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LEFT-HAND QUOTIENTS LIPSCHITZ IDEALS BY OPERATOR IDEALS

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Abstract. Expanding on the concept of left-hand quotients of operator ideals, we analyze the known notion of Lipschitz left-hand quotient $\mathcal{A}^{-1} \circ \mathcal{I}^{\mathrm{Lip}}$, where $\mathcal{A}$ denotes an operator ideal and $\mathcal{I}^{\mathrm{Lip}}$ represents a Lipschitz ideal. We establish that these quotients provide a structured approach to generating new Lipschitz ideals. Specifically, if a Lipschitz ideal $\mathcal{I}^{\mathrm{Lip}}$ satisfies the linearization property within a given operator ideal $\mathcal{I}$, then the left-hand quotient $\mathcal{A}^{-1} \circ \mathcal{I}^{\mathrm{Lip}}$ can be expressed as a Lipschitz composition ideal in the form $(\mathcal{A}^{-1} \circ \mathcal{I}) \circ \mathcal{I}^{\mathrm{Lip}}$. Furthermore, we study Lipschitz mappings whose Lipschitz images are Grothendieck sets or Rosenthal sets, and demonstrate that both form Lipschitz ideals that can be constructed as Lipschitz left-hand quotients.

1. Introduction

The study of left- and right-hand quotients of two operator ideals has had limited attention from researchers since Puhl introduced this idea in [13] and Pietsch developed their main properties and presented some examples in [12]. Among others, we can cite the study of quotients of (s,p)-mixing operator ideals by Carl and Defant [5], quotients of (p,σ)-absolutely continuous operator ideals by Sánchez-Pérez [16], and quotients of completely continuous operator ideals by Johnson et al. [10].

According to the framework introduced in [13, p. 132], for two operator ideals $\mathcal{A}$ and $\mathcal{I}$, a bounded linear operator between Banach spaces $T : E \to F$ is said to belong to the left-hand quotient $\mathcal{A}^{-1} \circ \mathcal{I}(E,F)$ if $A \circ T \in \mathcal{I}(E,G)$ for all $A \in \mathcal{A}(F,G)$, where G is an arbitrary Banach space.

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As discussed in [12, Sections 3.2 and 7.2], it is well known that $\mathcal{A}^{-1}\circ\mathcal{I}$ yields an operator ideal called left-hand quotient ideal. Moreover, if $[\mathcal{A},\|\cdot\|_{\mathcal{A}}]$ and $[\mathcal{I},\|\cdot\|_{\mathcal{I}}]$ are Banach operator ideals, then the usual norm of the operator $T\in\mathcal{A}^{-1}\circ\mathcal{I}(E,F)$ comes given by

$$\|T\|_{\mathcal{A}^{-1}\circ\mathcal{I}} = \sup\left\{\|A\circ T\|_{\mathcal{I}} : A\in\mathcal{A}(F,G),\ \|A\|_{\mathcal{A}}\leq 1\right\},$$

where the supremum is taken over all Banach spaces G and all operators A as above, and the pair $[\mathcal{A}^{-1}\circ\mathcal{I},\|\cdot\|_{\mathcal{A}^{-1}\circ\mathcal{I}}]$ generates a Banach operator ideal.

Motivated by this classical construction, we study in this paper the known notion of Lipschitz left-hand quotient ideal which involves not only Lipschitz ideals but also operator ideals. For this study, we will consider pointed metric spaces X and Y, each endowed with a distance denoted by d and a specified base point which we designate as 0, and Banach spaces E over the real or complex field $\mathbb{K}$.

A mapping $f: X\to Y$ is referred to as Lipschitz when there exists a constant $C>0$ such that $d(f(x),f(z))\leq Cd(x,z)$ for all $x,z\in X$. The set of all mappings $f: X\to Y$ such that $f(0)=0$ satisfying the above inequality is denoted by $\mathrm{Lip}_0(X,Y)$, and for any such mapping f, its Lipschitz constant $\mathrm{Lip}(f)$ is the smallest possible value of C fulfilling the inequality. It is a well-known fact that $(\mathrm{Lip}_0(X,E),\mathrm{Lip})$ constitutes a Banach space whenever E is so. In the particular case where $E=\mathbb{K}$, we denote $\mathrm{Lip}_0(X,\mathbb{K})$ simply by $X^{\#}$ referring to it as the Lipschitz dual of X.

The notion of left-hand quotient operator ideal was extended to the Lipschitz setting by Saleh [17, Section 2.9] with the introduction of the concept of Lipschitz left-hand quotient ideals between metric spaces and Banach spaces. Later, it was generalized to Lipschitz left-hand quotient ideals between arbitrary metric spaces by Achour, Dahia, and Saleh [1, Definition 5.3]. As an example, in [1, Corollary 5.9], they showed that the concept of $(s;q,\theta)$-mixing Lipschitz maps provides a Lipschitz extension of the classical division theorem for (s,p)-mixing operators [12, Theorem 20.3.1].

We will begin recalling the concept of Lipschitz ideals (see [2, 17]) and then re-introducing the main notion of this paper.

Let X be a pointed metric space and let E be a Banach space. Consider a set $\mathcal{I}^{\mathrm{Lip}}(X,E)\subseteq\mathrm{Lip}_0(X,E)$ and a function $\|\cdot\|_{\mathcal{I}^{\mathrm{Lip}}}:\mathcal{I}^{\mathrm{Lip}}(X,E)\to\mathbb{R}_0^+$ satisfying the following properties:

- **(N1):** $(\mathcal{I}^{\mathrm{Lip}}(X,E),\|\cdot\|_{\mathcal{I}^{\mathrm{Lip}}})$ is a Banach space and the inequality $\mathrm{Lip}(f)\leq\|f\|_{\mathcal{I}^{\mathrm{Lip}}}$ holds for all $f\in\mathcal{I}^{\mathrm{Lip}}(X,E)$.
- **(N2):** For any $g\in X^{\#}$ and $e\in E$, the function $g\cdot e: X\to E$, defined by $(g\cdot e)(x)=g(x)e$ for all $x\in X$, belongs to $\mathcal{I}^{\mathrm{Lip}}(X,E)$ and $\|g\cdot e\|_{\mathcal{I}^{\mathrm{Lip}}}=\mathrm{Lip}(g)\|e\|$.
- **(N3):** (Ideal property) Suppose that Y is a pointed metric space and that F is a Banach space. For any $h\in\mathrm{Lip}_0(Y,X)$, $f\in\mathcal{I}^{\mathrm{Lip}}(X,E)$ and $T\in\mathcal{L}(E,F)$, the map $T\circ f\circ h$ is in $\mathcal{I}^{\mathrm{Lip}}(Y,F)$ and $\|T\circ f\circ h\|_{\mathcal{I}^{\mathrm{Lip}}}\leq\|T\|\,\|f\|_{\mathcal{I}^{\mathrm{Lip}}}\,\mathrm{Lip}(h)$.

Then $[\mathcal{I}^{\mathrm{Lip}}, \|\cdot\|_{\mathcal{I}^{\mathrm{Lip}}}]$ is said to be a Banach ideal of Lipschitz maps or a Banach Lipschitz ideal. Moreover, $[\mathcal{I}^{\mathrm{Lip}}, \|\cdot\|_{\mathcal{I}^{\mathrm{Lip}}}]$ is said to be:

(S): Surjective if for every $f \in \mathrm{Lip}_0(X,E)$ and any surjective map $\pi \in \mathrm{Lip}_0(Y,X)$ with $\mathrm{Lip}(\pi) \leq 1$, where Y is a pointed metric space, it holds that $f \in \mathcal{I}^{\mathrm{Lip}}(X,E)$ with $\|f\|_{\mathcal{I}^{\mathrm{Lip}}} = \|f\circ\pi\|_{\mathcal{I}^{\mathrm{Lip}}}$ whenever $f\circ\pi \in \mathcal{I}^{\mathrm{Lip}}(Y,E)$.

