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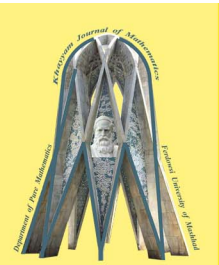

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MODEL CATEGORY STRUCTURE ON SIMPLICIAL ALGEBRAS VIA DOLD-KAN CORRESPONDENCE PART I

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Abstract. This article is the first part of a two-part project whose goal is to provide a self-contained and purely algebraic proof of a deep result of Quillen stating that the category of simplicial commutative algebras over a commutative ring admits a model structure. In the present paper, we recall the necessary foundations from the theory of model categories and establish several technical lemmas required for the transfer of model structures. The principal result is a detailed and self-contained proof of the existence of a cofibrantly generated model structure on the category of connective chain complexes over an arbitrary ring. This algebraic model structure will serve, in the second part of the project, as the basis for transferring a model structure to the category of simplicial commutative algebras.

1. Introduction

Homological algebra originated in the study of algebraic invariants of topological spaces and was later reformulated in the language of categories. Its central objective is the construction of derived functors, which provide homotopically meaningful refinements of classical functors. In abelian categories, derived functors are obtained by replacing objects with projective or injective resolutions and passing to homology. This leads naturally to derived categories, where homological constructions acquire a conceptual and unified interpretation.

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However, many situations of interest in algebra fall outside the abelian framework. In particular, the category of commutative algebras over a commutative ring is not abelian and does not admit resolutions in the classical sense. Nevertheless, one still wishes to construct derived functors in this context, for example in order to extend the theory of Kähler differentials and derivations beyond short exact sequences and to obtain long exact sequences measuring obstructions.

Quillen's theory of model categories provides a powerful framework for carrying out homological constructions in nonabelian settings. A model category is a bicomplete category equipped with three distinguished classes of morphisms — weak equivalences, fibrations, and cofibrations— satisfying certain axioms that abstract the basic properties of homotopy theory. This structure allows one to define homotopy categories and to construct derived functors via fibrant and cofibrant replacements, thereby generalizing the classical theory of resolutions.

In the case of commutative algebra, simplicial methods provide the appropriate nonabelian analogue of chain complexes. The Dold–Kan correspondence identifies connective chain complexes of modules with simplicial modules, suggesting that simplicial objects may serve as replacements for complexes even when abelian structure is absent. In particular, simplicial commutative algebras form the natural ambient category for developing a homotopical approach to commutative algebra. Once equipped with a suitable model structure, this category supports the construction of the cotangent complex and, consequently, André–Quillen homology and cohomology, which play a central role in deformation theory and in the characterization of important classes of rings and algebras; see for example [1, 2, 3, 4, 6, 8, 9, 15, 16, 18, 20, 19, 21].

The existing constructions of the model structure on simplicial commutative algebras rely, directly or indirectly, on the classical model structure on simplicial sets and hence ultimately on topological foundations. From a purely algebraic perspective, it is desirable to develop an approach that avoids this dependence and is formulated entirely in algebraic terms.

The purpose of the present work is to provide such a purely algebraic construction. Our strategy proceeds in two stages. First, we establish the model structure on the category of connective chain complexes by means of direct algebraic arguments grounded in classical homological algebra. We then transfer this model structure, via suitable adjunctions, to the category of simplicial commutative algebras. A crucial step in this transfer is the verification of the so-called acyclicity condition; the new contribution here is to verify this condition by entirely algebraic methods, using the shuffle product and its compatibility with the Dold–Kan correspondence in order to construct appropriate mapping spaces, cylinder objects, and path objects.

This article constitutes the first part of the project. After recalling the necessary background on model categories and fixing notation, we establish several technical lemmas required for the transfer argument developed in the second part. The main result of this paper is Theorem 3.1, in which we provide a complete and self-contained proof of the existence of a cofibrantly generated model structure on the category of connective chain complexes. Although this model structure is well known, the available proofs in the literature are either only sketched or

dispersed; see, for example, [12, Theorem 1.5] and [7, Theorem 7.2]). We present a detailed and unified treatment that makes the construction entirely algebraic and suitable for subsequent transfer arguments. The continuation of this work, devoted to the transfer of this model structure to simplicial commutative algebras will appear in a second article; see [10].

Although the homotopy category of connective chain complexes (in the sense of Quillen) is isomorphic to the classical derived category, the model structure provides additional advantages that are not captured by derived categories alone. In particular, the cofibrantly generated nature of the model structure allows the application of Quillen's "small object argument" to produce functorial factorizations of morphisms which in turn provide functorial fibrant and cofibrant replacements (not merely functorial up to homotopy). Moreover, this feature allows for the systematic transfer of the model structure along adjunctions that is essential when moving from connective chain complexes to nonabelian categories such as simplicial commutative algebras. This further facilitates constructions such as homotopy limits and colimits, and the cotangent complex, in a way that is impossible to encode using classical derived categories. The interested reader should refer to [13, Chapters 11 and 19]. Thus, the algebraic model structure on connective chain complexes serves both as a rigorous prototype of a model category and as a foundational tool enabling the developments carried out in the second part of this work.

2. Preliminaries on model categories

Quillen's theory of model categories provides a general framework to practice homological algebra far beyond the realm of abelian categories and their corresponding derived categories. In this section, we collect some basic definitions and facts from the theory of model categories and also prove a few lemmas that will be needed later. For general references on model categories, refer to [5, 7, 11, 13, 14, 19, 22].

Definition 2.1. A *model category* is a locally small bicomplete category $\mathcal{C}$ endowed with three classes of morphisms called *fibrations*, *cofibrations*, and *weak equivalences*. A morphism that is a fibration as well as a weak equivalence is called a *trivial fibration*, and a morphism that is a cofibration as well as a weak equivalence is called a *trivial cofibration*. Moreover, these morphisms should satisfy the following conditions:

(i) *Lifting*: Any commutative diagram on the left in which f is a trivial cofibration and g is a fibration, or f is a cofibration and g is a trivial fibration, can be completed to the commutative diagram on the right:

$$\begin{array}{ccc} A & \longrightarrow & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ B & \longrightarrow & D \end{array} \quad \Longrightarrow \quad \begin{array}{ccc} A & \longrightarrow & C \\ {\scriptstyle f}\downarrow & \nearrow & \downarrow{\scriptstyle g} \\ B & \longrightarrow & D \end{array}$$

(ii) *Retract*: If f is a retract of g in the arrow category $\mathrm{Arr}(\mathcal{C})$, that is, there is a commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{h} & C & \xrightarrow{l} & A \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ B & \xrightarrow{p} & D & \xrightarrow{q} & B \end{array}$$

in which $lh = 1^A$ and $qp = 1^B$, and if g is a fibration (cofibration, or weak equivalence), then f is a fibration (cofibration, or weak equivalence).

(iii) *Two-Out-of-Three*: If $f = gh$, and any two of f, g, or h are weak equivalences, then so is the third.

(iv) *Factorization*: For any morphism f, there is a factorization $f = gh$ for a fibration g and a trivial cofibration h, and also a factorization $f = g'h'$ for a trivial fibration g' and a cofibration h'.

A model category supports a notion of homotopy, which is the bedrock of constructing derived functors. In order to define this notion, we need the definitions of cylinder and path objects which could be shown to exist in every model category.

Definition 2.2. Let $\mathcal{C}$ be a model category, and let $A \in \mathrm{Obj}(\mathcal{C})$. Then

(i) A *cylinder object* for A is an object $\mathrm{Cyl}(A)$ through which the codiagonal morphism $\nabla_A : A \sqcup A \to A$ factors as

$$\begin{array}{ccc} A \sqcup A & \xrightarrow{\nabla_A} & A \\ {\scriptstyle \kappa_A}\downarrow & \nearrow_{\xi_A} & \\ \mathrm{Cyl}(A) & & \end{array}$$

in which κ_A is a cofibration and ξ_A is a weak equivalence.

(ii) A *path object* for A is an object $\mathrm{Path}(A)$ through which the diagonal morphism $\Delta_A : A \to A \sqcap A$ factors as

$$\begin{array}{ccc} A & \xrightarrow{\Delta_A} & A \sqcap A \\ {\scriptstyle \zeta_A}\downarrow & \nearrow_{\nu_A} & \\ \mathrm{Path}(A) & & \end{array}$$

in which ν_A is a fibration and ζ_A is a weak equivalence.

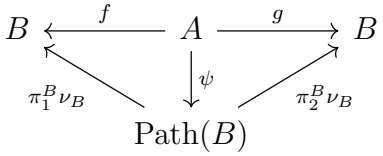
Definition 2.3. Let $\mathcal{C}$ be a model category, and let $f, g : A \to B$ be two morphisms in $\mathcal{C}$. Then:

(i) f is said to be *left homotopic* to g, denoted by $f \sim_l g$, if there is a cylinder object $\mathrm{Cyl}(A)$ for A, as in Definition 2.2, such that f and g factor through $\mathrm{Cyl}(A)$ as

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xleftarrow{g} & A \\ & {\scriptstyle \kappa_A \iota_1^A}\searrow & \uparrow{\scriptstyle \phi} & \swarrow{\scriptstyle \kappa_A \iota_2^A} & \\ & & \mathrm{Cyl}(A) & & \end{array}$$

in which $\iota_1^A : A \to A \sqcup A$ and $\iota_2^A : A \to A \sqcup A$ are canonical injections. Moreover, in this case, ϕ is said to be a *left homotopy* from f to g.

(ii) f is said to be *right homotopic* to g, denoted by $f \sim_r g$, if there is a path object $\mathrm{Path}(B)$ for B, as in Definition 2.2, such that f and g factor through $\mathrm{Path}(B)$ as



in which $\pi_1^B : B \sqcap B \to B$ and $\pi_2^B : B \sqcap B \to B$ are canonical projections. Moreover, in this case, ψ is said to be a *right homotopy* from f to g.

(iii) f is said to be *homotopic* to g, denoted by $f \sim g$, if $f \sim_l g$ and $f \sim_r g$.

Recall that a model category is bicomplete, that is, it has all small direct and inverse limits. In particular, a model category has initial and terminal objects. If $\mathcal{C}$ is a model category, $\circ$ is its initial object, and $*$ is its terminal object, then an object A in $\mathcal{C}$ is said to be *fibrant* if the unique morphism $A \to *$ is a fibration, and is said to be *cofibrant* if the unique morphism $\circ \to A$ is a cofibration. Now we can see that homotopy is an equivalence relation on morphisms.

Proposition 2.4. *Let $\mathcal{C}$ be a model category, and let $f, g : A \to B$ be two morphisms in $\mathcal{C}$. Then the following assertions hold:*

(i) *If $f \sim_l g$ and A is cofibrant, then $f \sim_r g$ and $\sim_l$ is an equivalence relation on $\mathrm{Mor}_{\mathcal{C}}(A, B)$.*

(ii) *If $f \sim_r g$ and B is fibrant, then $f \sim_l g$ and $\sim_r$ is an equivalence relation on $\mathrm{Mor}_{\mathcal{C}}(A, B)$.*

(iii) *If A is cofibrant and B is fibrant, then $f \sim_l g$ if and only if $f \sim_r g$, so $\sim$ is an equivalence relation on $\mathrm{Mor}_{\mathcal{C}}(A, B)$.*

Proof. See [13, Proposition 7.4.5]. □

A model category enjoys a concept of replacement, which can be thought of as a generalization as well as a nonabelian analogue of a resolution in abelian homological algebra. If $\mathcal{C}$ is a model category, and A is an object in $\mathcal{C}$, then a *fibrant replacement* of A is a trivial cofibration $\varrho_A : A \to A_{\mathrm{f}}$ in which A_{f} is fibrant, and a *cofibrant replacement* of A is a trivial fibration $\rho_A : A_{\mathrm{c}} \to A$ in which A_{c} is cofibrant. Now we have a comparison lemma for replacements as expected.

Proposition 2.5. *Let $\mathcal{C}$ be a model category. Then the following assertions hold:*

(i) *Every object A of $\mathcal{C}$ has a fibrant replacement $\varrho_A : A \to A_{\mathrm{f}}$. Choosing a fibrant replacement for each object of $\mathcal{C}$, if $f : A \to B$ is a morphism in $\mathcal{C}$, then there is a morphism $f_{\mathrm{f}} : A_{\mathrm{f}} \to B_{\mathrm{f}}$, which is unique up to homotopy and makes the following diagram commutative:*

$$\begin{array}{ccc} A & \xrightarrow{\varrho_A} & A_{\mathrm{f}} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_{\mathrm{f}}} \\ B & \xrightarrow{\varrho_B} & B_{\mathrm{f}} \end{array}$$

Moreover, if $g : B \to C$ is another morphism in $\mathcal{C}$, then $(gf)_{\mathrm{f}} \sim g_{\mathrm{f}} f_{\mathrm{f}}$, and $(1^A)_{\mathrm{f}} \sim 1^{A_{\mathrm{f}}}$.

