \Omega_n^{\alpha_a, 1}(x^1,\, x^2) \right\|_1 \leq \frac{\left\| D_{2^B}(x^1) \right\|_1 \left\| D_{2^B}(x^2) \right\|_1}{A_n^{\alpha_a}} \sum_{j=0}^{2^B - 1} A_{n^{(B)} + j}^{\alpha_a - 1}$$

$$+ \frac{\alpha_a(1 - \alpha_a)}{n^{\alpha_a}} \sum_{j=0}^{2^k - 1} \gamma(j) \Big[\left\| D_{2^B}(x^1) \right\|_1 \left\| K_j(x^2) \right\|_1$$

$$+ \left\| D_{2^B}(x^2) \right\|_1 \left\| K_j(x^1) \right\|_1 + \left\| \Omega_j(x^1,\, x^2) \right\|_1 \Big]$$

$$
\begin{aligned}
&+\frac{\alpha_a(1-\alpha_a)}{n^{\alpha_a}}\sum_{j=2^k}^{2^B-2}\gamma(j)\Bigg[\left\|D_{2^B}(x^1)\right\|_1\left\|K_j(x^2)\right\|_1\\
&+\left\|D_{2^B}(x^2)\right\|_1\left\|K_j(x^1)\right\|_1+\left\|\Omega_j(x^1,\,x^2)\right\|_1\Bigg]\\
&+\frac{A_n^{\alpha_a-1}}{A_n^{\alpha_a}}\Bigg[\left\|(2^B)D_{2^B}(x^1)\right\|_1\left\|K_{2^B-1}(x^2)\right\|_1\\
&+\left\|2^BD_{2^B}(x^2)\right\|_1\left\|K_{2^B-1}(x^1)\right\|_1+\left\|(2^B)\Omega_{2^B-1}(x^1,x^2)\right\|_1\Bigg]\\
\leq\,& C. \qquad (2.19)
\end{aligned}
$$

Thus,

$$\left\|t_n^{\alpha_a,1}f\right\|_1\leq C\left\|f\right\|_1. \qquad (2.20)$$

Similarly,

$$\left\|t_n^{\alpha_a,1}g\right\|_\infty\leq C\left\|g\right\|_\infty.$$

Thus, the corollary follows. □

Lemma 2.3. *Let* $1>\alpha_n>0$. *Then,*

$$\left|\Omega_n^{\alpha_n}(x^1,x^2)\right|\leq\tilde{\Omega}_n^{\alpha_n}(x^1,x^2), \qquad (2.21)$$

where

$$
\begin{aligned}
\tilde{\Omega}_n^{\alpha_n}(x^1,\,x^2):=&\left|\Omega_n^{\alpha_a,1}(x^1,\,x^2)\right|+\frac{A_{n^{(B-1)}}^{\alpha_n}}{A_n^{\alpha_a}}\Bigg[D_{2^B}(x^1)D_{2^B}(x^2)\Bigg]\\
&+\frac{\alpha_n(1-\alpha_n)}{n^{\alpha_n}}\sum_{j=0}^{2^k-1}\gamma(j)\Bigg[\left|D_{2^B}(x^1)K_j(x^2)\right|\\
&+\left|D_{2^B}(x^2)K_j(x^1)\right|+\left|\Omega_j(x^1,\,x^2)\right|\Bigg]\\
&+\frac{\alpha_n(1-\alpha_n)}{n^{\alpha_n}}\sum_{j=2^k}^{n^{(B-1)}}\gamma(j)\Bigg[\left|D_{2^B}(x^1)K_j(x^2)\right|\\
&+\left|D_{2^B}(x^2)K_j(x^1)\right|+\left|\Omega_j(x^1,\,x^2)\right|\Bigg]\\
&+\frac{A_n^{\alpha_n-1}}{A_n^{\alpha_n}}\Bigg[(n^{(B-1)})D_{2^B}(x^1)\left|K_{n^{(B-1)}}(x^2)\right|\\
&+(n^{(B-1)})D_{2^B}(x^2)\left|K_{n^{(B-1)}}(x^1)\right|\\
&+(n^{(B-1)})\left|\Omega_{n^{(B-1)}}(x^1,x^2)\right|\Bigg]. \qquad (2.22)
\end{aligned}
$$