Definition 1.1. [17, Section 2.9]. Let $\mathcal{A}$ be an operator ideal and let $\mathcal{I}^{\mathrm{Lip}}$ be a Lipschitz ideal. Given a pointed metric space X and a Banach space E, a mapping $f \in \mathrm{Lip}_0(X,E)$ is said to lie in the Lipschitz left-hand quotient $\mathcal{A}^{-1}\circ\mathcal{I}^{\mathrm{Lip}}$, written as $f \in \mathcal{A}^{-1}\circ\mathcal{I}^{\mathrm{Lip}}(X,E)$, if $A\circ f \in \mathcal{I}^{\mathrm{Lip}}(X,G)$ for all $A \in \mathcal{A}(E,G)$, where G is an arbitrary Banach space. Moreover, when $\mathcal{A}$ is endowed with a norm $\|\cdot\|_{\mathcal{A}}$ and $\mathcal{I}^{\mathrm{Lip}}$ with a norm $\|\cdot\|_{\mathcal{I}^{\mathrm{Lip}}}$, for any $f \in \mathcal{A}^{-1}\circ\mathcal{I}^{\mathrm{Lip}}(X,E)$, we define

$$\|f\|_{\mathcal{A}^{-1}\circ\mathcal{I}^{\mathrm{Lip}}} := \sup\left\{\|A\circ f\|_{\mathcal{I}^{\mathrm{Lip}}} : A \in \mathcal{A}(E,G),\ \|A\|_{\mathcal{A}} \leq 1\right\},$$

where the supremum is taken over all Banach spaces G and all operators A as above.

We now fix some notation. We denote the closed unit ball in E by B_E, and the Banach space of all bounded linear operators from E to F, equipped with the operator norm, by $\mathcal{L}(E,F)$. In the special case where $F = \mathbb{K}$, it is usual to denote it by E^*. Given a set $A \subseteq E$, $\overline{\mathrm{lin}}(A)$ and $\overline{\mathrm{abco}}(A)$ represent the norm-closed linear hull and the norm-closed absolutely convex hull of A in E, respectively.

The concepts of Grothendieck Lipschitz maps and Rosenthal Lipschitz maps were introduced by Saleh in [17, Section 2.9] in terms of their descriptions as Lipschitz left-hand quotient ideals. With the purpose of characterizing such Lipschitz ideals, it is important to note that here we do not use the global range of a map between metric spaces $f\colon X \to Y$ (as it happens for the concepts of Grothendieck and Rosenthal linear operators between Banach spaces E and F on the ball B_E), but its Lipschitz image given by

$$\mathrm{Im}_{\mathrm{Lip}}(f) := \{d(f(x),f(z))/d(x,z)\colon x,z \in X,\ x \neq z\} \subseteq Y.$$

It is immediate that f is Lipschitz if and only if $\mathrm{Im}_{\mathrm{Lip}}(f)$ is a bounded subset of Y.

Our approach in this paper requires the linearization of Lipschitz maps on X, using its canonical Banach predual called the Lipschitz-free Banach space of X, denoted by $\mathcal{F}(X)$ (see [6]) and known also as the Arens–Eells space of X (see [4]). We next gather some needed properties of this space which can be found in the Weaver's monograph [18] (see also [9, Lemma 1.1]).

Theorem 1.2. [18]. *Let X be a pointed metric space. Consider the Banach space:*

$$\mathcal{F}(X) := \overline{\mathrm{lin}}(\{\delta_x\colon x \in X\}) \subseteq (X^{\#})^*,$$

where $\delta_x\colon X^{\#} \to \mathbb{K}$ is the functional defined by $\delta_x(f) = f(x)$ for all $f \in X^{\#}$.

(i) *The map $\delta_X\colon X \to \mathcal{F}(X)$, given by $\delta_X(x) = \delta_x$ for all $x \in X$, is an isometric embedding.*

(ii) *The map $Q_X\colon X^{\#} \to \mathcal{F}(X)^*$, defined by $Q_X(f)(\delta_x) = f(x)$ for all $f \in X^{\#}$ and $x \in X$, is an isometric isomorphism.*

(iii) $B_{\mathcal{F}(X)} = \overline{\mathrm{abco}}(\mathrm{Im}_{\mathrm{Lip}}(\delta_X))$.
(iv) *For every Banach space E and every map $f \in \mathrm{Lip}_0(X, E)$, there exists a unique operator $T_f \in \mathcal{L}(\mathcal{F}(X), E)$ such that $T_f \circ \delta_X = f$. Moreover, $\|T_f\| = \mathrm{Lip}(f)$.*
(v) *The map $f \mapsto T_f$ is an isometric isomorphism from $(\mathrm{Lip}_0(X, E), \mathrm{Lip})$ onto $(\mathcal{L}(\mathcal{F}(X), E), \|\cdot\|)$.*

Our analysis heavily relies on this linearization process applied to the elements of the involved Lipschitz ideal $\mathcal{I}^{\mathrm{Lip}}$ in the composition $\mathcal{A}^{-1} \circ \mathcal{I}^{\mathrm{Lip}}$.

Definition 1.3. Let $[\mathcal{I}, \|\cdot\|_{\mathcal{I}}]$ be a normed operator ideal and let $[\mathcal{I}^{\mathrm{Lip}}, \|\cdot\|_{\mathcal{I}^{\mathrm{Lip}}}]$ be a normed Lipschitz ideal. We say that $[\mathcal{I}^{\mathrm{Lip}}, \|\cdot\|_{\mathcal{I}^{\mathrm{Lip}}}]$ satisfies the linearization property with respect to $[\mathcal{I}, \|\cdot\|_{\mathcal{I}}]$ if for every pointed metric space X, every Banach space E and every map $f \in \mathrm{Lip}_0(X, E)$, we have $f \in \mathcal{I}^{\mathrm{Lip}}(X, E)$ if and only if $T_f \in \mathcal{I}(\mathcal{F}(X), E)$, and, in such a case, $\|f\|_{\mathcal{I}^{\mathrm{Lip}}} = \|T_f\|_{\mathcal{I}}$.

More generally, if $\mathcal{I}$ is a closed operator ideal that is injective or surjective in a suitable sense, then the composition ideal $\mathcal{I} \circ \mathrm{Lip}_0$ often inherits the linearization property via the canonical identification $f \mapsto T_f$. Further examples and details can be found in [2, Section 3] and [15].

This article is organized as follows: After this introductory section, the work is divided into two parts. In section 2, we analyze the foundational characteristics of Lipschitz left-hand quotient ideals $\mathcal{A}^{-1} \circ \mathcal{I}^{\mathrm{Lip}}$, whenever $\mathcal{A}$ is an operator ideal and $\mathcal{I}^{\mathrm{Lip}}$ is a Lipschitz ideal. When both $\mathcal{A}$ and $\mathcal{I}^{\mathrm{Lip}}$ are endowed with complete norms, we show that the space $\mathcal{A}^{-1} \circ \mathcal{I}^{\mathrm{Lip}}$, equipped with the norm $\|\cdot\|_{\mathcal{A}^{-1} \circ \mathcal{I}^{\mathrm{Lip}}}$, forms a Banach Lipschitz ideal. Moreover, if $\mathcal{I}^{\mathrm{Lip}}$ is surjective, then so is the Lipschitz ideal $\mathcal{A}^{-1} \circ \mathcal{I}^{\mathrm{Lip}}$. This result positions Lipschitz left-hand quotients as an effective method for constructing new Lipschitz ideals.

In addition, we demonstrate that if $\mathcal{I}^{\mathrm{Lip}}$ has the linearization property relative to an operator ideal $\mathcal{I}$, then for a mapping $f \in \mathrm{Lip}_0(X, E)$, the condition $f \in \mathcal{A}^{-1} \circ \mathcal{I}^{\mathrm{Lip}}(X, E)$ is equivalent to the condition $T_f \in \mathcal{A}^{-1} \circ \mathcal{I}(\mathcal{F}(X), E)$. In this situation, the Lipschitz left-hand quotient ideal $\mathcal{A}^{-1} \circ \mathcal{I}^{\mathrm{Lip}}$ can be viewed as a composition ideal of the form $(\mathcal{A}^{-1} \circ \mathcal{I}) \circ \mathrm{Lip}_0$. For the study of composition ideals of Lipschitz maps, we refer to the papers by Achour et al. [2] and Saadi [15].

In section 3, we address two specific examples of Lipschitz left-hand quotient ideals: The spaces of Lipschitz mappings whose Lipschitz images are Grothendieck sets or Rosenthal sets. Here, we establish some interesting characterizations of such ideals that allow us to advance our understanding of them, which until now has been limited to their definitions in [17, Section 2.9].

2. Lipschitz left-hand quotient ideals

From now on, unless otherwise stated, we assume that X is a pointed metric space and that E is a Banach space.

We begin by confirming that the supremum involved in Definition 1.1 is finite for every mapping in $\mathcal{A}^{-1} \circ \mathcal{I}^{\mathrm{Lip}}$ when the norm on $\mathcal{A}$ is complete.

