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NONLINEAR MAPS MIXED JORDAN TRIPLE $*$-DERIVATIONS ON FACTOR VON NEUMANN ALGEBRAS

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Abstract. Let $\mathcal{A}$ be a factor von Neumann algebra with $\dim(\mathcal{A}) \geq 2$. In this paper, we prove that a map $\Phi : \mathcal{A} \longrightarrow \mathcal{A}$ (not necessarily linear) satisfies $\Phi(A \blacklozenge B \lozenge C) = \Phi(A) \blacklozenge B \lozenge C + A \blacklozenge \Phi(B) \lozenge C + A \blacklozenge B \lozenge \Phi(C)$ for all $A, B, C \in \mathcal{A}$, if and only if Φ is an additive $*$-derivation.

1. Introduction and preliminaries

Let $\mathcal{A}$ be an associative $*$-algebra over the complex field $\mathbb{C}$. A map $\Phi : \mathcal{A} \longrightarrow \mathcal{A}$ is called an additive $*$-derivation if it is an additive derivation and satisfies $\Phi(A^*) = \Phi(A)^*$ for all $A \in \mathcal{A}$.

For $A, B \in \mathcal{A}$, we define the Lie product $[A, B]$, skew Lie product $[A, B]_*$ and the Jordan $*$-product $A \bullet B$ by

$$[A, B] = AB - BA, \quad [A, B]_* = AB - BA^*, \text{ and } A \bullet B = AB + BA^*, \text{ respectively.}$$

We also define $A \blacklozenge B$ and $A \lozenge B$, by For $A, B \in \mathcal{A}$,

$$A \blacklozenge B = A^*B + B^*A \text{ and, } A \lozenge B = A^*B - B^*A, \text{ respectively.}$$

These products have recently attracted the attention of many authors (see for example, [1, 2, 3, 4]). Liu and Ji [7] proved that a bijective map Φ on factor von Neumann algebras preserves the product $\blacklozenge$ if and only if Φ is a $*$-isomorphism. Zhou et al. [9] studied the structure of nonlinear mixed Lie triple derivations on prime $*$-algebras. They proved that any map Φ on a unital $*$-algebra $\mathcal{A}$, containing a nontrivial projection, satisfying

$$\Phi([[A, B]_*, C]) = [[\Phi(A), B]_*, C] + [[A, \Phi(B)]_*, C] + [[A, B]_*, \Phi(C)] \ (A, B, C \in \mathcal{A}),$$

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is an additive $*$-derivation. Li et al. [6] studied the nonlinear maps preserving the mixed product $[A\bullet B, C]_*$ on von Neumann algebras. Zhang [10] studied nonlinear maps that preserve the mixed product $[A, B]_*\bullet C$ on factor von Neumann algebras. Recently, Li and Zhang [5] proved that Φ is a nonlinear mixed Jordan triple $*$-derivation on factor von Neumann algebras if and only if Φ is an additive $*$-derivation.

In this paper, we prove that a map $\Phi : \mathcal{A} \longrightarrow \mathcal{A}$ satisfies

$$\Phi\left(A\blacklozenge B\lozenge C\right) = \Phi\left(A\right)\blacklozenge B\lozenge C + A\blacklozenge\Phi\left(B\right)\lozenge C + A\blacklozenge B\lozenge\Phi\left(C\right) \;\; \left(A, B, C \in \mathcal{A}\right),$$

if and only if Φ is an additive $*$-derivation.

2. Main Results

To prove our main result we need to quote the following lemma from [8].

Lemma 2.1. *Let $\mathcal{A}$ be a von Neumann algebra with no central abelian projections. Then there exists a projection $P \in \mathcal{A}$ such that $\underline{P} = 0$ and $\overline{P} = I$.*

Theorem 2.2. *Let $\mathcal{A}$ be a factor von Neumann algebra with $dim\left(\mathcal{A}\right) \geq 2$. Then a map $\Phi : \mathcal{A} \longrightarrow \mathcal{A}$ satisfies*

$$\Phi\left(A\blacklozenge B\lozenge C\right) = \Phi\left(A\right)\blacklozenge B\lozenge C + A\blacklozenge\Phi\left(B\right)\lozenge C + A\blacklozenge B\lozenge\Phi\left(C\right) \;\; \left(A, B, C \in \mathcal{A}\right),$$

if and only if Φ is an additive $*$-derivation.