(ii) *Every object A of $\mathcal{C}$ has a cofibrant replacement $\rho_A : A_{\mathrm{c}} \to A$. Choosing a cofibrant replacement for each object of $\mathcal{C}$, if $f : A \to B$ is a morphism in $\mathcal{C}$, then there is a morphism $f_{\mathrm{c}} : A_{\mathrm{c}} \to B_{\mathrm{c}}$, which is unique up to homotopy and makes the following diagram commutative:*

$$\begin{array}{ccc} A_{\mathrm{c}} & \xrightarrow{\rho_A} & A \\ {\scriptstyle f_{\mathrm{c}}}\downarrow & & \downarrow{\scriptstyle f} \\ B_{\mathrm{c}} & \xrightarrow{\rho_B} & B \end{array}$$

Moreover, if $g : B \to C$ is another morphism in $\mathcal{C}$, then $(gf)_{\mathrm{c}} \sim g_{\mathrm{c}} f_{\mathrm{c}}$, and $(1^A)_{\mathrm{c}} \sim 1^{A_{\mathrm{c}}}$.

Proof. See [13, Corollary 8.1.8]. □

The notions of fibrant and cofibrant replacements also allow us to define the homotopy category of a model category which can be thought of as a generalization as well as a nonabelian analogue of the derived category of an abelian category.

Definition 2.6. Let $\mathcal{C}$ be a model category. Choose a fibrant replacement $\varrho_A : A \to A_{\mathrm{f}}$ as well as a cofibrant replacement $\rho_A : A_{\mathrm{c}} \to A$ for every $A \in \mathrm{Obj}(\mathcal{C})$. Then the *homotopy category* $\mathrm{Ho}(\mathcal{C})$ of $\mathcal{C}$ is defined as follows:

(i) $\mathrm{Obj}\,(\mathrm{Ho}(\mathcal{C})) := \mathrm{Obj}(\mathcal{C})$.
(ii) $\mathrm{Mor}_{\mathrm{Ho}(\mathcal{C})}(A, B) := \mathrm{Mor}_{\mathcal{C}}(A_{\mathrm{fc}}, B_{\mathrm{fc}})/\sim$ for every $A, B \in \mathrm{Obj}\,(\mathrm{Ho}(\mathcal{C}))$.

We next have the universal property of the homotopy category, which in turn shows that the homotopy category is the localization of the model category with respect to the class of weak equivalences.

Proposition 2.7. *Let $\mathcal{C}$ be a model category. Choose a fibrant replacement $\varrho_A : A \to A_{\mathrm{f}}$ as well as a cofibrant replacement $\rho_A : A_{\mathrm{c}} \to A$ for every $A \in \mathrm{Obj}(\mathcal{C})$. Then there is a covariant functor $\mathcal{L}_{\mathcal{C}} : \mathcal{C} \to \mathrm{Ho}(\mathcal{C})$ defined as $\mathcal{L}_{\mathcal{C}}(A) := A$ for every $A \in \mathrm{Obj}(\mathcal{C})$, and $\mathcal{L}_{\mathcal{C}}(f) := [f_{\mathrm{fc}}]$ for every $f \in \mathrm{Mor}_{\mathcal{C}}(A, B)$. Moreover, $\mathcal{L}_{\mathcal{C}}$ has the following properties:*

(i) *A morphism f in $\mathcal{C}$ is a weak equivalence if and only if $\mathcal{L}_{\mathcal{C}}(f)$ is an isomorphism in $\mathrm{Ho}(\mathcal{C})$.*
(ii) *If $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ is a functor that maps weak equivalences in $\mathcal{C}$ to isomorphisms in $\mathcal{D}$, then $\mathcal{F}$ factors uniquely through $\mathcal{L}_{\mathcal{C}}$, that is, there is a unique functor $\check{\mathcal{F}} : \mathrm{Ho}(\mathcal{C}) \to \mathcal{D}$ that makes the following diagram commutative:*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\mathcal{F}} & \mathcal{D} \\ {\scriptstyle \mathcal{L}_{\mathcal{C}}}\downarrow & \nearrow_{\check{\mathcal{F}}} & \\ \mathrm{Ho}(\mathcal{C}) & & \end{array}$$

Proof. See [13, Proposition 8.3.7 and Theorem 8.3.10]. □

We next focus on cofibrantly generated model categories whose importance stems from the transferability of their model structure along adjunctions.

Let $\mathcal{C}$ be a category, and let $f : A \to B$ and $g : C \to D$ be two morphisms in $\mathcal{C}$. Suppose that any commutative diagram as the one on the left can be completed to the commutative diagram on the right

$$\begin{array}{ccc} A & \longrightarrow & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ B & \longrightarrow & D \end{array} \quad \Longrightarrow \quad \begin{array}{ccc} A & \longrightarrow & C \\ {\scriptstyle f}\downarrow & \nearrow & \downarrow{\scriptstyle g} \\ B & \longrightarrow & D \end{array}$$

Then we say that f has the *left lifting property against* g and g has the *right lifting property against* f. Given a class $\mathcal{X}$ of morphisms in $\mathcal{C}$, let $\mathrm{LLP}(\mathcal{X})$ denote the class of morphisms in $\mathcal{C}$ that have the left lifting property against all morphisms in $\mathcal{X}$, and $\mathrm{RLP}(\mathcal{X})$ denote the class of morphisms in $\mathcal{C}$ that have the right lifting property against all morphisms in $\mathcal{X}$. If $\mathcal{C}$ is a model category, then by [13, Proposition 7.2.3], g is a fibration if and only if it has the right lifting property against all trivial cofibrations; g is a trivial fibration if and only if it has the right lifting property against all cofibrations; f is a cofibration if and only if it has the left lifting property against all trivial fibrations; and f is a trivial cofibration if and only if it has the left lifting property against all fibrations.

Recall that if $\mathcal{C}$ is a locally small cocomplete category and $\mathcal{X}$ is a class of morphisms in $\mathcal{C}$, then an object A of $\mathcal{C}$ is said to be *small* with respect to $\mathcal{X}$ if the functor $\mathrm{Mor}_{\mathcal{C}}(A, -) : \mathcal{C} \to \mathcal{S}et$ preserves sequential direct limits of direct systems in $\mathcal{X}$, that is, for any sequence $\{g_n : B_n \to B_{n+1}\}_{n\geq 1}$ of morphisms in $\mathcal{X}$, the natural morphism

$$\varinjlim_{n\geq 1} \mathrm{Mor}_{\mathcal{C}}(A, B_n) \to \mathrm{Mor}_{\mathcal{C}}\left(A, \varinjlim_{n\geq 1} B_n\right)$$

is an isomorphism.

Quillen devised a method to transfer model structures along adjunctions, which is primarily based on his well-known "small object argument". A set $\mathcal{X}$ of morphisms in a locally small cocomplete category $\mathcal{C}$ is said to *permit the small object argument* if the source of any morphism in $\mathcal{X}$ is small with respect to $\mathrm{LLP}(\mathrm{RLP}(\mathcal{X}))$. This condition is necessary in order to be able to run the small object argument. We now recall the notion of a cofibrantly generated model category.

Definition 2.8. A model category $\mathcal{C}$ is said to be *cofibrantly generated* if there exist two sets $\mathcal{X}$ and $\mathcal{Y}$ of morphisms in C that satisfy the following conditions:

(i) $\mathcal{X}$ and $\mathcal{Y}$ permit the small object argument.
(ii) $\mathrm{RLP}(\mathcal{X})$ is the class of trivial fibrations.
(iii) $\mathrm{RLP}(\mathcal{Y})$ is the class of fibrations.

The next theorem is known as Quillen–Kan transfer machine.

Theorem 2.9. *Let $\mathcal{C}$ be a cofibrantly generated model category with the corresponding sets of morphisms $\mathcal{X}$ and $\mathcal{Y}$, let $\mathcal{D}$ be a locally small bicomplete category, and let $(\mathcal{F}, \mathcal{G}) : \mathcal{C} \leftrightarrows \mathcal{D}$ be an adjoint pair of functors. Then:*

(i) *Let a morphism $f : A \to B$ in $\mathcal{D}$ be a fibration if $\mathcal{G}(f) : \mathcal{G}(A) \to \mathcal{G}(B)$ is a fibration in $\mathcal{C}$.*
(ii) *Let a morphism $f : A \to B$ in $\mathcal{D}$ be a weak equivalence if $\mathcal{G}(f) : \mathcal{G}(A) \to \mathcal{G}(B)$ is a weak equivalence in $\mathcal{C}$.*
(iii) *Let a morphism $f : A \to B$ in $\mathcal{D}$ be a cofibration if it has the left lifting property against trivial fibrations defined in (i) and (ii).*

With the above definitions of fibrations, cofibrations, and weak equivalences in place, if the sets of morphisms $\mathcal{F}(\mathcal{X})$ and $\mathcal{F}(\mathcal{Y})$ permit the small object argument, and any cofibration in $\mathcal{D}$ that has the left lifting property against fibrations is also a weak equivalence, then $\mathcal{D}$ becomes a cofibrantly generated model category.

Proof. See [13, Theorem 11.3.2]. □

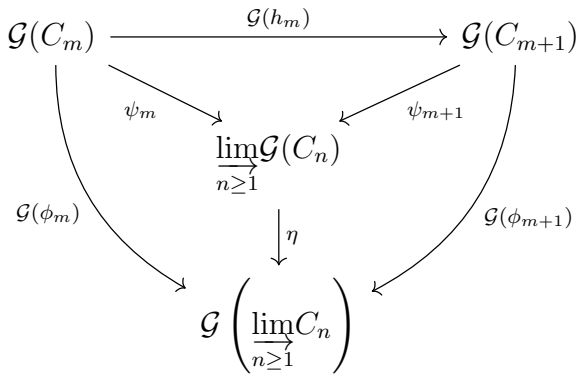
The last condition in Theorem 2.9 is commonly known as the "acyclicity condition" which is in practice notoriously hard to establish. We next prove a lemma that facilitates the permission of small object argument in our cases of interest.

Lemma 2.10. *Let $\mathcal{C}$ and $\mathcal{D}$ be two locally small cocomplete categories, and let $(\mathcal{F}, \mathcal{G}) : \mathcal{C} \leftrightarrows \mathcal{D}$ be an adjoint pair of functors in which $\mathcal{G}$ preserves sequential direct limits. Suppose that $\mathcal{X}$ is a set of morphisms in $\mathcal{C}$ such that the source of any morphism in $\mathcal{X}$ is small with respect to $\mathcal{G}\left(\mathrm{LLP}\left(\mathrm{RLP}\left(\mathcal{F}(\mathcal{X})\right)\right)\right)$. Then $\mathcal{F}(\mathcal{X}) = \{F(f) \mid f \in \mathcal{X}\}$ permits the small object argument.*

Proof. Let $f : A \to B$ be a morphism in $\mathcal{X}$, and consider the morphism $\mathcal{F}(f) : \mathcal{F}(A) \to \mathcal{F}(B)$ in $\mathcal{F}(\mathcal{X})$. We need to show that $\mathcal{F}(A)$ is small with respect to $\mathrm{LLP}\left(\mathrm{RLP}\left(\mathcal{F}(\mathcal{X})\right)\right)$. To this end, let $\{h_n : C_n \to C_{n+1}\}_{n\geq 1}$ be a sequence of morphisms in $\mathrm{LLP}\left(\mathrm{RLP}\left(\mathcal{F}(\mathcal{X})\right)\right)$. Let $\phi_m : C_m \to \varinjlim_{n\geq 1} C_n$ be the canonical injection for every $m \geq 1$. Then there is a unique morphism

$$\eta : \varinjlim_{n\geq 1} \mathcal{G}(C_n) \to \mathcal{G}\left(\varinjlim_{n\geq 1} C_n\right)$$

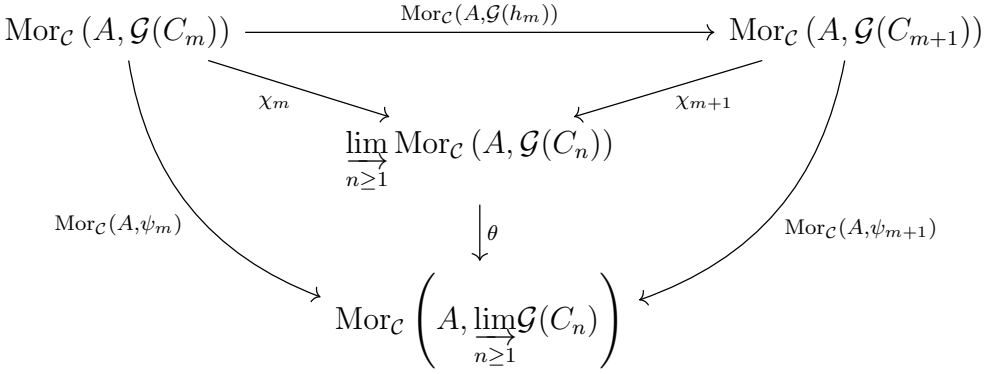
that makes the following diagram commutative for every $m \geq 1$:



