



TENSORIAL REPRESENTATIONS OF POSITIVE WEAKLY (q, r) -DOMINATED MULTILINEAR OPERATORS

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ABSTRACT. We introduce the class of positive weakly (q, r) -dominated multilinear operators between Banach lattices. This notion extends classical domination and summability concepts to the positive multilinear setting and generates a new positive multi-ideal. A Pietsch domination theorem and a polynomial version are established. Finally, we provide a tensorial representation that yields an isometric identification with the dual of an appropriate completed tensor product.

1. INTRODUCTION AND PRELIMINARIES

The theory of absolutely summing operators, initiated by Pietsch in the 1960s, has played a central role in the development of operator ideals and their applications to Banach space theory. Since then, several nonlinear extensions have been studied, especially for multilinear operators, homogeneous polynomials, and Lipschitz mappings, leading to a rich framework that unifies summability, domination, and factorization properties. This area of research has provided a unified approach to extending linear results to nonlinear settings, with significant contributions from works of Pietsch [14, 15], Cohen [7], Kwapien [10, 11], and others. In recent years, increasing attention has focused on the positive versions of these operators. In fact, positive operator theory, which uses the lattice structure of

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Banach spaces, has become a powerful tool for strengthening and extending classical results. In [8], the basic elements of positive linear and multilinear operator ideals were established. Thereafter, in [4], positive polynomial ideals were introduced as a natural extension of the linear and multilinear cases. These ideals not only encompass the corresponding positive classes, which fail to satisfy the conditions of classical ideals, but also provide a unifying framework for their study. A positive multilinear ideal $\mathcal{M}^+$ (or polynomial ideal $\mathcal{P}^+$) is a class of multilinear operators (or polynomials) between Banach lattices that is stable under composition with positive linear operators. The theory of summability in the positive setting not only produces sharper inequalities but also exposes phenomena absent in the purely linear framework. The concept of absolutely $(p, q_1, \dots, q_m, r)$ -summing multilinear operators, introduced by Achour [1], provides a natural extension of the classical absolutely (p, q, r) -summing operators of Pietsch [14]. When $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$, this class is referred to as the (q, r) -dominated operators. In this paper, we introduce the class of positive weakly (q, r) -dominated multilinear operators, which combines lattice positivity with weak absolute summability. This class is stable under composition with positive operators and fits naturally into the framework of positive multi-ideals. We also establish the Pietsch domination theorem in this setting, characterizing these operators through vector measures on the positive balls of the dual space. This result extends the classical Pietsch theorem to the positive setting, based on Banach lattice spaces. Furthermore, we also examine the polynomial version, showing that the structure extends to m -homogeneous polynomials, giving rise to positive polynomial ideals. Finally, we establish a tensorial representation for positive weakly (q, r) -dominated multilinear operators. By introducing a suitable tensor norm, we obtain an isometric identification of these operators with the dual of a completed tensor product, thereby providing the natural tensorial framework for the theory. The same approach applies to polynomials, where a tensor norm is constructed using $\widehat{\otimes}_{s, |\pi|}^m E$, the m -fold positive projective symmetric tensor product of E .

The paper is organized as follows. We recall standard notations used throughout the paper. We present Banach lattice spaces and some of their key properties. We provide the definition of the regular multilinear space $\mathcal{L}^r(E_1, \dots, E_m; F)$ and regular polynomials space $\mathcal{P}^r({}^m E; F)$, which are needed for defining positive weakly (q, r) -summing operators. Section 2 introduces the class of positive weakly (q, r) -dominated multilinear operators and establishes their basic properties. We then naturally extend this to define positive weakly (q, r) -dominated polynomials. Both classes form positive ideals. Section 3 is devoted to the tensorial representation, which leads to the desired isometric identification. In the case where $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$, we show that the space $\mathcal{L}_{w, (q, r)}^{m+}(E, \dots, E_m; F)$ of positive weakly (q, r) -dominated multilinear operators can be identified with the dual of

$$E_1 \widehat{\otimes}_{\mu_{(q, r)}^{m+}} \cdots \widehat{\otimes}_{\mu_{(q, r)}^{m+}} E_m \widehat{\otimes}_{\mu_{(q, r)}^{m+}} F^*,$$

where $\mu_{(q,r)}^{m+}$ is a tensor norm that we define below. Similarly, we identify the space $\mathcal{P}_{w,(q,r)}^{m+}({}^mE; F)$ of positive weakly (q, r) -dominated polynomials with

$$((\widehat{\otimes}_{s,|\pi|}^m E) \widehat{\otimes}_{\lambda_{(q,r)}^{m+}} F^*)^*,$$

where we show that $\lambda_{(q,r)}^{m+}$ is $\mu_{(q,r)}^{1+}$.