Proposition 2.1. *Let $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ be a Banach operator ideal, let $[\mathcal{I}^{\mathrm{Lip}}, \|\cdot\|_{\mathcal{I}^{\mathrm{Lip}}}]$ be a normed Lipschitz ideal, and let $f \in \mathcal{A}^{-1} \circ \mathcal{I}^{\mathrm{Lip}}(X, E)$. Then, for any Banach space G, we have*

$$\sup \{\|A \circ f\|_{\mathcal{I}^{\mathrm{Lip}}} : A \in \mathcal{A}(E, G),\ \|A\|_{\mathcal{A}} \leq 1\} < \infty.$$

Proof. Our proof is based on [12, Theorems 7.2.2], let us assume that this supremum is not finite. Then, for each $n \in \mathbb{N}$, there are a Banach space G_n and an operator $A_n \in \mathcal{A}(E, G_n)$ so that $\|A_n\|_{\mathcal{A}} \leq 1/2^n$ and $\|A_n \circ f\|_{\mathcal{I}^{\mathrm{Lip}}} \geq n$. Let $G = (G_n)_{n\in\mathbb{N}}$ be the sequence of such spaces and consider the Banach space $\ell_1(\mathbb{N}, G)$ formed by all sequences $(x_n)_{n\in\mathbb{N}}$ with $x_n \in G_n$ for each $n \in \mathbb{N}$ such that $\sum\limits_{n=1}^{\infty} \|x_n\| < \infty$, where $\|.\|_1$ is the norm given by $\|(x_n)_{n\in\mathbb{N}}\|_1 = \sum\limits_{n=1}^{\infty} \|x_n\|$.

For each $n \in \mathbb{N}$, define the bounded linear operators $J_n \colon G_n \to \ell_1(\mathbb{N}, G)$ and $Q_n \colon \ell_1(\mathbb{N}, G) \to G_n$ by $J_n(x) = (\delta_{in} x)_{i\in\mathbb{N}}$ and $Q_n((x_i)_{i\in\mathbb{N}}) = x_n$, respectively, where δ_{in} is the Kronecker delta. Clearly, $\|J_n\| = 1$ and $\|Q_n\| \leq 1$. Since each operator $J_k \circ A_k$ belongs to $\mathcal{A}(E, \ell_1(\mathbb{N}, G))$ with $\|J_k \circ A_k\| \leq \|J_k\| \, \|A_k\|_{\mathcal{A}} \leq \|A_k\|_{\mathcal{A}}$, we deduce that

$$\begin{aligned} \left\| \sum_{k=n+1}^{n+h} J_k \circ A_k \right\|_{\mathcal{A}} &\leq \sum_{k=n+1}^{n+h} \|J_k \circ A_k\|_{\mathcal{A}} = \sum_{j=1}^{h} \|J_{n+j} \circ A_{n+j}\|_{\mathcal{A}} \\ &\leq \sum_{j=1}^{h} \|A_{n+j}\|_{\mathcal{A}} \leq \sum_{j=1}^{h} \frac{1}{2^{n+j}} = \frac{1}{2^n} \sum_{j=1}^{h} \frac{1}{2^j} \leq \frac{1}{2^n} \sum_{j=1}^{\infty} \frac{1}{2^j} = \frac{1}{2^n}, \end{aligned}$$

for all $n, h \in \mathbb{N}$. Hence the sequence $(\sum_{k=1}^{n} J_k \circ A_k)_{n\in\mathbb{N}}$ is Cauchy in the complete norm $\|\cdot\|_{\mathcal{A}}$, and therefore it converges in this norm to some operator $A \in \mathcal{A}(E, \ell_1(\mathbb{N}, G))$. Now observe that

$$n \leq \|A_n \circ f\|_{\mathcal{I}^{\mathrm{Lip}}} = \|Q_n \circ A \circ f\|_{\mathcal{I}^{\mathrm{Lip}}} \leq \|A \circ f\|_{\mathcal{I}^{\mathrm{Lip}}},$$

for all $n \in \mathbb{N}$, and we arrive at a contradiction. Thus, the supremum must be finite. □

Given two normed Lipschitz ideals $[\mathcal{I}_1^{\mathrm{Lip}}, \|\cdot\|_{\mathcal{I}_1^{\mathrm{Lip}}}]$ and $[\mathcal{I}_2^{\mathrm{Lip}}, \|\cdot\|_{\mathcal{I}_2^{\mathrm{Lip}}}]$, we w write

$$[\mathcal{I}_1^{\mathrm{Lip}}, \|\cdot\|_{\mathcal{I}_1^{\mathrm{Lip}}}] \leq [\mathcal{I}_2^{\mathrm{Lip}}, \|\cdot\|_{\mathcal{I}_2^{\mathrm{Lip}}}]$$

to indicate that $\mathcal{I}_1^{\mathrm{Lip}}(X, E) \subseteq \mathcal{I}_2^{\mathrm{Lip}}(X, E)$ and $\|f\|_{\mathcal{I}_2^{\mathrm{Lip}}} \leq \|f\|_{\mathcal{I}_1^{\mathrm{Lip}}}$ for all $f \in \mathcal{I}_1^{\mathrm{Lip}}(X, E)$.

An inclusion relationship between Lipschitz left-hand quotient ideals naturally arises from the inclusion of the underlying Lipschitz ideals as the following easy result establishes.

Proposition 2.2. *Let $[\mathcal{I}_1^{\mathrm{Lip}}, \|\cdot\|_{\mathcal{I}_1^{\mathrm{Lip}}}]$ and $[\mathcal{I}_2^{\mathrm{Lip}}, \|\cdot\|_{\mathcal{I}_2^{\mathrm{Lip}}}]$ be normed Lipschitz ideals such that*

$$[\mathcal{I}_1^{\mathrm{Lip}}, \|\cdot\|_{\mathcal{I}_1^{\mathrm{Lip}}}] \leq [\mathcal{I}_2^{\mathrm{Lip}}, \|\cdot\|_{\mathcal{I}_2^{\mathrm{Lip}}}].$$

Then, for any Banach operator ideal $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$, we have the relation

$$[\mathcal{A}^{-1} \circ \mathcal{I}_1^{\mathrm{Lip}}, \|\cdot\|_{\mathcal{A}^{-1}\circ\mathcal{I}_1^{\mathrm{Lip}}}] \leq [\mathcal{A}^{-1} \circ \mathcal{I}_2^{\mathrm{Lip}}, \|\cdot\|_{\mathcal{A}^{-1}\circ\mathcal{I}_2^{\mathrm{Lip}}}].$$

Since $[\mathrm{Lip}_0, \mathrm{Lip}]$ is the biggest normed Lipschitz ideal containing any normed Lipschitz ideal $[\mathcal{I}^{\mathrm{Lip}}, \|\cdot\|_{\mathcal{I}^{\mathrm{Lip}}}]$, we deduce the following result.

Corollary 2.3. *Let $[\mathcal{I}^{\mathrm{Lip}}, \|\cdot\|_{\mathcal{I}^{\mathrm{Lip}}}]$ be a normed Lipschitz ideal. Then, for every Banach operator ideal $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$, we have*

$$[\mathcal{A}^{-1} \circ \mathcal{I}^{\mathrm{Lip}}, \|\cdot\|_{\mathcal{A}^{-1}\circ\mathcal{I}^{\mathrm{Lip}}}] \leq [\mathcal{A}^{-1} \circ \mathrm{Lip}_0, \|\cdot\|_{\mathcal{A}^{-1}\circ\mathrm{Lip}_0}].$$

In some sense, the above corollary assures that $\mathcal{A}^{-1} \circ \mathrm{Lip}_0$ can be seen as the maximal Lipschitz left-hand quotient associated to any Banach operator ideal $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$.

Under a suitable condition on the Lipschitz ideal $\mathcal{I}^{\mathrm{Lip}}$, we derive a result that complements Corollary 2.3.

