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ON NORMAL IDEAL TOPOLOGICAL SPACES

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ABSTRACT. An ideal on a set X is defined as a non-empty collection of subsets of X that is closed under taking subsets (i.e., hereditary) and finite unions. Extensions of classical separation axioms on topological spaces can be defined using ideals. A notable and interesting extension of the normality axiom, termed $\mathcal{P}$-normality, was recently introduced by Pachón Rubiano. In this paper, we investigate the properties of this separation axiom and provide several characterizations. In particular, we establish generalizations of both Urysohn's lemma and the Tietze extension theorem within this framework.

1. Introduction and Preliminaries

Ideals are fundamental concepts in mathematics, playing an important role in the study of topological spaces. The initial development of topological spaces enhanced by an ideal can be traced back to Kuratowski [7, 8] in 1933, who introduced the concept of a local function as a generalization of the closure operation. Subsequently, ideals in topological spaces were systematically studied by Vaidyanathaswamy [19] in 1944. Further significant contributions include Freud's generalization of the Cantor-Bendixson theorem using ideal topological spaces in 1958 [4], and Scheinberg's application of ideals to measure theory in 1971 [14]. In 1990, Janković and Hamlett [5] published a comprehensive survey paper on the topic of ideal topological spaces.

Recall that a topological space X is normal if any two disjoint closed subsets of X possess disjoint neighborhoods [2]. The primary significance of normal spaces stems from Urysohn's Lemma, which states that if A and B are two disjoint

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closed subsets of X, there exists a continuous function $f : X \to [0,1]$ such that $f(x) = 0$ for all $x \in A$ and $f(x) = 1$ for all $x \in B$. Essentially, this means that disjoint closed sets in a normal space are not merely separated by neighborhoods, but are also separated by a continuous function.

Several extensions of classical separation axioms utilizing the concept of ideals have been investigated in the literature [3, 10, 11, 12, 15, 16]. Specifically, the $\mathcal{I}$-separation axioms were initially introduced by Renuka Devi and Sivaraj [12], while the $\mathcal{J}$-separation axioms were defined independently by Suriyakala and Vembu [16]. A particularly useful and interesting set of extensions - namely the $\mathcal{P}$-Hausdorff, $\mathcal{P}$-regular, and $\mathcal{P}$-normal axioms - was recently introduced by Pachón Rubiano [11]. We commence our discussion by formally stating the definitions of some of these generalizations of normal spaces.

Let $(X, \tau, \mathcal{I})$ be an ideal space, where (X, τ) is a topological space and $\mathcal{I}$ is an ideal on X. Then X is said to be:

- [12] $\mathcal{I}$-normal if for every two disjoint closed sets A and B, there exist disjoint open sets U and V such that $A - U \in \mathcal{I}$ and $B - V \in \mathcal{I}$,
- [16] $\mathcal{J}$-normal if (X, τ) is T_1 and if, for every two disjoint closed sets A and B, there exist open sets U and V such that $A \subseteq U$, $B \subseteq V$ and $U \cap V \in \mathcal{I}$,
- [11] $\mathcal{P}$-normal if for every two disjoint closed sets A and B, there exist open sets U and V such that $A \subseteq (U - V)$, $B \subseteq (V - U)$ and $U \cap V \in \mathcal{I}$.

It is immediately clear that $(X, \tau, \{\emptyset\})$ is $\mathcal{P}$-normal if and only if (X, τ) is normal. Furthermore, if $(X, \tau, \mathcal{I})$ is $\mathcal{J}$-normal, then it is also $\mathcal{P}$-normal. This is because, if A and B are disjoint closed sets, the $\mathcal{J}$-normality guarantees the existence of open sets U and V such that $A \subseteq U$, $B \subseteq V$ and $U \cap V \in \mathcal{I}$. By setting $U' = U - B$ and $V' = V - A$, we observe that $A \subseteq (U' - V')$, $B \subseteq (V' - U')$,and importantly, $U' \cap V' \in \mathcal{I}$.

On the other hand, Rubiano demonstrated that $\mathcal{I}$-normality and $\mathcal{P}$-normality are independent concepts. In the context of $\mathcal{J}$-normality, there can exist a situation where, for some pair of disjoint closed sets A and B, either $A \cup B \subseteq U$ or $A \cup B \subseteq V$ (for the corresponding U and V), a property that is often considered less desirable.

In this paper, we study and investigate several properties of $\mathcal{P}$-normal spaces. In particular, we provide a generalization of Urysohn's lemma and the Tietze extension theorem. Subsequently, we recall some basic concepts and fundamental results concerning ideals and ideal spaces; for further details, we refer the reader to [5, 7].

Definition 1.1. [5, 7] An ideal $\mathcal{I}$ on a nonempty set X is a family of subsets of X that satisfies the following two conditions:

(1) $\mathcal{I}$ is closed under taking subsets: $A \subseteq B$ and $B \in \mathcal{I}$ implies $A \in \mathcal{I}$,
(2) $\mathcal{I}$ is closed under finite unions: if $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$.

It is worth noting that conditions (1) and (2) in the definition of an ideal can be consolidated into a single, equivalent condition: $A \in \mathcal{I}$ and $B \in \mathcal{I} \iff A \cup B \in \mathcal{I}$. Furthermore, it is well-established that the classical concepts of a grill and a primal represent a form of duality for ideals, while a filter serves as a dual counterpart to a proper ideal. Consequently, numerous topological concepts expressible in terms of these structures have been shown to be equivalent [6, 9, 17, 1, 18].