Proof. Recall that $B=|n|$ and $n^{(B)}=\sum_{j=0}^{B-1}n_j2^j$, $n^{(B)}+2^B=n$. We have

$$\begin{aligned}\Omega_n^{\alpha_n}(x^1,x^2)=&\frac{1}{A_n^{\alpha_n}}\sum_{j=0}^{2^B-1}A_{2^B+n^{(B)}-j}^{\alpha_n-1}D_{j,j}(x^1,x^2)\\&+\frac{1}{A_n^{\alpha_n}}\sum_{j=2^B}^{n}A_{2^B+n^{(B)}-j}^{\alpha_n-1}D_{j,j}(x^1,x^2)\\=&\Omega_n^{\alpha_a,1}(x^1,x^2)+\frac{1}{A_n^{\alpha_n}}\sum_{j=1}^{n^{(B-1)}}A_{n^{(B-1)}-j}^{\alpha_n-1}D_{j+2^B,j+2^B}(x^1,x^2).\end{aligned}\tag{2.23}$$

Since (see Lemma 1.1)

$$D_{j+2^B}(x)=D_{2^B}(x)+\psi_{2^B-1}D_j(x),\tag{2.24}$$

we have

$$\begin{aligned}D_{j+2^B,j+2^B}(x^1,x^2)=&D_{2^B}(x^1)D_{2^B}(x^2)\\&+\psi^1_{2^B-1}(x^1)D_{2^B}(x^2)D_j(x^1)\\&+\psi^2_{2^B-1}(x^2)D_{2^B}(x^1)D_j(x^2)\\&+\psi^1_{2^B-1}(x^1)\psi^2_{2^B-1}(x^2)D_{j,j}(x^1,x^2).\end{aligned}\tag{2.25}$$

Thus,

$$\begin{aligned}&\frac{1}{A_n^{\alpha_n}}\sum_{j=1}^{n^{(B-1)}}A_{n^{(B-1)}-j}^{\alpha_n-1}D_{j+2^B,j+2^B}(x^1,x^2)\\
&=\frac{D_{2^B}(x^1)D_{2^B}(x^2)}{A_n^{\alpha_n}}\sum_{j=1}^{n^{(B-1)}}A_{n^{(B)}+j}^{\alpha_n-1}\\
&\quad+\frac{\psi_{2^B-1}(x^1)D_{2^B}(x^2)}{A_n^{\alpha_n}}\sum_{j=1}^{n^{(B-1)}}A_{n^{(B)}+j}^{\alpha_n-1}D_j(x^1)+\frac{\psi_{2^B-1}(x^2)D_{2^B}(x^1)}{A_n^{\alpha_n}}\\
&\quad\times\sum_{j=1}^{n^{(B-1)}}A_{n^{(B)}+j}^{\alpha_n-1}D_j(x^2)+\frac{\psi_{2^B-1}(x^1)\psi_{2^B-1}(x^2)}{A_n^{\alpha_n}}\sum_{j=1}^{n^{(B-1)}}A_{n^{(B)}+j}^{\alpha_n-1}D_{j,j}(x^1,x^2)\\
&=\frac{D_{2^B}(x^1)D_{2^B}(x^2)}{A_n^{\alpha_n}}\sum_{j=1}^{n^{(B-1)}}A_{n^{(B)}+j}^{\alpha_n-1}+\frac{\psi_{2^B-1}(x^1)D_{2^B}(x^2)}{A_n^{\alpha_n}}\\
&\quad\times\sum_{j=1}^{n^{(B-1)}}A_{n^{(B)}+j}^{\alpha_n-1}D_j(x^1)+\frac{\psi_{2^B-1}(x^2)D_{2^B}(x^1)}{A_n^{\alpha_n}}\sum_{j=1}^{n^{(B-1)}}A_{n^{(B)}+j}^{\alpha_n-1}D_j(x^2)\\
&\quad+\frac{\psi_{2^B-1}(x^1)\psi_{2^B-1}(x^2)}{A_n^{\alpha_n}}\sum_{j=1}^{n^{(B-1)}}A_{n^{(B)}+j}^{\alpha_n-1}D_{j,j}(x^1,x^2).\end{aligned}\tag{2.26}$$