Throughout the paper, E, F , and G denote Banach lattices and X, Y denote Banach spaces. Our spaces are over the field of real scalars $\mathbb{R}$. By B_X we denote the closed unit ball of X and by X^* its topological dual. We use the symbol $\mathcal{L}(X; Y)$ for the space of all bounded linear operators from X into Y . For $1 \leq p \leq \infty$, we denote by p^* its conjugate, that is, $1/p + 1/p^* = 1$. Let E be a Banach lattice with norm $\|\cdot\|$ and order $\leq$. We denote by E^+ the positive cone of E , that is, $E^+ = \{x \in E : x \geq 0\}$. Let $x \in E$ that its positive part be defined by $x^+ := \sup\{x, 0\} \geq 0$ and its negative part is defined by $x^- := \sup\{-x, 0\} \geq 0$. We have $x = x^+ - x^-$, $|x| = x^+ + x^-$, and the inequalities $x \leq |x|$, $x^+ \leq |x|$ and $x^- \leq |x|$. The dual E^* of a Banach lattice E is a Banach lattice with the natural order $x_1^* \leq x_2^* \Leftrightarrow \langle x, x_1^* \rangle \leq \langle x, x_2^* \rangle$, for all $x \in E^+$. Since E is a sublattice of E^{**} , we have for $x_1, x_2 \in E$, $x_1 \leq x_2 \iff \langle x_1, x^* \rangle \leq \langle x_2, x^* \rangle$, $\forall x^* \in E^{*+}$. We have $|\langle x^*, x \rangle| \leq \langle |x^*|, |x| \rangle$, for every $x^* \in E^*$ and $x \in E$. We denote by $\ell_p^n(X)$ the Banach space of all absolutely p -summable sequences $(x_i)_{i=1}^n \subset X$ with the norm $\|(x_i)_{i=1}^n\|_p = (\sum_{i=1}^n \|x_i\|^p)^{\frac{1}{p}}$, and by $\ell_{p,w}^n(X)$ the Banach space of all weakly p -summable sequences $(x_i)_{i=1}^n \subset X$ with the norm, $\|(x_i)_{i=1}^n\|_{p,w} = \sup_{x^* \in B_{X^*}} (\sum_{i=1}^n |\langle x^*, x_i \rangle|^p)^{\frac{1}{p}}$. Consider the case where X is replaced by a Banach lattice E , and define

$$\ell_{p,|w|}^n(E) = \{(x_i)_{i=1}^n \subset E : (|x_i|)_{i=1}^n \in \ell_{p,w}^n(E)\} \text{ and } \|(x_i)_{i=1}^n\|_{p,|w|} = \|(|x_i|)_{i=1}^n\|_{p,w}.$$

Let $B_{E^*}^+ = \{x^* \in B_{E^*} : x^* \geq 0\} = B_{E^*} \cap E^{*+}$. If $(x_i)_{i=1}^n \subset E^+$, then

$$\|(x_i)_{i=1}^n\|_{p,|w|} = \|(x_i)_{i=1}^n\|_{p,w} = \sup_{x^* \in B_{E^*}^+} \left(\sum_{i=1}^n \langle x^*, x_i \rangle^p \right)^{\frac{1}{p}}.$$

For every $(x_i)_{i=1}^n \subset E$, it is straightforward to show that

$$\|(x_i^+)_{i=1}^n\|_{p,|w|} \leq \|(x_i)_{i=1}^n\|_{p,|w|} \text{ and } \|(x_i^-)_{i=1}^n\|_{p,|w|} \leq \|(x_i)_{i=1}^n\|_{p,|w|}. \quad (1.1)$$

Given $m \in \mathbb{N}^*$, we denote by $\mathcal{L}(E_1, \dots, E_m; F)$ the Banach space of all bounded multilinear operators from $E_1 \times \dots \times E_m$ into F endowed with the supremum norm $\|T\| = \sup_{\|x_j\| \leq 1} \|T(x_1, \dots, x_m)\|$. An operator $T \in \mathcal{L}(E_1, \dots, E_m; F)$ is

called positive if $T(x_1, \dots, x_m) \geq 0$ for every $x_j \in E_j^+$ ($1 \leq j \leq m$). We denote by $\mathcal{L}^+(E_1, \dots, E_m; F)$ the set of all positive m -linear operators. For every $T \in \mathcal{L}^+(E_1, \dots, E_m; F)$ and $x_j \in E_j$ ($1 \leq j \leq m$), we have

$$|T(x_1, \dots, x_m)| \leq T(|x_1|, \dots, |x_m|).$$

An m -linear operator $T : E_1 \times \dots \times E_m \rightarrow F$ is a lattice m -morphism if

$$|T(x_1, \dots, x_m)| = T(|x_1|, \dots, |x_m|)$$

for all $x_j \in E_j$. An m -linear operator $T : E_1 \times \cdots \times E_m \rightarrow F$, is called regular if it can be written as $T = T_1 - T_2$ with $T_1, T_2 \in \mathcal{L}^+(E_1, \dots, E_m; F)$. We denote by $\mathcal{L}^r(E_1, \dots, E_m; F)$ the space of all regular m -linear operators from $E_1 \times \cdots \times E_m$ into F . In [5], if F is Dedekind complete, then $\mathcal{L}^r(E_1, \dots, E_m; F)$ is a Banach lattice with the norm $\|T\|_{\mathcal{L}^r} = \| |T| \|$. In this case, $\mathcal{L}^{r+}(E_1, \dots, E_m; F) = \mathcal{L}^+(E_1, \dots, E_m; F)$. For every $x_j^* \in E_j^*$ ($1 \leq j \leq m$), we have $x_1^* \otimes \cdots \otimes x_m^* \in \mathcal{L}^r(E_1, \dots, E_m)$, and

$$\|x_1^* \otimes \cdots \otimes x_m^*\|_{\mathcal{L}^r} = \|x_1^*\| \cdots \|x_m^*\|.$$

Let $E_1, \dots, E_m$ be Banach lattices, and let $E_1 \otimes \cdots \otimes E_m$ denote their algebraic tensor product. Fremlin [9] introduced the vector lattice tensor product $E_1 \overline{\otimes} \cdots \overline{\otimes} E_m$, defined so that

$$|x_1 \otimes \cdots \otimes x_m| = |x_1| \otimes \cdots \otimes |x_m|$$

for all $x_j \in E_j$ ($1 \leq j \leq m$). He also introduced the positive projective tensor product $E_1 \otimes_{|\pi|} \cdots \otimes_{|\pi|} E_m$, where for every $\theta \in E_1 \overline{\otimes} \cdots \overline{\otimes} E_m$

$$\|\theta\|_{|\pi|} = \left\{ \sum_{i=1}^k \prod_{j=1}^m \|x_i^j\| : x_i^j \in E_j^+, k \in \mathbb{N}^*, |\theta| \leq \sum_{i=1}^k x_i^1 \otimes \cdots \otimes x_i^m \right\}.$$