Example 1.2. The following families are ideals on a nonempty set X:

(1) the power set $P(X)$; and the one point set $\{\emptyset\}$,
(2) the finite subsets of X; and the countable subsets of X,
(3) the bounded subsets of a metric space X,
(4) the totally bounded subsets of a metric space X,
(5) the subsets of a metric space X with compact closure,
(6) the closed and discrete sets of a space X,
(7) the relatively compact sets of a space X,
(8) the nowhere dense sets of a space X,
(9) the meager sets of a space X,
(10) the sets of measure zero of a measure space X.

Let (X, τ) be a topological space. For any subset $A \subseteq X$, we denote its τ-closure and τ-interior by $cl_\tau(A)$ (or simply $cl(A)$) and $int_\tau(A)$ (or $int(A)$), respectively. Furthermore, for any point $x \in X$, the system of open neighborhoods of x is denoted by $\tau(x)$, defined as$\tau(x) := \{U \in \tau \mid x \in U\}$. Finally, when X is endowed with multiple topologies, we use the notation τ-open (τ-closed) to specify openness (closedness) with respect to the topology τ.

Recall that a map $\lambda : P(X) \to P(X)$ is said to be the following properties:

(1) *Monotone* if $A \subseteq B \subseteq X$ implies $\lambda A \subseteq \lambda B$, where we write λA for $\lambda(A)$;
(2) *Idempotent* if $\lambda^2 A = \lambda\lambda A = \lambda A$ for all $A \subseteq X$;
(3) *Enlarging* if $A \subseteq \lambda A$ for all $A \subseteq X$;
(4) *Additive* if$\lambda(A \cup B) = \lambda A \cup \lambda B$ for all $A, B \subseteq X$.

A map $c : P(X) \to P(X)$ such that is enlarging, idempotent, additive and satisfies $c(\emptyset) = \emptyset$ is called a Kuratowski closure operator. If c is a Kuratowski closure operator, then the collection $\tau := \{A \mid c(X - A) = X - A\}$ is a topology on X such that $cl(A) = c(A)$ for all $A \subseteq X$ [7].

Definition 1.3. [5] Let $(X, \tau, \mathcal{I})$ be an ideal space. Then

$$A^*(\mathcal{I}, \tau) := \{x \in X \mid U \cap A \notin \mathcal{I} \quad \text{for every } U \in \tau(x)\}$$

is called the local function of A with respect to τ and $\mathcal{I}$ and when no ambiguity is present we will simply write A^*.

Theorem 1.4. [5] *Let $(X, \tau, \mathcal{I})$ be an ideal space. Then the operator $cl^* : P(X) \to P(X)$ defined by $cl^*(A) = A \cup A^*$ is a Kuratowski closure operator. Hence the collection $\tau^*(\mathcal{I}) := \{A \subseteq X \mid cl^*(X - A) = X - A\}$ is a topology on X, called the topology induced by τ and $\mathcal{I}$.*

When no ambiguity arises, we shall simply write τ^*.

Theorem 1.5. [5] *Let $(X, \tau, \mathcal{I})$ be an ideal space. Then the following statements hold:*

(1) *$\tau \subseteq \tau^*$ and $\tau^{**} = \tau^*$.*

(2) *If* $\mathcal{I} = \{\emptyset\}$*, then* $cl^*(A) = cl(A) = A^*$ *and* $\tau = \tau^*$*.*
(3) *The collection* $\beta := \{V - I \mid V \in \tau, I \in \mathcal{I}\}$ *is a base for* τ^**.*
(4) *For every* $I \in \mathcal{I}$*,* $I^* = \emptyset$ *and hence* I *is* τ^**-closed.*

Definition 1.6. [5] Let $(X, \tau, \mathcal{I})$ be an ideal space. Then τ is said to be compatible with $\mathcal{I}$, denoted $\tau \sim \mathcal{I}$ if the following condition holds: for every $A \subseteq X$, if for every $x \in A$ there exists a $U \in \tau(x)$ such that $U \cap A \in \mathcal{I}$, then $A \in \mathcal{I}$.

Theorem 1.7. [5] *Let* $(X, \tau, \mathcal{I})$ *be an ideal space. Then the following statements hold:*
(1) $\tau \sim \mathcal{I}$ *if and only if* $A - A^* \in \mathcal{I}$*, for each* $A \subseteq X$*.*
(2) *If* $\tau \sim \mathcal{I}$*, then* β *is a topology, and hence* $\beta = \tau^*$*.*

2. $\mathcal{P}$-normal Spaces

In [11], Pachón Rubiano defined and studied $\mathcal{P}$-Hausdorff, $\mathcal{P}$-regular, and $\mathcal{P}$-normal spaces, and investigated various relationships between these concepts and other separation axioms in the context of ideal spaces. In this section, we study $\mathcal{P}$-normal spaces in more detail, provide several characterizations, and present a generalization of Urysohn's lemma and the Tietze extension theorem.

