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RATE OF CONVERGENCE OF A COMPOSITION OPERATOR

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Abstract. Very recently, some composition operators were introduced by means of Laguerre polynomials. Here we extend the results of Gupta by establishing some of the direct estimates for such operators on $[0,\infty)$.

Moreover, we introduce a new version of these operators that preserves linear functions and in comparison to their original version, has better approximation outcomes on $[\alpha+2,\infty)(\alpha>-1)$.

1. A composition operator

Sucu, Icoz, and Varma [10] proposed an important operator based on Laguerre polynomials for $x\geq 0$ and $\alpha>-1$ as

$$(G_n^\alpha f)(t)=e^{-nt/2}2^{-\alpha-1}\sum_{\ell=0}^{\infty}2^{-\ell}L_\ell^\alpha\left(\frac{-nt}{2}\right)f\left(\frac{\ell}{n}\right),$$

where

$$L_\ell^\alpha(-t):=\sum_{s=0}^{\ell}\frac{(\alpha+\ell)!}{(\ell-s)!\,(\alpha+s)!\,s!}t^s,\quad \alpha>-1.$$

The Durrmeyer variant of Szász-Mirakyan operator (see [8] and references therein) is defined by

$$(S_nf)(t)=n\sum_{\nu=0}^{\infty}s_\nu(nt)\int_0^\infty s_\nu(nu)f(u)du,$$

where $s_\nu(r)=e^{-r}r^\nu/\nu!$.

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Very recently, Gupta [6] captured a new operator by composition of G_n^α and S_n, as

$$(Q_n^\alpha f)(t) := (S_n \circ G_n^\alpha f)(t).$$

In [6], a term was inadvertently omitted, which we now correct as follows:

$$(Q_n^\alpha f)(t) = \sum_{\ell=0}^{\infty} S_{n,\ell}^\alpha(t) f\left(\frac{\ell}{n}\right), \tag{1.1}$$

where

$$S_{n,\ell}^\alpha(t) = \frac{e^{-nt}}{2^{\alpha+\ell}} \sum_{s=0}^{\ell} \frac{(\alpha+\ell)!}{(\ell-s)!(\alpha+s)!} \frac{1}{3^{s+1}} \; {}_1F_1\left(1+s;1;\frac{2nt}{3}\right).$$

For simplicity, we write the aforementioned sequence $Q = (Q_n^\alpha)$ and for $f \in Dom(Q) = \bigcap_{n\in\mathbb{N}} Dom(Q_n^\alpha)$, where $Dom(Q_n^\alpha)$ denotes the set of all functions $f : [0,\infty) \to \mathbb{R}$, for which the operators (1.1) are well defined.

Numerous integral and discrete operators of exponential and nonexponential types are known to exist; see, for instance, [2, 1, 3, 5, 7, 9] and so on. The operator provided in (1.1) is not of the exponential kind; rather, it is just a discrete operator. Furthermore, it is noted that these operators do not maintain linear functions. We examine asymptotic expansion in limiting and quantitative form in the current paper. Ultimately, we adjust these operators appropriately to maintain linear functions.

2. Moments

Lemma 2.1. *The moments of Q_n^α are given by*

$$\begin{aligned}
(Q_n^\alpha e_0)(t) &= 1,\\
(Q_n^\alpha e_1)(t) &= t + \frac{\alpha+2}{n},\\
(Q_n^\alpha e_2)(t) &= t^2 + \frac{10+6\alpha+\alpha^2+9nt+2\alpha nt}{n^2},\\
(Q_n^\alpha e_3)(t) &= t^3 + \frac{74+48\alpha+12\alpha^2+(\alpha^3+97+33\alpha+3\alpha^2)nt+(21+3\alpha)n^2t^2}{n^3},\\
(Q_n^\alpha e_4)(t) &= t^4 + \frac{1}{n^4}\Big[730+490\alpha+144\alpha^2+20\alpha^3+\alpha^4+1257nt+520\alpha nt\\
&\qquad + 78\alpha^2 nt + 4\alpha^3 nt + (403+96\alpha+6\alpha^2)n^2t^2 + 38n^3t^3 + 4\alpha n^3t^3\Big].
\end{aligned}$$

Proof. By moment generating function of the operator Q_n^α given in [6], we have

$$(Q_n^\alpha e^{Au})(t) = \frac{\left(2-e^{\frac{A}{n}}\right)^{-\alpha}}{(3-2e^{\frac{A}{n}})} \exp\left(nt\left(\frac{e^{\frac{A}{n}}-1}{3-2e^{\frac{A}{n}}}\right)\right).$$