Its completion $E_1 \widehat{\otimes}_{|\pi|} \cdots \widehat{\otimes}_{|\pi|} E_m$ is again a Banach lattice, and the canonical mapping $\otimes(x_1, \dots, x_m) \mapsto x_1 \otimes \cdots \otimes x_m$ is a lattice m -morphism. If F is Dedekind complete, according to [5, Proposition 3.3], every regular m -linear operator $T : E_1 \times \cdots \times E_m \rightarrow F$ admits a unique linearization $T^\otimes \in \mathcal{L}^r(E_1 \widehat{\otimes}_{|\pi|} \cdots \widehat{\otimes}_{|\pi|} E_m; F)$ such that $T^\otimes(x_1 \otimes \cdots \otimes x_m) = T(x_1, \dots, x_m)$, yielding an isometric lattice isomorphism between $\mathcal{L}^r(E_1 \widehat{\otimes}_{|\pi|} \cdots \widehat{\otimes}_{|\pi|} E_m; F)$ and $\mathcal{L}^r(E_1, \dots, E_m; F)$. In particular, when $F = \mathbb{R}$, we have the isometrically isomorphic and lattice homomorphic identification

$$\mathcal{L}^r(E_1, \dots, E_m) = (E_1 \widehat{\otimes}_{|\pi|} \cdots \widehat{\otimes}_{|\pi|} E_m)^*.$$

Consequently,

$$B_{(E_1 \widehat{\otimes}_{|\pi|} \cdots \widehat{\otimes}_{|\pi|} E_m)^*}^+ = B_{\mathcal{L}^r(E_1, \dots, E_m)}^+.$$

Moreover, for every $\varphi \in \mathcal{L}^r(E_1, \dots, E_m)$ and $x_1 \otimes \cdots \otimes x_m \in E_1 \otimes \cdots \otimes E_m$, we have

$$|\langle \varphi, x_1 \otimes \cdots \otimes x_m \rangle| \leq \langle |\varphi|, |x_1| \otimes \cdots \otimes |x_m| \rangle,$$

For $\epsilon_1, \dots, \epsilon_m \in \{+, -\}$, we have

$$\begin{aligned} & \sup_{\varphi \in B_{\mathcal{L}^r(E_1, \dots, E_m)}^+} \left(\sum_{i=1}^n \varphi(x_i^{1\epsilon_1}, \dots, x_i^{m\epsilon_m})^q \right)^{\frac{1}{q}} \\ & \leq \sup_{\varphi \in B_{\mathcal{L}^r(E_1, \dots, E_m)}^+} \left(\sum_{i=1}^n \varphi(|x_i^1|, \dots, |x_i^m|)^q \right)^{\frac{1}{q}}. \end{aligned} \tag{1.2}$$

A map $P : X \rightarrow Y$ is an m -homogeneous polynomial if there exists a unique symmetric m -linear operator $\widehat{P} : X \times \cdots \times X \rightarrow Y$ such that $P(x) = \widehat{P}\left(x, \overset{(m)}{\cdot}, x\right)$.

We denote by $\mathcal{P}(^mX; Y)$, the Banach space of all continuous m -homogeneous polynomials from X into Y endowed with the norm

$$\|P\| = \sup_{\|x\| \leq 1} \|P(x)\| = \inf \{C : \|P(x)\| \leq C \|x\|^m, x \in X\}.$$

We denote by $\mathcal{P}_f(^mX; Y)$ the space of all m -homogeneous polynomials of finite type, that is

$$\mathcal{P}_f(^mX; Y) = \left\{ \sum_{i=1}^k \varphi_i^m(x) y_i : k \in \mathbb{N}, \varphi_i \in X^*, y_i \in Y, 1 \leq i \leq k \right\}.$$

Let E and F be Banach lattices. An m -homogeneous polynomials $\mathcal{P}(^mE; F)$ is positive if $P(x) \geq 0$ for all $x \in E^+$. It is called regular if its associated symmetric m -linear operator $\hat{P}$ is regular. We denote by $\mathcal{P}^r(^mE; F)$ the space of all regular polynomials from E into F . It is easy to see that P is regular if and only if there exist $P_1, P_2 \in \mathcal{P}^+(^mE; F)$ such that $P = P_1 - P_2$. For a Banach lattice E , the positive projective symmetric tensor norm on $\overline{\otimes}_s^m E$ is defined by

$$\|u\|_{s,|\pi|} = \inf \left\{ \sum_{i=1}^k \|x_i\|^m : x_i \in E^+, k \in \mathbb{N}^*, |u| \leq \sum_{i=1}^k x_i \otimes \cdots \otimes x_i \right\}$$

for each $u \in \overline{\otimes}_s^m E$. We denote by $\widehat{\otimes}_{s,|\pi|}^m E$ the completion of $\overline{\otimes}_s^m E$ under the lattice norm $\|\cdot\|_{s,|\pi|}$. Then $\widehat{\otimes}_{s,|\pi|}^m E$ is a Banach lattice, called the m -fold positive projective symmetric tensor product of E . Moreover, if F is Dedekind complete Banach lattice, then for any regular m -homogeneous polynomial $P : E \rightarrow F$, there exists a unique regular linear operator $P^\otimes : \widehat{\otimes}_{s,|\pi|}^m E \rightarrow F$, called the linearization of P , such that $P(x) = P^\otimes \left(x \otimes \cdots \otimes x \right)$ for every $x \in E$. Moreover, in [5, Proposition 3.4], the correspondence $P \mapsto P^\otimes$ is isometrically isomorphic and lattice homomorphic between the Banach lattices $\mathcal{P}^r(^mE; F)$ and $\mathcal{L}^r(\widehat{\otimes}_{s,|\pi|}^m E; F)$. If $F = \mathbb{R}$, then