Therefore, from (2.26) and with the help of Abel's transform with analogous steps to the proof of Lemma 2.1, we get

$$\begin{aligned}
&\frac{1}{A_n^{\alpha_a}}\Big|\sum_{j=1}^{n^{(B-1)}} A_{n^{(B-1)}-j}^{\alpha_a-1} D_{j+2^B,\,j+2^B}(x^1,x^2)\Big| \\
&\leq \frac{A_{n^{(B-1)}}^{\alpha_n}}{A_n^{\alpha_a}}\Bigg[D_{2^B}(x^1)D_{2^B}(x^2)\Bigg] \\
&\quad + \frac{\alpha_a(1-\alpha_n)}{n^{\alpha_n}}\sum_{j=0}^{2^k-1}\gamma(j)\Bigg[\Bigg|D_{2^B}(x^1)K_j(x^2)\Bigg| + \Bigg|D_{2^B}(x^2)K_j(x^1)\Bigg| \\
&\quad + \Bigg|\Omega_j(x^1,\,x^2)\Bigg|\Bigg] + \frac{\alpha_n(1-\alpha_n)}{n^{\alpha_n}}\sum_{j=2^k}^{n^{(B-1)}}\gamma(j)\Bigg[\Bigg|D_{2^B}(x^1)K_j(x^2)\Bigg| \\
&\quad + \Bigg|D_{2^B}(x^2)K_j(x^1)\Bigg| + \Bigg|\Omega_j(x^1,\,x^2)\Bigg|\Bigg] + \frac{A_n^{\alpha_n-1}}{A_n^{\alpha_n}}\Bigg[(n^{(B-1)})D_{2^B}(x^1) \\
&\quad \times \Big|K_{n^{(B-1)}}(x^2)\Big| + n_B(n^{(B-1)})D_{2^B}(x^2)\Big|K_{n^{(B-1)}}(x^1)\Big| \\
&\quad + (n^{(B-1)})\Big|\Omega_{n^{(B-1)}}(x^1,x^2)\Big|\Bigg]. \qquad (2.27)
\end{aligned}$$

Thus, (2.27) implies that

$$\Big|\Omega_n^{\alpha_n}(x^1,\,x^2)\Big| \leq \tilde{\Omega}_n^{\alpha_n}(x^1,\,x^2). \qquad (2.28)$$

□

Define the maximal operators

$$\sigma_*^{\alpha_n} f := \sup_{n\in\mathbb{N}_{\alpha,q}} |\sigma_n^{\alpha_n} f| = \sup_{n\in\mathbb{N}_{\alpha,q}} \Big|f * \Omega_n^{\alpha_n}(x^1,\,x^2)\Big|$$

and

$$\tilde{\sigma}_*^{\alpha_n} f := \sup_{n\in\mathbb{N}_{\alpha,q}} |\tilde{\sigma}_n^{\alpha_n} f| = \sup_{n\in\mathbb{N}_{\alpha,q}} \Big|f * \tilde{\Omega}_n^{\alpha_n}(x^1,\,x^2)\Big|.$$

We prove Lemma 2.4, which is essential tool for the proof of Theorem 2.5. The way we get this is by the investigation of kernel functions, its maximal function on the 2-adic group. This is the very base for the proof of main results.