Proposition 2.4. *Let $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ be a Banach operator ideal, and let $[\mathcal{I}^{\mathrm{Lip}}, \|\cdot\|_{\mathcal{I}^{\mathrm{Lip}}}]$ be a normed Lipschitz ideal. Then*

$$[\mathcal{A}^{-1} \circ \mathcal{I}^{\mathrm{Lip}}, \|\cdot\|_{\mathcal{A}^{-1}\circ\mathcal{I}^{\mathrm{Lip}}}] \leq [\mathrm{Lip}_0, \mathrm{Lip}].$$

Moreover, if $\mathcal{I}^{\mathrm{Lip}}$ enjoys the linearization property with respect to $\mathcal{A}$, we have

$$[\mathcal{A}^{-1} \circ \mathcal{I}^{\mathrm{Lip}}, \|\cdot\|_{\mathcal{A}^{-1}\circ\mathcal{I}^{\mathrm{Lip}}}] = [\mathrm{Lip}_0, \mathrm{Lip}].$$

Proof. Let $f \in \mathcal{A}^{-1}\circ\mathcal{I}^{\mathrm{Lip}}(X, E)$. Hence $f \in \mathrm{Lip}_0(X, E)$ and $A\circ f \in \mathcal{I}^{\mathrm{Lip}}(X, G)$ for any $A \in \mathcal{A}(E, G)$, where G is an arbitrary Banach space. We now make an observation: Since $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ is a Banach operator ideal, any functional $e^* \in E^*$ belongs to $\mathcal{A}(E, \mathbb{K})$ with $\|e^*\|_{\mathcal{A}} = \|e^*\|$. In order to prove that $\mathrm{Lip}(f) \leq \|f\|_{\mathcal{A}^{-1}\circ\mathcal{I}^{\mathrm{Lip}}}$, let $x, z \in X$, and take a functional $e^* \in B_{E^*}$ so that $|e^*(f(x) - f(z))| = \|f(x) - f(z)\|$. So we have

$$\begin{aligned} \|f(x) - f(z)\| &= |(e^* \circ f)(x) - (e^* \circ f)(z)| \\ &\leq \mathrm{Lip}(e^* \circ f)d(x, z) \\ &\leq \|e^* \circ f\|_{\mathcal{I}^{\mathrm{Lip}}} d(x, z) \\ &\leq \|f\|_{\mathcal{A}^{-1}\circ\mathcal{I}^{\mathrm{Lip}}} \|e^*\|_{\mathcal{A}} d(x, z) \\ &\leq \|f\|_{\mathcal{A}^{-1}\circ\mathcal{I}^{\mathrm{Lip}}} d(x, z), \end{aligned}$$

and taking the supremum over all pairs $x, z \in X$, we conclude that $\mathrm{Lip}(f) \leq \|f\|_{\mathcal{A}^{-1}\circ\mathcal{I}^{\mathrm{Lip}}}$.

Now, suppose that $\mathcal{I}^{\mathrm{Lip}}$ satisfies the linearization property with respect to $\mathcal{A}$. Let $f \in \mathrm{Lip}_0(X, E)$, and take a Banach space G and an operator $A \in \mathcal{A}(E, G)$. Clearly, $A \circ f \in \mathrm{Lip}_0(X, G)$. Theorem 1.2 provides two unique operators $T_f \in \mathcal{L}(\mathcal{F}(X), E)$ and $T_{A\circ f} \in \mathcal{L}(\mathcal{F}(X), G)$ such that $T_f \circ \delta_X = f$ and $T_{A\circ f} \circ \delta_X = A \circ f$. Since $(A \circ T_f) \circ \delta_X = A \circ f$ and $A \circ T_f \in \mathcal{L}(\mathcal{F}(X), G)$, it follows that $T_{A\circ f} = A \circ T_f$ by the cited uniqueness. By the ideal property of $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$, we get that $T_{A\circ f} \in \mathcal{A}(\mathcal{F}(X), G)$ and

$$\|T_{A\circ f}\|_{\mathcal{A}} = \|A \circ T_f\|_{\mathcal{A}} \leq \|A\|_{\mathcal{A}} \|T_f\| = \|A\|_{\mathcal{A}} \mathrm{Lip}(f).$$

As $\mathcal{I}^{\mathrm{Lip}}$ has the linearization property with respect to $\mathcal{A}$, we obtain that $A \circ f \in \mathcal{I}^{\mathrm{Lip}}(X, G)$ with $\|A \circ f\|_{\mathcal{I}^{\mathrm{Lip}}} = \|T_{A\circ f}\|_{\mathcal{A}}$, and thus $\|A \circ f\|_{\mathcal{I}^{\mathrm{Lip}}} \leq \|A\|_{\mathcal{A}} \mathrm{Lip}(f)$. Therefore, $f \in \mathcal{A}^{-1} \circ \mathcal{I}^{\mathrm{Lip}}(X, E)$ and $\|f\|_{\mathcal{A}^{-1}\circ\mathcal{I}^{\mathrm{Lip}}} \leq \mathrm{Lip}(f)$ by taking the supremum over all Banach spaces G and all operators $A \in \mathcal{A}(E, G)$ with $\|A\|_{\mathcal{A}} \leq$

1. Combining this with the earlier inequality, we conclude that $\|f\|_{\mathcal{A}^{-1}\circ\mathcal{I}^{\mathrm{Lip}}} = \mathrm{Lip}(f)$, as required. $\square$

Next, we demonstrate that Lipschitz left-hand quotients serve as a method to construct Lipschitz ideals. Compare the following result to its linear version stated in [12, Theorems 3.2.2 and 7.2.2].

Theorem 2.5. *Let $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ be a Banach operator ideal and let $[\mathcal{I}^{\mathrm{Lip}}, \|\cdot\|_{\mathcal{I}^{\mathrm{Lip}}}]$ be a (Banach) normed Lipschitz ideal. Then $[\mathcal{A}^{-1}\circ\mathcal{I}^{\mathrm{Lip}}, \|\cdot\|_{\mathcal{A}^{-1}\circ\mathcal{I}^{\mathrm{Lip}}}]$ forms a (Banach) normed Lipschitz ideal. Moreover, $[\mathcal{A}^{-1}\circ\mathcal{I}^{\mathrm{Lip}}, \|\cdot\|_{\mathcal{A}^{-1}\circ\mathcal{I}^{\mathrm{Lip}}}]$ is surjective whenever $[\mathcal{I}^{\mathrm{Lip}}, \|\cdot\|_{\mathcal{I}^{\mathrm{Lip}}}]$ is so.*

Proof. (N1) Consider scalars $\lambda_i \in \mathbb{K}$ and maps $f_i \in \mathcal{A}^{-1}\circ\mathcal{I}^{\mathrm{Lip}}(X,E)$ for $i = 1, 2$. If $\|f_1\|_{\mathcal{A}^{-1}\circ\mathcal{I}^{\mathrm{Lip}}} = 0$, then $\mathrm{Lip}(f_1) = 0$ by Proposition 2.4, and so $f_1 = 0$. Let $A \in \mathcal{A}(E,G)$, where G is an arbitrary Banach space. Since each $A\circ f_i \in \mathcal{I}^{\mathrm{Lip}}(X,G)$ and $\mathcal{I}^{\mathrm{Lip}}(X,G)$ is a linear space, it follows that

$$A\circ(\lambda_1 f_1 + \lambda_2 f_2) = \lambda_1(A\circ f_1) + \lambda_2(A\circ f_2) \in \mathcal{I}^{\mathrm{Lip}}(X,G),$$

which implies that $\lambda_1 f_1 + \lambda_2 f_2 \in \mathcal{A}^{-1}\circ\mathcal{I}^{\mathrm{Lip}}(X,E)$, and an easy calculation yields

$$\begin{aligned}
\|\lambda_1 f_1\|_{\mathcal{A}^{-1}\circ\mathcal{I}^{\mathrm{Lip}}} &= \sup\left\{\|A\circ(\lambda_1 f_1)\|_{\mathcal{I}^{\mathrm{Lip}}} : A \in \mathcal{A}(E,G),\ \|A\|_{\mathcal{A}} \le 1\right\}\\
&= \sup\left\{\|\lambda_1(A\circ f_1)\|_{\mathcal{I}^{\mathrm{Lip}}} : A \in \mathcal{A}(E,G),\ \|A\|_{\mathcal{A}} \le 1\right\}\\
&= |\lambda_1|\,\|f_1\|_{\mathcal{A}^{-1}\circ\mathcal{I}^{\mathrm{Lip}}},
\end{aligned}$$

and

$$\begin{aligned}
\|f_1 + f_2\|_{\mathcal{A}^{-1}\circ\mathcal{I}^{\mathrm{Lip}}} &= \sup\left\{\|A\circ(f_1+f_2)\|_{\mathcal{I}^{\mathrm{Lip}}} : A \in \mathcal{A}(E,G),\ \|A\|_{\mathcal{A}} \le 1\right\}\\
&= \sup\left\{\|(A\circ f_1) + (A\circ f_2)\|_{\mathcal{I}^{\mathrm{Lip}}} : A \in \mathcal{A}(E,G),\ \|A\|_{\mathcal{A}} \le 1\right\}\\
&\le \|f_1\|_{\mathcal{A}^{-1}\circ\mathcal{I}^{\mathrm{Lip}}} + \|f_2\|_{\mathcal{A}^{-1}\circ\mathcal{I}^{\mathrm{Lip}}}.
\end{aligned}$$

Therefore $\|\cdot\|_{\mathcal{A}^{-1}\circ\mathcal{I}^{\mathrm{Lip}}}$ defines a norm on $\mathcal{A}^{-1}\circ\mathcal{I}^{\mathrm{Lip}}(X,E)$.