$$\mathcal{P}^r(^mE) = \left(\widehat{\otimes}_{s,|\pi|}^m E \right)^*.$$

In [8], the authors gave the following definition: A positive multi-ideal is a subclass $\mathcal{M}^+$ of all continuous multilinear operators between Banach lattices such that for all $m \in \mathbb{N}^*$ and Banach lattices $E_1, \dots, E_m$ and F , the components

$$\mathcal{M}^+(E_1, \dots, E_m; F) := \mathcal{L}(E_1, \dots, E_m; F) \cap \mathcal{M}^+$$

satisfy the following conditions:

(i) $\mathcal{M}^+(E_1, \dots, E_m; F)$ is a linear subspace of $\mathcal{L}(E_1, \dots, E_m; F)$, which contains the m -linear mappings of finite rank.

(ii) The positive ideal property: If $T \in \mathcal{M}^+(E_1, \dots, E_m; F)$, $u_j \in \mathcal{L}^+(G_j; E_j)$ for $j = 1, \dots, m$ and $v \in \mathcal{L}^+(F; G)$, then $v \circ T \circ (u_1, \dots, u_m)$ is in $\mathcal{M}^+(G_1, \dots, G_m; G)$. Let $\|\cdot\|_{\mathcal{M}^+} : \mathcal{M}^+ \rightarrow \mathbb{R}^+$ satisfy the following conditions:

a) $(\mathcal{M}^+(E_1, \dots, E_m; F), \|\cdot\|_{\mathcal{M}^+})$ is a Banach space for all Banach lattices $E_1, \dots, E_m, F$.

b) The canonical m -linear form $T^m : \mathbb{R}^m \rightarrow \mathbb{R}$ given by $T^m(\lambda^1, \dots, \lambda^m) =$

$\lambda^1 \cdots \lambda^m$ satisfies $\|T^m\|_{\mathcal{M}^+} = 1$ for all m .

c) $T \in \mathcal{M}^+(E_1, \dots, E_m; F)$, $u_j \in \mathcal{L}^+(G_j; E_j)$ for $j = 1, \dots, m$ and $v \in \mathcal{L}^+(F; G)$.

Then

$$\|v \circ T \circ (u_1, \dots, u_m)\|_{\mathcal{M}^+} \leq \|v\| \|T\|_{\mathcal{M}^+} \|u_1\| \cdots \|u_m\|.$$

The class $(\mathcal{M}^+, \|\cdot\|_{\mathcal{M}})$ is referred to as a positive Banach multi-ideal. In particular, when $m = 1$, we specifically refer to it as a positive Banach ideal. Replacing the class $\mathcal{M}^+$ with the polynomial class $\mathcal{P}^+$, and condition c) with $P \in \mathcal{P}^+({}^m E; F)$, $u \in \mathcal{L}^+(G; E)$ and $v \in \mathcal{L}^+(F; G)$ together with

$$\|v \circ P \circ u\|_{\mathcal{P}^+} \leq \|v\| \|P\|_{\mathcal{P}^+} \|u\|^m,$$

we obtain the definition of positive polynomial ideals.

2. POSITIVE WEAKLY (q, r) -DOMINATED MULTILINEAR OPERATORS

The notion of absolutely (p, q, r) -summing linear operators was first introduced and studied by Pietsch [14]. Later, Achour in [1] extended this concept to the multilinear setting by defining absolutely $(p, q_1, \dots, q_m; r)$ -summing multilinear operators. A positive linear version was subsequently introduced and analyzed by Chen Belacel and Domínguez [6]. Building on these ideas, a positive multilinear version was proposed in [8], adapting Achour's definition to the ordered context. When $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$, this notion is referred to as (q, r) -dominated linear operators. In this section, we present another version of positive multilinear operators. Let us start with the definition in the positive linear case.

Definition 2.1. [6, Definition 3.1] Consider $1 \leq r, p, q \leq \infty$ such that $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. Let E and F be Banach lattices. A mapping $u \in \mathcal{L}(E; F)$ is said to be positive (q, r) -dominated if there is a constant $C > 0$ such that for every $x_1, \dots, x_n \in E$ and $y_1^*, \dots, y_n^* \in F^*$, the following inequality holds:

$$\|(\langle u(x_i), y_i^* \rangle)_{i=1}^n\|_p \leq C \|(x_i)_{i=1}^n\|_{q, |w|} \|(y_i^*)_{i=1}^n\|_{r, |w|}. \quad (2.1)$$

The space consisting of all such mappings is denoted by $\Psi_{(q, r)}(E; F)$. In this case, we define

$$\|u\|_{\Psi_{(q, r)}} = \inf\{C > 0 : C \text{ satisfies (2.1)}\}.$$

There are several ways to generalize this definition in the positive multilinear setting. In [8], the authors introduced a positive multilinear version by adapting each variable to its corresponding weakly q_j -summable norm. In our approach, we adjust the variables to a single regular form, which allows us to employ weakly q -summable sequences in the space $\mathcal{L}^r(E_1, \dots, E_m)$. This idea was used by Bougoutaia, Belacel and Hamdi [3] to define the concept of positive weakly Cohen p -nuclear operators. We show that this new class forms a concrete example of a positive multi-ideal, satisfies Pietsch's domination theorem, and admits a representation via a tensor product equipped with a norm tailored to this class.