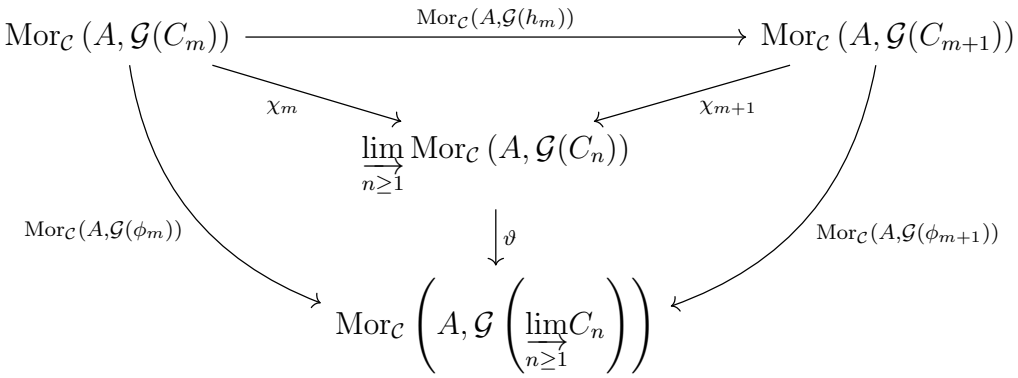
Since $\mathcal{G}$ preserves sequential direct limits, η is an isomorphism. Moreover, there is a unique morphism

$$\theta : \varinjlim_{n\geq 1} \mathrm{Mor}_{\mathcal{C}}\left(A, \mathcal{G}(C_n)\right) \to \mathrm{Mor}_{\mathcal{C}}\left(A, \varinjlim_{n\geq 1} \mathcal{G}(C_n)\right)$$

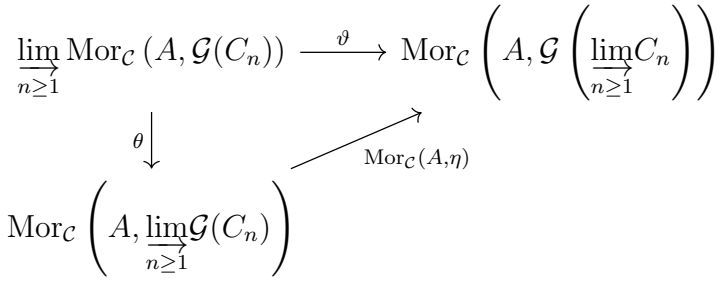
that makes the following diagram commutative for every $m \geq 1$:



Also, there is a unique morphism $\vartheta : \varinjlim_{n\geq 1} \mathrm{Mor}_{\mathcal{C}}\left(A, \mathcal{G}(C_n)\right) \to \mathrm{Mor}_{\mathcal{C}}\left(A, \mathcal{G}\left(\varinjlim_{n\geq 1} C_n\right)\right)$ that makes the following diagram commutative for every $m \geq 1$:



Now, we see that the following diagram is commutative:



Indeed, we have for every $m \geq 1$:

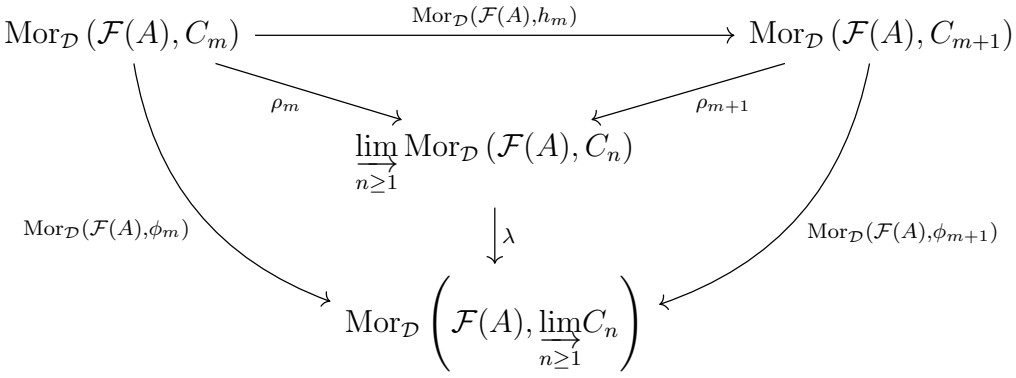
$\mathrm{Mor}_{\mathcal{C}}(A,\eta)\theta\chi_m = \mathrm{Mor}_{\mathcal{C}}(A,\eta)\,\mathrm{Mor}_{\mathcal{C}}(A,\psi_m) = \mathrm{Mor}_{\mathcal{C}}(A,\eta\psi_m) = \mathrm{Mor}_{\mathcal{C}}\left(A,\mathcal{G}(\phi_m)\right) = \vartheta\chi_m$

It follows that $\mathrm{Mor}_{\mathcal{C}}(A,\eta)\theta = \vartheta$ as desired. We next note that for any $n \geq 1$, we have $h_n \in \mathrm{LLP}\left(\mathrm{RLP}\left(\mathcal{F}(\mathcal{X})\right)\right)$, so $\mathcal{G}(h_n) \in \mathcal{G}\left(\mathrm{LLP}\left(\mathrm{RLP}\left(\mathcal{F}(\mathcal{X})\right)\right)\right)$. By the

hypothesis, A is small with respect to $\mathcal{G}\left(\mathrm{LLP}\left(\mathrm{RLP}\left(\mathcal{F}(\mathcal{X})\right)\right)\right)$, so we conclude that θ is an isomorphism. We had also observed above that η was an isomorphism, so $\mathrm{Mor}_{\mathcal{C}}(A,\eta)$ is an isomorphism. Therefore, $\vartheta = \mathrm{Mor}_{\mathcal{C}}(A,\eta)\theta$ is an isomorphism. On the other hand, there is a unique morphism

$$\lambda : \varinjlim_{n\geq 1} \mathrm{Mor}_{\mathcal{D}}\left(\mathcal{F}(A), C_n\right) \to \mathrm{Mor}_{\mathcal{D}}\left(\mathcal{F}(A), \varinjlim_{n\geq 1} C_n\right)$$

that makes the following diagram commutative for every $m \geq 1$:



Let

$$\sigma_{U,V} : \mathrm{Mor}_{\mathcal{D}}\left(\mathcal{F}(U), V\right) \to \mathrm{Mor}_{\mathcal{C}}\left(U, \mathcal{G}(V)\right)$$

be the natural bijection of the adjunction for every $U \in \mathrm{Obj}(\mathcal{C})$ and $V \in \mathrm{Obj}(\mathcal{D})$. Then the following diagram is commutative:

$$\begin{array}{ccc}
\varinjlim_{n\geq 1} \mathrm{Mor}_{\mathcal{D}}\left(\mathcal{F}(A), C_n\right) & \xrightarrow{\varinjlim_{n\geq 1} \sigma_{A,C_n}} & \varinjlim_{n\geq 1} \mathrm{Mor}_{\mathcal{C}}\left(A, \mathcal{G}(C_n)\right) \\
\downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle \vartheta} \\
\mathrm{Mor}_{\mathcal{D}}\left(\mathcal{F}(A), \varinjlim_{n\geq 1} C_n\right) & \xrightarrow{\sigma_{A,\varinjlim_{n\geq 1} C_n}} & \mathrm{Mor}_{\mathcal{C}}\left(A, \mathcal{G}\left(\varinjlim_{n\geq 1} C_n\right)\right)
\end{array}$$

Indeed, the commutativity of the above diagram follows from the naturality of the vertical morphisms applied to the natural transformation $\sigma_{A,-} : \mathrm{Mor}_{\mathcal{D}}\left(\mathcal{F}(A), -\right) \to \mathrm{Mor}_{\mathcal{C}}\left(A, \mathcal{G}(-)\right)$. Now since all the morphisms in the above diagram are isomorphisms except possibly for λ, we infer that λ must also be an isomorphism. This means that $\mathcal{F}(A)$ is small with respect to $\mathrm{LLP}\left(\mathrm{RLP}\left(\mathcal{F}(\mathcal{X})\right)\right)$. Therefore, $\mathcal{F}(\mathcal{X})$ permits the small object argument. □

Corollary 2.11. *Let $\mathcal{C}$ and $\mathcal{D}$ be two locally small cocomplete categories, and $(\mathcal{F},\mathcal{G}) : \mathcal{C} \leftrightarrows \mathcal{D}$ an adjoint pair of functors in which $\mathcal{G}$ preserves sequential direct limits. Suppose that $\mathcal{X}$ is a set of morphisms in $\mathcal{C}$ such that the source of any morphism in $\mathcal{X}$ is small with respect to $\mathrm{Mor}(\mathcal{C})$. Then the source of any morphism in $\mathcal{F}(\mathcal{X})$ is small with respect to $\mathrm{Mor}(\mathcal{D})$. In particular, $\mathcal{F}(\mathcal{X})$ permits the small object argument.*

Proof. Follows from the proof of Lemma 2.10. □

The following lemma might be of independent interest.

Lemma 2.12. *Let $\mathcal{C}$ and $\mathcal{D}$ be two locally small categories, and $(\mathcal{F},\mathcal{G}):\mathcal{C}\leftrightarrows\mathcal{D}$ an adjoint pair of functors. Let $\sigma_{A,B}:\mathrm{Mor}_{\mathcal{D}}(\mathcal{F}(A),B)\to\mathrm{Mor}_{\mathcal{C}}(A,\mathcal{G}(B))$ be the natural bijection of the adjunction for every $A\in\mathrm{Obj}(\mathcal{C})$ and $B\in\mathrm{Obj}(\mathcal{D})$. Then the following assertions hold:*

(i) *For any $f\in\mathrm{Mor}_{\mathcal{D}}(\mathcal{F}(A),B)$ and $g\in\mathrm{Mor}_{\mathcal{D}}(B,B')$, we have $\sigma_{A,B'}(gf)=\mathcal{G}(g)\sigma_{A,B}(f)$.*

(ii) *For any $f\in\mathrm{Mor}_{\mathcal{C}}(A,\mathcal{G}(B))$ and $g\in\mathrm{Mor}_{\mathcal{D}}(B,B')$, we have $\sigma^{-1}_{A,B'}(\mathcal{G}(g)f)=g\sigma^{-1}_{A,B}(f)$.*

(iii) *For any $g\in\mathrm{Mor}_{\mathcal{C}}(A,\mathcal{G}(B))$ and $f\in\mathrm{Mor}_{\mathcal{C}}(A',A)$, we have $\sigma^{-1}_{A',B}(gf)=\sigma^{-1}_{A,B}(g)\mathcal{F}(f)$.*

(iv) *For any $g\in\mathrm{Mor}_{\mathcal{D}}(\mathcal{F}(A),B)$ and $f\in\mathrm{Mor}_{\mathcal{C}}(A',A)$, we have $\sigma_{A',B}(g\mathcal{F}(f))=\sigma_{A,B}(g)f$.*

Proof. (i): Let $f\in\mathrm{Mor}_{\mathcal{D}}(\mathcal{F}(A),B)$ and $g\in\mathrm{Mor}_{\mathcal{D}}(B,B')$. Let $u_A:A\to\mathcal{G}(\mathcal{F}(A))$ be the unit morphism of A. Then using [17, p. 82, Eq. (5)], we have $\sigma_{A,B'}(gf)=\mathcal{G}(gf)u_A=\mathcal{G}(g)\mathcal{G}(f)u_A=\mathcal{G}(g)\sigma_{A,B}(f)$.

(ii): Let $f\in\mathrm{Mor}_{\mathcal{C}}(A,\mathcal{G}(B))$ and $g\in\mathrm{Mor}_{\mathcal{D}}(B,B')$. Set $h=\sigma^{-1}_{A,B'}(\mathcal{G}(g)f)$ and $l=\sigma^{-1}_{A,B}(f)$. Using (i), we get $\sigma_{A,B'}(gl)=\mathcal{G}(g)\sigma_{A,B}(l)=\mathcal{G}(g)f=\sigma_{A,B'}(h)$. Since $\sigma_{A,B'}$ is bijective, we conclude that $g\sigma^{-1}_{A,B}(f)=gl=h=\sigma^{-1}_{A,B'}(\mathcal{G}(g)f)$.

(iii): Similar to (i) using the counit morphism.

(iv): Similar to (ii). □

We can now compare the lifting properties along an adjunction. Note that the next lemma is in [13, Theorem 7.2.17], but the proof there skips some crucial steps which we include here for clarity.