Lemma 2.4. *Let $1 > \alpha_n > 0$, $f \in L^1(\mathbb{I}\times\mathbb{I})$ such that*

$$\operatorname{supp} f \subset I_k(u^1)\times I_k(u^2),\ \int_{I_k(u^1)\times I_k(u^2)} f\,d\lambda = 0$$

for some 2-adic rectangle $I_k(u^1)\times I_k(u^2) \subset \mathbb{I}\times\mathbb{I}$. Then, we have

$$\int_{\overline{I_k(u^1)\times I_k(u^2)}} \sup_{n\in\mathbb{N}_{\alpha,q}} |\tilde{\sigma}_n^{\alpha_n} f|\,d\lambda \leq C_q\,\|f\|_1\,. \qquad (2.29)$$

Proof. From Lemma 2.3, we have

$$
\begin{aligned}
\left|\tilde{\Omega}_n^{\alpha_n}(x^1,\, x^2)\right| =&\left|\Omega_n^{\alpha_a,1}(x^1,\, x^2)\right| + \frac{A^{\alpha_n}_{n^{(B-1)}}}{A^{\alpha_n}_{n^{(B)}}} D_{2^B}(x^1) D_{2^B}(x^2)\\
&+ \frac{\alpha_n(1-\alpha_n)}{n^{\alpha_n}} \sum_{j=0}^{2^k-1} \gamma(j)\left|D_{2^B}(x^1)K_j(x^2)\right|\\
&+ \frac{\alpha_n(1-\alpha_n)}{n^{\alpha_n}} \sum_{j=0}^{2^k-1} \gamma(j)\left|D_{2^B}(x^2)K_j(x^1)\right|\\
&+ \frac{\alpha_n(1-\alpha_n)}{n^{\alpha_n}} \sum_{j=0}^{2^k-1} \gamma(j)\left|\Omega_j(x^1,\, x^2)\right|\\
&+ \frac{\alpha_n(1-\alpha_n)}{n^{\alpha_n}} \sum_{j=2^k}^{n^{(j-1)}} \gamma(j)\left|D_{2^B}(x^1)K_j(x^2)\right|\\
&+ \frac{\alpha_n(1-\alpha_n)}{n^{\alpha_n}} \sum_{j=2^k}^{n^{(j-1)}} \gamma(j)\left|D_{2^B}(x^2)K_j(x^1)\right|\\
&+ \frac{\alpha_n(1-\alpha_n)}{n^{\alpha_n}} \sum_{j=2^k}^{n^{(j-1)}} \gamma(j)\left|\Omega_j(x^1,\, x^2)\right|\\
&+ \frac{A_n^{\alpha_n-1}}{A_n^{\alpha_n}}(n^{(t-1)}) D_{2^t}(x^1)\left|K_{n^{(t-1)}}(x^2)\right|\\
&+ (n^{(t-1)}) D_{2^t}(x^2)\left|K_{n^{(t-1)}}(x^1)\right|\\
&+ (n^{(t-1)})\left|\Omega_{n^{(t-1)}}(x^1, x^2)\right|\\
:=& \sum_{r=1}^{11} \Phi_r.
\end{aligned}
\tag{2.30}
$$

From Lemma 2.1, the estimation for integral

$$
\begin{aligned}
&\int_{\overline{I_k(u^1)\times I_k(u^2)}} \sup_{n\in\mathbb{N}} \left| \int_{I_k(u^1)\times I_k(u^2)} f(x^1,x^2)\Phi_1(y^1+x^1, y^2+x^2)d\mu(x^1,x^2)\right| d\mu(y^1,y^2)\\
&\leq C\left\|f\right\|_1 .
\end{aligned}
\tag{2.31}
$$

Since $f * D_t = 0$ for $t < s \leq k$ because of the A_k measurablity of $D_{2^t,2^t}$ and $\int f = 0$.
Besides, $D_{2^t,2^t}(y^1+x^1, y^2+x^2) = 0$, for $s > k, (y^1+x^1, y^2+x^2) \notin \overline{I_k \times I_k}$, then

$$
\int_{\overline{I_k(u^1)\times I_k(u^2)}} \sup_{n\in\mathbb{N}} \left| \int_{I_k(u^1)\times I_k(u^2)} f(x^1,x^2)\Phi_2(y^1+x^1, y^2+x^2)d\mu(x^1,x^2)\right| d\mu(y^1,y^2)
$$

$$= 0. \tag{2.32}$$

Likewise in the proof of Lemma 2.1, for $r = 3, 4, 9, 10, 11$, we have the estimation for the integral

$$\int_{I_k(u^1)\times I_k(u^2)} \sup_{n\in\mathbb{N}} \left| \int_{I_k(u^1)\times I_k(u^2)} f(x^1, x^2)\Phi_r(y^1 + x^1, y^2 + x^2) d\mu(x^1, x^2) \right| d\mu(y^1, y^2)$$
$$\leq C \|f\|_1 . \tag{2.33}$$