Differentiation partially repeatedly with respect to A and substituting $A=0$, the moments can be obtained. $\square$

Lemma 2.2. *The central moments of* $\mu_{n,s}^{\alpha}(t) := (Q_n^{\alpha}(u-t)^s)(t), s = 0, 1, 2, \ldots,$ *are given by*

$$\begin{aligned}
&\mu_{n,0}^{\alpha}(t) = 1,\\
&\mu_{n,1}^{\alpha}(t) = \frac{\alpha+2}{n},\\
&\mu_{n,2}^{\alpha}(t) = \frac{10+6\alpha+\alpha^2+5nt}{n^2},\\
&\mu_{n,3}^{\alpha}(t) = \frac{74+48\alpha+12\alpha^2+\alpha^3+67nt+15\alpha nt}{n^3},\\
&\mu_{n,4}^{\alpha}(t) = \frac{730+490\alpha+144\alpha^2+20\alpha^3+\alpha^4+961nt+328\alpha nt+30\alpha^2 nt+75n^2t^2}{n^4}.
\end{aligned} \tag{2.1}$$

In view of the Cauchy–Schwarz–Bunyakovsky inequality, clearly note that the absolute moments satisfy

$$\left|\mu_{n,1}^{\alpha}(t)\right| \leq \sqrt{\mu_{n,2}^{\alpha}(t)}. \tag{2.2}$$

Moreover, for all $t \in [0, \infty)$, one can observe that

$$\mu_{n,r}^{\alpha}(t) = O\left(n^{-\left\lfloor \frac{r+1}{2} \right\rfloor}\right), \tag{2.3}$$

where $\lfloor m \rfloor$ stands for the integral part of m.

3. Asymptotic expansions

This section deals with the asymptotic formulae and other direct results.

Theorem 3.1. *Let us consider* $f \in Dom(Q) \bigcap L^{\infty}([0,\infty))$ *having first derivative at a fixed point* t. *Then*

$$(Q_n^{\alpha} f)(t) = f(t) + f'(t)\frac{\alpha+2}{n} + o\left(n^{-1}\right) \qquad (n \to \infty).$$

Proof. By Taylor's series expansion, with a bounded function $h_{\ell,n}(t)$ satisfying $\lim\limits_{\frac{\ell}{n}\to t} h_{\ell,n}(t) = 0$ and Lemma 2.2, we can write

$$\begin{aligned}
(Q_n^{\alpha} f)(t) &= \sum_{\ell=0}^{\infty} S_{n,\ell}^{\alpha}(t) f\left(\frac{\ell}{n}\right)\\
&= \sum_{\ell=0}^{\infty} S_{n,\ell}^{\alpha}(t)\left[f(t) + \left(\frac{\ell}{n} - t\right) f'(t) + h_{\ell,n}(t)\left(\frac{\ell}{n} - t\right)\right]\\
&= f(t) + f'(t)\,\mu_{n,1}^{\alpha}(t) + \sum_{\ell=0}^{\infty} S_{n,\ell}^{\alpha}(t) h_{\ell,n}(t)\left(\frac{\ell}{n} - t\right)\\
&:= f(t) + \frac{\alpha+2}{n} f'(t) + R.
\end{aligned}$$

We estimate the remainder term R; let $\varepsilon > 0$ be fixed. Since $h_{\ell,n}(t)$ is a bounded with $\lim\limits_{\ell/n\to t} h_{\ell,n}(t) = 0$, there exists $\delta > 0$ such that $|h_{\ell,n}(t)| \leq \varepsilon$ for each $|\frac{\ell}{n} - t| \leq \delta$.