Proof. The necessity is clear. The proof of sufficiency is divided into several steps.

Step 1: It is clear that $\Phi\left(0\right) = 0$.

Step 2: $\Phi\left(\frac{I}{2}\right)^* = \Phi\left(\frac{I}{2}\right) \in Z\left(\mathcal{A}\right)$.
For every $A \in \mathcal{A}$, we have

$$0 = \Phi\left(A\blacklozenge\frac{I}{2}\lozenge\frac{I}{2}\right) = A\blacklozenge\frac{I}{2}\lozenge\Phi\left(\frac{I}{2}\right) = \left(\frac{A+A^*}{2}\right)\Phi\left(\frac{I}{2}\right) - \Phi\left(\frac{I}{2}\right)^*\left(\frac{A+A^*}{2}\right).$$

By putting $A = I$ in the latter identity, we can conclude that $\Phi\left(\frac{I}{2}\right) = \Phi\left(\frac{I}{2}\right)^*$.

Therefore, $\Phi\left(\frac{I}{2}\right) \in Z\left(\mathcal{A}\right)$.

Step 3: $\Phi\left(A\right) = \Phi\left(A\right)^*$, for every self-adjoint $A \in \mathcal{A}$.
Let $A \in \mathcal{A}$, then

$$\begin{aligned}
\Phi\left(\frac{1}{2}\left(A - A^*\right)\right) &= \Phi\left(\frac{I}{2}\blacklozenge\frac{I}{2}\lozenge A\right)\\
&= \Phi\left(\frac{I}{2}\right)\blacklozenge\frac{I}{2}\lozenge A + \frac{I}{2}\blacklozenge\Phi\left(\frac{I}{2}\right)\lozenge A + \frac{I}{2}\blacklozenge\frac{I}{2}\lozenge\Phi\left(A\right)\\
&= 2\left(\Phi\left(\frac{I}{2}\right)A - A^*\Phi\left(\frac{I}{2}\right)\right) + \frac{I}{2}\left(\Phi\left(A\right) - \Phi\left(A\right)^*\right).
\end{aligned}$$

If $A = A^*$, then from the latter identity we arrive at $\Phi\left(A\right) = \Phi\left(A\right)^*$.

Step 4: $\Phi\left(\frac{iI}{2}\right)^{*}=-\Phi\left(\frac{iI}{2}\right)\in Z(\mathcal{A})$.
For every $A\in\mathcal{A}_s$ we have

$$\begin{aligned}0=\Phi\left(\frac{iI}{2}\blacklozenge A\lozenge\frac{iI}{2}\right)&=\Phi\left(\frac{iI}{2}\right)\blacklozenge A\lozenge\frac{iI}{2}+\frac{iI}{2}\blacklozenge\Phi(A)\lozenge\frac{iI}{2}+\frac{iI}{2}\blacklozenge A\lozenge\Phi\left(\frac{iI}{2}\right)\\&=\Phi\left(\frac{iI}{2}\right)\blacklozenge A\lozenge\frac{iI}{2}\\&=\left(\Phi\left(\frac{iI}{2}\right)^{*}A+A\Phi\left(\frac{iI}{2}\right)\right)i.\end{aligned}$$

Putting $A=I$ we get $\Phi\left(\frac{iI}{2}\right)^{*}=-\Phi\left(\frac{iI}{2}\right)$. Thus, $\Phi\left(\frac{iI}{2}\right)^{*}A=A\Phi\left(\frac{iI}{2}\right)^{*}$, which holds true for all $A\in\mathcal{A}_s$. Therefore, $\Phi\left(\frac{iI}{2}\right)\in Z(\mathcal{A})$.