Since (see Lemma 1.2), we note the estimation for integral

$$\int_{I_k(u^1)\times I_k(u^2)} \sup_{j\geq 2^k} |\Omega_j| \leq C < \infty$$

and the fact that

$$\frac{\alpha_n(1-\alpha_n)}{n^{\alpha_n}} \sum_{j=2^k}^{n^{(j-1)}} \gamma(j) \leq \frac{\alpha_n(1-\alpha_n)}{n^{\alpha_n}} \sum_{j=1}^{n^{(j-1)}} \gamma(j)$$
$$\leq C \frac{(n^{(j-1)})^{\alpha_n}}{(n)^{\alpha_n}} \leq C \frac{(2^j)^{\alpha_n}}{(n)^{\alpha_n}} \leq C_q \tag{2.34}$$

we have the estimation for integral

$$\int_{I_k(u^1)\times I_k(u^2)} \sup_{n\in\mathbb{N}} \left| \int_{I_k(u^1)\times I_k(u^2)} f(x^1, x^2)\Phi_8(y^1 + x^1, y^2 + x^2) d\mu(x^1, x^2) \right| d\mu(y^1, y^2)$$
$$\leq C_q \|f\|_1 . \tag{2.35}$$

Similarly, we get

$$\int_{I_k(u^1)\times I_k(u^2)} \sup_{n\in\mathbb{N}} \left| \int_{I_k(u^1)\times I_k(u^2)} f(x^1, x^2) \Big[\Phi_6(y^1 + x^1, y^2 + x^2)\right.$$
$$\left. + \Phi_7(y^1 + x^1, y^2 + x^2)\Big] d\mu(x^1, x^2) \right| d\mu(y^1, y^2)$$
$$\leq C_q \|f\|_1 + C_q \|f\|_1 \leq C_q \|f\|_1 . \tag{2.36}$$

Besides,

$$\int_{I_k(u^1)\times I_k(u^2)} \sup_{n\in\mathbb{N}} \left| \int_{I_k(u^1)\times I_k(u^2)} f(x^1, x^2)\Phi_5(y^1 + x^1, y^2 + x^2) d\mu(x^1, x^2) \right| d\mu(y^1, y^2)$$
$$= 0, \tag{2.37}$$

as similar to the proof of Lemma 2.1, for $j < 2^k$. Thus, (2.31)–(2.36) imply the lemma holds true. □

In the following theorem, we prove that $\sigma_*^{\alpha_n}$, $\tilde{\sigma}_*{}^{\alpha_n}$ are of strong type (L^∞, L^∞) and operators $\sigma_*^{\alpha_n}$, $\tilde{\sigma}_*{}^{\alpha_n}$ are of weak type (L^1, L^1).

Theorem 2.5. *Let $\lambda > 0$ and $f \in L^1(\mathbb{I}^2)$, $g \in L^\infty(\mathbb{I}^2)$. Then,*

$$\|\tilde{\sigma_*}^{\alpha_n} g\|_\infty \le C_q \|g\|_\infty, \; \|\sigma_*^{\alpha_n} g\|_\infty \le C_q \|g\|_\infty$$

and

$$\mu(\{x : |(\sigma_*^{\alpha_n} f)(x)| > \lambda\}) \le \frac{C_q \|f\|_1}{\lambda},$$

$$\mu(\{x : |(\sigma_*^{\alpha_n} f)(x)| > \lambda\}) \le \frac{C_q \|f\|_1}{\lambda}.$$