Lemma 2.13. *Let $\mathcal{C}$ and $\mathcal{D}$ be two locally small categories, and $(\mathcal{F},\mathcal{G}):\mathcal{C}\leftrightarrows\mathcal{D}$ an adjoint pair of functors. Let $f:A\to B$ be a morphism in $\mathcal{C}$, and $g:C\to D$ a morphism in $\mathcal{D}$. Then $\mathcal{F}(f)$ has the left lifting property against g if and only if f has the left lifting property against $\mathcal{G}(g)$.*

Proof. Let $\sigma_{U,V}:\mathrm{Mor}_{\mathcal{D}}(\mathcal{F}(U),V)\to\mathrm{Mor}_{\mathcal{C}}(U,\mathcal{G}(V))$ be the natural bijection of the adjunction for every $U\in\mathrm{Obj}(\mathcal{C})$ and $V\in\mathrm{Obj}(\mathcal{D})$. Suppose that $\mathcal{F}(f)$ has the left lifting property against g. Consider a commutative diagram as follows:

$$\begin{array}{ccc} A & \xrightarrow{h} & \mathcal{G}(C) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathcal{G}(g)} \\ B & \xrightarrow{l} & \mathcal{G}(D) \end{array}$$

Then the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{F}(A) & \xrightarrow{\sigma^{-1}_{A,C}(h)} & C \\ {\scriptstyle \mathcal{F}(f)}\downarrow & & \downarrow{\scriptstyle g} \\ \mathcal{F}(B) & \xrightarrow{\sigma^{-1}_{B,D}(l)} & D \end{array}$$

Indeed, using Lemma 2.12 (ii) and (iii), we have:

$$\sigma_{B,D}^{-1}(l)\mathcal{F}(f) = \sigma_{A,D}^{-1}(lf) = \sigma_{A,D}^{-1}\left(\mathcal{G}(g)h\right) = g\sigma_{A,C}^{-1}(h)$$

By the assumption, there is a morphism $u : \mathcal{F}(B) \to C$ that makes the following diagram commutative:

$$\begin{array}{ccc} \mathcal{F}(A) & \xrightarrow{\sigma_{A,C}^{-1}(h)} & C \\ {\scriptstyle \mathcal{F}(f)}\downarrow & \overset{u}{\nearrow} & \downarrow{\scriptstyle g} \\ \mathcal{F}(B) & \xrightarrow{\sigma_{B,D}^{-1}(l)} & D \end{array}$$

Then the following diagram is commutative:

$$\begin{array}{ccc} A & \xrightarrow{h} & \mathcal{G}(C) \\ {\scriptstyle f}\downarrow & \overset{\sigma_{B,C}(u)}{\nearrow} & \downarrow{\scriptstyle \mathcal{G}(g)} \\ B & \xrightarrow{l} & \mathcal{G}(D) \end{array}$$

Indeed, using Lemma 2.12 (i) and (iv), we have:

$$\sigma_{B,C}(u)f = \sigma_{A,C}\left(u\mathcal{F}(f)\right) = \sigma_{A,C}\left(\sigma_{A,C}^{-1}(h)\right) = h$$

and

$$\mathcal{G}(g)\sigma_{B,C}(u) = \sigma_{B,D}(gu) = \sigma_{B,D}\left(\sigma_{B,D}^{-1}(l)\right) = l$$

This shows that f has the left lifting property against $\mathcal{G}(g)$. The converse is similarly established. □

Corollary 2.14. *Let $\mathcal{C}$ and $\mathcal{D}$ be two locally small categories, and $(\mathcal{F}, \mathcal{G}) : \mathcal{C} \leftrightarrows \mathcal{D}$ an adjoint pair of functors. Let $\mathcal{X}$ be a class of morphisms in $\mathcal{C}$, and $\mathcal{Y}$ a class of morphisms in $\mathcal{D}$. Then the following assertions hold:*

(i) *For any morphism f in $\mathcal{C}$, we have $\mathcal{F}(f) \in \mathrm{LLP}(\mathcal{Y})$ if and only if $f \in \mathrm{LLP}\left(\mathcal{G}(\mathcal{Y})\right)$.*

(ii) *For any morphism g in $\mathcal{D}$, we have $g \in \mathrm{RLP}\left(\mathcal{F}(\mathcal{X})\right)$ if and only if $\mathcal{G}(g) \in \mathrm{RLP}\left(\mathcal{X}\right)$.*

Proof. (i): Let f be a morphism in $\mathcal{C}$. By Lemma 2.13, $\mathcal{F}(f) \in \mathrm{LLP}(\mathcal{Y})$ if and only if $\mathcal{F}(f)$ has the left lifting property against every $g \in \mathcal{Y}$ if and only if f has the left lifting property against every $\mathcal{G}(g) \in \mathcal{G}(\mathcal{Y})$ if and only if $f \in \mathrm{LLP}\left(\mathcal{G}(\mathcal{Y})\right)$.

(ii): Similar to (i). □

The next proposition transfers model structures along equivalences.

Proposition 2.15. *Let $\mathcal{C}$ be a model category, $\mathcal{D}$ a category, and $\mathcal{G} : \mathcal{D} \to \mathcal{C}$ a covariant equivalence of categories. Then $\mathcal{D}$ becomes a model category by letting fibrations (cofibrations, or weak equivalences) be morphisms $f : A \to B$ in $\mathcal{D}$ such that $\mathcal{G}(f) : \mathcal{G}(A) \to \mathcal{G}(B)$ is a fibration (cofibration, or weak equivalence) in $\mathcal{C}$, respectively.*

Proof. One can directly check the conditions bearing in mind that $\mathcal{G}$ is fully faithful. □

Corollary 2.16. *Let $\mathcal{C}$ be a cofibrantly generated model category, $\mathcal{D}$ a category, and $(\mathcal{F},\mathcal{G}) : \mathcal{C} \leftrightarrows \mathcal{D}$ an adjoint equivalence of categories. Then $\mathcal{D}$ becomes a cofibrantly generated model category by letting fibrations (cofibrations, or weak equivalences) be morphisms $f : A \to B$ in $\mathcal{D}$ such that $\mathcal{G}(f) : \mathcal{G}(A) \to \mathcal{G}(B)$ is a fibration (cofibration, or weak equivalence) in $\mathcal{C}$, respectively.*

Proof. By Proposition 2.15, $\mathcal{D}$ is a model category with the specified classes of fibrations, cofibrations, and weak equivalences. Moreover, since $\mathcal{G}$ is a covariant equivalence of categories, $\mathcal{G}$ preserves direct limits. Now let $\mathcal{X}$ and $\mathcal{Y}$ be sets of morphisms as in Definition 2.8. Then $\mathrm{RLP}(\mathcal{X})$ is the class of trivial fibrations and $\mathrm{RLP}(\mathcal{Y})$ is the class of fibrations in $\mathcal{C}$. Since $\mathcal{C}$ is a model category, [13, Proposition 7.2.3] implies that $\mathrm{LLP}\left(\mathrm{RLP}(\mathcal{X})\right)$ is the class of cofibrations and $\mathrm{LLP}\left(\mathrm{RLP}(\mathcal{Y})\right)$ is the class of trivial cofibrations in $\mathcal{C}$. By Corollary 2.14 (ii), for a given morphism f in $\mathcal{D}$, we have $f \in \mathrm{RLP}\left(\mathcal{F}(\mathcal{X})\right)$ if and only if $\mathcal{G}(f) \in \mathrm{RLP}(\mathcal{X})$ if and only if $\mathcal{G}(f)$ is a trivial fibration in C if and only if f is a trivial fibration in $\mathcal{D}$. As a result, $\mathrm{RLP}\left(\mathcal{F}(\mathcal{X})\right)$ is the class of trivial fibrations in $\mathcal{D}$. A similar argument shows that $\mathrm{RLP}\left(\mathcal{F}(\mathcal{Y})\right)$ is the class of fibrations in $\mathcal{D}$. As $\mathcal{D}$ is a model category, $\mathrm{LLP}\left(\mathrm{RLP}\left(\mathcal{F}(\mathcal{X})\right)\right)$ is the class of cofibrations and $\mathrm{LLP}\left(\mathrm{RLP}\left(\mathcal{F}(\mathcal{Y})\right)\right)$ is the class of trivial cofibrations in $\mathcal{D}$. By definition, if f is a cofibration in $\mathcal{D}$, then $\mathcal{G}(f)$ is a cofibration in $\mathcal{C}$, meaning that $\mathcal{G}\left(\mathrm{LLP}\left(\mathrm{RLP}\left(\mathcal{F}(\mathcal{X})\right)\right)\right) \subseteq \mathrm{LLP}\left(\mathrm{RLP}\left(\mathcal{X}\right)\right)$. But $\mathcal{C}$ is cofibrantly generated, so $\mathcal{X}$ permits the small object argument, whence the source of any morphism in $\mathcal{X}$ is small with respect to $\mathrm{LLP}\left(\mathrm{RLP}\left(\mathcal{X}\right)\right)$, hence with respect to $\mathcal{G}\left(\mathrm{LLP}\left(\mathrm{RLP}\left(\mathcal{F}(\mathcal{X})\right)\right)\right)$. By Lemma 2.10, $\mathcal{F}(\mathcal{X})$ permits the small object argument. A similar argument shows that $\mathcal{F}(\mathcal{Y})$ permits the small object argument. Therefore, $\mathcal{D}$ is cofibrantly generated with the corresponding sets $\mathcal{F}(\mathcal{X})$ and $\mathcal{F}(\mathcal{Y})$. □

3. Model Structure on Connective Chain Complexes

In this section, we use some ideas from [14, Chapter 2, Section 2.3] and [7, Section 7] to cook up a straightforward and clear-cut proof of the model structure of connective chain complexes.

Let R be a ring. Recall that an R-complex is a $\mathbb{Z}$-indexed sequence

$$X : \cdots \to X_{i+1} \xrightarrow{\partial^X_{i+1}} X_i \xrightarrow{\partial^X_i} X_{i-1} \to \cdots$$

of left R-modules and R-homomorphisms such that $\partial^X_i \partial^X_{i+1} = 0$ for every $i \in \mathbb{Z}$. Given an R-complex X, we utilize the conventional notations $Z_i(X) = \mathrm{Ker}\left(\partial^X_i\right)$ and $B_i(X) = \mathrm{Im}\left(\partial^X_{i+1}\right)$ for every $i \in \mathbb{Z}$. An R-complex X is said to be *connective* if $X_i = 0$ for every $i < 0$. The category of connective R-complexes is denoted by $\mathcal{C}_{\geq 0}(R)$ which is a full subcategory of the category $\mathcal{C}(R)$ of unbounded R-complexes. In this section, we directly show that $\mathcal{C}_{\geq 0}(R)$ is a cofibrantly generated model category. The cofibrant generation property of this model structure allows us to transfer it along adjunctions.

We first recall the Eilenberg-MacLane complexes as follows. For any $n \in \mathbb{Z}$, we set $S(n) := \Sigma^n R$, that is, the R-complex

$$\cdots \to 0 \to R \to 0 \to \cdots$$

where R is located in degree n. Then it is easy to verify that if X is an R-complex, then the map $\phi_n^X : \mathrm{Mor}_{\mathcal{C}(R)}(S(n), X) \to Z_n(X)$, given by $f \mapsto f_n(1)$, is a natural isomorphism. Similarly, for any $n \in \mathbb{Z}$, we set $D(n) := \Sigma^{n-1} \mathrm{Cone}\left(1^R\right)$, that is, the R-complex

$$\cdots \to 0 \to R \xrightarrow{1^R} R \to 0 \to \cdots$$

where R is located in degrees n and $n-1$. Then it is easy to verify that if X is an R-complex, then the map $\psi_n^X : \mathrm{Mor}_{\mathcal{C}(R)}(D(n), X) \to X_n$, given by $f \mapsto f_n(1)$, is a natural isomorphism. We make repeated use of these facts in the proof of the following theorem.

Theorem 3.1. *Let R be a ring. Then $\mathcal{C}_{\geq 0}(R)$ is a cofibrantly generated model category with the model structure given as follows:*

(i) *Fibrations are morphisms $f : X \to Y$ in $\mathcal{C}_{\geq 0}(R)$ such that $f_i : X_i \to Y_i$ is surjective for every $i \geq 1$.*
(ii) *Cofibrations are morphisms $f : X \to Y$ in $\mathcal{C}_{\geq 0}(R)$ such that $f_i : X_i \to Y_i$ is injective and $\mathrm{Coker}(f_i)$ is a projective left R-module for every $i \geq 0$.*
(iii) *Weak equivalences are quasi-isomorphisms, that is, morphisms $f : X \to Y$ in $\mathcal{C}_{\geq 0}(R)$ such that $H_i(f) : H_i(X) \to H_i(Y)$ is an isomorphism for every $i \geq 0$.*

Proof. We first note that $\mathcal{C}_{\geq 0}(R)$ is a locally small bicomplete category. Let

$$\mathcal{X} = \{\iota^n : S(n-1) \to D(n) \mid n \geq 1\} \cup \{\lambda : 0 \to S(0)\}$$

where ι^n is the inclusion morphism for every $n \geq 1$. Also, let

$$\mathcal{Y} = \{\kappa^n : 0 \to D(n) \mid n \geq 1\}.$$

We show that $\mathcal{C}_{\geq 0}(R)$ is a cofibrantly generated model category with the corresponding sets $\mathcal{X}$ and $\mathcal{Y}$ of morphisms as in Definition 2.8. For any $n \geq 0$, we have $\mathrm{Mor}_{\mathcal{C}_{\geq 0}(R)}(S(n), -) \cong Z_n(-)$, so since the functor $Z_n(-)$ preserves direct limits of direct systems, we conclude that the functor $\mathrm{Mor}_{\mathcal{C}_{\geq 0}(R)}(S(n), -)$ preserves direct limits of direct systems as well, thereby $S(n)$ is small with respect to $\mathrm{Mor}(\mathcal{C}_{\geq 0}(R))$. Also, the source of λ is 0, so it is clearly small with respect to $\mathrm{Mor}(\mathcal{C}_{\geq 0}(R))$. In particular, the sources of morphisms in $\mathcal{X}$ are small with respect to $\mathrm{LLP}(\mathrm{RLP}(\mathcal{X}))$, so $\mathcal{X}$ permits the small object argument. Similarly, since the source of any morphism in $\mathcal{Y}$ is 0, we see that $\mathcal{Y}$ permits the small object argument.