Theorem 2.1. [11] *An ideal space* $(X, \tau, \mathcal{I})$ *is* $\mathcal{P}$*-normal if and only if for each closed set* A *and each open set* U*, if* $A \subseteq U$*, then there are an open set* G *and a closed set* $\hat{G}$*, such that* $A \subseteq G \subseteq U$*,* $A \subseteq \hat{G} \subseteq U$ *and* $G - \hat{G} \in \mathcal{I}$*.*

An ideal $\mathcal{I}$ on a topological space (X, τ) is called τ-codense if $\tau \cap \mathcal{I} = \{\emptyset\}$. It has been shown that $\mathcal{I}$ is τ-codense if and only if $X = X^*$, which is equivalent to the condition that $I \in \mathcal{I}$ implies $int_\tau(I) = \emptyset$ [5].

Theorem 2.2. *Let* $(X, \tau, \mathcal{I})$ *be an ideal space such that* $\mathcal{I}$ *is* τ*-codense. Then* X *is* $\mathcal{P}$*-normal if and only if it is normal.*

Proof. Suppose X is $\mathcal{P}$-normal, A and B are two disjoint τ-closed subsets of X. By assumption there exist τ-open sets U and V such that $A \subseteq (U - V)$, $B \subseteq (V - U)$ and $U \cap V \in \mathcal{I}$. Then we have $U \cap V \in \mathcal{I} \cap \tau = \{\emptyset\}$. Thus $A \subseteq U$, $B \subseteq V$ and $U \cap V = \emptyset$, which completes the proof. □

The following example demonstrates that $\mathcal{P}$-normality does not imply normality.

Example 2.3. Let $(\mathbb{Q}, \tau)$ be the rational numbers with the cofinite topology and $\mathcal{I}_c$ be the ideal of all countable sets. Since $\mathbb{Q}$ is countable, it follows that $\mathcal{I}_c = P(\mathbb{Q})$. Then it is easy to show that the ideal space $(\mathbb{Q}, \tau, \mathcal{I}_c)$ is $\mathcal{P}$-normal but $(\mathbb{Q}, \tau)$ is not normal.

Example 2.4. Let $X = \{a, b, c, d, e\}$, $\tau = \{\emptyset, \{c, e\}, \{c, d, e\}, \{a, b, c, e\}, X\}$ and $I = P(\{c, e\})$. Then the only nonempty and disjoint closed sets are $\{a, b\}$ and $\{d\}$. If we put $U = \{a, b, c, e\}$ and $V = \{c, d, e\}$, then we have that $\{a, b\} \subseteq U - V$, $\{d\} \subseteq V - U$ and $U \cap V = \{c, e\} \in \mathcal{I}$. Hence $(X, \tau, \mathcal{I})$ is $\mathcal{P}$-normal. But it is clear that (X, τ) is not normal.

Let (X, τ) be a space and $A \subseteq X$. Then the subspace topology on A is denoted by τ_A. If an ideal space $(X, \tau, \mathcal{I})$ is $\mathcal{P}$-normal and $A \subseteq X$ is τ-closed, then it is easy to show that the ideal space $(A, \tau_A, \mathcal{I}_A)$ is $\mathcal{P}$-normal, where $\mathcal{I}_A = \{I \cap A \mid I \in \mathcal{I}\}$.

Theorem 2.5. *Let $(X, \tau, \mathcal{I})$ be a $\mathcal{P}$-normal ideal space. Then the subspace (X^*, τ_{X^*}) is normal.*

Proof. Since X^* is τ-closed, it follows that $(X^*, \tau_{X^*}, \mathcal{I}_{X^*})$ is $\mathcal{P}$-normal. On the other hand, we have $X^* = \{x \in X \mid \tau(x) \cap \mathcal{I} = \emptyset\}$. So $\tau_{X^*} \cap \mathcal{I}_{X^*} = \{\emptyset\}$, and hence $\mathcal{I}_{X^*}$ is τ_{X^*}-codense. Therefore by Theorem 2.2, the result holds. □

Definition 2.6. Let (X, τ) be a space and μ be a family of subsets of X such that $\tau \subseteq \mu$. Then we say that X is a (τ, μ)-normal space if for every two disjoint τ-closed subsets A and B of X, there exist $U, V \in \mu$ such that $A \subseteq U$, $B \subseteq V$ and $U \cap V = \emptyset$.

It is clear that a topological space (X, τ) is normal if and only if it is (τ, τ)-normal. Furthermore, if (X, τ) is normal, then it is (τ, μ)-normal for any topology μ finer than τ, and consequently, it is (τ, τ^*)-normal.

Theorem 2.7. *An ideal space $(X, \tau, \mathcal{I})$ is $\mathcal{P}$-normal if and only if it is (τ, β)-normal.*

Proof. Suppose X is $\mathcal{P}$-normal and A and B are two disjoint τ-closed subsets of X. By assumption there exist τ-open sets U and V such that $A \subseteq (U - V)$, $B \subseteq (V - U)$ and $U \cap V \in \mathcal{I}$. Put $I = U \cap V$, $U' = U - I$ and $V' = V - I$, then we have $A \subseteq U'$, $B \subseteq V'$ and $U' \cap V' = \emptyset$. On the other hand $U', V' \in \beta$, so X is (τ, β)-normal. Conversely, assume that X is (τ, β)-normal and A and B are two disjoint τ-closed subsets of X. Then there exist $U, V \in \tau$ and $I_1, I_2 \in \mathcal{I}$ such that $A \subseteq (U - I_1)$, $B \subseteq (V - I_2)$ and $(U - I_1) \cap (V - I_2) = \emptyset$. Put $U' = U - B$ and $V' = V - A$, then we have $A \subseteq (U' - V')$, $B \subseteq (V' - U')$ and $U' \cap V' \subseteq U \cap V \subseteq I_1 \cup I_2 \in \mathcal{I}$. Since U' and V' are τ-open, it follows that X is $\mathcal{P}$-normal. □

By the preceding theorem and Theorem 1.7, we establish the following result.