Hence

$$R=\sum_{\left|\frac{\ell}{n}-t\right|\geq\delta}S_{n,\ell}^{\alpha}(t)\,h_{\ell,n}(t)\left(\frac{\ell}{n}-t\right)+\sum_{\left|\frac{\ell}{n}-t\right|<\delta}S_{n,\ell}^{\alpha}(t)\,h_{\ell,n}(t)\left(\frac{\ell}{n}-t\right)$$
$$:=R_1+R_{2.}$$

In view of (2.2), we obtain

$$|R_2|=\left|\sum_{\left|\frac{\ell}{n}-t\right|<\delta}S_{n,\ell}^{\alpha}(t)\,h_{\ell,n}(t)\left(\frac{\ell}{n}-t\right)\right|\leq\varepsilon\sum_{\left|\frac{\ell}{n}-t\right|<\delta}S_{n,\ell}^{\alpha}(t)\left|\frac{\ell}{n}-t\right|$$
$$\leq\varepsilon\sqrt{\mu_{n,2}^{\alpha}(t)}=\varepsilon\sqrt{\frac{10+6\alpha+\alpha^2+5nt}{n^2}}=o\left(n^{-1}\right)$$

for $n\to\infty$. Moreover, with a constant $B>0$ such that $|h_{\ell,n}(t)|\leq B$, we get

$$|R_1|\leq B\sum_{\left|\frac{\ell}{n}-t\right|\geq\delta}S_{n,\ell}^{\alpha}(x)\left|\frac{\ell}{n}-t\right|=o\left(n^{-1}\right)$$

as $n\to\infty$, and hence the claim follows. □

Theorem 3.2. *Let us take $f\in Dom(Q)\bigcap L^{\infty}([0,\infty))$, and suppose that the rth derivative of f exists at fixed t. Then the following asymptotic formula*

$$(Q_n^{\alpha}f)(t)=f(t)+\sum_{i=1}^{r}\frac{f^{(i)}(t)}{i!}\mu_{n,i}^{\alpha}(t)+o\left(n^{-\left\lfloor\frac{r+1}{2}\right\rfloor}\right)\qquad(n\to\infty),$$

holds true, where $\mu_{n,i}^{\alpha}$ is the ith order algebraic moment.

Proof. Since function is r times differentiable at a point t by the local Taylor's formula with a bounded function $h_{\ell,n}(t)$ such that $\lim\limits_{\frac{\ell}{n}\to t}h_{\ell,n}(t)=0$, we immediately have

$$(Q_n^{\alpha}f)(t)=\sum_{\ell=0}^{\infty}S_{n,\ell}^{\alpha}(t)f\left(\frac{\ell}{n}\right)$$
$$=\sum_{\ell=0}^{\infty}S_{n,\ell}^{\alpha}(t)\left(\sum_{i=0}^{r}\frac{f^{(i)}(t)}{i!}\left(\frac{\ell}{n}-t\right)^{i}+h_{\ell,n}(t)\left(\frac{\ell}{n}-t\right)^{r}\right)$$
$$=\sum_{\ell=0}^{\infty}S_{n,\ell}^{\alpha}(t)\sum_{i=0}^{r}\frac{f^{(i)}(t)}{i!}\left(\frac{\ell}{n}-t\right)^{i}+\sum_{\ell=0}^{\infty}S_{n,\ell}^{\alpha}(t)\,h_{\ell,n}(t)\left(\frac{\ell}{n}-t\right)^{r}$$
$$=:I_1+R.$$

Let us consider the term I_1. Then

$$\begin{aligned} I_1 &= \sum_{\ell=0}^{\infty} S_{n,\ell}^{\alpha}(t) \sum_{i=0}^{r} \frac{f^{(i)}(t)}{i!} \left(\frac{\ell}{n} - t\right)^i \\ &= f(t) + \sum_{\ell=0}^{\infty} S_{n,\ell}^{\alpha}(t) \sum_{i=1}^{r} \frac{f^{(i)}(t)}{i!} \left(\frac{\ell}{n} - t\right)^i \\ &= f(t) + \sum_{i=1}^{r} \frac{f^{(i)}(t)}{i!} \mu_{n,i}^{\alpha}(t). \end{aligned}$$

Now, we evaluate the term remainder term R.

Let $\varepsilon > 0$ be fixed. Since $h(y)$ is a bounded function such that $\lim_{\ell/n \to t} h_{\ell,n}(t) = 0$, there exists $\delta > 0$ such that $|h_{\ell,n}(t)| \leq \varepsilon$ for every $|\frac{\ell}{n} - t| \leq \delta$. We have

$$\begin{aligned} R &= \sum_{\left|\frac{\ell}{n} - t\right| \geq \delta} S_{n,\ell}^{\alpha}(t)\, h_{\ell,n}(t) \left(\frac{\ell}{n} - t\right)^r + \sum_{\left|\frac{\ell}{n} - t\right| < \delta} S_{n,\ell}^{\alpha}(t)\, h_{\ell,n}(t) \left(\frac{\ell}{n} - t\right)^r \\ &:= R_1 + R_2. \end{aligned}$$