To establish its completeness when the norms on $\mathcal{A}$ and $\mathcal{I}^{\mathrm{Lip}}$ are complete, suppose that $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(\mathcal{A}^{-1}\circ\mathcal{I}^{\mathrm{Lip}}(X,E), \|\cdot\|_{\mathcal{A}^{-1}\circ\mathcal{I}^{\mathrm{Lip}}})$. Let $A \in \mathcal{A}(E,G)$, where G is any Banach space. By Proposition 2.4, we know that $\mathrm{Lip} \le \|\cdot\|_{\mathcal{A}^{-1}\circ\mathcal{I}^{\mathrm{Lip}}}$ over $\mathcal{A}^{-1}\circ\mathcal{I}^{\mathrm{Lip}}(X,E)$, so $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(\mathrm{Lip}_0(X,E), \mathrm{Lip})$ and therefore there exists a map $f \in \mathrm{Lip}_0(X,E)$ such that $\mathrm{Lip}(f_n - f) \to 0$ as $n \to \infty$. Consequently,

$$\mathrm{Lip}(A\circ f_n - A\circ f) \le \|A\|\,\mathrm{Lip}(f_n - f) \to 0.$$

Also, for any $p, q \in \mathbb{N}$, we have

$$\|A\circ f_p - A\circ f_q\|_{\mathcal{I}^{\mathrm{Lip}}} = \|A\circ(f_p - f_q)\|_{\mathcal{I}^{\mathrm{Lip}}} \le \|A\|_{\mathcal{A}}\,\|f_p - f_q\|_{\mathcal{A}^{-1}\circ\mathcal{I}^{\mathrm{Lip}}},$$

showing $(A\circ f_n)$ is a Cauchy sequence in $(\mathcal{I}^{\mathrm{Lip}}(X,G), \|\cdot\|_{\mathcal{I}^{\mathrm{Lip}}})$. As this last space is complete, there exists $g \in \mathcal{I}^{\mathrm{Lip}}(X,G)$ such that $\|A\circ f_n - g\|_{\mathcal{I}^{\mathrm{Lip}}} \to 0$ as $n \to \infty$, and since $\mathrm{Lip} \le \|\cdot\|_{\mathcal{I}^{\mathrm{Lip}}}$ over $\mathcal{I}^{\mathrm{Lip}}(X,G)$, it follows that $A\circ f = g$, and so $f \in \mathcal{A}^{-1}\circ\mathcal{I}^{\mathrm{Lip}}(X,E)$ and $\|A\circ f_n - A\circ f\|_{\mathcal{I}^{\mathrm{Lip}}} \to 0$ as $n \to \infty$.

To prove that $(f_n)_{n\in\mathbb{N}}$ converges to f in $(\mathcal{A}^{-1}\circ\mathcal{I}^{\mathrm{Lip}}(X,E), \|\cdot\|_{\mathcal{A}^{-1}\circ\mathcal{I}^{\mathrm{Lip}}})$, let $\varepsilon > 0$. Then there exists $m \in \mathbb{N}$ such that $\|f_p - f_q\|_{\mathcal{A}^{-1}\circ\mathcal{I}^{\mathrm{Lip}}} < \varepsilon/2$ for all $p, q \ge m$. Hence we have that

$$\|A\circ f_p - A\circ f_{p+n}\|_{\mathcal{I}^{\mathrm{Lip}}} < \|A\|_{\mathcal{A}}\frac{\varepsilon}{2}$$

for all $p \geq m$, $n \in \mathbb{N}$ and $A \in \mathcal{A}(F,G)$. Taking limits with $n \to \infty$, it follows that

$$\|A\circ (f_p - f)\|_{\mathcal{I}^{\mathrm{Lip}}} = \|A\circ f_p - A\circ f\|_{\mathcal{I}^{\mathrm{Lip}}} \leq \|A\|_{\mathcal{A}}\frac{\varepsilon}{2}$$

for all $p \geq m$ and $A \in \mathcal{A}(E,G)$. Taking supremum over all $A \in \mathcal{A}(E,G)$ such that $\|A\|_{\mathcal{A}} \leq 1$, we get that $\|f_p - f\|_{\mathcal{A}^{-1}\circ\mathcal{I}^{\mathrm{Lip}}} < \varepsilon$ for all $p \geq m$, as desired.

(N2) Let $g \in X^{\#}$ and $e \in E$. For any $A \in \mathcal{A}(E,G)$, where G is an arbitrary Banach space, note that $A\circ(g\cdot e) = g\cdot A(e)$. Since $[\mathcal{I}^{\mathrm{Lip}}, \|\cdot\|_{\mathcal{I}^{\mathrm{Lip}}}]$ is a normed Lipschitz ideal, the map $g\cdot A(e)$ belongs to $\mathcal{I}^{\mathrm{Lip}}(X,G)$ and $\|g\cdot A(e)\|_{\mathcal{I}^{\mathrm{Lip}}} = \mathrm{Lip}(g)\|A(e)\|$. Therefore $A\circ(g\cdot e) \in \mathcal{I}^{\mathrm{Lip}}(X,G)$ and $\|A\circ(g\cdot e)\|_{\mathcal{I}^{\mathrm{Lip}}} = \mathrm{Lip}(g)\,\|A(e)\|$. Hence $g\cdot e \in \mathcal{A}^{-1}\circ\mathcal{I}^{\mathrm{Lip}}(X,E)$ and since

$$\|A\circ(g\cdot e)\|_{\mathcal{I}^{\mathrm{Lip}}} \leq \mathrm{Lip}(g)\,\|A\|\,\|e\| \leq \mathrm{Lip}(g)\,\|A\|_{\mathcal{A}}\,\|e\|,$$

it follows that $\|g\cdot e\|_{\mathcal{A}^{-1}\circ\mathcal{I}^{\mathrm{Lip}}} \leq \mathrm{Lip}(g)\,\|e\|$. For the converse inequality, note that $\mathrm{Lip}(g)\,\|e\| = \mathrm{Lip}(g\cdot e) \leq \|g\cdot e\|_{\mathcal{A}^{-1}\circ\mathcal{I}^{\mathrm{Lip}}}$ by Proposition 2.4.

(N3) Let Y be a pointed metric space, let F be a Banach space, let $h \in \mathrm{Lip}_0(Y,X)$, $f \in \mathcal{A}^{-1}\circ\mathcal{I}^{\mathrm{Lip}}(X,E)$, and let $T \in \mathcal{L}(E,F)$. Clearly, $T\circ f\circ h \in \mathrm{Lip}_0(Y,F)$. If $A \in \mathcal{A}(F,H)$, where H is an arbitrary Banach space, then $A\circ T \in \mathcal{A}(E,H)$ and $\|A\circ T\|_{\mathcal{A}} \leq \|A\|_{\mathcal{A}}\,\|T\|$ by the ideal property of $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$. Hence $A\circ T\circ f \in \mathcal{I}^{\mathrm{Lip}}(X,H)$ and

$$\|A\circ T\circ f\|_{\mathcal{I}^{\mathrm{Lip}}} \leq \|A\circ T\|_{\mathcal{A}}\,\|f\|_{\mathcal{A}^{-1}\circ\mathcal{I}^{\mathrm{Lip}}}.$$

Applying the ideal property of $[\mathcal{I}^{\mathrm{Lip}}, \|\cdot\|_{\mathcal{I}^{\mathrm{Lip}}}]$, we deduce that $A\circ T\circ f\circ h \in \mathcal{I}^{\mathrm{Lip}}(Y,H)$ with

$$\|A\circ T\circ f\circ h\|_{\mathcal{I}^{\mathrm{Lip}}} \leq \|A\circ T\circ f\|_{\mathcal{I}^{\mathrm{Lip}}}\,\mathrm{Lip}(h),$$

and so $T\circ f\circ h \in \mathcal{A}^{-1}\circ\mathcal{I}^{\mathrm{Lip}}(Y,F)$. Finally, since

$$\|A\circ T\circ f\circ h\|_{\mathcal{I}^{\mathrm{Lip}}} \leq \|A\|_{\mathcal{A}}\,\|T\|\,\|f\|_{\mathcal{A}^{-1}\circ\mathcal{I}^{\mathrm{Lip}}}\,\mathrm{Lip}(h)$$

for all $A \in \mathcal{A}(F,H)$, we conclude that

$$\|T\circ f\circ h\|_{\mathcal{A}^{-1}\circ\mathcal{I}^{\mathrm{Lip}}} \leq \|T\|\,\|f\|_{\mathcal{A}^{-1}\circ\mathcal{I}^{\mathrm{Lip}}}\,\mathrm{Lip}(h).$$