Definition 2.2. Consider $1 \leq p, q, r \leq \infty$ such that $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. Let $E_1, \dots, E_m$ and F be Banach lattices. A mapping $T \in \mathcal{L}(E_1, \dots, E_m; F)$ is said to be positive weakly (q, r) -dominated if there is a constant $C > 0$ such that for every

$(x_i^1, \dots, x_i^m) \in E_1^+ \times \dots \times E_m^+$ ($1 \leq i \leq n$) and $y_1^*, \dots, y_n^* \in F^{*+}$, the following inequality holds:

$$\|(\langle T(x_i^1, \dots, x_i^m), y_i^* \rangle)_{i=1}^n\|_p \leq C \sup_{\varphi \in B_{\mathcal{L}^r(E_1, \dots, E_m)}^+} \left(\sum_{i=1}^n \varphi(x_i^1, \dots, x_i^m)^q \right)^{\frac{1}{q}} \| (y_i^*)_{i=1}^n \|_{r, |w|}. \quad (2.2)$$

The space consisting of all such mappings is denoted by $\mathcal{L}_{w, (q, r)}^{m+}(E_1, \dots, E_m; F)$. In this case, we define

$$d_{w, (q, r)}^{m+}(T) = \inf\{C > 0 : C \text{ satisfies (2.2)}\}.$$

It is straightforward to verify that every finite type multilinear operator is positive weakly (q, r) -dominated. Hence,

$$\mathcal{L}_f(E_1, \dots, E_m; F) \subset \mathcal{L}_{w, (q, r)}^{m+}(E_1, \dots, E_m; F). \quad (2.3)$$

For $m = 1$, we obtain the following coincidence $\mathcal{L}_{w, (q, r)}^{1+}(E_1; F) = \Psi_{(q, r)}(E_1; F)$. In the next result, we give the following equivalent definition.

Proposition 2.3. *Let $1 \leq p, q, r \leq \infty$ with $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$ and $T \in \mathcal{L}(E_1, \dots, E_m; F)$. The following properties are equivalent:*

- 1) *The operator T is positive weakly (q, r) -dominated.*
- 2) *There is a constant $C > 0$ such that for any $(x_i^1, \dots, x_i^m) \in E_1 \times \dots \times E_m$ ($1 \leq i \leq n$) and $y_1^*, \dots, y_n^* \in F^*$, we have*

$$\begin{aligned} & \|(\langle T(x_i^1, \dots, x_i^m), y_i^* \rangle)_{i=1}^n\|_p \\ & \leq C \sup_{\varphi \in B_{\mathcal{L}^r(E_1, \dots, E_m)}^+} \left(\sum_{i=1}^n \varphi(|x_i^1|, \dots, |x_i^m|)^q \right)^{\frac{1}{q}} \| (y_i^*)_{i=1}^n \|_{r, |w|}. \end{aligned} \quad (2.4)$$

In this case, we define

$$d_{w, (q, r)}^{m+}(T) = \inf\{C > 0 : C \text{ satisfies (2.4)}\}.$$

Proof. 2) $\Rightarrow$ 1) : Immediately applying Definition 2.2 for $(x_i^1, \dots, x_i^m) \in E_1^+ \times \dots \times E_m^+$, $1 \leq i \leq n$ and $y_1^*, \dots, y_n^* \in F^{*+}$.

1) $\Rightarrow$ 2) : Suppose that T is positive weakly (q, r) -dominated. For convenience, we prove only the inequality for the case when $m = 2$. Let $(x_i^1, x_i^2) \in E_1 \times E_2$, ($1 \leq$

$i \leq n$) $y_1^*, \dots, y_n^* \in F^*$; then one has

$$\begin{aligned}
& \left(\sum_{i=1}^n |\langle T(x_i^1, x_i^2), y_i^* \rangle|^p \right)^{\frac{1}{p}} \\
&= \left(\sum_{i=1}^n |\langle T(x_i^{1+} - x_i^{1-}, x_i^{2+} - x_i^{2-}), y_i^* \rangle|^p \right)^{\frac{1}{p}} \\
&\leq \left(\sum_{i=1}^n |\langle T(x_i^{1+}, x_i^{2+}), y_i^* \rangle|^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^n |\langle T(x_i^{1+}, x_i^{2-}), y_i^* \rangle|^p \right)^{\frac{1}{p}} \\
&\quad + \left(\sum_{i=1}^n |\langle T(x_i^{1-}, x_i^{2+}), y_i^* \rangle|^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^n |\langle T(x_i^{1-}, x_i^{2-}), y_i^* \rangle|^p \right)^{\frac{1}{p}} \\
&\leq \left(\sum_{i=1}^n |\langle T(x_i^{1+}, x_i^{2+}), y_i^{*+} \rangle|^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^n |\langle T(x_i^{1+}, x_i^{2+}), y_i^{*-} \rangle|^p \right)^{\frac{1}{p}} \\
&\quad + \left(\sum_{i=1}^n |\langle T(x_i^{1+}, x_i^{2-}), y_i^{*+} \rangle|^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^n |\langle T(x_i^{1+}, x_i^{2-}), y_i^{*-} \rangle|^p \right)^{\frac{1}{p}} \\
&\quad + \left(\sum_{i=1}^n |\langle T(x_i^{1-}, x_i^{2+}), y_i^{*+} \rangle|^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^n |\langle T(x_i^{1-}, x_i^{2+}), y_i^{*-} \rangle|^p \right)^{\frac{1}{p}} \\
&\quad + \left(\sum_{i=1}^n |\langle T(x_i^{1-}, x_i^{2-}), y_i^{*+} \rangle|^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^n |\langle T(x_i^{1-}, x_i^{2-}), y_i^{*-} \rangle|^p \right)^{\frac{1}{p}}.
\end{aligned}$$