Step 5: $\Phi\left(\frac{iI}{2}\right)=0$.
It follows from

$$\begin{aligned}\Phi\left(\frac{iI}{2}\right)&=\Phi\left(\frac{iI}{2}\blacklozenge\frac{iI}{2}\lozenge\frac{iI}{2}\right)\\&=\Phi\left(\frac{iI}{2}\right)\blacklozenge\frac{iI}{2}\lozenge\frac{iI}{2}+\frac{iI}{2}\blacklozenge\Phi\left(\frac{iI}{2}\right)\lozenge\frac{iI}{2}+\frac{iI}{2}\blacklozenge\frac{iI}{2}\lozenge\Phi\left(\frac{iI}{2}\right)=3\Phi\left(\frac{iI}{2}\right).\end{aligned}$$

Step 6: $\Phi(iA)=i\Phi(A)$ for all $A\in\mathcal{A}_s$.
From Steps 4 and 5, we have

$$\Phi(iA)=\Phi\left(A\blacklozenge\frac{I}{2}\lozenge\frac{iI}{2}\right)=\Phi(A)\blacklozenge\frac{I}{2}\lozenge\frac{iI}{2}+A\blacklozenge\Phi\left(\frac{I}{2}\right)\lozenge\frac{iI}{2}+A\blacklozenge\frac{I}{2}\lozenge\Phi\left(\frac{iI}{2}\right).$$

Employing the latter identity for $A=\frac{I}{2}$, we get $\Phi\left(\frac{I}{2}\right)=0$. Therefore, $\Phi(iA)=i\Phi(A)$.

For the following steps we need to fix some notations. Suppose that $P\in\mathcal{A}$ is the projection with $\underline{P}=0$ and $\overline{P}=I$, whose existence confirmed by Lemma 2.1. Set $P_1=P,\ P_2=I-P,$ and $\mathcal{A}_{ij}=P_i\mathcal{A}P_j\ (i,j=1,2)$. Then, $\mathcal{A}=\sum_{i,j=1}^{2}\mathcal{A}_{ij}$. When we write A_{ij}, it indicates that $A_{ij}\in\mathcal{A}_{ij},(i,j=1,2)$. We also define $\boldsymbol{A}_{12}$ by $\boldsymbol{A}_{12}=P_1AP_2+P_2AP_1$, for every $A\in\mathcal{A}_s$. Hence, for every $A\in\mathcal{A}_s$, we get $A=A_{11}+\boldsymbol{A}_{12}+A_{22}$.

Step 7: For every $A,B\in\mathcal{A}_s$, we have $\Phi(A_{11}+\boldsymbol{B}_{12})=\Phi(A_{11})+\Phi(\boldsymbol{B}_{12})$, and $\Phi(\boldsymbol{B}_{12}+A_{22})=\Phi(\boldsymbol{B}_{12})+\Phi(A_{22})$.
We set

$$T=\Phi(A_{11}+\boldsymbol{B}_{12})-\Phi(A_{11})-\Phi(\boldsymbol{B}_{12}).$$

We need to show that $T=0$. To this end, since $I\blacklozenge A_{11}\lozenge iP_2=0$, $\Phi\left(\frac{I}{2}\right)=0$, and $\Phi(0)=0$, we get

$$\begin{aligned}
&\frac{I}{2}\blacklozenge\Phi\left(A_{11}+\boldsymbol{B}_{12}\right)\lozenge iP_2+\frac{I}{2}\blacklozenge\left(A_{11}+\boldsymbol{B}_{12}\right)\lozenge\Phi\left(iP_2\right)\\
&=\Phi\left(\frac{I}{2}\blacklozenge\left(A_{11}+\boldsymbol{B}_{12}\right)\lozenge iP_2\right)\\
&=\Phi\left(\frac{I}{2}\blacklozenge\boldsymbol{B}_{12}\lozenge iP_2\right)\\
&=\frac{I}{2}\blacklozenge\Phi\left(\boldsymbol{B}_{12}\right)\lozenge iP_2+\frac{I}{2}\blacklozenge\boldsymbol{B}_{12}\lozenge\Phi\left(iP_2\right).
\end{aligned}$$

From this, we conclude that

$$\frac{I}{2}\blacklozenge T\lozenge iP_2=0.$$

Thus, $\boldsymbol{T}_{12}=T_{22}=0$. Similarly, we can conclude $T_{11}=0$. Therefore,

$$\Phi\left(A_{11}+\boldsymbol{B}_{12}\right)=\Phi\left(A_{11}\right)+\Phi\left(\boldsymbol{B}_{12}\right).$$

Likewise, we obtain

$$\Phi\left(\boldsymbol{B}_{12}+A_{22}\right)=\Phi\left(\boldsymbol{B}_{12}\right)+\Phi\left(A_{22}\right).$$

Step 8: $\Phi\left(A_{11}+\boldsymbol{B}_{12}+C_{22}\right)=\Phi\left(A_{11}\right)+\Phi\left(\boldsymbol{B}_{12}\right)+\Phi\left(C_{22}\right)$, for every $A,B,\ C\in\mathcal{A}_s$.