In view of (2.3), we obtain

$$|R_2| \leq \varepsilon \left|\mu_{n,r}^{\alpha}(t)\right| = o\left(n^{-\left\lfloor \frac{r+1}{2} \right\rfloor}\right)$$

for $n \to \infty$. Moreover, for a constant $B > 0$ such that $|h_{\ell,n}(t)| \leq B$, we have

$$|R_1| \leq B \left|\mu_{n,r}^{\alpha}(t)\right| = o\left(n^{-\left\lfloor \frac{r+1}{2} \right\rfloor}\right)$$

for $n \to \infty$. Thus

$$\lim_{n \to \infty} |R| = 0,$$

and hence the claim follows. □

Consequences of above Theorems 3.1 and 3.2 provide the following first and second order Voronovskaya type theorems, respectively.

Theorem 3.3. *Let $f \in Dom(Q) \bigcap L^{\infty}([0,\infty))$ and have first derivative at a fixed point t. Then*

$$\lim_{n \to \infty} n\left[(Q_n^{\alpha} f)(t) - f(t)\right] = (\alpha + 2)\, f'(t).$$

Proof. Applying the asymptotic formula of Theorem 3.2 with $r = 1$, and using (2.1), we can write

$$(Q_n^{\alpha} f)(t) = f(t) + \sum_{i=1}^{1} \frac{f^{(i)}(t)}{i!} \mu_{n,i}^{\alpha}(t) + o\left(n^{-1}\right) \qquad (n \to \infty).$$

Then the proof follows by passing to the limit for $n \to \infty$. □

Theorem 3.4. *Suppose that $f \in Dom(Q) \bigcap L^{\infty}([0,\infty))$ is a function having second derivative at a point t. Then we have*

$$\lim_{n \to \infty} n\left[(Q_n^{\alpha} f)(t) - f(t)\right] = (\alpha + 2)\, f'(t) + \frac{5t}{2} f''(t).$$

Proof. As in the proof of Theorem 3.3, applying the asymptotic formula of Theorem 3.2 with $r = 2$, and using (2.1), we get

$$(Q_n^\alpha f)(t) = f(t) + \sum_{i=1}^{2} \frac{f^{(i)}(t)}{i!}\mu_{n,i}^\alpha(t) + o\left(n^{-\left\lfloor \frac{2+1}{2} \right\rfloor}\right) \qquad (n \to \infty).$$

Then the proof follows by passing to the limit for $n \to \infty$.

The above theorems show that the point-wise approximation order is at least $O(n^{-1})$, as $n \to +\infty$.

□

4. Quantitative Voronovskaya type estimates

In this section, we give some quantitative estimates for the Voronovskaya type results given in the previous section.

Let $f : I \to \mathbb{R}$ be function. By $C^0(I)$, we denote the space of all uniformly continuous and bounded functions $f : I \to \mathbb{R}$. For $m \geq 1$, by $C^m(I)$ the subspace of $C^0(I)$ with functions $f : I \to \mathbb{R}$ is m-times continuously differentiable and $f^{(\ell)} \in C^0(I)$.

At first, we recall some important relation and estimation on the Taylor remainder term obtained by Gonska, Pitul, and Raşa [4].

By

$$\omega(f, \delta) = \sup_{0<h\leq\delta} \sup_{x\in[0,\infty)} \mid f(x+h) - f(x) \mid,$$

we denote the usual modulus of continuity of $f \in C_B[0, \infty)$.

For $C_B[0, \infty)$, let us consider the following Peetre's K-functionals:

$$K_s(f; \delta) := \inf_{g\in W^1} \{\|f - g\|_\infty + \delta \left\|g^{(s)}\right\|_\infty\}, \tag{4.1}$$

where $\delta > 0$ and

$$W^s = \{g \in C_B[0, \infty) : g', g'', \ldots, g^{(s)} \in C_B[0, \infty)\}.$$

Then there exists an absolute constant $C > 0$ such that

$$C^{-1}\omega(f; \sqrt{\delta}) \leq K_1(f; \delta) \leq C\omega(f; \sqrt{\delta}),$$
$$K_2(f, \delta) \leq C\omega_2(f, \sqrt{\delta}), \tag{4.2}$$

where

$$\omega_2(f, \sqrt{\delta}) = \sup_{0<h\leq\sqrt{\delta}} \sup_{t\in[0,\infty)} \mid f(t+2h) - 2f(t+h) + f(t) \mid \tag{4.3}$$

is the second order modulus of smoothness of f.