We now show that $\mathrm{RLP}(\mathcal{X})$ is the class of trivial fibrations in $\mathcal{C}_{\geq 0}(R)$. Let $f : X \to Y$ be a morphism in $\mathcal{C}_{\geq 0}(R)$. Suppose that $f \in \mathrm{RLP}(\mathcal{X})$. We show that f is a trivial fibration, that is, f is a quasi-isomorphism and f_n is surjective for every $n \geq 1$. Let $n \geq 0$. We first show that $H_n(f) : H_n(X) \to H_n(Y)$ is an isomorphism. To see that $H_n(f)$ is injective, let $x \in Z_n(X)$ be such that $H_n(f)(x + B_n(X)) = f_n(x) + B_n(Y) = 0$. Then $f_n(x) \in B_n(Y)$, so $f_n(x) = \partial_{n+1}^Y(y)$ for some $y \in Y_{n+1}$. Consider the natural isomorphisms $\phi_n^X : \mathrm{Mor}_{\mathcal{C}_{\geq 0}(R)}(S(n), X) \to Z_n(X)$ and $\psi_{n+1}^Y : \mathrm{Mor}_{\mathcal{C}_{\geq 0}(R)}(D(n+1), Y) \to Y_{n+1}$. There is a $g \in \mathrm{Mor}_{\mathcal{C}_{\geq 0}(R)}(S(n), X)$ such that $\phi_n^X(g) = g_n(1) = x$, and an $h \in \mathrm{Mor}_{\mathcal{C}_{\geq 0}(R)}(D(n+1), Y)$ such that

$\psi_{n+1}^{Y}(h) = h_{n+1}(1) = y$. Then the following diagram is commutative:

$$\begin{array}{ccc} S(n) & \xrightarrow{g} & X \\ {\scriptstyle \iota^{n+1}}\downarrow & & \downarrow{\scriptstyle f} \\ D(n+1) & \xrightarrow{h} & Y \end{array}$$

Indeed, it is clear from the structure of $S(n)$ that $\iota_i^{n+1} = 0 = g_i$ for every $i \neq n$. Moreover, we have:

$$h_n\left(\iota_n^{n+1}(1)\right) = h_n(1) = \partial_{n+1}^{Y}\left(h_{n+1}(1)\right) = \partial_{n+1}^{Y}(y) = f_n(x) = f_n\left(g_n(1)\right)$$

By the choice of f, the above diagram can be completed to the following commutative diagram:

$$\begin{array}{ccc} S(n) & \xrightarrow{g} & X \\ {\scriptstyle \iota^{n+1}}\downarrow & \overset{l}{\nearrow} & \downarrow{\scriptstyle f} \\ D(n+1) & \xrightarrow{h} & Y \end{array}$$

Thus $\partial_{n+1}^{X}\left(l_{n+1}(1)\right) = l_n(1) = l_n\left(\iota_n^{n+1}(1)\right) = g_n(1) = x$, so $x \in B_n(X)$. This means that $H_n(f)$ is injective. To see that $H_n(f)$ is surjective, let $z \in Z_n(Y)$. If $n = 0$, then consider the isomorphism $\phi_0^Y : \mathrm{Mor}_{\mathcal{C}_{\geq 0}(R)}\left(S(0), Y\right) \to Z_0(Y)$. There is an $r \in \mathrm{Mor}_{\mathcal{C}_{\geq 0}(R)}\left(S(0), Y\right)$ such that $\phi_0^Y(r) = r_0(1) = z$. Then it is clear that the following diagram is commutative:

$$\begin{array}{ccc} 0 & \longrightarrow & X \\ {\scriptstyle \lambda}\downarrow & & \downarrow{\scriptstyle f} \\ S(0) & \xrightarrow{r} & Y \end{array}$$

By the choice of f, the above diagram can be completed to the following commutative diagram:

$$\begin{array}{ccc} 0 & \longrightarrow & X \\ {\scriptstyle \lambda}\downarrow & \overset{s}{\nearrow} & \downarrow{\scriptstyle f} \\ S(0) & \xrightarrow{r} & Y \end{array}$$

Set $w = s_0(1) \in X_0 = Z_0(X)$. Then $f_0(w) = f_0\left(s_0(1)\right) = r_0(1) = z$. If $n \geq 1$, then consider the isomorphism $\psi_n^Y : \mathrm{Mor}_{\mathcal{C}_{\geq 0}(R)}\left(D(n), Y\right) \to Y_n$. There is an $r \in \mathrm{Mor}_{\mathcal{C}_{\geq 0}(R)}\left(D(n), Y\right)$ such that $\psi_n^Y(r) = r_n(1) = z$. Then the following diagram is commutative:

$$\begin{array}{ccc} S(n-1) & \xrightarrow{0} & X \\ {\scriptstyle \iota^{n}}\downarrow & & \downarrow{\scriptstyle f} \\ D(n) & \xrightarrow{r} & Y \end{array}$$

Indeed, it is clear that $\iota_i^n = 0$ for every $i \neq n-1$. Moreover, since $z \in Z_n(Y)$, we have

$$r_{n-1}\left(\iota_{n-1}^{n}(1)\right) = r_{n-1}(1) = \partial_n^Y\left(r_n(1)\right) = \partial_n^Y(z) = 0.$$

By the choice of f, the above diagram can be completed to the following commutative diagram:

$$\begin{array}{ccc} S(n-1) & \xrightarrow{0} & X \\ {\scriptstyle \iota^n}\downarrow & \overset{s}{\nearrow} & \downarrow{\scriptstyle f} \\ D(n) & \xrightarrow{r} & Y \end{array}$$

Set $w = s_n(1) \in X_n$. Then $f_n(w) = f_n\left(s_n(1)\right) = r_n(1) = z$. Moreover,

$$\partial_n^X(w) = \partial_n^X\left(s_n(1)\right) = s_{n-1}(1) = s_{n-1}\left(\iota_{n-1}^n(1)\right) = 0,$$

so $w \in Z_n(X)$. In other words, for any $n \geq 0$ and $z \in Z_n(Y)$, we have found an element $w \in Z_n(X)$ such that $f_n(w) = z$. In particular, we have $H_n(f)\left(w + B_n(X)\right) = f_n(w) + B_n(Y) = z + B_n(Y)$. This shows that $H_n(f)$ is surjective. Therefore, $H_n(f)$ is an isomorphism. Next let $n \geq 1$. We show that f_n is surjective. Let $v \in Y_n$. Then $\partial_n^Y(v) \in B_{n-1}(Y) \subseteq Z_{n-1}(Y)$, so as noted above, there is an element $u \in Z_{n-1}(X)$ such that $f_{n-1}(u) = \partial_n^Y(v)$. Consider the isomorphisms $\phi_{n-1}^X : \mathrm{Mor}_{\mathcal{C}_{\geq 0}(R)}\left(S(n-1), X\right) \to Z_{n-1}(X)$ and $\psi_n^Y : \mathrm{Mor}_{\mathcal{C}_{\geq 0}(R)}\left(D(n), Y\right) \to Y_n$. There is a $p \in \mathrm{Mor}_{\mathcal{C}_{\geq 0}(R)}\left(S(n-1), X\right)$ such that $\phi_{n-1}^X(p) = p_{n-1}(1) = u$, and a $q \in \mathrm{Mor}_{\mathcal{C}_{\geq 0}(R)}\left(D(n), Y\right)$ such that $\psi_n^Y(q) = q_n(1) = v$. Then the following diagram is commutative:

$$\begin{array}{ccc} S(n-1) & \xrightarrow{p} & X \\ {\scriptstyle \iota^n}\downarrow & & \downarrow{\scriptstyle f} \\ D(n) & \xrightarrow{q} & Y \end{array}$$

Indeed, it is clear that $\iota_i^n = 0 = p_i$ for every $i \neq n-1$. Moreover, we have:

$$q_{n-1}\left(\iota_{n-1}^n(1)\right) = q_{n-1}(1) = \partial_n^Y\left(q_n(1)\right) = \partial_n^Y(v) = f_{n-1}(u) = f_{n-1}\left(p_{n-1}(1)\right)$$

By the choice of f, the above diagram can be completed to the following commutative diagram:

$$\begin{array}{ccc} S(n-1) & \xrightarrow{p} & X \\ {\scriptstyle \iota^n}\downarrow & \overset{t}{\nearrow} & \downarrow{\scriptstyle f} \\ D(n) & \xrightarrow{q} & Y \end{array}$$

Then $f_n\left(t_n(1)\right) = q_n(1) = v$. Thus f_n is surjective. As a result, f is a trivial fibration.

Conversely, suppose that f is a trivial fibration, that is, f is a quasi-isomorphism and f_n is surjective for every $n \geq 1$. We first show that f_0 is also surjective. Consider the following commutative diagram:

$$\begin{array}{ccc} X_1 & \xrightarrow{\overline{\partial_1^X}} & B_0(X) \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle B_0(f)} \\ Y_1 & \xrightarrow{\overline{\partial_1^Y}} & B_0(Y) \end{array}$$

As f_1 is surjective, the above diagram implies that $B_0(f)$ is surjective. Now consider the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B_0(X) & \longrightarrow & X_0 & \longrightarrow & H_0(X) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle B_0(f)} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle H_0(f)} & & \\ 0 & \longrightarrow & B_0(Y) & \longrightarrow & Y_0 & \longrightarrow & H_0(Y) & \longrightarrow & 0 \end{array}$$

Since $B_0(f)$ is surjective and $H_0(f)$ is an isomorphism, the Short Five Lemma implies that f_0 is surjective. Therefore, f is a surjective quasi-isomorphism. We next show that $f \in \mathrm{RLP}(\mathcal{X})$. Let $n \geq 1$, and consider a commutative diagram as follows:

$$\begin{array}{ccc} S(n-1) & \xrightarrow{p} & X \\ {\scriptstyle \iota^n}\downarrow & & \downarrow{\scriptstyle f} \\ D(n) & \xrightarrow{q} & Y \end{array}$$

Let $x = p_{n-1}(1) \in X_{n-1}$ and $y = q_n(1) \in Y_n$. Then we have:

$$f_{n-1}(x) = f_{n-1}\left(p_{n-1}(1)\right) = q_{n-1}\left(\iota^n_{n-1}(1)\right) = q_{n-1}(1) = \partial^Y_n\left(q_n(1)\right) = \partial^Y_n(y)$$

Moreover, $\partial^X_{n-1}(x) = \partial^X_{n-1}\left(p_{n-1}(1)\right) = 0$, so $x \in Z_{n-1}(X)$. But then

$$H_{n-1}(f)\left(x + B_{n-1}(X)\right) = f_{n-1}(x) + B_{n-1}(Y) = \partial^Y_n(y) + B_{n-1}(Y) = 0,$$

so since $H_{n-1}(f)$ is an isomorphism, we get $x + B_{n-1}(X) = 0$, that is, $x \in B_{n-1}(X)$, so $x = \partial^X_n(u)$ for some $u \in X_n$. As f is a morphism of R-complexes, we have $\partial^Y_n(y) = f_{n-1}(x) = f_{n-1}\left(\partial^X_n(u)\right) = \partial^Y_n\left(f_n(u)\right)$, so $y - f_n(u) \in Z_n(Y)$. But f is a surjective quasi-isomorphism, so it is easy to see that $Z(f) : Z(X) \to Z(Y)$ is surjective. In particular, there is an element $v \in Z_n(X)$ such that $f_n(v) = y - f_n(u)$. Set $z = u + v \in X_n$. Then $\partial^X_n(z) = \partial^X_n(u) + \partial^X_n(v) = \partial^X_n(u) = x$ and $f_n(z) = f_n(u) + f_n(v) = y$. Consider the isomorphism $\psi^X_n : \mathrm{Mor}_{\mathcal{C}_{\geq 0}(R)}\left(D(n), X\right) \to X_n$. There is an $l \in \mathrm{Mor}_{\mathcal{C}_{\geq 0}(R)}\left(D(n), X\right)$ such that $\psi^X_n(l) = l_n(1) = z$. Then the following diagram is commutative:

$$\begin{array}{ccc} S(n-1) & \xrightarrow{p} & X \\ {\scriptstyle \iota^n}\downarrow & \overset{l}{\nearrow} & \downarrow{\scriptstyle f} \\ D(n) & \xrightarrow{q} & Y \end{array}$$

Indeed, $\iota^n_i = 0$ for every $i \neq n-1$. Furthermore, we have:

$$l_{n-1}\left(\iota^n_{n-1}(1)\right) = l_{n-1}(1) = \partial^X_n\left(l_n(1)\right) = \partial^X_n(z) = x = p_{n-1}(1)$$

On the other hand, $q_i = 0 = l_i$ for every $i \neq n, n-1$. Moreover, we have $f_n(l_n(1)) = f_n(z) = y = q_n(1)$ and

$$\begin{aligned} f_{n-1}(l_{n-1}(1)) &= f_{n-1}\left(\partial_n^X(l_n(1))\right) \\ &= f_{n-1}\left(\partial_n^X(z)\right) \\ &= f_{n-1}(x) = \partial_n^Y(y) \\ &= \partial_n^Y(q_n(1)) \\ &= q_{n-1}(1). \end{aligned}$$

Next consider a commutative diagram as follows:

$$\begin{array}{ccc} 0 & \longrightarrow & X \\ {\scriptstyle\lambda}\downarrow & & \downarrow{\scriptstyle f} \\ S(0) & \xrightarrow{r} & Y \end{array}$$

By the surjectivity of f_0, there is an element $w \in X_0 = Z_0(X)$ such that $f_0(w) = r_0(1)$. Consider the isomorphism $\phi_0^X : \mathrm{Mor}_{\mathcal{C}_{\geq 0}(R)}(S(0), X) \to Z_0(X)$. There is an $s \in \mathrm{Mor}_{\mathcal{C}_{\geq 0}(R)}(S(0), X)$ such that $\phi_0^X(s) = s_0(1) = w$. Then the following diagram is commutative:

$$\begin{array}{ccc} 0 & \longrightarrow & X \\ {\scriptstyle\lambda}\downarrow & \overset{s}{\nearrow} & \downarrow{\scriptstyle f} \\ S(0) & \xrightarrow{r} & Y \end{array}$$

Indeed, it is clear that the upper triangle is commutative. Moreover, for the lower triangle, we note that $s_i = 0 = r_i$ for every $i \neq 0$. Also, $f_0(s_0(1)) = f_0(w) = r_0(1)$. This means that $f \in \mathrm{RLP}(\mathcal{X})$. Therefore, $\mathrm{RLP}(\mathcal{X})$ is the class of trivial fibrations.