Corollary 2.8. *Let $(X, \tau, \mathcal{I})$ be an ideal space such that $\tau \sim \mathcal{I}$. Then X is $\mathcal{P}$-normal if and only if it is (τ, τ^*)-normal.*

Definition 2.9. Let (X, τ) be a space and $A, B \subseteq X$. We say that A and B are τ-completely separated in X if there exists a continuous function $f : (X, \tau) \to [0, 1]$ such that $f(x) = 0$ for all $x \in A$ and $f(x) = 1$ for all $x \in B$, where topology on $[0, 1]$ is the usual topology.

Recall that an ideal $\mathcal{I}$ on a set X is called a σ-ideal if it is countably additive, meaning it is closed under countable unions [5]. Now, we present an extension of Urysohn's Lemma for σ-ideals.

Theorem 2.10. *($\mathcal{P}$-Urysohn's lemma) Let $(X, \tau, \mathcal{I})$ be a $\mathcal{P}$-normal ideal space such that $\mathcal{I}$ is a σ-ideal. Then every two disjoint τ-closed subsets of X are τ^*-completely separated in X.*

Proof. Suppose X is $\mathcal{P}$-normal and A and B are two disjoint τ-closed subsets of X. Put $U := X - B$, then $A \subseteq U$ and so by Theorem 2.1, there exist a τ-open set $G_{\frac{1}{2}}$ and a τ-closed set $\hat{G}_{\frac{1}{2}}$, such that:

$$A \subseteq G_{\frac{1}{2}} \subseteq U, \qquad A \subseteq \hat{G}_{\frac{1}{2}} \subseteq U, \qquad G_{\frac{1}{2}} - \hat{G}_{\frac{1}{2}} \in \mathcal{I} \tag{2.1}$$

Again by the equation (1) and Theorem 2.1, there exist τ-open sets $G_{\frac{1}{4}}$ and $G_{\frac{3}{4}}$; and τ-closed sets $\hat{G}_{\frac{1}{4}}$ and $\hat{G}_{\frac{3}{4}}$ such that:

$$A \subseteq G_{\frac{1}{4}} \subseteq G_{\frac{1}{2}} \subseteq U, \qquad A \subseteq \hat{G}_{\frac{1}{4}} \subseteq G_{\frac{1}{2}} \subseteq U, \qquad G_{\frac{1}{4}} - \hat{G}_{\frac{1}{4}} \in \mathcal{I} \tag{2.2}$$

$$A \subseteq \hat{G}_{\frac{1}{2}} \subseteq G_{\frac{3}{4}} \subseteq U, \qquad A \subseteq \hat{G}_{\frac{1}{2}} \subseteq \hat{G}_{\frac{3}{4}} \subseteq U, \qquad G_{\frac{3}{4}} - \hat{G}_{\frac{3}{4}} \in \mathcal{I} \tag{2.3}$$

Thus by induction, for every $r \in \{\frac{k}{2^n} \mid 0 < k < 2^n\}$, there exist a τ-open set G_r and a τ-closed set $\hat{G}_r$, such that satisfying the following conditions:

(1) If $I_r := G_r - \hat{G}_r$, then $I_r \in \tau \cap \mathcal{I}$,
(2) $r < s$ implies $A \subseteq G_r \subseteq \hat{G}_r \cup I_r \subseteq G_s \cup I_r \subseteq \hat{G}_s \cup I_r \cup I_s$.

If we take $D_r := \bigcup\{I_t \mid t \leq r\}$, then since $\mathcal{I}$ is σ-ideal, $D_r \in \tau \cap \mathcal{I}$. On the other hand, by Theorem 1.5, D_r is τ^*-closed. Hence for $r < s$ we have:

$$A \subseteq G_r \cup D_r \subseteq \hat{G}_r \cup D_r \subseteq G_s \cup D_s \subseteq \hat{G}_s \cup D_s \subseteq U.$$

Thus if we define $F_r := G_r \cup D_r$, then for every $r \in \{\frac{k}{2^n} \mid 0 < k < 2^n\}$, there exist a τ-open set F_r such that $r < s$ implies that:

$$A \subseteq F_r \subseteq cl^*(F_r) \subseteq F_s \subseteq cl^*(F_s) \subseteq U.$$

Now, we define a map $f : (X, \tau^*) \to [0, 1]$ as follows:

$$f(x) = \begin{cases} 1, & \text{if } x \in B, \\ \inf\{r \mid x \in F_r\}, & \text{if } x \in U. \end{cases}$$

Similar to the proof of the Urysohn's lemma for spaces, we have f is a continuous map such that $f(x) = 0$ for all $x \in A$ and $f(x) = 1$ for all $x \in B$, which completes the proof. □

Recall that an ideal $\mathcal{I}$ on a set X is called principal if there exists a subset $A \subseteq X$ such that $\mathcal{I} = P(A)$. It is straightforward to verify that an ideal $\mathcal{I}$ is principal if and only if it is closed under arbitrary unions.