By using (1.1) and (1.2), we obtain

$$\begin{aligned}
& \left(\sum_{i=1}^n |\langle T(x_i^1, x_i^2), y_i^* \rangle|^p \right)^{\frac{1}{p}} \\
&\leq 8d_{w,(q,r)}^{2+}(T) \sup_{\varphi \in B_{\mathcal{L}^r(E_1, E_2)}^+} \left(\sum_{i=1}^n \varphi(|x_i^1|, |x_i^2|)^q \right)^{\frac{1}{q}} \|(y_i^*)_{i=1}^n\|_{r, |w|}.
\end{aligned}$$

□

Proposition 2.4. *Let $E_1, \dots, E_m, F, G_j$ ($1 \leq j \leq m$) and H be Banach lattices. Let $T \in \mathcal{L}_{w,(q,r)}^{m+}(E_1, \dots, E_m; F)$, $u_j \in \mathcal{L}^+(G_j; E_j)$ ($1 \leq j \leq m$) and $v \in \mathcal{L}^+(F; H)$. Then*

$$v \circ T \circ (u_1, \dots, u_m) \in \mathcal{L}_{w,(q,r)}^{m+}(G_1, \dots, G_m; H).$$

Moreover,

$$d_{w,(q,r)}^{m+}(v \circ T \circ (u_1, \dots, u_m)) \leq d_{w,(q,r)}^{m+}(T) \|u_1\| \cdots \|u_m\| \|v\|.$$

Proof. Let $(x_i^1, \dots, x_i^m) \in G_1^+ \times \cdots \times G_m^+$ ($1 \leq i \leq n$) and $y_1^*, \dots, y_n^* \in G^{*+}$. Since $T \in \mathcal{L}_{w,(q,r)}^{m+}(E_1, \dots, E_m; F)$, $u_j(x_i^j) \geq 0$ and $v^*(y_i^*) \geq 0$ ($1 \leq j \leq m, 1 \leq i \leq n$)

we have

$$\begin{aligned}
& \left\| \left(\langle v \circ T \circ (u_1, \dots, u_m) (x_i^1, \dots, x_i^m), y_i^* \rangle \right)_{i=1}^n \right\|_p \\
&= \left\| \left(\langle T (u_1 (x_i^1), \dots, u_m (x_i^m)), v^* (y_i^*) \rangle \right)_{i=1}^n \right\|_p \\
&\leq d_{w,(q,r)}^{m+}(T) \sup_{\varphi \in B_{\mathcal{L}^r(E_1, \dots, E_m)}^+} \left(\sum_{i=1}^n \varphi (u_1 (x_i^1), \dots, u_m (x_i^m))^q \right)^{\frac{1}{q}} \| (v^* (y_i^*))_{i=1}^n \|_{r,|w|} \\
&\leq d_{w,(q,r)}^{m+}(T) \|u_1\| \cdots \|u_m\| \|v\| \\
&\quad \sup_{\varphi \in B_{\mathcal{L}^r(E_1, \dots, E_m)}^+} \left(\sum_{i=1}^n \varphi (x_i^1, \dots, x_i^m)^q \right)^{\frac{1}{q}} \| (y_i^*)_{i=1}^n \|_{r,|w|}.
\end{aligned}$$

Thus $v \circ T \circ (u_1, \dots, u_m)$ is in $\mathcal{L}_{w,(q,r)}^{m+}(G_1, \dots, G_m; G)$ and we have

$$d_{w,(q,r)}^{m+}(v \circ T \circ (u_1, \dots, u_m)) \leq d_{w,(q,r)}^{m+}(T) \|u_1\| \cdots \|u_m\| \|v\|.$$

□

The pair $(\mathcal{L}_{w,(q,r)}^{m+}, d_{w,(q,r)}^{m+})$ defines a positive Banach multilinear ideal. The proof follows directly from the previous Proposition and the inclusion (2.3), while the remaining details are straightforward. Now, we characterize the positive weakly (q, r) -dominated multilinear operators by the Pietsch domination theorem. For this purpose, we use the full general Pietsch domination theorem given by Pellegrino, Santos and Sepúlveda in [13, Theorem 4.6] and [12]. For simplify, we denote by

$$\widehat{E}_{|\pi|} = E_1 \widehat{\otimes}_{|\pi|} \cdots \widehat{\otimes}_{|\pi|} E_m.$$

Theorem 2.5 (Pietsch Domination Theorem). *Let $1 \leq p, q, r \leq \infty$ with $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. Let $E_1, \dots, E_m$ and F be Banach lattices. The following statements are equivalent:*

- 1) *The operator $T \in \mathcal{L}(E_1, \dots, E_m; F)$ is positive weakly (q, r) -dominated.*
- 2) *There are a constant $C > 0$ and Borel probability measures μ on $B_{(\widehat{E}_{|\pi|})^*}^+$ and η on $B_{F^{**}}^+$ such that*

$$\begin{aligned}
& \left| \langle T(x^1, \dots, x^m), y^* \rangle \right| \\
& \leq C \left(\int_{B_{(\widehat{E}_{|\pi|})^*}^+} \varphi(|x_i^1|, \dots, |x_i^m|)^q d\mu \right)^{\frac{1}{q}} \left(\int_{B_{F^{**}}^+} \langle |y^*|, y^{**} \rangle^r d\eta \right)^{\frac{1}{r}},
\end{aligned} \tag{2.5}$$

for all $(x^1, \dots, x^m, y^*) \in E_1 \times \cdots \times E_m \times F^*$. Therefore, we have

$$d_{w,(q,r)}^{m+}(T) = \inf \{ C > 0 : C \text{ satisfies inequality (2.5)} \}.$$