We next show that $\mathrm{RLP}(\mathcal{Y})$ is the class of fibrations in $\mathcal{C}_{\geq 0}(R)$. Let $f : X \to Y$ be a morphism in $\mathcal{C}_{\geq 0}(R)$. Suppose that $f \in \mathrm{RLP}(\mathcal{Y})$. We show that f is a fibration, that is, f_n is surjective for every $n \geq 1$. Let $n \geq 1$ and $v \in Y_n$. Consider the isomorphism $\psi_n^Y : \mathrm{Mor}_{\mathcal{C}_{\geq 0}(R)}(D(n), Y) \to Y_n$. There is a $q \in \mathrm{Mor}_{\mathcal{C}_{\geq 0}(R)}(D(n), Y)$ such that $\psi_n^Y(q) = q_n(1) = v$. By the choice of f, the commutative diagram

$$\begin{array}{ccc} 0 & \longrightarrow & X \\ {\scriptstyle\kappa^n}\downarrow & & \downarrow{\scriptstyle f} \\ D(n) & \xrightarrow{q} & Y \end{array}$$

can be completed to the following commutative diagram:

$$\begin{array}{ccc} 0 & \longrightarrow & X \\ {\scriptstyle\kappa^n}\downarrow & \overset{t}{\nearrow} & \downarrow{\scriptstyle f} \\ D(n) & \xrightarrow{q} & Y \end{array}$$

Then $f_n(t_n(1)) = q_n(1) = v$. This shows that f_n is surjective. As a result, f is a fibration.

Conversely, suppose that f is a fibration, that is, f_n is surjective for every $n \geq 1$. Let $n \geq 1$, and consider a commutative diagram as follows:

$$\begin{array}{ccc} 0 & \longrightarrow & X \\ {\scriptstyle \kappa^n}\downarrow & & \downarrow{\scriptstyle f} \\ D(n) & \xrightarrow{q} & Y \end{array}$$

Since f_n is surjective, there is an element $x \in X_n$ such that $f_n(x) = q_n(1)$. Consider the isomorphism $\psi_n^X : \mathrm{Mor}_{\mathcal{C}_{\geq 0}(R)}(D(n), X) \to X_n$. There is an $l \in \mathrm{Mor}_{\mathcal{C}_{\geq 0}(R)}(D(n), X)$ such that $\psi_n^X(l) = l_n(1) = x$. Then the following diagram is commutative:

$$\begin{array}{ccc} 0 & \longrightarrow & X \\ {\scriptstyle \kappa^n}\downarrow & \overset{l}{\nearrow} & \downarrow{\scriptstyle f} \\ D(n) & \xrightarrow{q} & Y \end{array}$$

Indeed, it is obvious that the upper triangle is commutative. For the lower triangle, we note that $q_i = 0 = l_i$ for every $i \neq n, n-1$. Moreover, using the fact that f is a morphism of R-complexes, we have $f_n(l_n(1)) = f_n(x) = q_n(1)$ and

$$\begin{aligned} f_{n-1}(l_{n-1}(1)) &= f_{n-1}\left(\partial_n^X(l_n(1))\right) \\ &= f_{n-1}\left(\partial_n^X(x)\right) \\ &= \partial_n^Y(f_n(x)) \\ &= \partial_n^Y(q_n(1)) \\ &= q_{n-1}(1). \end{aligned}$$

This implies that $f \in \mathrm{RLP}(\mathcal{Y})$. Therefore, $\mathrm{RLP}(\mathcal{Y})$ is the class of fibrations.

We are now ready to check the axioms of the model category as follows:

Retract: Suppose that f is a retract of g, that is, there is a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{h} & Z & \xrightarrow{l} & X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{p} & W & \xrightarrow{q} & Y \end{array}$$

in $\mathcal{C}_{\geq 0}(R)$ in which $lh = 1^X$ and $qp = 1^Y$. If g is a fibration, then g_i is surjective for every $i \geq 1$. On the other hand, the relation $q_i p_i = 1^{Y_i}$ implies that q_i is surjective for every $i \geq 0$. Therefore, $q_i g_i$ is surjective for every $i \geq 1$. Now the relation $f_i l_i = q_i g_i$ yields that f_i is surjective for every $i \geq 1$. That means that f is a fibration.

If g is a cofibration, then g_i is injective and $\mathrm{Coker}(g_i)$ is projective for every $i \geq 0$. The relation $l_i h_i = 1^{X_i}$ implies that h_i is injective for every $i \geq 0$. Therefore, $g_i h_i$ is injective for every $i \geq 0$. Now the relation $p_i f_i = g_i h_i$ yields that f_i is injective for every $i \geq 0$. Furthermore, we get the following induced

commutative diagram:

$$\begin{array}{ccccc} X & \xrightarrow{h} & Z & \xrightarrow{l} & X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{p} & W & \xrightarrow{q} & Y \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Coker}(f) & \xrightarrow{\bar{p}} & \mathrm{Coker}(g) & \xrightarrow{\bar{q}} & \mathrm{Coker}(f) \end{array}$$

It is also evident that $\bar{q}\bar{p} = 1^{\mathrm{Coker}(f)}$. It follows that $\mathrm{Coker}(f)$ is isomorphic to a direct summand of $\mathrm{Coker}(g)$. Therefore, $\mathrm{Coker}(f_i)$ is projective for every $i \geq 0$. Thus f is a cofibration.

Finally, if g is a weak equivalence, then g is a quasi-isomorphism. Fix $i \geq 0$. The functoriality of homology yields the following commutative diagram:

$$\begin{array}{ccccc} H_i(X) & \xrightarrow{H_i(h)} & H_i(Z) & \xrightarrow{H_i(l)} & H_i(X) \\ \downarrow{\scriptstyle H_i(f)} & & \downarrow{\scriptstyle H_i(g)} & & \downarrow{\scriptstyle H_i(f)} \\ H_i(Y) & \xrightarrow{H_i(p)} & H_i(W) & \xrightarrow{H_i(q)} & H_i(Y) \end{array}$$

Setting $\phi = H_i(l)H_i(g)^{-1}H_i(p) : H_i(Y) \to H_i(X)$, we observe from the above commutative diagram that ϕ is the inverse of $H_i(f)$, so $H_i(f)$ is an isomorphism. That is to say, f is a quasi-isomorphism, hence a weak equivalence.

Two-Out-of-Three: Suppose that we have morphisms $f = gh$ in $\mathcal{C}_{\geq 0}(R)$ in which two of f, g, or h are quasi-isomorphisms. Then by the functoriality of homology, it is clear that the third morphism must be a quasi-isomorphism as well.

First Half of Lifting: Consider a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{h} & Z \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow{l} & W \end{array}$$

in $\mathcal{C}_{\geq 0}(R)$ in which f is a cofibration and g is a trivial fibration. By definition, $f_i : X_i \to Y_i$ is injective and $\mathrm{Coker}(f_i)$ is projective for every $i \geq 0$. As a result, for any $i \geq 0$, the short exact sequence

$$0 \to X_i \xrightarrow{f_i} Y_i \to \mathrm{Coker}(f_i) \to 0$$

is split, resulting in a decomposition $Y_i = \mathrm{Im}(f_i) \oplus P_i$ for some projective left R-module $P_i \cong \mathrm{Coker}(f_i)$. Moreover, as noted before, $g : Z \to W$ is a surjective quasi-isomorphism, so there is a short exact sequence

$$0 \to \mathrm{Ker}(g) \xrightarrow{\iota} Z \xrightarrow{g} W \to 0$$

of R-complexes in which $\mathrm{Ker}(g)$ is exact. Now we construct a morphism $\phi = (\phi_i)_{i \geq 0} : Y \to Z$ of R-complexes inductively. Let $i = 0$, and consider the following

commutative diagram:

$$\begin{array}{ccc} X_0 & \xrightarrow{h_0} & Z_0 \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle g_0} \\ Y_0 & \xrightarrow{l_0} & W_0 \end{array}$$

Since g_0 is surjective and P_0 is projective, there is an R-homomorphism $\rho_0 : P_0 \to Z_0$ that makes the following diagram commutative:

$$\begin{array}{ccc} & & P_0 \\ & {\scriptstyle \rho_0}\swarrow & \downarrow{\scriptstyle l_0|_{P_0}} \\ Z_0 & \xrightarrow{g_0} & W_0 \end{array}$$

If $y \in Y_0 = \mathrm{Im}(f_0) \oplus P_0$, then we can uniquely write $y = f_0(x) + z$ for some $x \in X_0$ and $z \in P_0$. Define a map $\phi_0 : Y_0 \to Z_0$ by setting $\phi_0(y) = h_0(x) + \rho_0(z)$. Then it is clear that ϕ_0 is an R-homomorphism. Moreover, the following diagram is commutative:

$$\begin{array}{ccc} X_0 & \xrightarrow{h_0} & Z_0 \\ {\scriptstyle f_0}\downarrow & {\scriptstyle \phi_0}\nearrow & \downarrow{\scriptstyle g_0} \\ Y_0 & \xrightarrow{l_0} & W_0 \end{array}$$

Indeed, if $x \in X_0$, then using the definition, we have $\phi_0 (f_0(x)) = h_0(x)$. Further, if $y = f_0(x) + z \in Y_0 = \mathrm{Im}(f_0) \oplus P_0$, then we have:

$$g_0 (\phi_0(y)) = g_0 (h_0(x)) + g_0 (\rho_0(z)) = l_0 (f_0(x)) + l_0(z) = l_0 (f_0(x) + z) = l_0(y)$$

Now suppose that $n \geq 1$, and for any $0 \leq i \leq n-1$, the R-homomorphism $\phi_i : Y_i \to Z_i$ is constructed in a way that the following diagrams are commutative:

$$\begin{array}{ccc} X_i & \xrightarrow{h_i} & Z_i \\ {\scriptstyle f_i}\downarrow & {\scriptstyle \phi_i}\nearrow & \downarrow{\scriptstyle g_i} \\ Y_i & \xrightarrow{l_i} & W_i \end{array} \quad \text{and} \quad \begin{array}{ccc} Y_i & \xrightarrow{\partial_i^Y} & Y_{i-1} \\ {\scriptstyle \phi_i}\downarrow & & \downarrow{\scriptstyle \phi_{i-1}} \\ Z_i & \xrightarrow{\partial_i^Z} & Z_{i-1} \end{array}$$

That is, $\phi_i f_i = h_i$, $g_i \phi_i = l_i$, and $\partial_i^Z \phi_i = \phi_{i-1} \partial_i^Y$ for every $0 \leq i \leq n-1$. By the same argument as in the case $i = 0$, we can obtain an R-homomorphism $\rho_n : P_n \to Z_n$ that makes the diagram

$$\begin{array}{ccc} & & P_n \\ & {\scriptstyle \rho_n}\swarrow & \downarrow{\scriptstyle l_n|_{P_n}} \\ Z_n & \xrightarrow{g_n} & W_n \end{array}$$

commutative, and we can construct an R-homomorphism $\psi_n : Y_n \to Z_n$, given by $\psi_n(y) = h_n(x) + \rho_n(z)$ for every $y = f_n(x) + z \in Y_n = \mathrm{Im}(f_n) \oplus P_n$, that makes

the following diagram commutative:

$$\begin{array}{ccc} X_n & \xrightarrow{h_n} & Z_n \\ {\scriptstyle f_n}\downarrow & \overset{\psi_n}{\nearrow} & \downarrow{\scriptstyle g_n} \\ Y_n & \xrightarrow{l_n} & W_n \end{array}$$

That is, $\psi_n f_n = h_n$ and $g_n \psi_n = l_n$. However, the following diagram need not be commutative:

$$\begin{array}{ccc} Y_n & \xrightarrow{\partial_n^Y} & Y_{n-1} \\ {\scriptstyle \psi_n}\downarrow & & \downarrow{\scriptstyle \phi_{n-1}} \\ Z_n & \xrightarrow{\partial_n^Z} & Z_{n-1} \end{array}$$

To fix this, we set $\zeta_n = \partial_n^Z \psi_n - \phi_{n-1} \partial_n^Y : Y_n \to Z_{n-1}$. Using the fact that $l : Y \to W$ and $g : Z \to W$ are morphisms of R-complexes, we have for every $z \in P_n \subseteq Y_n$:

$$\begin{aligned} g_{n-1}\left(\zeta_n(z)\right) &= g_{n-1}\left(\partial_n^Z\left(\psi_n(z)\right)\right) - g_{n-1}\left(\phi_{n-1}\left(\partial_n^Y(z)\right)\right) \\ &= g_{n-1}\left(\partial_n^Z\left(\rho_n(z)\right)\right) - l_{n-1}\left(\partial_n^Y(z)\right) \\ &= \partial_n^W\left(g_n\left(\rho_n(z)\right)\right) - \partial_n^W\left(l_n(z)\right) \\ &= \partial_n^W\left(l_n(z)\right) - \partial_n^W\left(l_n(z)\right) \\ &= 0 \end{aligned}$$