Definition 2.11. Let $(X, \tau, \mathcal{I})$ be an ideal space. We say that τ^* is β-finitely generated if every τ^*-open subset of X can be expressed as a finite union of elements of β.

Example 2.12. Let $(X, \tau, \mathcal{I})$ be an ideal space such that $\tau \sim \mathcal{I}$. Then τ^* is β-finitely generated, since in this case $\beta = \tau^*$.

Theorem 2.13. *Suppose that $(X, \tau, \mathcal{I})$ is an ideal space such that every two disjoint τ-closed subsets of X are τ^*-completely separated in X. If either of the following conditions holds:*

(1) *$\mathcal{I}$ is a principal ideal,*
(2) *τ^* is β-finitely generated.*

Then X is $\mathcal{P}$-normal.

Proof. Let A and B be two disjoint τ-closed subsets of X. By assumption there exists a continuous function $f : (X, \tau^*) \to [0,1]$ such that $f(x) = 0$ for all $x \in A$ and $f(x) = 1$ for all $x \in B$. Then $A \subseteq f^{-1}([0, \frac{1}{2}))$ and $B \subseteq f^{-1}((\frac{1}{2}, 1]))$. Since β is a base for τ^*, there exist families $\{U_j \mid j \in J\}$ and $\{V_k \mid k \in K\}$ of τ-open sets; and families $\{I_j \mid j \in J\}$ and $\{D_k \mid k \in K\}$ of the elements of $\mathcal{I}$ such that:

$$f^{-1}([0, \frac{1}{2})) = \bigcup\{U_j - I_j \mid j \in J\} \qquad f^{-1}((\frac{1}{2}, 1]) = \bigcup\{V_k - D_k \mid k \in K\}.$$

If we take $U := \bigcup\{U_j \mid j \in J\}$, $V := \bigcup\{V_k \mid k \in K\}$, $U' := U - B$ and $V' := V - A$, then U' and V' are τ-open subsets of X such that $A \subseteq (U' - V')$ and $B \subseteq (V' - U')$. On the other hand, we have

$$U' \cap V' \subseteq U \cap V = \bigcup\{U_j \cap V_k \mid j \in J,\ k \in K\}.$$

Since $f^{-1}([0, \frac{1}{2})) \bigcap f^{-1}((\frac{1}{2}, 1]) = \emptyset$, it follows that $U_j \cap V_k \subseteq I_j \cup D_k \in \mathcal{I}$ for every $j \in J$ and $k \in K$. Hence

$$U' \cap V' \subseteq \bigcup\{I_j \cup D_k \mid j \in J,\ k \in K\}.$$

Thus by assumption $U' \cap V' \in \mathcal{I}$, which completes the proof. □

We are now in a position to present a generalization of the classical Tietze extension theorem.

Definition 2.14. Let $(X, \tau, \mathcal{I})$ be an ideal space, Y be an arbitrary space and $A \subseteq X$. Then we say that a continuous map $f : (A, \tau_A) \to Y$ has a τ^*-continuous extension, if there exists a continuous map $F : (X, \tau^*) \to Y$ such that $F(a) = f(a)$ for all $a \in A$.

Theorem 2.15. *Let $(X, \tau, \mathcal{I})$ be a $\mathcal{P}$-normal ideal space such that $\mathcal{I}$ is a σ-ideal. If $f : (A, \tau_A) \to [-1, 1]$ is a continuous map from a τ-closed subspace A of X into $[-1, 1]$ equipped with the usual topology, then f has a τ^*-continuous extension.*

Proof. Let $c_0 = \sup\{|f(a)| \mid a \in A\}$, $E_0 = \{a \in A \mid f(a) \geq \frac{c_0}{3}\}$, $F_0 = \{a \in A \mid f(a) \leq -\frac{c_0}{3}\}$. Then E_0 and F_0 are τ_A-closed disjoint subsets of A and hence are τ-closed disjoint subsets of X. Since every interval $[a, b]$ is homeomorphic to $[0, 1]$, by the Urysohn's lemma, there exists a continuous map $g_0 : (X, \tau^*) \to [-\frac{c_0}{3}, \frac{c_0}{3}]$ such that $g_0(x) = \frac{c_0}{3}$ for all $x \in E_0$ and $g_0(x) = -\frac{c_0}{3}$ for all $x \in F_0$. Moreover, $|g_0| \leq \frac{c_0}{3}$ and $| f - g_0 | \leq \frac{2c_0}{3}$ on A.

Now by induction, we have a sequence of τ^*-continuous maps $\{g_n \mid n \geq 0\}$ such that $|g_n| \leq \frac{2^n c_0}{3^{n+1}}$ and $|f - g_0 - ... - g_n| \leq \frac{2^{n+1} c_0}{3^{n+1}}$ on A. Put $F_n = g_0 + ... + g_n$. Then $F_n : (X, \tau^*) \to [-1, 1]$ is a continuous map such that $|F_n - F_m| \leq \frac{2}{3}^{m+1} c_0$. Therefore the sequence $\{F_n\}$ is Cauchy and hence there exists a continuous map $F : (X, \tau^*) \to [-1, 1]$ such that $\{F_n\}$ converges uniformly to F. Since $|F_n - f| \leq \frac{2^n c_0}{3^{n+1}}$ on A, it follows that $F = f$ on A, which completes the proof. □

Definition 2.16. Let $(X, \tau, \mathcal{I})$ be an ideal space and A be a subset of X. Then we say that A is $C(\tau^*)$-embedded in X if every continuous map $f : (A, \tau_A) \to R$ has a τ^*-continuous extension.