(S) Let $f \in \mathrm{Lip}_0(X,E)$ and let $\pi \in \mathrm{Lip}_0(Y,X)$ be a surjective map with $\mathrm{Lip}(\pi) \leq 1$ for some pointed metric space Y such that $f\circ\pi \in \mathcal{A}^{-1}\circ\mathcal{I}^{\mathrm{Lip}}(Y,E)$. Then, for any Banach space G and any $A \in \mathcal{A}(E,G)$, one has that $A\circ f\circ\pi \in \mathcal{I}^{\mathrm{Lip}}(Y,G)$. Since the Lipschitz ideal $[\mathcal{I}^{\mathrm{Lip}}, \|\cdot\|_{\mathcal{I}^{\mathrm{Lip}}}]$ is surjective, it follows that $A\circ f \in \mathcal{I}^{\mathrm{Lip}}(X,G)$ and $\|A\circ f\|_{\mathcal{I}^{\mathrm{Lip}}} = \|A\circ f\circ\pi\|_{\mathcal{I}^{\mathrm{Lip}}}$. Hence $f \in \mathcal{A}^{-1}\circ\mathcal{I}^{\mathrm{Lip}}(X,E)$ and

$$\|A\circ f\|_{\mathcal{I}^{\mathrm{Lip}}} = \|A\circ f\circ\pi\|_{\mathcal{I}^{\mathrm{Lip}}} \leq \|A\|_{\mathcal{A}}\,\|f\circ\pi\|_{\mathcal{A}^{-1}\circ\mathcal{I}^{\mathrm{Lip}}}.$$

Taking the supremum over all Banach spaces G and all operators $A \in \mathcal{A}(E,G)$ with $\|A\|_{\mathcal{A}} \leq 1$, we deduce that $\|f\|_{\mathcal{A}^{-1}\circ\mathcal{I}^{\mathrm{Lip}}} \leq \|f\circ\pi\|_{\mathcal{A}^{-1}\circ\mathcal{I}^{\mathrm{Lip}}}$. The converse inequality follows from (N3). $\square$

The next theorem establishes a closely connection between the elements of a Lipschitz left-hand quotient ideal $\mathcal{A}^{-1}\circ\mathcal{I}^{\mathrm{Lip}}$ with their linearizations under a mild condition on $\mathcal{I}^{\mathrm{Lip}}$.

Theorem 2.6. *Let $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ be a Banach operator ideal, let $[\mathcal{I}, \|\cdot\|_{\mathcal{I}}]$ a normed operator ideal, and let $[\mathcal{I}^{\mathrm{Lip}}, \|\cdot\|_{\mathcal{I}^{\mathrm{Lip}}}]$ be a normed Lipschitz ideal that satisfies the linearization property with respect to $\mathcal{I}$. For every $f \in \mathrm{Lip}_0(X, E)$, the following conditions are equivalent:*

(i) *f belongs to $\mathcal{A}^{-1}\circ\mathcal{I}^{\mathrm{Lip}}(X, E)$.*
(ii) *T_f belongs to $\mathcal{A}^{-1}\circ\mathcal{I}(\mathcal{F}(X), E)$.*

In such a case, $\|f\|_{\mathcal{A}^{-1}\circ\mathcal{I}^{\mathrm{Lip}}} = \|T_f\|_{\mathcal{A}^{-1}\circ\mathcal{I}}$. Furthermore, the map $f \mapsto T_f$ is an isometric isomorphism from $(\mathcal{A}^{-1}\circ\mathcal{I}^{\mathrm{Lip}}(X, E), \|\cdot\|_{\mathcal{A}^{-1}\circ\mathcal{I}^{\mathrm{Lip}}})$ onto $(\mathcal{A}^{-1}\circ\mathcal{I}(\mathcal{F}(X), E), \|\cdot\|_{\mathcal{A}^{-1}\circ\mathcal{I}})$.

Proof. $(i) \Rightarrow (ii)$: Let $f \in \mathcal{A}^{-1}\circ\mathcal{I}^{\mathrm{Lip}}(X, E)$. For any $A \in \mathcal{A}(E, G)$, where G is an arbitrary Banach space, we have that $A \circ f \in \mathcal{I}^{\mathrm{Lip}}(X, G)$. As shown in the proof of Proposition 2.4, we can ensure the existence of two operators $T_f \in \mathcal{L}(\mathcal{F}(X), E)$ and $T_{A\circ f} \in \mathcal{L}(\mathcal{F}(X), G)$ such that $T_{A\circ f} = A \circ T_f$. Since $\mathcal{I}^{\mathrm{Lip}}$ satisfies the linearization property in $\mathcal{I}$, we deduce that $T_{A\circ f} \in \mathcal{I}(\mathcal{F}(X), G)$ and $\|T_{A\circ f}\|_{\mathcal{I}} = \|A \circ f\|_{\mathcal{I}^{\mathrm{Lip}}}$. Because A was chosen arbitrarily from $\mathcal{A}(E, G)$, it follows that $T_f \in \mathcal{A}^{-1}\circ\mathcal{I}(\mathcal{F}(X), G)$. Moreover, we have

$$\|A \circ T_f\|_{\mathcal{I}} = \|T_{A\circ f}\|_{\mathcal{I}} = \|A \circ f\|_{\mathcal{I}^{\mathrm{Lip}}} \leq \|A\|_{\mathcal{A}} \|f\|_{\mathcal{A}^{-1}\circ\mathcal{I}^{\mathrm{Lip}}},$$

and passing to the supremum over all Banach spaces G and all operators $A \in \mathcal{A}(E, G)$ with $\|A\|_{\mathcal{A}} \leq 1$, we deduce that $\|T_f\|_{\mathcal{A}^{-1}\circ\mathcal{I}} \leq \|f\|_{\mathcal{A}^{-1}\circ\mathcal{I}^{\mathrm{Lip}}}$.

$(ii) \Rightarrow (i)$: Assume that $T_f \in \mathcal{A}^{-1}\circ\mathcal{I}(\mathcal{F}(X), E)$. Then, for all $A \in \mathcal{A}(E, G)$, where G is an arbitrary Banach space, we have $A \circ T_f \in \mathcal{I}(\mathcal{F}(X), G)$. Consequently, since $T_{A\circ f} = A \circ T_f$, it follows that $T_{A\circ f} \in \mathcal{I}(\mathcal{F}(X), G)$. Given that $\mathcal{I}^{\mathrm{Lip}}$ satisfies the linearization property in $\mathcal{I}$, we deduce that $A \circ f \in \mathcal{I}^{\mathrm{Lip}}(X, G)$. Therefore $f \in \mathcal{A}^{-1}\circ\mathcal{I}^{\mathrm{Lip}}(X, E)$, and since

$$\|A \circ f\|_{\mathcal{I}^{\mathrm{Lip}}} = \|T_{A\circ f}\|_{\mathcal{I}} = \|A \circ T_f\|_{\mathcal{I}} \leq \|A\|_{\mathcal{A}} \|T_f\|_{\mathcal{A}^{-1}\circ\mathcal{I}},$$

we conclude that $\|f\|_{\mathcal{A}^{-1}\circ\mathcal{I}^{\mathrm{Lip}}} \leq \|T_f\|_{\mathcal{A}^{-1}\circ\mathcal{I}}$.