Set $T=\Phi\left(A_{11}+\boldsymbol{B}_{12}+C_{22}\right)-\Phi\left(A_{11}+\boldsymbol{B}_{12}\right)-\Phi\left(C_{22}\right)$. We shall show that $T=0$. Since $C_{22}\blacklozenge\frac{I}{2}\lozenge iP_1=0$, we get $\boldsymbol{T}_{12}=T_{11}=0$. It follows from Step 7 that

$$\begin{aligned}
&\Phi\left(A_{11}+\boldsymbol{B}_{12}+C_{22}\right)\blacklozenge\frac{I}{2}\lozenge iP_2+\left(A_{11}+\boldsymbol{B}_{12}+C_{22}\right)\blacklozenge\frac{I}{2}\lozenge\Phi\left(iP_2\right)\\
&=\Phi\left(\left(A_{11}+\boldsymbol{B}_{12}+C_{22}\right)\blacklozenge\frac{I}{2}\lozenge iP_2\right)=\Phi\left(\boldsymbol{B}_{12}\blacklozenge\frac{I}{2}\lozenge iP_2\right)+\Phi\left(C_{22}\blacklozenge\frac{I}{2}\lozenge iP_2\right)\\
&=\Phi\left(\left(A_{11}+\boldsymbol{B}_{12}\right)\blacklozenge\frac{I}{2}\lozenge iP_2\right)+\Phi\left(C_{22}\blacklozenge\frac{I}{2}\lozenge iP_2\right)\\
&=\left(A_{11}+\boldsymbol{B}_{12}+C_{22}\right)\blacklozenge\frac{I}{2}\lozenge\Phi\left(iP_2\right)+\Phi\left(A_{11}+\boldsymbol{B}_{12}\right)\blacklozenge\frac{I}{2}\lozenge iP_2+\Phi\left(C_{22}\right)\blacklozenge\frac{I}{2}\lozenge iP_2.
\end{aligned}$$

This implies that $T\blacklozenge\frac{I}{2}\lozenge iP_2=0$. From this we obtain that $T_{22}=0$, therefore $T=0$, as claimed.

Step 9: $\Phi\left(\boldsymbol{A}_{jk}+\boldsymbol{B}_{jk}\right)=\Phi\left(\boldsymbol{A}_{jk}\right)+\Phi\left(\boldsymbol{B}_{jk}\right)$, for every $A,B\in\mathcal{A}_s,\ 1\leq j\neq k\leq 2$.

Since

$$\frac{I}{2}\blacklozenge\left(P_j+\boldsymbol{A}_{jk}\right)\lozenge i\left(P_k+\boldsymbol{B}_{jk}\right)=i\left(\boldsymbol{A}_{jk}+\boldsymbol{B}_{jk}\right)+\left(i\boldsymbol{A}_{jk}\boldsymbol{B}_{jk}+i\boldsymbol{B}_{jk}\boldsymbol{A}_{jk}\right),$$

we get from Step 6 and Step 8 that

$$\Phi\left(i\left(\boldsymbol{A}_{jk}+\boldsymbol{B}_{jk}\right)\right)+\Phi\left(i\boldsymbol{A}_{jk}\boldsymbol{B}_{jk}+i\boldsymbol{B}_{jk}\boldsymbol{A}_{jk}\right)=\Phi\left(\frac{I}{2}\blacklozenge\left(P_j+\boldsymbol{A}_{jk}\right)\lozenge i\left(P_k+\boldsymbol{B}_{jk}\right)\right)$$

$$=\frac{I}{2}\blacklozenge\left(\Phi\left(P_j\right)+\Phi\left(\boldsymbol{A}_{jk}\right)\right)\lozenge\left(i\left(P_k+\boldsymbol{B}_{jk}\right)\right)+\frac{I}{2}\blacklozenge\left(P_j+\boldsymbol{A}_{jk}\right)\lozenge\left(\Phi\left(iP_k\right)+\Phi\left(i\boldsymbol{B}_{jk}\right)\right)$$