Let $f \in C^m(I)$, and consider the following version of the Taylor formula:

$$f(t) = \sum_{i=0}^{m} \frac{f^{(i)}(t_0)}{i!}(t - t_0)^i + R_m(f; t, t_0),$$

where $t, t_0 \in I$ and $R_m(f; t, t_0) = h(t_0 - t)(t_0 - t)^m$ is the remainder term satisfying

$$|R_m(f; t, t_0)| \leq \frac{|t - t_0|^m}{m!}\omega\left(f^{(m)}; |t - t_0|\right) \tag{4.4}$$

and

$$|R_m(f;t,t_0)| \leq 2\frac{|t-t_0|^m}{m!}K_1\left(f^{(m)};\frac{|t-t_0|}{2(m+1)}\right) \tag{4.5}$$

(see [4]).

Here we study quantitative estimates of the convergence results.

Theorem 4.1. *Let $f \in C^r$ and $t \in (0,\infty)$ be fixed. Then we have*

$$\begin{aligned}&\left|n^{\left\lfloor\frac{r+1}{2}\right\rfloor}\left[\left(Q_n^\alpha f\right)(t)-f(t)\right]-\sum_{i=1}^{r}\frac{f^{(i)}(t)}{i!}\mu_{n,i}^\alpha(t)\right|\\&\leq \frac{2n^{\left\lfloor\frac{r+1}{2}\right\rfloor}\left|\mu_{n,r}^\alpha(t)\right|}{r!}K_1\left(f^{(r)};\frac{\left|\mu_{n,r+1}^\alpha(t)\right|}{2(r+1)\left|\mu_{n,r}^\alpha(t)\right|}\right).\end{aligned}$$

Proof. Using the Taylor formula of the rth order as in Theorem 3.2, we get

$$\begin{aligned}&\left|n^{\left\lfloor\frac{r+1}{2}\right\rfloor}\left[\left(Q_n^\alpha f\right)(t)-f(t)\right]-\sum_{i=1}^{r}\frac{f^{(i)}(t)}{i!}\mu_{n,i}^\alpha(t)\right|\\&\leq n^{\left\lfloor\frac{r+1}{2}\right\rfloor}\sum_{\ell=0}^{\infty}S_{n,\ell}^\alpha(t)\left|R_r\left(f;\frac{\ell}{n},t\right)\right|.\end{aligned}$$

Since the remainder term $R_r\left(f;\frac{\ell}{n},t\right) = h_{\ell,n}(t)\left(\frac{\ell}{n}-t\right)^r$ satisfies (4.5), we can write

$$\begin{aligned}&\left|n^{\left\lfloor\frac{r+1}{2}\right\rfloor}\left[\left(Q_n^\alpha f\right)(t)-f(t)\right]-\sum_{i=1}^{r}\frac{f^{(i)}(t)}{i!}\mu_{n,i}^\alpha(t)\right|\\&\leq n^{\left\lfloor\frac{r+1}{2}\right\rfloor}\sum_{\ell=0}^{\infty}S_{n,\ell}^\alpha(t)2\frac{\left|t-\frac{\ell}{n}\right|^r}{r!}K_1\left(f^{(r)};\frac{\left|t-\frac{\ell}{n}\right|}{2(r+1)}\right).\end{aligned}$$

Let $g \in C^r$ be fixed. Then there holds

$$\begin{aligned}&\left|n^{\left\lfloor\frac{r+1}{2}\right\rfloor}\left[\left(Q_n^\alpha f\right)(t)-f(t)\right]-\sum_{i=1}^{r}\frac{f^{(i)}(t)}{i!}\mu_{n,i}^\alpha(t)\right|\\&\leq n^{\left\lfloor\frac{r+1}{2}\right\rfloor}\sum_{\ell=0}^{\infty}S_{n,\ell}^\alpha(t)2\frac{\left|t-\frac{\ell}{n}\right|^r}{r!}K_1\left(f^{(r)};\frac{\left|t-\frac{\ell}{n}\right|}{2(r+1)}\right)\\&\leq 2n^{\left\lfloor\frac{r+1}{2}\right\rfloor}\sum_{\ell=0}^{\infty}S_{n,\ell}^\alpha(t)\frac{\left|t-\frac{\ell}{n}\right|^r}{r!}\left\{\left\|f^{(r)}-g^{(r)}\right\|_\infty+\frac{\left|t-\frac{\ell}{n}\right|^r}{2(r+1)}\left\|g^{(r+1)}\right\|_\infty\right\}.\end{aligned}$$