3) There is a constant $C > 0$ and Borel probability measures μ on $B_{(\widehat{E}_{|\pi|})^*}^+$ and η on $B_{F^{**}}^+$ such that

$$|\langle T(x^1, \dots, x^m), y^* \rangle| \leq C \left(\int_{B_{(\widehat{E}_{|\pi|})^*}^+} \varphi(x^1, \dots, x^m)^q d\mu \right)^{\frac{1}{q}} \times \left(\int_{B_{F^{**}}^+} \langle y^*, y^{**} \rangle^r d\eta \right)^{\frac{1}{r}}. \quad (2.6)$$

for all $(x^1, \dots, x^m, y^*) \in E_1^+ \times \dots \times E_m^+ \times F^{*+}$. Therefore, we have

$$d_{w,(q,r)}^{m+}(T) = \inf\{C > 0 : C \text{ satisfies (2.6)}\}.$$

Proof. 1) $\iff$ 2) : We will choose the parameters as specified in [13, Theorem 4.6]

$$\begin{cases} S : \mathcal{L}(E_1, \dots, E_m; F) \times (E_1 \times \dots \times E_m \times F^*) \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^+ : \\ S(T, (x^1, \dots, x^m, y^*), \lambda_1, \lambda_2) = |\lambda_2| |\langle T(x^1, \dots, x^m), y^* \rangle| \\ R_1 : B_{(\widehat{E}_{|\pi|})^*}^+ \times (E_1 \times \dots \times E_m \times F^*) \times \mathbb{R} \rightarrow \mathbb{R}^+ : \\ R_1(\varphi, (x^1, \dots, x^m, y^*), \lambda_1) = \varphi(|x^1|, \dots, |x^m|) \\ R_2 : B_{F^{**}}^+ \times (E_1 \times \dots \times E_m \times F^*) \times \mathbb{R} \rightarrow \mathbb{R}^+ : \\ R_2(y^{**}, (x^1, \dots, x^m, y^*), \lambda_2) = |\lambda_2| \langle |y^*|, y^{**} \rangle. \end{cases}$$

These maps satisfy conditions (1) and (2) from [13, p. 1255]. We can easily conclude that $T : E_1 \times \dots \times E_m \rightarrow F$ is positive weakly (q, r) -dominated if and only if

$$\begin{aligned} & \left(\sum_{i=1}^n S(T, (x_i^1, \dots, x_i^m, y_i^*), \lambda_{i,1}, \lambda_{i,2})^p \right)^{\frac{1}{p}} \\ & \leq C \sup_{\varphi \in B_{(\widehat{E}_{|\pi|})^*}^+} \left(\sum_{i=1}^n R_1(\varphi, (x_i^1, \dots, x_i^m, y_i^*), \lambda_{i,1})^q \right)^{\frac{1}{q}} \\ & \quad \times \sup_{y^{**} \in B_{F^{**}}^+} \left(\sum_{i=1}^n R_2(y^{**}, (x_i^1, \dots, x_i^m, y_i^*), \lambda_{i,2})^r \right)^{\frac{1}{r}}, \end{aligned}$$

that is, T is R_1, R_2 - S -abstract (q, r) -summing. As outlined in [13, Theorem 4.6], this implies that T is R_1, R_2 - S -abstract (q, r) -summing if and only if there exists a positive constant C and probability measures μ on $B_{(\widehat{E}_{|\pi|})^*}^+$ and η on $B_{F^{**}}^+$, such that

$$\begin{aligned} & S(T, (x^1, \dots, x^m, y^*), \lambda_1, \lambda_2) \\ & \leq C \left(\int_{B_{(\widehat{E}_{|\pi|})^*}^+} R_1(\varphi, (x^1, \dots, x^m, y^*), \lambda_1)^q d\mu \right)^{\frac{1}{q}} \\ & \quad \left(\int_{B_{F^{**}}^+} R_2(y^{**}, (x^1, \dots, x^m, y^*), \lambda_2)^r d\eta \right)^{\frac{1}{r}}. \end{aligned}$$

Consequently,

$$\begin{aligned} & |\langle T(x^1, \dots, x^m), y^* \rangle| \\ & \leq C \left(\int_{B^+_{(\widehat{E}_{|\pi|})^*}} \varphi(|x^1|, \dots, |x^m|)^q d\mu \right)^{\frac{1}{q}} \left(\int_{B^{+**}_{F^{**}}} \langle |y^*|, y^{**} \rangle^r d\eta \right)^{\frac{1}{r}}. \end{aligned}$$

The implications 2) $\implies$ 3) and 3) $\implies$ 1 are immediate. $\square$

Proposition 2.6. *Let $T \in \mathcal{L}(E_1, \dots, E_m; F)$ and $1 \leq p, q, r \leq \infty$. Consider the following statements:*

- 1) $T^\otimes : E_1 \widehat{\otimes}_{|\pi|} \dots \widehat{\otimes}_{|\pi|} E_m \rightarrow F$ is positive weakly (q, r) -dominated;
 - 2) There exist a Banach space G , a positive (Dimant) strongly p -summing m -linear operator $S : E_1 \times \dots \times E_m \rightarrow G$ and a Cohen positive strongly p -summing linear operator $u : G \rightarrow F$ such that $T = u \circ S$;
 - 3) T is positive weakly (q, r) -dominated.
- Then, the statement 1) implies 2), which implies 3).

Proof. 1) $\implies$ 2) Since $T^\otimes$ is positive (q, r) -dominated, by [6, Theorem 3.7] there exist a Banach space G , a positive q -summing linear operator $v : E_1 \widehat{\otimes}_{|\pi|} \dots \widehat{\otimes}_{|\pi|} E_m \rightarrow G$ and a Cohen positive strongly q^* -summing linear operator $u : G \rightarrow F$ such that $T = u \circ v$. Let $S = v \circ \otimes$. Then $T = u \circ v \circ \otimes = u \circ S$ and the diagram

$$\begin{array}{ccc} E_1 \times \dots \times E_m & \xrightarrow{T} & F \\ \downarrow \otimes & \searrow S & \uparrow u \\ \widehat{E}_{|\pi|} & \xrightarrow{v} & G \end{array}$$

commutes. Since $S^\otimes = v$, by [2, Corollary 1], it follows that S is positive Dimant strongly p -summing.