Theorem 2.17. *($\mathcal{P}$-Tietze extension theorem) Let $(X, \tau, \mathcal{I})$ be a $\mathcal{P}$-normal ideal space such that $\mathcal{I}$ is a σ-ideal. Then every τ-closed subset of X is $C(\tau^*)$-embedded in X.*

Proof. By Theorem 2.15, similar to the proof of Tietze extension theorem for spaces we can replace the interval $[-1, 1]$ by $\mathbb{R}$. □

Theorem 2.18. *Suppose that $(X, \tau, \mathcal{I})$ is an ideal space such that every τ-closed subset of X is $C(\tau^*)$-embedded in X. If either of the following conditions holds:*

(1) *$\mathcal{I}$ is a principal ideal,*
(2) *τ^* is β-finitely generated.*

Then X is $\mathcal{P}$-normal.

Proof. Let B and C be two disjoint τ-closed subsets of X. Put $A = B \cup C$ and define a map $f : A \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0, & \text{if } x \in B, \\ 1, & \text{if } x \in C. \end{cases}$$

Then $f : (A, \tau_A) \to \mathbb{R}$ is a continuous map, so by assumption f has a τ^*-continuous extension $F : (X, \tau^*) \to \mathbb{R}$. Now, we have $B \subseteq F^{-1}((-\frac{1}{2}, \frac{1}{2}))$ and $C \subseteq F^{-1}((\frac{1}{2}, \frac{3}{2}))$. Since β is a base for τ^*, there exist families $\{U_j \mid j \in J\}$ and $\{V_k \mid k \in K\}$ of τ-open sets; and families $\{I_j \mid j \in J\}$ and $\{D_k \mid k \in K\}$ of the elements of $\mathcal{I}$ such that:

$$F^{-1}((-\frac{1}{2}, \frac{1}{2})) = \bigcup\{U_j - I_j \mid j \in J\} \qquad F^{-1}((\frac{1}{2}, \frac{3}{2})) = \bigcup\{V_k - D_k \mid k \in K\}.$$

If we take $U := \bigcup\{U_j \mid j \in J\}$, $V := \bigcup\{V_k \mid k \in K\}$, $U' := U - C$ and $V' := V - B$, then U' and V' are τ-open subsets of X such that $B \subseteq (U' - V')$ and $C \subseteq (V' - U')$. Similar to the proof of Theorem 2.13, we have $U' \cap V' \in \mathcal{I}$, which completes the proof. □

Now, by Corollary 2.8 and Theorems 2.10, 2.13, 2.17 and 2.18 we have the following results.

Corollary 2.19. *Let $(X, \tau, \mathcal{I})$ be an ideal space such that $\tau \sim \mathcal{I}$ and $\mathcal{I}$ be a σ-ideal. Then the following statements are equivalent:*

(1) *X is $\mathcal{P}$-normal,*
(2) *X is (τ, τ^*)-normal,*
(3) *every two disjoint τ-closed subsets of X are τ^*-completely separated in X,*
(4) *every τ-closed subset of X is $C(\tau^*)$-embedded in X.*

Corollary 2.20. *Let $(X, \tau, \mathcal{I})$ be an ideal space such that $\mathcal{I}$ be a principal ideal. Then the following statements are equivalent:*

(1) *X is $\mathcal{P}$-normal,*
(2) *every two disjoint τ-closed subsets of X are τ^*-completely separated in X,*
(3) *every τ-closed subset of X is $C(\tau^*)$-embedded in X.*

Example 2.21. Let $(\mathbb{R}, \tau_0)$ be the real numbers with the particular point topology with point 0, i.e., the open sets are exactly those subsets of $\mathbb{R}$ that contain 0, together with the empty set; and $\mathcal{I}_0 = P(\{0\})$. It is clear that this space is not normal. If A and B are two disjoint closed subsets and put $U = A \cup \{0\}$, $V = B \cup \{0\}$, then $A \subseteq U - V$, $B \subseteq V - U$ and $U \cap V = \{0\} \in \mathcal{I}_0$. Hence $(\mathbb{R}, \tau_0, \mathcal{I}_0)$ is $\mathcal{P}$-normal. In general, if Y is a nonempty proper subset of $\mathbb{R}$, $\tau_Y = \{A \subseteq \mathbb{R} \mid A \supseteq Y\}$ and $\mathcal{I}_Y = P(Y)$, then the ideal space $(\mathbb{R}, \tau_Y, \mathcal{I}_Y)$ is $\mathcal{P}$-normal but it is not normal. Since $\mathcal{I}_Y$ is a principal ideal, by Corollary 2.20, $\mathcal{P}$-Urysohn's lemma and $\mathcal{P}$-Tietze extension theorem hold for the ideal space $(\mathbb{R}, \tau_Y, \mathcal{I}_Y)$.

In the following, we present an application of $\mathcal{P}$-normality concerning the ring of real-valued continuous functions. For any topological space (X, τ), we denote $C(X, \tau)$ as the ring of real-valued continuous functions defined on X.