That is, $\zeta_n(z) \in \operatorname{Ker}(g_{n-1})$. Therefore, we have for every $z \in P_n \subseteq Y_n$:

$$\begin{aligned} \partial_{n-1}^{\operatorname{Ker}(g)}\left(\zeta_n(z)\right) &= \partial_{n-1}^Z\left(\zeta_n(z)\right) \\ &= \partial_{n-1}^Z\left(\partial_n^Z\left(\psi_n(z)\right)\right) - \partial_{n-1}^Z\left(\phi_{n-1}\left(\partial_n^Y(z)\right)\right) \\ &= \partial_{n-1}^Z\left(\partial_n^Z\left(\psi_n(z)\right)\right) - \phi_{n-2}\left(\partial_{n-1}^Y\left(\partial_n^Y(z)\right)\right) \\ &= 0 \end{aligned}$$

That is, $\zeta_n(z) \in Z_{n-1}\left(\operatorname{Ker}(g)\right)$. But $\operatorname{Ker}(g)$ is exact, so $\zeta_n(z) \in B_{n-1}\left(\operatorname{Ker}(g)\right)$. Therefore, ζ_n induces an R-homomorphism

$$\overline{\zeta_n \mid_{P_n}} : P_n \to B_{n-1}\left(\operatorname{Ker}(g)\right).$$

By the projectivity of P_n, there is an R-homomorphism $\mu_n : P_n \to \operatorname{Ker}(g_n)$ that makes the following diagram commutative:

$$\begin{array}{ccc} & \overset{\mu_n}{\swarrow} & P_n \\ & & \downarrow{\scriptstyle \overline{\zeta_n|_{P_n}}} \\ \operatorname{Ker}(g_n) & \xrightarrow{\overline{\partial_n^{\operatorname{Ker}(g)}}} & B_{n-1}\left(\operatorname{Ker}(g)\right) \end{array}$$

Let $\pi_n : Y_n = \operatorname{Im}(f_n) \oplus P_n \to P_n$ be the canonical projection. Also, let $\chi_n : Y_n \to Z_n$ be the following composition:

$$Y_n \xrightarrow{\pi_n} P_n \xrightarrow{\mu_n} \operatorname{Ker}(g_n) \xrightarrow{\iota_n} Z_n$$

Set $\phi_n = \psi_n - \chi_n : Y_n \to Z_n$. Then the following diagram is commutative:

$$\begin{array}{ccc} X_n & \xrightarrow{h_n} & Z_n \\ {\scriptstyle f_n}\downarrow & \overset{\phi_n}{\nearrow} & \downarrow{\scriptstyle g_n} \\ Y_n & \xrightarrow{l_n} & W_n \end{array}$$

Indeed, we have for every $x \in X_n$:

$$\phi_n\left(f_n(x)\right) = \psi_n\left(f_n(x)\right) - \chi_n\left(f_n(x)\right) = h_n(x) - \iota_n\left(\mu_n\left(\pi_n\left(f_n(x)\right)\right)\right) = h_n(x)$$

Also, we have for every $y = f_n(x) + z \in Y_n = Im(f_n) \oplus P_n$:

$$\begin{aligned} g_n\left(\phi_n(y)\right) &= g_n\left(\psi_n(y)\right) - g_n\left(\chi_n(y)\right) \\ &= l_n(y) - g_n\left(\iota_n\left(\mu_n\left(\pi_n(y)\right)\right)\right) \\ &= l_n(y) - g_n\left(\mu_n(z)\right) \\ &= l_n(y) \end{aligned}$$

In addition, the following diagram is commutative:

$$\begin{array}{ccc} Y_n & \xrightarrow{\partial_n^Y} & Y_{n-1} \\ {\scriptstyle \phi_n}\downarrow & & \downarrow{\scriptstyle \phi_{n-1}} \\ Z_n & \xrightarrow{\partial_n^Z} & Z_{n-1} \end{array}$$

Indeed, noting that $f : X \to Y$ and $h : X \to Z$ are morphisms of R-complexes, we have for every $y = f_n(x) + z \in Y_n = \mathrm{Im}(f_n) \oplus P_n$:

$$\begin{aligned} \partial_n^Z\left(\phi_n(y)\right) &= \partial_n^Z\left(\psi_n(y)\right) - \partial_n^Z\left(\chi_n(y)\right) \\ &= \partial_n^Z\left(\psi_n(y)\right) - \partial_n^Z\left(\iota_n\left(\mu_n\left(\pi_n(y)\right)\right)\right) \\ &= \partial_n^Z\left(\psi_n(y)\right) - \partial_n^Z\left(\mu_n(z)\right) \\ &= \partial_n^Z\left(\psi_n(y)\right) - \partial_n^{\mathrm{Ker}(g)}\left(\mu_n(z)\right) \\ &= \partial_n^Z\left(\psi_n(y)\right) - \zeta_n(z) \\ &= \partial_n^Z\left(\psi_n(y)\right) - \partial_n^Z\left(\psi_n(z)\right) + \phi_{n-1}\left(\partial_n^Y(z)\right) \\ &= \partial_n^Z\left(\psi_n(y-z)\right) + \phi_{n-1}\left(\partial_n^Y\left(y - f_n(x)\right)\right) \\ &= \partial_n^Z\left(\psi_n\left(f_n(x)\right)\right) + \phi_{n-1}\left(\partial_n^Y(y)\right) - \phi_{n-1}\left(\partial_n^Y\left(f_n(x)\right)\right) \\ &= \partial_n^Z\left(\psi_n\left(f_n(x)\right)\right) + \phi_{n-1}\left(\partial_n^Y(y)\right) - \phi_{n-1}\left(f_{n-1}\left(\partial_n^X(x)\right)\right) \\ &= \partial_n^Z\left(h_n(x)\right) + \phi_{n-1}\left(\partial_n^Y(y)\right) - h_{n-1}\left(\partial_n^X(x)\right) \\ &= \partial_n^Z\left(h_n(x)\right) + \phi_{n-1}\left(\partial_n^Y(y)\right) - \partial_n^Z\left(h_n(x)\right) \\ &= \phi_{n-1}\left(\partial_n^Y(y)\right) \end{aligned}$$

As a consequence, we can continue this construction inductively and find a morphism $\phi = (\phi_i)_{i\geq 0} : Y \to Z$ of R-complexes that makes the following diagram

commutative:

$$\begin{array}{ccc} X & \xrightarrow{h} & Z \\ {\scriptstyle f}\downarrow & \overset{\phi}{\nearrow} & \downarrow{\scriptstyle g} \\ Y & \xrightarrow{l} & W \end{array}$$

First Half of Factorization: Let $f : X \to Y$ be a morphism in $\mathcal{C}_{\geq 0}(R)$. For any $n \geq 1$, consider the isomorphism $\psi_n^Y : \mathrm{Mor}_{\mathcal{C}_{\geq 0}(R)}(D(n), Y) \to Y_n$. For any $n \geq 1$ and $x \in Y_n$, there is a $p^{n,x} \in \mathrm{Mor}_{\mathcal{C}_{\geq 0}(R)}(D(n), Y)$ such that $\psi_n^Y(p^{n,x}) = p_n^{n,x}(1) = x$. Let $P = \bigoplus_{n=1}^{\infty} \bigoplus_{x \in Y_n} D(n)$. By the universal property of direct sum, there is a unique morphism $p : P \to Y$ of R-complexes that makes the diagram

$$\begin{array}{ccc} D(n) & \xrightarrow{\iota^{n,x}} & P \\ {\scriptstyle p^{n,x}}\downarrow & \swarrow{\scriptstyle p} & \\ Y & & \end{array}$$

commutative in which $\iota^{n,x}$ is the canonical injection for every $n \geq 1$ and $x \in Y_n$. Thus if $n \geq 1$ and $x \in Y_n$, then $p_n(\iota_n^{n,x}(1)) = p_n^{n,x}(1) = x$. This shows that p_n is surjective for every $n \geq 1$. In addition, there is a unique morphism $\eta : X \oplus P \to U$ of R-complexes that makes the diagram

$$\begin{array}{ccccc} X & \xrightarrow{\kappa} & X \oplus P & \xleftarrow{\nu} & P \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle \eta} & \swarrow{\scriptstyle p} & \\ & & Y & & \end{array}$$

commutative in which κ and ν are canonical injections. For any $i \geq 1$, $\eta_i \nu_i = p_i$ is surjective, so η_i is surjective. This implies that η is a fibration. On the other hand, for any $i \geq 0$, κ_i is injective, and $\mathrm{Coker}(\kappa_i) \cong P_i$ is a direct sum of copies of R, so $\mathrm{Coker}(\kappa_i)$ is projective. That is, κ is a cofibration. Also, for any $i \geq 0$, we have

$$\begin{aligned} H_i(P) &= H_i\left(\bigoplus_{n=1}^{\infty} \bigoplus_{x \in Y_n} D(n)\right) \\ &\cong \bigoplus_{n=1}^{\infty} \bigoplus_{x \in Y_n} H_i(D(n)) \\ &= 0, \end{aligned}$$

which implies that

$$H_i(\kappa) : H_i(X) \to H_i(X \oplus P)$$

is an isomorphism, so κ is a quasi-isomorphism. Therefore, κ is a trivial cofibration. Now $f = \eta\kappa$ is a factorization of f in which η is a fibration and κ is a trivial cofibration. We further show that κ has the left lifting property against all fibrations. To this end, let $t : U \to V$ be a fibration in $\mathcal{C}_{\geq 0}(R)$. Consider a commutative diagram as follows:

$$\begin{array}{ccc} X & \xrightarrow{r} & U \\ {\scriptstyle \kappa}\downarrow & & \downarrow{\scriptstyle t} \\ X \oplus P & \xrightarrow{s} & V \end{array}$$

For any $n \geq 1$ and $x \in Y_n$, let $u^{n,x} : D(n) \to V$ be the following composition:

$$D(n) \xrightarrow{\iota^{n,x}} P \xrightarrow{\nu} X \oplus P \xrightarrow{s} V$$

As we observed above, $\mathrm{RLP}(\mathcal{Y})$ is the class of fibrations, so $t \in \mathrm{RLP}(\mathcal{Y})$. It follows that for any $n \geq 1$ and $x \in Y_n$, there is a morphism $q^{n,x} : D(n) \to U$ that makes the following diagram commutative:

$$\begin{array}{ccc} 0 & \longrightarrow & U \\ {\scriptstyle \kappa^n}\downarrow & \nearrow_{q^{n,x}} & \downarrow{\scriptstyle t} \\ D(n) & \xrightarrow{u^{n,x}} & V \end{array}$$

By the universal property of direct sum, there is a unique morphism $q : P \to U$ of R-complexes that makes the following diagram commutative for every $n \geq 1$ and $x \in Y_n$:

$$\begin{array}{ccc} D(n) & \xrightarrow{\iota^{n,x}} & P \\ {\scriptstyle q^{n,x}}\downarrow & \swarrow_{q} & \\ U & & \end{array}$$

In addition, there is a unique morphism $\theta : X \oplus P \to U$ of R-complexes that makes the following diagram commutative:

$$\begin{array}{ccccc} X & \xrightarrow{\kappa} & X \oplus P & \xleftarrow{\nu} & P \\ & {\scriptstyle r}\searrow & \downarrow{\scriptstyle \theta} & \swarrow{\scriptstyle q} & \\ & & U & & \end{array}$$

Then the following diagram is commutative:

$$\begin{array}{ccc} X & \xrightarrow{r} & U \\ {\scriptstyle \kappa}\downarrow & \nearrow_{\theta} & \downarrow{\scriptstyle t} \\ X \oplus P & \xrightarrow{s} & V \end{array}$$

Clearly, $\theta\kappa = r$. Also, $tq\iota^{n,x} = tq^{n,x} = u^{n,x} = s\nu\iota^{n,x}$ for every $n \geq 1$ and $x \in Y_n$, implying that $tq = s\nu$. Further, $t\theta\kappa = tr = s\kappa$ and $t\theta\nu = tq = s\nu$, showing that $t\theta = s$. As a consequence, κ has the left lifting property against all fibrations.