Proof. Analogous with methods of proving Corollary 2.2, we can get upper estimate for the L^1-norm of $\tilde{\Omega}_n^{\alpha_n}$. That is,

$$\begin{aligned}
||\tilde{\Omega}_n^{\alpha_n}(x^1,\, x^2)||_1 \le{} & ||\Omega_n^{\alpha_a,1}(x^1,\, x^2)||_1 + \frac{A^{\alpha_n}_{n(B-1)}}{A^{\alpha_n}_{n_{(j)}}} ||D_{2^B}(x^1)||_1 ||D_{2^B}(x^2)||_1 \\
& + \frac{\alpha_n(1-\alpha_n)}{n^{\alpha_n}} \sum_{j=0}^{2^k-1} \gamma(j) \Bigg[||D_{2^B}(x^1)||_1 ||K_j(x^2)||_1 \\
& + n_B ||D_{2^B}(x^2)||_1 ||K_j(x^1)||_1 + ||\Omega_j(x^1,\, x^2)||_1 \Bigg] \\
& + \frac{\alpha_n(1-\alpha_n)}{n^{\alpha_n}} \sum_{j=2^k}^{n_{(j-1)}} \gamma(j) \Bigg[||D_{2^B}(x^1)||_1 ||K_j(x^2)||_1 \\
& + ||D_{2^B}(x^2)||_1 ||K_j(x^1)||_1 + ||\Omega_j(x^1,\, x^2)||_1 \Bigg] \\
& + \frac{A_n^{\alpha_n-1}}{A_n^{\alpha_n}} \Bigg[(n^{(B-1)}) \times ||D_{2^B}(x^1)||_1 ||K_{n^{(B-1)}}(x^2)||_1 \\
& + (n_B n^{(B-1)}) ||D_{2^B}(x^2)||_1 ||K_{n^{(B-1)}}(x^1)||_1 \\
& + n^{(B-1)}) ||\Omega_{n^{(B-1)}}(x^1, x^2)||_1 \Bigg] \\
\le{} & C + C\frac{(2^k)^{\alpha_n}}{(n)^{\alpha_n}} \le C_q
\end{aligned} \tag{2.38}$$

since $n \in \mathbb{N}_{\alpha,q}$. This, implies that $\tilde{\sigma_*}^{\alpha_n}$ is of strong type (L^∞, L^∞). By the fact that $|\Omega_n^{\alpha_n}(x^1, x^2)| \le \tilde{\Omega}_n^{\alpha_n}(x^1, x^2)$, we have the result that $\sigma_*^{\alpha_n}$ is of strong type (L^∞, L^∞). To proof $\tilde{\sigma_*}^{\alpha_n}$ is of weak type (L^1, L^1), we follow the applications of Calderon–Zygmund decomposition lemma in [11]. That is, let $f \in L^1(\mathbb{I}^2)$ and $\|f\|_1 < \eta$. Then there is a decomposition

$$f = f_0 + \sum_{j=1}^{\infty} f_j$$

such that

$$\|f_0\|_\infty \le C\eta, \; \|f_0\|_1 \le C\|f\|_1$$

and

$$\mathbb{I}^j \times \mathbb{I}^j = I_{k_j}(u^{j,1}) \times I_{k_j}(u^{j,2})$$

are disjoint 2-adic rectangles for which

$$\operatorname{supp} f_j \subset \mathbb{I}^j \times \mathbb{I}^j,\ \int_{\mathbb{I}^j \times \mathbb{I}^j} f_j d\lambda = 0, \quad (u^{j,1}, u^{j,2}) \in \mathbb{I} \times \mathbb{I},\ k_j \in \mathbb{N},\ j \in \mathbb{P},$$
$$\lambda(F) \leq \frac{C \left\| f_1 \right\|}{\eta}, \quad \textit{where} \quad F = \cup_{j=1}^{\infty} (\mathbb{I}^j \times \mathbb{I}^j). \tag{2.39}$$

By the σ-sublinearity of the maximal operator with an appropriate constant C_q and since $\tilde{\sigma}_*^{\alpha_n}$ is of strong type (L^∞, L^∞), we have $\left\|\tilde{\sigma}_* f_0\right\|_\infty \leq C_q \left\|f_0\right\|_\infty \leq C_q \eta$.