This yields

$$\begin{aligned}
&\left| n^{\left\lfloor \frac{r+1}{2} \right\rfloor} \left[\left(Q_n^{\alpha} f \right)(t) - f(t) \right] - \sum_{i=1}^{r} \frac{f^{(i)}(t)}{i!} \mu_{n,i}^{\alpha}(t) \right| \\
&\leq \frac{2\left\| f^{(r)} - g^{(r)} \right\|_{\infty}}{r!} n^{\left\lfloor \frac{r+1}{2} \right\rfloor} \left| \mu_{n,r}^{\alpha}(t) \right| \\
&\quad + \frac{\left\| g^{(r+1)} \right\|_{\infty}}{(r+1)!} n^{\left\lfloor \frac{r+1}{2} \right\rfloor} \left| \mu_{n,r+1}^{\alpha}(t) \right|.
\end{aligned}$$

Finally, the quantitative estimate that follows, is

$$\begin{aligned}
&\left| n^{\left\lfloor \frac{r+1}{2} \right\rfloor} \left[\left(Q_n^{\alpha} f \right)(t) - f(t) \right] - \sum_{i=1}^{r} \frac{f^{(i)}(t)}{i!} \mu_{n,i}^{\alpha}(t) \right| \\
&\leq \frac{2n^{\left\lfloor \frac{r+1}{2} \right\rfloor} \left| \mu_{n,r}^{\alpha}(t) \right|}{r!} \left\{ \left\| f^{(r)} - g^{(r)} \right\|_{\infty} + \frac{\left| \mu_{n,r+1}^{\alpha}(t) \right|}{2(r+1) \left| \mu_{n,r}^{\alpha}(t) \right|} \left\| g^{(r+1)} \right\|_{\infty} \right\} \\
&\leq \frac{2n^{\left\lfloor \frac{r+1}{2} \right\rfloor} \left| \mu_{n,r}^{\alpha}(t) \right|}{r!} K_1 \left(f^{(r)}; \frac{\left| \mu_{n,r+1}^{\alpha}(t) \right|}{2(r+1) \left| \mu_{n,r}^{\alpha}(t) \right|} \right).
\end{aligned}$$

□

Corollary 4.2. *Let $f \in C^1$ and $t \in (0, \infty)$ be fixed. Then*

$$\left| n \left[\left(Q_n^{\alpha} f \right) - f - (\alpha + 2) f' \right](t) \right| \leq \frac{2\mu_{n,2}^{\alpha}(t)}{\sqrt{\mu_{n,2}^{\alpha}(t)}} K_1 \left(f'; \frac{\sqrt{\mu_{n,2}^{\alpha}(t)}}{4} \right).$$

As corollaries of Theorem 4.1 for $r = 1$ and $r = 2$, we obtain quantitative estimates for the first and second order Voronovskaya results given in Theorems 3.4 and 3.3 proved in the previous section, respectively.

5. Construction of an operator preserving linear functions

It is obvious from Lemma 2.1 that our new operators (Q_n^{α}) defined as (1.1) do not preserve the linear functions. As the operator Q_n^{α} reproduces only constant functions, this inspired us to propose modifying it to retain both constant and linear functions.

To preserve the linear functions, we will restrict the domain of these operators $t \geq \alpha + 2$, $\alpha > -1$ and then the modification of the operators provided in (1.1) is defined as

$$(Q_n^{\alpha^*} f)(t) := (Q_n^{\alpha} f) \left(t - \frac{\alpha + 2}{n} \right) = \sum_{\ell=0}^{\infty} S_{n,\ell}^{\alpha} \left(t - \frac{\alpha + 2}{n} \right) f \left(\frac{\ell}{n} \right), \tag{5.1}$$

where

$$S_{n,\ell}^{\alpha}(t) = \frac{e^{-nt}}{2^{\alpha+\ell}} \sum_{s=0}^{\ell} \frac{(\alpha + \ell)!}{(\ell - s)!(\alpha + s)!} \frac{1}{3^{s+1}} \; {}_1F_1 \left(1 + s; 1; \frac{2nt}{3} \right),$$

for $t \in I_n = [\frac{\alpha+2}{n}, \infty)$ $(n \geq 1)$, and then the corresponding moment generating function turns out to be

$$(Q_n^{\alpha^*} e^{Au})(t) = \frac{\left(2 - e^{\frac{A}{n}}\right)^{-\alpha}}{(3 - 2e^{\frac{A}{n}})} \exp\left(n\left(t - \frac{\alpha+2}{n}\right)\left(\frac{e^{\frac{A}{n}} - 1}{3 - 2e^{\frac{A}{n}}}\right)\right).$$

As in Lemma 2.1, one has the following result.