Second Half of Lifting: Consider a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{h} & Z \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow{l} & W \end{array}$$

in $\mathcal{C}_{\geq 0}(R)$ in which f is a trivial cofibration and g is a fibration. Consider the factorization

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \kappa}\downarrow & \nearrow_{\eta} & \\ X \oplus P & & \end{array}$$

of f that was established above in which

$$P = \bigoplus_{n=1}^{\infty} \bigoplus_{x \in Y_n} D(n),$$

η is a fibration, and κ is a trivial cofibration that further has the left lifting property against all fibrations. Since f and κ are quasi-isomorphisms, we deduce that η is also a quasi-isomorphism, so η is a trivial fibration. Consider the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\kappa} & X \oplus P \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \eta} \\ Y & \xrightarrow{1^Y} & Y \end{array}$$

As f is a cofibration and η is a trivial fibration, the first half of lifting that was established above provides a morphism $r : Y \to X \oplus P$ that makes the following diagram commutative:

$$\begin{array}{ccc} X & \xrightarrow{\kappa} & X \oplus P \\ {\scriptstyle f}\downarrow & \overset{r}{\nearrow} & \downarrow{\scriptstyle \eta} \\ Y & \xrightarrow{1^Y} & Y \end{array}$$

This yields the following commutative diagram:

$$\begin{array}{ccccc} X & \xrightarrow{1^X} & X & \xrightarrow{1^X} & X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \kappa} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{r} & X \oplus P & \xrightarrow{\eta} & Y \end{array}$$

This shows that f is a retract of κ. But κ has the left lifting property against all fibrations, so the proof of [13, Lemma 7.2.8] that makes no use of the model category assumption, shows that f has the left lifting property against all fibrations. This implies that the above diagram can be completed as follows:

$$\begin{array}{ccc} X & \xrightarrow{h} & Z \\ {\scriptstyle f}\downarrow & \nearrow & \downarrow{\scriptstyle g} \\ Y & \xrightarrow{l} & W \end{array}$$

Second Half of Factorization: Let $f : X \to Y$ be a morphism in $\mathcal{C}_{\geq 0}(R)$. We construct the desired factorization inductively. Let $\rho_0 : P_0 \to Y_0$ be an R-epimorphism in which P_0 is a projective left R-module, and set $Q_0 = X_0 \oplus P_0$. By the universal property of direct sum, there is a unique morphism $\eta_0 : Q_0 \to Y_0$ of R-complexes that makes the diagram

$$\begin{array}{ccccc} X_0 & \xrightarrow{\kappa_0} & Q_0 & \xleftarrow{\nu_0} & P_0 \\ & {\scriptstyle f_0}\searrow & \downarrow{\scriptstyle \eta_0} & \swarrow{\scriptstyle \rho_0} & \\ & & Y_0 & & \end{array}$$

commutative in which κ_0 and ν_0 are canonical injections. Then $f_0 = \eta_0\kappa_0$ and $\rho_0 = \eta_0\nu_0$. It is clear that κ_0 is injective whose $\mathrm{Coker}(\kappa_0) \cong P_0$ is projective.

On the other hand, since ρ_0 is surjective, we infer that η_0 is surjective. Now suppose that $n \geq 1$, and for any $0 \leq i \leq n-1$, the R-module Q_i and the R-homomorphisms $\partial_i^Q : Q_i \to Q_{i-1}$, $\kappa_i : X_i \to Q_i$, and $\eta_i : Q_i \to Y_i$ are constructed in a way that $\partial_{i-1}^Q \partial_i^Q = 0$, $f_i = \eta_i \kappa_i$, κ_i is injective whose $\operatorname{Coker}(\kappa_i)$ is projective, η_i is surjective with $H_{i-1}(\eta)$ an isomorphism, and the following diagram is commutative:

$$\begin{array}{ccc} X_i & \xrightarrow{\partial_i^X} & X_{i-1} \\ {\scriptstyle \kappa_i}\downarrow & & \downarrow{\scriptstyle \kappa_{i-1}} \\ Q_i & \xrightarrow{\partial_i^Q} & Q_{i-1} \\ {\scriptstyle \eta_i}\downarrow & & \downarrow{\scriptstyle \eta_{i-1}} \\ Y_i & \xrightarrow{\partial_i^Y} & Y_{i-1} \end{array}$$

Using the above diagram, we see that if $x \in X_n$, then

$$\partial_{n-1}^Q \left(\kappa_{n-1}\left(\partial_n^X(x)\right)\right) = \kappa_{n-2}\left(\partial_{n-1}^X\left(\partial_n^X(x)\right)\right) = 0,$$

so $\kappa_{n-1}\left(\partial_n^X(x)\right) \in Z_{n-1}(Q)$. Therefore, we get an induced R-homomorphism $\overline{\kappa_{n-1}\partial_n^X} : X_n \to Z_{n-1}(Q)$. Consider the following commutative diagram:

$$\begin{array}{ccc} X_n & \xrightarrow{f_n} & Y_n \\ {\scriptstyle \overline{\kappa_{n-1}\partial_n^X}}\downarrow & & \downarrow{\scriptstyle \overline{\partial_n^Y}} \\ Z_{n-1}(Q) & \xrightarrow{Z_{n-1}(\eta)} & Z_{n-1}(Y) \end{array}$$

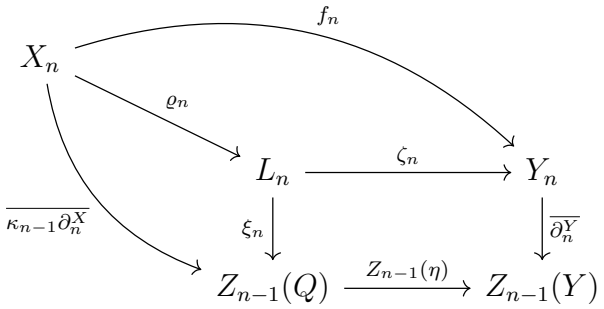
Indeed, using the fact that f is a morphism of R-complexes, we have

$$\eta_{n-1}\left(\kappa_{n-1}\left(\partial_n^X(x)\right)\right) = f_{n-1}\left(\partial_n^X(x)\right) = \partial_n^Y\left(f_n(x)\right)$$

for every $x \in X_n$. Set:

$$L_n = Z_{n-1}(Q) \sqcap_{Z_{n-1}(Y)} Y_n = \left\{(x, y) \in Z_{n-1}(Q) \oplus Y_n \;\middle|\; \eta_{n-1}(x) = \partial_n^Y(y)\right\}$$

Then there is a unique morphism $\varrho_n : X_n \to L_n$ that makes the following pullback diagram commutative:



As in the case $i = 0$, let $\rho_n : P_n \to L_n$ be an R-epimorphism in which P_n is a projective left R-module, and set $Q_n = X_n \oplus P_n$. By the universal property of

direct sum, there is a unique morphism $\theta_n : Q_n \to L_n$ of R-complexes that makes the diagram

$$\begin{array}{ccccc} X_n & \xrightarrow{\kappa_n} & Q_n & \xleftarrow{\nu_n} & P_n \\ & \searrow^{\varrho_n} & \downarrow{\scriptstyle \theta_n} & \swarrow_{\rho_n} & \\ & & L_n & & \end{array}$$

commutative in which κ_n and ν_n are canonical injections. Set $\eta_n = \zeta_n\theta_n : Q_n \to Y_n$. Then $f_n = \zeta_n\varrho_n = \zeta_n\theta_n\kappa_n = \eta_n\kappa_n$. It is also clear that κ_n is injective whose $\mathrm{Coker}(\kappa_n) \cong P_n$ is projective.

Consider the following compositions of R-homomorphisms:

$$X_n \xrightarrow{\partial_n^X} X_{n-1} \xrightarrow{\kappa_{n-1}} Q_{n-1} \quad \text{and} \quad P_n \xrightarrow{\rho_n} L_n \xrightarrow{\xi_n} Z_{n-1}(Q) \xrightarrow{\iota_{n-1}^Q} Q_{n-1}$$

By the universal property of direct sum, there is a unique morphism $\partial_n^Q : Q_n \to Q_{n-1}$ of R-complexes that makes the following diagram commutative:

$$\begin{array}{ccccc} X_n & \xrightarrow{\kappa_n} & Q_n & \xleftarrow{\nu_n} & P_n \\ & \searrow^{\kappa_{n-1}\partial_n^X} & \downarrow{\scriptstyle \partial_n^Q} & \swarrow_{\iota_{n-1}^Q\xi_n\rho_n} & \\ & & Q_{n-1} & & \end{array}$$

Then we have

$$\partial_{n-1}^Q\partial_n^Q\kappa_n = \partial_{n-1}^Q\kappa_{n-1}\partial_n^X = \kappa_{n-2}\partial_{n-1}^X\partial_n^X = 0$$

and $\partial_{n-1}^Q\partial_n^Q\nu_n = \partial_{n-1}^Q\iota_{n-1}^Q\xi_n\rho_n = 0$ as $\mathrm{Im}(\xi_n) \subseteq Z_{n-1}(Q)$. It follows that $\partial_{n-1}^Q\partial_n^Q = 0$.

We next show that the following diagram is commutative:

$$\begin{array}{ccc} X_n & \xrightarrow{\partial_n^X} & X_{n-1} \\ {\scriptstyle \kappa_n}\downarrow & & \downarrow{\scriptstyle \kappa_{n-1}} \\ Q_n & \xrightarrow{\partial_n^Q} & Q_{n-1} \\ {\scriptstyle \eta_n}\downarrow & & \downarrow{\scriptstyle \eta_{n-1}} \\ Y_n & \xrightarrow{\partial_n^Y} & Y_{n-1} \end{array}$$

Indeed, we have $\partial_n^Q\kappa_n = \kappa_{n-1}\partial_n^X$ from the previous commutative diagram. Moreover, we have:

$$\partial_n^Y\eta_n\kappa_n = \partial_n^Y\zeta_n\theta_n\kappa_n = \partial_n^Y\zeta_n\varrho_n = \partial_n^Y f_n = f_{n-1}\partial_n^X = \eta_{n-1}\kappa_{n-1}\partial_n^X = \eta_{n-1}\partial_n^Q\kappa_n$$

and

$$\begin{aligned} \partial_n^Y\eta_n\nu_n &= \partial_n^Y\zeta_n\theta_n\nu_n = \partial_n^Y\zeta_n\rho_n = \iota_{n-1}^Y\overline{\partial_n^Y}\zeta_n\rho_n = \iota_{n-1}^Y Z_{n-1}(\eta)\xi_n\rho_n = \eta_{n-1}\iota_{n-1}^Q\xi_n\rho_n \\ &= \eta_{n-1}\partial_n^Q\nu_n \end{aligned}$$

It follows that $\partial_n^Y\eta_n = \eta_{n-1}\partial_n^Q$.

We now show that $H_{n-1}(\eta)$ is injective. Suppose that $H_{n-1}(\eta)\,(x + B_{n-1}(Q)) = \eta_{n-1}(x) + B_{n-1}(Y) = 0$ for some $x \in Z_{n-1}(Q)$. Then $\eta_{n-1}(x) \in B_{n-1}(Y)$, so $\eta_{n-1}(x) = \partial_n^Y(y)$ for some $y \in Y_n$. This implies that $(x, y) \in L_n$. On the other

hand, $\rho_n = \theta_n\nu_n$ and ρ_n is surjective, so $\theta_n : Q_n \to L_n$ is surjective. Therefore, there is an element $(z, w) \in Q_n = X_n \oplus P_n$ such that $\theta_n(z, w) = (x, y)$. Then we have:

$$x = \xi_n(x, y) = \xi_n(\theta_n(z, w)) = \xi_n(\varrho_n(z)) + \xi_n(\rho_n(w)) = \kappa_{n-1}\left(\partial_n^X(z)\right) + \partial_n^Q(\nu_n(w))$$
$$= \partial_n^Q(\kappa_n(z)) + \partial_n^Q(\nu_n(w)) = \partial_n^Q(\kappa_n(z) + \nu_n(w)) = \partial_n^Q(z, w)$$

Hence $x \in B_{n-1}(Q)$, so $x + B_{n-1}(Q) = 0$. This shows that $H_{n-1}(\eta)$ is injective.

We next show that $H_{n-1}(\eta)$ is surjective. Let $y \in Z_n(Y)$. Then $\eta_{n-1}(0) = 0 = \partial_n^Y(y)$, so $(0, y) \in L_n$. By the surjectivity of $\theta_n : Q_n \to L_n$, there is an element $(z, w) \in Q_n = X_n \oplus P_n$ such that $\theta_n(z, w) = (0, y)$. Then $y = \zeta_n(0, y) = \zeta_n(\theta_n(z, w)) = \eta_n(z, w)$. Also, a similar calculation as in the previous paragraph shows that $\partial_n^Q(z, w) = 0$, so $(z, w) \in Z_n(Q)$. This shows that $Z_n(\eta) : Z_n(Q) \to Z_n(Y)$ is surjective. Since we have constructed all these R-homomorphisms in the same way in each step, we can infer that $Z_{n-1}(\eta)$ is surjective. It follows that $H_{n-1}(\eta)$ is surjective. Therefore, $H_{n-1}(\eta)$ is an isomorphism.

We finally show that η_n is surjective. Let $y \in Y_n$. Then $\partial_n^Y(y) \in B_{n-1}(Y) \subseteq Z_{n-1}(Y)$, so by the surjectivity of $Z_{n-1}(\eta)$, there is an element $x \in Z_{n-1}(Q)$ such that $\eta_{n-1}(x) = \partial_n^Y(y)$. Then $(x, y) \in L_n$, so by the surjectivity of θ_n, there is an element $(z, w) \in Q_n = X_n \oplus P_n$ such that $\theta_n(z, w) = (x, y)$. It follows that $y = \zeta_n(x, y) = \zeta_n(\theta_n(z, w)) = \eta_n(z, w)$. Thus η_n is surjective.

As a consequence, we can inductively construct the morphisms $\kappa : X \to Q$ and $\eta : Q \to Y$ of R-complexes such that $f = \eta\kappa$, η is a trivial fibration, and κ is a cofibration.

All in all, we have a cofibrantly generated model structure on $\mathcal{C}_{\geq 0}(R)$. □

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