$$=\Phi\left(\frac{I}{2}\blacklozenge\boldsymbol{A}_{jk}\lozenge iP_K\right)+\Phi\left(\frac{I}{2}\blacklozenge\boldsymbol{P}_j\lozenge i\boldsymbol{B}_{jk}\right)+\Phi\left(\frac{I}{2}\blacklozenge\boldsymbol{A}_{jk}\lozenge i\boldsymbol{B}_{jk}\right)$$

$$=\Phi\left(i\boldsymbol{A}_{jk}\right)+\Phi\left(i\boldsymbol{B}_{jk}\right)+\Phi\left(i\boldsymbol{A}_{jk}\boldsymbol{B}_{jk}+i\boldsymbol{B}_{jk}\boldsymbol{A}_{jk}\right).$$

It follows $\Phi\left(i\boldsymbol{A}_{jk}+i\boldsymbol{B}_{jk}\right)=\Phi\left(i\boldsymbol{A}_{jk}\right)+\Phi\left(i\boldsymbol{B}_{jk}\right)$, from which we get

$$\Phi\left(\boldsymbol{A}_{jk}+\boldsymbol{B}_{jk}\right)=\Phi\left(\boldsymbol{A}_{jk}\right)+\Phi\left(\boldsymbol{B}_{jk}\right).$$

Step 10: For every $A_{jj}, B_{jj}\in\mathcal{A}_{jj}$, $\Phi\left(A_{jj}+B_{jj}\right)=\Phi\left(A_{jj}\right)+\Phi\left(B_{jj}\right)$.
Set $T=\Phi\left(A_{jj}+B_{jj}\right)-\Phi\left(A_{jj}\right)-\Phi\left(B_{jj}\right)\ (1\le j\ne k\le 2)$. We see that $T=0$. Since $\frac{I}{2}\blacklozenge A_{jj}\lozenge iP_k=\frac{I}{2}\blacklozenge B_{jj}\lozenge iP_k=0$. We conclude that $\frac{I}{2}\blacklozenge T\lozenge iP_k=0$. Therefore $T_{kk}=\boldsymbol{T}_{jk}=0$. For every $C\in\mathcal{A}_s$, it follows from Step 9 that

$$\Phi\left(P_j\blacklozenge A_{jj}+B_{jj}\lozenge iC_{jk}\right)=\Phi\left(P_j\blacklozenge A_{jj}\lozenge iC_{jk}\right)+\Phi\left(P_j\blacklozenge B_{jj}\lozenge iC_{jk}\right).$$

Hence $P_j\blacklozenge T\lozenge iC_{jk}=0$. Thus $2iT_{jj}C_{jk}=0$, for all $C\in\mathcal{A}_s$. Primeness of $\mathcal{A}$ now implies that $T_{jj}=0$, so $T=0$.

Step 11: Φ is additive on $\mathcal{A}_s$.
Let $A=\sum_{i,j=1}^{2}A_{ij}$ and $B=\sum_{i,j=1}^{2}$ be two arbitrary elements of $\mathcal{A}$. It follows from Steps 8, 9, and 10 that

$$\begin{aligned}\Phi\left(A+B\right)&=\Phi\left(\sum_{i,j=1}^{2}A_{ij}+\sum_{i,j=1}^{2}B_{ij}\right)=\Phi\left(\sum_{i,j=1}^{2}\left(A_{ij}+B_{ij}\right)\right)\\&=\sum_{i,j=1}^{2}\Phi\left(A_{ij}+B_{ij}\right)=\sum_{i,j=1}^{2}\Phi\left(A_{ij}\right)+\sum_{i,j=1}^{2}\Phi\left(B_{ij}\right)\\&=\Phi\left(\sum_{i,j=1}^{2}A_{ij}\right)+\Phi\left(\sum_{i,j=1}^{2}B_{ij}\right)=\Phi\left(A\right)+\Phi\left(B\right).\end{aligned}$$

Step 12: Φ is additive on $\mathcal{A}_{sk}=\{A\in\mathcal{A}:A^*=-A\}$.
Follows from Step 6 and Step 11.