Lemma 5.1. *For every $t \in [\alpha+2, \infty), \alpha \leq 2.5$ the algebraic moments of the operators $Q_n^{\alpha^*}$ defined in (5.1) are given by*

$$(Q_n^{\alpha^*} e_0)(t) = 1,$$
$$(Q_n^{\alpha^*} e_1)(t) = t,$$
$$(Q_n^{\alpha^*} e_2)(t) = t^2 + \frac{nt(5-2\alpha) + 6\alpha - 4}{n^2}$$
$$(Q_n^{\alpha^*} e_3)(t) = t^3 + \frac{-31\alpha + 15t^2n^2 + (25-9\alpha)tn - 44}{n^3},$$
$$(Q_n^{\alpha^*} e_4)(t) = t^4 + \frac{27\alpha^2 - 267\alpha + 30t^3n^3 + (199-18\alpha)t^2n^2 + (69-214\alpha)tn - 460}{n^4}.$$

Lemma 5.2. *Let $f \in C_B[0, \infty)$ (the class of all uniform continuous and bounded functions on $[0, \infty)$ and with norm $\| f \| = \sup\{|f(t)| : t \in [0, \infty)\}$). Then one has for $t \in [\alpha+2, \infty)$,*

$$\| Q_n^{\alpha^*} f \| \leq \| f \| .$$

Proof. In view of (5.1) and Lemma 5.1, the proof easily follows. □

As a consequence of the above Lemma 5.1, one has the following result.

Lemma 5.3. *For every $t \in [\alpha+2, \infty)$, the central moments of $\mu_{n,s}^{\alpha^*}(t) := (Q_n^{\alpha^*}(u-t)^s)(t)$, $s = 0, 1, 2, \ldots$, are given by*

$$\mu_{n,0}^{\alpha^*}(t) = 1,$$
$$\mu_{n,1}^{\alpha^*}(t) = 0,$$
$$\mu_{n,2}^{\alpha^*}(t) = \frac{nt(5-2\alpha) + 6\alpha - 4}{n^2}, \tag{5.2}$$
$$\mu_{n,3}^{\alpha^*}(t) = -\frac{31\alpha - 37nt - 6n^2t^2\alpha + 27nt\alpha + 44}{n^3},$$
$$\mu_{n,4}^{\alpha^*}(t) = -\frac{12n^3t^3\alpha - 54n^2t^2\alpha - 75n^2t^2 + 90nt\alpha - 245nt - 27\alpha^2 + 267\alpha + 460}{n^4}.$$

We may observe here that, for $-1 \leq \alpha \leq 2.5$, all of these moments are positive for sufficiently large n.

Theorem 5.4. *Let $f \in C_B[0, \infty)$. Then one has for $t > \alpha+2$ and $-1 \leq \alpha \leq 2.5$,*

$$| (Q_n^{\alpha^*} f)(t) - f(t) | \leq 2\omega\left(f, \delta_{n,t}^{\alpha^*}\right), \tag{5.3}$$

where

$$\delta_{n,t}^{\alpha^*} = (\mu_{n,2}^{\alpha^*}(t))^{1/2} = \left(\frac{nt(5-2\alpha)+6\alpha-4}{n^2}\right)^{1/2}.$$

Proof. Clearly, in view of the monotonicity and linearity of the operators $Q_n^{\alpha^*}$, for every $\delta_{n,t}^{\alpha^*} > 0$, one gets

$$\mid (Q_n^{\alpha^*} f)(t) - f(t) \mid \leq \omega\left(f,\delta\right)\left(1+\frac{\sqrt{\mu_{n,2}^{\alpha^*}(t)}}{\delta}\right) \leq 2\omega\left(f,\delta\right).$$