For the final part of the statement, it is sufficient to prove the surjectivity of the mapping $f \mapsto T_f$ from $\mathcal{A}^{-1}\circ\mathcal{I}^{\mathrm{Lip}}(X, E)$ into $\mathcal{A}^{-1}\circ\mathcal{I}(\mathcal{F}(X), E)$. To do this, let $S \in \mathcal{A}^{-1}\circ\mathcal{I}(\mathcal{F}(X), E)$. For any $A \in \mathcal{A}(E, G)$, where G is a Banach space, we have that $A \circ S \in \mathcal{I}(\mathcal{F}(X), G)$. By Theorem 1.2, there exists $g \in \mathrm{Lip}_0(X, G)$ such that $A \circ S = T_g$. Since $\mathcal{I}^{\mathrm{Lip}}$ satisfies the linearization property with respect to $\mathcal{I}$, it follows that $g \in \mathcal{I}^{\mathrm{Lip}}(X, G)$. Clearly, the mapping $f := S \circ \delta_X$ belongs to $\mathrm{Lip}_0(X, E)$, and for any $A \in \mathcal{A}(E, G)$, we get that

$$A \circ f = A \circ S \circ \delta_X = T_g \circ \delta_X = g \in \mathcal{I}^{\mathrm{Lip}}(X, G).$$

Thus, we conclude that $f \in \mathcal{A}^{-1}\circ\mathcal{I}^{\mathrm{Lip}}(X, E)$ and that $T_f = S$, as desired. ☐

We now recall the method for generating Lipschitz ideals through composition ([2, 15]). Given an operator ideal $\mathcal{A}$, a map $f \in \mathrm{Lip}_0(X, E)$ belongs to the composition ideal $\mathcal{A}\circ\mathrm{Lip}_0$, denoted as $f \in \mathcal{A}\circ\mathrm{Lip}_0(X, E)$, if there exist a Banach space G, an operator $A \in \mathcal{A}(G, E)$, and a map $g \in \mathrm{Lip}_0(X, G)$, such that $f = A \circ g$. If $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ is a normed operator ideal and $f \in \mathcal{A} \circ \mathrm{Lip}_0(X, E)$, the formula

$$\|f\|_{\mathcal{A}\circ\mathrm{Lip}_0} = \inf \left\{\mathrm{Lip}(g) \, \|A\|_{\mathcal{A}}\right\},$$

defines a norm on $\mathcal{A} \circ \mathrm{Lip}_0(X, E)$, where the infimum is taken over all factorizations of f as described above.

Left-hand quotients of an operator ideal and a Lipschitz ideal that satisfies the linearization property in a particular operator ideal can be viewed as a composition ideal.

Proposition 2.7. *Let $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ be a Banach operator ideal, let $[\mathcal{I}, \|\cdot\|_{\mathcal{I}}]$ be a normed operator ideal, and let $[\mathcal{I}^{\mathrm{Lip}}, \|\cdot\|_{\mathcal{I}^{\mathrm{Lip}}}]$ be a normed Lipschitz ideal that satisfies the linearization property in $\mathcal{I}$. Then*

$$[\mathcal{A}^{-1} \circ \mathcal{I}^{\mathrm{Lip}}, \|\cdot\|_{\mathcal{A}^{-1}\circ\mathcal{I}^{\mathrm{Lip}}}] = [(\mathcal{A}^{-1} \circ \mathcal{I}) \circ \mathrm{Lip}_0, \|\cdot\|_{(\mathcal{A}^{-1}\circ\mathcal{I})\circ\mathrm{Lip}_0}].$$

Proof. Given $f \in \mathrm{Lip}_0(X, E)$, applying [2, Proposition 3.2] and Theorem 2.6, we obtain that

$$\begin{aligned} f \in (\mathcal{A}^{-1} \circ \mathcal{I}) \circ \mathrm{Lip}_0(X, E) \quad &\Leftrightarrow \quad T_f \in \mathcal{A}^{-1} \circ \mathcal{I}(\mathcal{F}(X), E) \\ &\Leftrightarrow \quad f \in \mathcal{A}^{-1} \circ \mathcal{I}^{\mathrm{Lip}}(X, E). \end{aligned}$$

and

$$\|f\|_{(\mathcal{A}^{-1}\circ\mathcal{I})\circ\mathrm{Lip}_0} = \|T_f\|_{\mathcal{A}^{-1}\circ\mathcal{I}} = \|f\|_{\mathcal{A}^{-1}\circ\mathcal{I}^{\mathrm{Lip}}}.$$

□

3. Examples of Lipschitz left-hand quotient ideals

In this section, we provide two examples of Lipschitz left-hand quotient ideals generated by both an operator ideal and a Lipschitz ideal, specifically the spaces of Lipschitz mappings from a pointed metric space into a Banach space whose Lipschitz images are Grothendieck sets or Rosenthal sets.

Let us recall that an operator $T \in \mathcal{L}(E, F)$ is considered compact, weakly compact, separable, Rosenthal, or Grothendieck if $T(B_E)$ is a relatively compact, relatively weakly compact, separable, Rosenthal, or Grothendieck subset of F, respectively. The corresponding operator ideals between E and F are denoted as $\mathcal{K}(E, F)$ for compact operators, $\mathcal{W}(E, F)$ for weakly compact operators, $\mathcal{S}(E, F)$ for separable bounded operators, $\mathcal{R}(E, F)$ for Rosenthal operators, and $\mathcal{G}(E, F)$ for Grothendieck operators. These operator ideals are known to satisfy the following well-established inclusions:

$$\mathcal{K}(E, F) \subseteq \mathcal{W}(E, F) \subseteq \mathcal{R}(E, F), \ \mathcal{W}(E, F) \subseteq \mathcal{G}(E, F), \ \mathcal{K}(E, F) \subseteq \mathcal{S}(E, F).$$

For comprehensive coverage of these operator ideals, Pietsch's monograph [12] provides an extensive analysis.

A set $K \subseteq E$ is called Grothendieck if for every operator $T \in \mathcal{L}(E, c_0)$, the image $T(K)$ is a relatively weakly compact subset of c_0. It is well known that $\mathcal{G}$ generates a closed surjective operator ideal (see for example [7]). For a detailed discussion on the Grothendieck property, the reader is referred to [8].

According to [9], the spaces $\mathrm{Lip}_{0\mathcal{W}}(X,E)$ and $\mathrm{Lip}_{0\mathcal{K}}(X,E)$ comprise all Lipschitz maps from X to E whose Lipschitz images are relatively weakly compact and relatively compact in E, respectively. As stated in [9, Propositions 2.1 and 2.2], both $\mathrm{Lip}_{0\mathcal{K}}$ and $\mathrm{Lip}_{0\mathcal{W}}$ are identified as Lipschitz ideals that uphold the linearization property within $\mathcal{K}$ and $\mathcal{W}$, respectively. This motivates the following concept (see [17, Section 2.9]).

Definition 3.1. A map $f \in \mathrm{Lip}_0(X,E)$ is said to be Grothendieck (respectively, Rosenthal) if its Lipschitz image $\mathrm{Im}_{\mathrm{Lip}}(f)$ is a Grothendieck (respectively, Rosenthal) subset of E. Let $\mathrm{Lip}_{0\mathcal{G}}(X,E)$ and $\mathrm{Lip}_{0\mathcal{R}}(X,E)$ denote the spaces of Grothendieck Lipschitz maps and Rosenthal Lipschitz maps from X to E, respectively.

We now proceed to establish that $\mathrm{Lip}_{0\mathcal{G}}$ also satisfies the linearization property with respect to the operator ideal $\mathcal{G}$.

Theorem 3.2. *Given a map $f \in \mathrm{Lip}_0(X,E)$, the following statements are equivalent:*

(i) *f belongs to $\mathrm{Lip}_{0\mathcal{G}}(X,E)$.*
(ii) *T_f belongs to $\mathcal{G}(\mathcal{F}(X),E)$.*

Moreover, $f \mapsto T_f$ is an isometric isomorphism from $(\mathrm{Lip}_{0\mathcal{G}}(X,E),\mathrm{Lip})$ onto $(\mathcal{G}(\mathcal{F}(X),E),\|\cdot\|)$.

Proof. Note first that a look at Theorem 1.2 yields the following relations:

$$\begin{aligned}\mathrm{Im}_{\mathrm{Lip}}(f) = T_f(\mathrm{Im}_{\mathrm{Lip}}(\delta_X)) &\subseteq T_f(\overline{\mathrm{abco}}(\mathrm{Im}_{\mathrm{Lip}}(\delta_X))) = T_f(B_{\mathcal{F}(X)})\\ &\subseteq \overline{\mathrm{abco}}(T_f(\mathrm{Im}_{\mathrm{Lip}}(\delta_X))) = \overline{\mathrm{abco}}(\mathrm{Im}_{\mathrm{Lip}}(f)).\end{aligned}$$

$(i) \Rightarrow (ii)$: If $f \in \mathrm{Lip}_{0\mathcal{G}}(X,E)$, then $\mathrm{Im}_{\mathrm{Lip}}(f)$ is a Grothendieck subset of E. Since the norm-closed absolutely convex hull of a Grothendieck set remains Grothendieck, the second inclusion in the above chain of relations shows that $T_f(B_{\mathcal{F}(X)})$ is Grothendieck in E and thus $T_f \in \mathcal{G}(\mathcal{F}(X),E)$.