Theorem 2.22. [13] *Let $(X, \tau, \mathcal{I})$ be an ideal space, Y be a regular space and $f : X \to Y$ be a function.*

(1) *If $f : (X, \tau^*) \to Y$ is continuous, then the restriction map $f|_{X^*} : (X^*, \tau_{X^*}) \to Y$ is continuous.*
(2) *If $X = X^*$, then $f : (X, \tau) \to Y$ is continuous if and only if $f : (X, \tau^*) \to Y$ is continuous.*

Since the real space $\mathbb{R}$ is regular, we have the following result.

Corollary 2.23. *Let $(X, \tau, \mathcal{I})$ be an ideal space. If $f \in C(X, \tau^*)$, then $f|_{X^*} \in C(X^*, \tau_{X^*})$. Also if $X = X^*$, then $C(X, \tau) = C(X, \tau^*)$.*

Theorem 2.24. *Let $(X, \tau, \mathcal{I})$ be a $\mathcal{P}$-normal ideal space such that $\mathcal{I}$ is a σ-ideal. Then the ring $C(X^*, \tau_{X^*})$ is a quotient ring of $C(X, \tau^*)$.*

Proof. Let the map $\varphi : C(X, \tau^*) \to C(X^*, \tau_{X^*})$ be defined by $\varphi(f) = f|_{X^*}$. It is easy to show that φ is a homomorphism. Now, let $g \in C(X^*, \tau_{X^*})$. Since X^* is a τ-closed subset of X, by Theorem 2.17, there exists $f \in C(X, \tau^*)$ such that $f|_{X^*} = g$. Thus φ is an epimorphism and hence $\dfrac{C(X, \tau^*)}{Ker(\varphi)} \cong C(X^*, \tau_{X^*})$, where $Ker(\varphi)$ is the kernel of φ. $\square$

Notice that in the above theorem we have:

$$Ker(\varphi) = \{f \in C(X, \tau^*) \mid \varphi(f) = 0\} = \{f \in C(X, \tau^*) \mid X^* \subseteq Z(f)\},$$

where $Z(f)$ is the zero-set of f.

3. $\mathcal{P}$-completely regular Spaces

An ideal space $(X, \tau, \mathcal{I})$ is defined as $\mathcal{P}$-regular if, for every τ-closed set F of X and each point $a \in X - F$, there exist τ-open sets U and V such that the separation conditions $(\{a\} \cup F) \cap U = \{a\}$ and $(\{a\} \cup F) \cap V = F$ are met, alongside the ideal condition $U \cap V \in \mathcal{I}$ (an equivalent formulation is $a \in U - V$, $F \subseteq V - U$ and $U \cap V \in \mathcal{I}$) [11]. Extending this notion to the realm of ideal spaces, we define $\mathcal{P}$-completely regularity in a manner analogous to its counterpart in traditional topological spaces, as follows:

Definition 3.1. An ideal space $(X, \tau, \mathcal{I})$ is called $\mathcal{P}$-completely regular if each τ-closed set A and each point $x \in X - A$, are τ^*-completely separated in X, i.e., there exists a continuous function $f : (X, \tau^*) \to \mathbb{R}$ such that $f(a) = 0$ for all $a \in A$ and $f(x) = 1$. Also, we say X is $\mathcal{P}$-Tychonoff if it is $\mathcal{P}$-completely regular and (X, τ) is T_1.

It is clear every regular space is $\mathcal{P}$-regular, and for T_1 spaces every $\mathcal{P}$-normal ideal space is $\mathcal{P}$-regular. Also, if $(\mathbb{R}, \tau_0, \mathcal{I}_0)$ is the ideal space given in Example 2.21, then it is $\mathcal{P}$-regular but not regular. In the following, we introduce some properties of $\mathcal{P}$-completely regular spaces in relation to the ring of real continuous functions.

Theorem 3.2. *An ideal space $(X, \tau, \mathcal{I})$ is $\mathcal{P}$-completely regular if and only if the zero sets of $C(X, \tau^*)$ form a basis for the closed sets of (X, τ), i.e., every τ-closed set can be written as the intersection of a family of zero sets in $C(X, \tau^*)$.*

Proof. Let F be a τ-closed set and $x \notin F$. Then there exists a $f \in C(X, \tau^*)$ such that $f(x) = 1$ and $f(F) = \{0\}$. Thus $F \subseteq Z(f)$ and $x \notin Z(f)$, which shows that the zero sets of $C(X, \tau^*)$ form a basis for the closed sets of (X, τ). Conversely, let F be a τ-closed set and $x \notin F$. Then there exists a $g \in C(X, \tau^*)$ such that $F \subseteq Z(g)$ and $x \notin Z(g)$. If we define $f = \frac{1}{g(x)} g$, then $f \in C(X, \tau^*)$ such that $f(x) = 1$ and $f(F) = \{0\}$. Thus the result holds. □

Theorem 3.3. *An ideal space $(X, \tau, \mathcal{I})$ is $\mathcal{P}$-completely regular if and only if τ is the induced topology by $C(X, \tau^*)$.*