If we choose $\delta = \delta_{n,t}^{\alpha^*}$, then the proof follows. □

Theorem 5.5. *Let $f \in C_B[0,\infty)$. Then one has for $t \in [0,\infty)$*

$$\mid (Q_n^{\alpha} f)(t) - f(t) \mid \leq 2\omega\left(f, \delta_{n,t}^{\alpha}\right),$$

where

$$\delta_{n,t}^{\alpha} = \sqrt{\mu_{n,2}^{\alpha}(t)} = \left(\frac{10+6\alpha+\alpha^2+5nt}{n^2}\right)^{1/2}.$$

Proof. Similarly, in view of the monotonicity and linearity of the operators Q_n^{α}, for every $\delta_{n,t}^{\alpha} > 0$, one has

$$\mid (Q_n^{\alpha} f)(t) - f(t) \mid \leq \omega\left(f,\delta\right)\left(1+\frac{\sqrt{\mu_{n,2}^{\alpha}(t)}}{\delta}\right) \leq 2\omega\left(f,\delta\right). \tag{5.4}$$

Choosing $\delta = \delta_{n,t}^{\alpha}$ then the proof follows. □

Theorem 5.6. *Let $f \in C_B[0,\infty)$ and let $t \in [\alpha+2,\infty)$, $-1 \leq \alpha \leq 2.5$. Then, there exists an absolute constant C such that*

$$|Q_n^{\alpha^*} f(t) - f(t)| \leq C\omega_2\left(f, \frac{[nt(5-2\alpha)+6\alpha-4]^{1/2}}{n}\right),$$

where $\mu_{n,2}^{\alpha^*}(t)$ is given by (5.2).

Proof. Let $g \in W^2$, let $t \in I_n$, and let $t \geq 0$. Using Taylor's theorem, we have

$$g(t) = g(t) + (u-t)g'(t) + \int_t^u (u-w)g''(w)dw. \tag{5.5}$$

Applying $Q_n^{\alpha^*}$ on both sides of (5.5) and using Lemma 5.3, we get

$$(Q_n^{\alpha^*} g)(t) - g(t) = (Q_n^{\alpha^*}(u-t))(t)g'(t) + \left(Q_n^{\alpha^*}\left(\int_t^u (u-w)g''(w)dw\right)\right)(t).$$

Obviously, we have $\left|\int_t^u (u-w)^2 g''(w)dw\right| \leq (u-t)^2\|g''\|$. Therefore

$$\mid (Q_n^{\alpha^*} g)(t) - g(t) \mid \leq (Q_n^{\alpha^*}(u-t)^2)(t) \parallel g'' \parallel = \mu_{n,2}^{\alpha^*}(t) \parallel g'' \parallel.$$

Since $\mid (Q_n^{\alpha^*} f)(t) \mid \leq \|f\|$, we get

$$\begin{aligned}\mid (Q_n^{\alpha^*} f)(t) - f(t) \mid &\leq \mid (Q_n^{\alpha^*}(f-g))(t) \mid + \mid (f-g)(t) \mid + \mid (Q_n^{\alpha^*} g)(t) - g(t) \mid \\ &\leq 2\|f-g\| + \mu_{n,2}^{\alpha^*}(t)\|g''\|.\end{aligned}$$

Finally, selecting the infimum over all $g \in W^2$ and making use of (4.1)–(4.3) and Lemma 5.3, we get

$$[| (Q_n^{\alpha^*} f) - f](t) | \leq C\omega_2 \left(f, \left[\frac{nt(5-2\alpha)+6\alpha-4}{n^2} \right]^{1/2} \right),$$

which proves the theorem. □

Remark 5.7. From (5.3) and (5.4), we see that the estimates of error given in Theorem 5.4 are better than those given in Theorem 5.5 provided $f \in C_B[0,\infty)$ and $t \in [\alpha+2,\infty)$.

In fact, for $t \in [\alpha+2,\infty)$,

$$\frac{nt(5-2\alpha)+6\alpha-4}{n^2} \leq \frac{10+6\alpha+\alpha^2+5nt}{n^2}$$

implies

$$3\alpha^2+4\alpha+14>0,$$

which is always true. This yields

$$0<\delta_{n,t}^{\alpha^*} \leq \delta_{n,t}^{\alpha}$$

for $t \in [\alpha+2,\infty)$ and so the claim follows.

We might expand the research in the future studies to include the preservation of exponential functions. The approaches have different analysis, which we will cover in another article.

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