Step 13: For every $A,B\in\mathcal{A}_s$, $\Phi\left(A+iB\right)=\Phi\left(A\right)+i\Phi\left(B\right)$.
Set $T=\Phi\left(A+iB\right)-\Phi\left(A\right)-\Phi\left(iB\right)$. We need to show that $T=0$. Since $iB\blacklozenge\frac{I}{2}\lozenge\frac{iI}{2}=0$, and $\Phi\left(\frac{I}{2}\right)=\Phi\left(\frac{iI}{2}\right)=\Phi\left(0\right)=0$, we conclude that

$$\begin{aligned}\Phi\left(A+iB\right)\blacklozenge\frac{I}{2}\lozenge\frac{iI}{2}=\Phi\left(\left(A+iB\right)\blacklozenge\frac{I}{2}\lozenge\frac{iI}{2}\right)&=\Phi\left(A\right)\blacklozenge\frac{I}{2}\lozenge\frac{iI}{2}+\Phi\left(iB\blacklozenge\frac{I}{2}\lozenge\frac{iI}{2}\right)\\&=\Phi\left(A\right)\blacklozenge\frac{I}{2}\lozenge\frac{iI}{2}+\Phi\left(iB\right)\blacklozenge\frac{I}{2}\lozenge\frac{iI}{2},\end{aligned}$$

and this implies that $T\blacklozenge\frac{I}{2}\lozenge\frac{iI}{2}=0$. Therefore, $T=0$. It follows from Step 6 that $\Phi\left(A+iB\right)=\Phi\left(A\right)+i\Phi\left(B\right)$.

Step 14: Φ preserves star.
Applying Steps 3, 6, and 13, one concludes that Φ preserves the star operation.

Step 15: For every $A,B\in\mathcal{A}_s$, $\Phi\left(AB\right)=\Phi\left(A\right)B+A\Phi\left(B\right)$.
For every $A,B\in\mathcal{A}_s$, we have

$$\begin{aligned}\Phi\left(i\left(AB+BA\right)\right)=\Phi\left(A\blacklozenge B\lozenge\frac{iI}{2}\right)&=\Phi\left(A\right)\blacklozenge B\lozenge\frac{iI}{2}+A\blacklozenge\Phi\left(B\right)\lozenge\frac{iI}{2}\\&=\left(\Phi\left(A\right)B+B\Phi\left(A\right)+A\Phi\left(B\right)+\Phi\left(B\right)A\right)i.\end{aligned}$$

By Step 6, we get

$$\Phi\left(AB+BA\right)=\Phi\left(A\right)B+B\Phi\left(A\right)+A\Phi\left(B\right)+\Phi\left(B\right)A. \tag{2.1}$$

Similarly we have

$$\Phi\left(AB-BA\right)=\Phi\left(A\right)B-B\Phi\left(A\right)+A\Phi\left(B\right)-\Phi\left(B\right)A. \tag{2.2}$$

It follows from (2.1), (2.2) and the additivity of Φ that

$$\Phi\left(AB\right)=\Phi\left(A\right)B+A\Phi\left(B\right).$$

Step 16: $\Phi\left(AB\right)=\Phi\left(A\right)B+A\Phi\left(B\right)$, for all $A,B\in\mathcal{A}$.
Since for every $A\in\mathcal{A}$, $A=A_1+iA_2$, where $A_1=\frac{A+A^*}{2}$ and $A_2=\frac{A-A^*}{2i}$ are self-adjoint, the identity $\Phi\left(AB\right)=\Phi\left(A\right)B+A\Phi\left(B\right)$ follows from Steps 6 and 14. $\square$

The following corollary is immediate by of [5, Theorems 2.1 and 2.4].

Corollary 2.3. *Let $\mathcal{A}$ be a factor von Neumann algebra with $\dim\left(\mathcal{A}\right)\geq 2$. Then for every map $\Phi:\mathcal{A}\longrightarrow\mathcal{A}$ the following assertions are equivalent:*

(1) $\Phi\left(\left[A,B\right]_*\bullet C\right)=\left[\Phi\left(A\right),B\right]_*\bullet C+\left[A,\Phi\left(B\right)\right]_*\bullet C+\left[A,B\right]_*\bullet\Phi\left(C\right)$, *for every $A,B,C\in\mathcal{A}$.*
(2) *Φ is an additive $*$-derivation.*
(3) $\Phi\left(A\blacklozenge B\lozenge C\right)=\Phi\left(A\right)\blacklozenge B\lozenge C+A\blacklozenge\Phi\left(B\right)\lozenge C+A\blacklozenge B\lozenge\Phi\left(C\right)$, *for every $A,B,C\in\mathcal{A}$.*

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