Since

$$\begin{aligned}
\lambda(\tilde{\sigma}_*^{\alpha_n} f > 2C_q\eta) &\leq \lambda(\tilde{\sigma}_*^{\alpha_n} f_0 > C_q\eta) + \lambda(\tilde{\sigma}_*^{\alpha}(\sum_{j=1}^{\infty} f_j) > C_q\eta) \\
&\leq 0 + \lambda(\tilde{\sigma_*}^{\alpha_n}(\sum_{j=1}^{\infty} f_j) > C_q\eta) \\
&\leq \lambda(F) + \lambda\Big[\bar{F} \cap \Big(\tilde{\sigma_*}^{\alpha_n}(\sum_{j=1}^{\infty} f_j) > C_q\eta\Big)\Big] \\
&\leq \frac{C\left\|f\right\|_1}{\eta} + \frac{C_q}{\eta} \sum_{j=1}^{\infty} \int_{\overline{\mathbb{I}^j \times \mathbb{I}^j}} \tilde{\sigma_*}^{\alpha_n} f_j d\lambda \\
&\leq \frac{C\left\|f\right\|_1}{\eta} \\
&\quad + \frac{C_q}{\eta} \sum_{j=1}^{\infty} \int_{\overline{I_{k_j}(u^j) \times I_{k_{k_j}}(u^j)}} \sup_{n \in \mathbb{N}_{\alpha_n, q}} \Bigg| \int_{I_{k_j}(u^j) \times I_{k_{k_j}}(u^j)} \Big| f_j(x^1, x^2) \Big| \\
&\quad \times \Big| \tilde{\Omega}_n^{\alpha_n}(y^1 + x^1, y^2 + x^2) \Big| d\lambda(x^1, x^2) \Bigg| d\lambda(y^1, y^2),
\end{aligned} \tag{2.40}$$

and with the help of Lemma 2.4 (that is, $\tilde{\sigma}_*^{\alpha_n}$ is quasi-local), we can estimate the boundedness of

$$\begin{aligned}
&\sum_{j=1}^{\infty} \int_{\overline{I_{k_j}(u^j) \times I_{k_{k_j}}(u^j)}} \sup_{n \in \mathbb{N}_{\alpha_n, q}} \Bigg| \int_{I_{k_j}(u^j) \times I_{k_{k_j}}(u^j)} \Big| f_j(x^1, x^2) \Big| \\
&\times \Big| \tilde{\Omega}_n^{\alpha_n}(y^1 + x^1, y^2 + x^2) \Big| d\lambda(x^1, x^2) \Bigg| d\lambda(y^1, y^2) \\
&\quad \leq C_q \left\| f_j \right\|_1 .
\end{aligned} \tag{2.41}$$

Hence, the operator $\tilde{\sigma}_*^{\alpha_n}$ is of weak type (L^1, L^1). As a result $\sigma_*^{\alpha_n}$ is of weak type (L^1, L^1). This completes the proof of Theorem 2.5. $\square$

Theorem 2.6. *Let $f \in L^1(\mathbb{I}\times\mathbb{I})$ and $1 > \alpha_n > 0$. Then,*

$$\alpha_n^{\alpha_n} f \to f \quad a.e. \quad for \quad n \to \infty, n \in \mathbb{N}_{\alpha,q}.$$

Proof. By the standard consequence of the fact that maximal operator $\sigma_*^{\alpha_n}$ is of weak type (L^1, L^1), the fact that it holds for each two-dimensional character polynomial of the group of 2-adic group (linear combinations of $\psi_i(x^1)\psi_i(x^2)$) and the fact that set of two-dimensional character polynomials is dense in $L^1(\mathbb{I}\times\mathbb{I})$ in which this density property obtained from the fact that one-dimensional kernel function D_{2^n} is equal to either 2^n or zero, the proof of this theorem follows. □

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