Proof. Let τ be the induced topology by $C(X, \tau^*)$. Therefore the collection $S = \{f^{-1}([r, \infty) \mid f \in C(X, \tau^*), r \in \mathbb{R}\} \cup \{f^{-1}((-\infty, r] \mid f \in C(X, \tau^*)\}$ is a closed subbases for τ. On the other hand, we have $f^{-1}([r, \infty)) = Z((f - r) \wedge 0)$ and $f^{-1}((-\infty, r]) = Z((f - r) \vee 0)$. Thus S is a collection of the zero sets in $C(X, \tau^*)$. Since zero sets are closed under finite unions, it follows that S is a basis of the zero sets in $C(X, \tau^*)$ for τ. Thus by Theorem 3.2, X is $\mathcal{P}$-completely regular. Conversely, let X is $\mathcal{P}$-completely regular and η be the induced topology on X by $C(X, \tau^*)$. Then the collection $S = \{Z((f - r) \wedge 0) \mid f \in C(X, \tau^*), r \in \mathbb{R}\} \cup \{Z((f - r) \vee 0) \mid f \in C(X, \tau^*), r \in \mathbb{R}\}$ is a closed basis for η. If we put $S' = \{Z(f) \mid f \in C(X, \tau^*)\}$, then $S \subseteq S'$. On the other hand, for every $f \in C(X, \tau^*)$, we have $Z(f) = Z((-|f| - 0) \vee 0)$, and hence $S = S'$. Thus by Theorem 3.2, the result holds. □

4. Some examples

There are interesting ideals on a space (X, τ) that are both a σ-ideal and compatible with τ. In this section, we introduce such examples of ideal spaces that satisfy the conditions of Corollary 2.19.

Recall that a subset A of a space (X, τ) is called nowhere dense if $int(cl(A)) = \emptyset$. A subset A of X is said to be meager if it is a countable union of nowhere dense subsets of X. Let $\mathcal{I}_n$ be the collection of nowhere dense subsets and $\mathcal{I}_m$ be the collection of meager subsets of X. Then $\mathcal{I}_n$ and $\mathcal{I}_m$ are ideals on X; and it is clear that $\mathcal{I}_m$ is a σ-ideal on X. Also, it has been show that τ is compatible with both $\mathcal{I}_n$ and $\mathcal{I}_m$. It is easy to show that $\tau \cap \mathcal{I}_n = \{\emptyset\}$ and hence $\mathcal{I}_n$ is τ-codense. However, if X is a Baire space (i.e., the intersection of every countable family of τ-open sets is dense), then $\mathcal{I}_m$ is also τ-codense [5].

Example 4.1. Let (X, τ) be a space. Then by Theorem 2.2, the ideal space $(X, \tau, \mathcal{I}_n)$ is $\mathcal{P}$-normal if and only if (X, τ) is normal.

Example 4.2. Let (X, τ) be a Baire space. Then by Theorem 2.2, the ideal space $(X, \tau, \mathcal{I}_m)$ is $\mathcal{P}$-normal if and only if (X, τ) is normal.

Example 4.3. Let (X, τ) be the cocountable space such that X be uncountable and $\mathcal{I}_c$ be the ideal of countable sets. Then it is clear that $\tau \cap \mathcal{I}_c = \{\emptyset\}$ and hence $\mathcal{I}_c$ is τ-codense. Since (X, τ) is not normal, by Theorem 2.2, the ideal space $(X, \tau, \mathcal{I}_c)$ is not $\mathcal{P}$-normal.

Example 4.4. Let (X, τ) be a space. Then the ideal space $(X, \tau, \mathcal{I}_m)$ is $\mathcal{P}$-normal if and only if it is (τ, τ^*)-normal if and only if every two disjoint τ-closed subsets of X are τ^*-completely separated in X.

A space is said to be hereditarily Lindelof if every subspace has the Lindelof property. For example, second countable spaces are hereditarily Lindelof. Let $(X, \tau, \mathcal{I})$ be an ideal space such that X is hereditarily Lindelof and $\mathcal{I}$ be a σ-ideal. Then $\tau \sim \mathcal{I}$ [5].

Example 4.5. Let (X, τ) be a hereditarily Lindelof space and $\mathcal{I}$ be a σ-ideal on X. Then the ideal space $(X, \tau, \mathcal{I})$ is $\mathcal{P}$-normal if and only if it is (τ, τ^*)-normal if and only if every two disjoint τ-closed subsets of X are τ^*-completely separated in X.

Example 4.6. Let (X, τ) be a hereditarily Lindelof space and $\mathcal{I}_c$ be the ideal of countable subsets of X. Then the ideal space $(X, \tau, \mathcal{I}_c)$ is $\mathcal{P}$-normal if and only if it is (τ, τ^*)-normal if and only if every two disjoint τ-closed subsets of X are τ^*-completely separated in X.

5. Conclusion

In this study, we have presented some characterizations of $\mathcal{P}$-normality in ideal topological spaces. Specifically, we have proved a generalization of the Tietze extension theorem and the Urysohn's lemma for ideal spaces, subsequently providing examples that are $\mathcal{P}$-normal but not classically normal. Consequently, the stated theorems hold for these examples, whereas they fail in the classical case. These findings, supported by fundamental properties and interrelationships, constitute a significant contribution to the theory of ideal topological spaces, offering a robust foundation for advanced research and applications in this field.
Future work could explore new results defined through grills, primals, and ideals. Furthermore, the properties of $\mathcal{P}$-normal spaces generated by maximal and minimal ideals remain open topics for further research.

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