$(ii) \Rightarrow (i)$: Assume that $T_f \in \mathcal{G}(\mathcal{F}(X),E)$, that is, $T_f(B_{\mathcal{F}(X)})$ is a Grothendieck subset of E. Now, by the first inclusion in the cited chain, it follows that $\mathrm{Im}_{\mathrm{Lip}}(f)$ is Grothendieck in E, that is, $f \in \mathrm{Lip}_{0\mathcal{G}}(X,E)$.

Applying what we have justly proved and that the correspondence $f \mapsto T_f$ is an isometric isomorphism from $(\mathrm{Lip}_0(X,E),\mathrm{Lip})$ onto $(\mathcal{L}(\mathcal{F}(X),E),\|\cdot\|)$ by Theorem 1.2, the last assertion of the statement holds. $\square$

Recall that a Lipschitz ideal $\mathcal{I}^{\mathrm{Lip}}$ is considered closed if every component $\mathcal{I}^{\mathrm{Lip}}(X,E)$ is a closed subspace of $\mathrm{Lip}_0(X,E)$ when equipped with the topology induced by the Lipschitz norm.

An application of [2, Proposition 3.2] shows that $[\mathrm{Lip}_{0\mathcal{G}},\mathrm{Lip}]$ generates a Lipschitz ideal of composition type which inherits the properties of $[\mathcal{G},\|\cdot\|]$ by [2, Corollary 3.3]

Corollary 3.3. $[\mathrm{Lip}_{0\mathcal{G}},\mathrm{Lip}] = [\mathcal{G}\circ\mathrm{Lip}_0,\|\cdot\|_{\mathcal{G}\circ\mathrm{Lip}_0}]$ *and, as a consequence,* $[\mathrm{Lip}_{0\mathcal{G}},\mathrm{Lip}]$ *is a closed surjective Lipschitz ideal.*

We are now in a position to explicitly characterize the space of all Grothendieck Lipschitz mappings using the framework of a Lipschitz left-hand quotient ideal.

Theorem 3.4. $[\mathrm{Lip}_{0\mathcal{G}}, \mathrm{Lip}] = [\mathcal{S}^{-1} \circ \mathrm{Lip}_{0\mathcal{W}}, \|\cdot\|_{\mathcal{S}^{-1}\circ\mathrm{Lip}_{0\mathcal{W}}}]$.

Proof. Let $f \in \mathrm{Lip}_0(X, E)$. By applying Theorem 3.2, [12, 3.2.6], and [9, Proposition 2.2] with Theorem 2.6, respectively, we can derive the following chain of equivalences:

$$\begin{aligned} f \in \mathrm{Lip}_{0\mathcal{G}}(X, E) \quad &\Leftrightarrow \quad T_f \in \mathcal{G}(\mathcal{F}(X), E) \\ &\Leftrightarrow \quad T_f \in \mathcal{S}^{-1} \circ \mathcal{W}(\mathcal{F}(X), E) \\ &\Leftrightarrow \quad f \in \mathcal{S}^{-1} \circ \mathrm{Lip}_{0\mathcal{W}}(X, E). \end{aligned}$$

Moreover, under these conditions, the following equalities of norms are satisfied:

$$\mathrm{Lip}(f) = \|T_f\| = \|T_f\|_{\mathcal{S}^{-1}\circ\mathcal{W}} = \|f\|_{\mathcal{S}^{-1}\circ\mathrm{Lip}_{0\mathcal{W}}}.$$

□

In order to address the Rosenthal case, let us recall that a subset $A \subseteq E$ is defined as conditionally weakly compact (or Rosenthal) if every sequence in A has a weak Cauchy subsequence. It is known (see, for example, [3]) that an operator $T \in \mathcal{L}(E, F)$ is Rosenthal if and only if it factors through a Banach space that does not contain an isomorphic copy of ℓ_1. We will now prove a similar result for Lipschitz maps whose Lipschitz image is a Rosenthal set, and the Rosenthal factorization theorem then allows us to factor these mappings through a Banach space not containing an isomorphic copy of ℓ_1.

Theorem 3.5. *Given a map $f \in \mathrm{Lip}_0(X, E)$, the following assertions are equivalent:*

(i) *f belongs to $\mathrm{Lip}_{0\mathcal{R}}(X, E)$.*
(ii) *T_f belongs to $\mathcal{R}(\mathcal{F}(X), E)$.*
(iii) *There exist a Banach space F which does not contain an isomorphic copy of ℓ_1, an operator $T \in \mathcal{L}(F, E)$ and a mapping $g \in \mathrm{Lip}_0(X, F)$ such that $f = T \circ g$.*

Proof. $(i) \Leftrightarrow (ii)$: It follows as the proof of Theorem 3.2 taking into account that the norm-closed absolutely convex hull of a Rosenthal subset of a Banach space is itself Rosenthal (see [11, p. 357]).

$(ii) \Rightarrow (iii)$: Rosenthal factorization theorem (see, e.g., [3]) gives a Banach space F not containing an isomorphic copy of ℓ_1, and both operators $T \in \mathcal{L}(F, E)$ and $S \in \mathcal{L}(\mathcal{F}(X), F)$ such that $T_f = T \circ S$. If we write $g := S \circ \delta_X \in \mathrm{Lip}_0(X, F)$, then $f = T_f \circ \delta_X = T \circ S \circ \delta_X = T \circ g$.

$(iii) \Rightarrow (i)$: Note that $\mathrm{Im}_{\mathrm{Lip}}(f) = T(\mathrm{Im}_{\mathrm{Lip}}(g))$, where T is weak-to-weak continuous and $\mathrm{Im}_{\mathrm{Lip}}(g)$ is Rosenthal in F by Rosenthal's ℓ_1-theorem [14]. Indeed, notice that $\mathrm{Im}_{\mathrm{Lip}}(g)$ is a bounded subset of a Banach space F not containing an isomorphic copy of ℓ_1. □

Since $\mathcal{R}$ is a closed surjective operator ideal (see [7]), the combination of Proposition 3.2 and Corollary 3.3 in [2] can be applied to establish the following.

Corollary 3.6. $[\mathrm{Lip}_{0\mathcal{R}}, \mathrm{Lip}] = [\mathcal{R} \circ \mathrm{Lip}_0, \|\cdot\|_{\mathcal{R}\circ\mathrm{Lip}_0}]$ *and, consequently,* $[\mathrm{Lip}_{0\mathcal{R}}, \mathrm{Lip}]$ *is a closed surjective Lipschitz ideal.*

Now, we recall that an operator $f \in \mathcal{L}(E, F)$ is referred to as completely continuous if every weakly convergent sequence $(x_n)_{n\in\mathbb{N}}$ is mapped to a norm-convergent sequence $(T(x_n))_{n\in\mathbb{N}}$. Let $\mathcal{V}(E, F)$ denote the space of all completely continuous operators from E to F. According to [12, 1.6.2 and 4.2.5], $\mathcal{V}$ constitutes a closed operator ideal.

We now characterize the subclass of Lipschitz mappings whose Lipschitz images are Rosenthal as a Lipschitz left-hand quotient ideal generated by the operator ideal $\mathcal{V}$ and the Lipschitz ideal $\mathrm{Lip}_{0\mathcal{K}}$.

Theorem 3.7. $[\mathrm{Lip}_{0\mathcal{R}}, \mathrm{Lip}] = [\mathcal{V}^{-1} \circ \mathrm{Lip}_{0\mathcal{K}}, \|\cdot\|_{\mathcal{V}^{-1}\circ\mathrm{Lip}_{0\mathcal{K}}}]$.

Proof. Take $f \in \mathrm{Lip}_0(X, E)$. By applying Theorem 3.5, [12, 3.2.4], and [9, Proposition 3.5] with Theorem 2.6, we derive the following:

$$\begin{aligned} f \in \mathrm{Lip}_{0\mathcal{R}}(X, E) &\Leftrightarrow T_f \in \mathcal{R}(\mathcal{F}(X), E) \\ &\Leftrightarrow T_f \in \mathcal{V}^{-1} \circ \mathcal{K}(\mathcal{F}(X), E) \\ &\Leftrightarrow f \in \mathcal{V}^{-1} \circ \mathrm{Lip}_{0\mathcal{K}}(X, E). \end{aligned}$$

In such a case, the equalities $\mathrm{Lip}(f) = \|T_f\| = \|T_f\|_{\mathcal{V}^{-1}\circ\mathcal{K}} = \|f\|_{\mathcal{V}^{-1}\circ\mathrm{Lip}_{0\mathcal{K}}}$ hold. □

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