2) $\implies$ 3) : There exist a Borel probability measures μ on $B^+_{(\widehat{E}_{|\pi|})^*}$ and η on $B^{+**}_{F^{**}}$ such that for every $(x^1, \dots, x^m) \in E_1^+ \times \dots \times E_m^+$ and $y^* \in F^{*+}$, we have

$$\begin{aligned} & |\langle T(x^1, \dots, x^m), y^* \rangle| \\ & = \langle u \circ S(x^1, \dots, x^m), y^* \rangle \\ & \leq d_p^+(u) \|S(x^1, \dots, x^m)\| \left(\int_{B^{+**}_{F^{**}}} \langle y^*, y^{**} \rangle^r d\eta \right)^{\frac{1}{r}} \\ & \leq d_p^+(u) d_{s,p}^{m+}(S) \left(\int_{B^+_{(\widehat{E}_{|\pi|})^*}} \varphi(x^1, \dots, x^m)^q d\mu \right)^{\frac{1}{q}} \left(\int_{B^{+**}_{F^{**}}} \langle y^*, y^{**} \rangle^r d\eta \right)^{\frac{1}{r}}. \end{aligned}$$

$\square$

In what follows, we develop and analyze the corresponding positive polynomial version. This approach allows us to establish that the class of positive weakly (q, r) -dominated polynomials forms a positive polynomial ideal, stable under composition with bounded positive operators. Moreover, we establish a

Pietsch domination theorem in the polynomial setting, showing that the domination inequalities extend naturally from the multilinear case.

Definition 2.7. Let $m \in \mathbb{N}^*$. Let $1 \leq p, q, r \leq \infty$ with $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. Let E and F be Banach lattices. A polynomial $P \in \mathcal{P}(^m E; F)$ is called positive weakly (q, r) -dominated if there exists a constant $C > 0$ such that for any $(x_i)_{i=1}^n \subset E^+$ and $(y_i^*)_{i=1}^n \subset F^{*+}$, the following inequality holds:

$$\|(\langle P(x_i), y_i^* \rangle)_{i=1}^n\|_p \leq C \sup_{\phi \in B_{\mathcal{P}^r(^m E)}^+} \left(\sum_{i=1}^n \phi(x_i)^q \right)^{\frac{1}{q}} \| (y_i^*)_{i=1}^n \|_{r, |w|}. \quad (2.7)$$

The space of all such polynomials is denoted by $\mathcal{P}_{w, (q, r)}^+(^m E; F)$. Its norm is given by

$$d_{w, (q, r)}^{m+}(P) = \inf \{ C > 0 : C \text{ satisfies (2.7)} \}.$$

An equivalent formulation of (2.7) is

$$\|(\langle P(x_i), y_i^* \rangle)_{i=1}^n\|_p \leq C \sup_{\phi \in B_{\mathcal{P}^r(^m E)}^+} \left(\sum_{i=1}^n \phi(|x_i|)^q \right)^{\frac{1}{q}} \|(|y_i^*|)_{i=1}^n\|_{r, w}$$

for every $(x_i)_{i=1}^n \subset E$ and $(y_i^*)_{i=1}^n \subset F^*$. It is straightforward to check that

$$\mathcal{P}_f(^m E; F) \subset \mathcal{P}_{w, (q, r)}^+(^m E; F). \quad (2.8)$$

Proposition 2.8. Let $P \in \mathcal{P}_{w, (p, r)}^+(^m E; F)$, $u \in \mathcal{L}^+(G; E)$ and $v \in \mathcal{L}^+(F; H)$. Then $v \circ P \circ u \in \mathcal{P}_{w, (q, r)}^+(^m G; H)$ and we have

$$d_{w, (q, r)}^{m+}(v \circ P \circ u) \leq \|v\| d_{w, (q, r)}^{m+}(P) \|u\|^m.$$

Proof. Let $(x_i)_{i=1}^n \subset E^+$ and $(y_i^*)_{i=1}^n \subset F^{*+}$. Then

$$\begin{aligned} & \left(\sum_{i=1}^n |\langle v \circ P \circ u(x_i), y_i^* \rangle|^p \right)^{\frac{1}{p}} \\ &= \left(\sum_{i=1}^n |\langle P \circ u(x_i), v^*(y_i^*) \rangle|^p \right)^{\frac{1}{p}} \\ &\leq d_{w, (q, r)}^{m+}(P) \sup_{\psi \in B_{\mathcal{P}^r(^m E)}^+} \left(\sum_{i=1}^n \psi(u(x_i))^q \right)^{\frac{1}{q}} \| (v^*(y_i^*))_{i=1}^n \|_{r, |w|} \\ &\leq d_{w, (q, r)}^{m+}(P) \|u\|^m \sup_{\psi \in B_{\mathcal{P}^r(^m E)}^+} \left(\sum_{i=1}^n \psi \left(\frac{u}{\|u\|}(x_i) \right)^q \right)^{\frac{1}{q}} \|v^*\| \| (y_i^*)_{i=1}^n \|_{r, |w|} \\ &\leq \|v\| d_{w, (q, r)}^{m+}(P) \|u\|^m \sup_{\phi \in B_{\mathcal{P}^r(^m E)}^+} \left(\sum_{i=1}^n \phi(x_i)^q \right)^{\frac{1}{q}} \| (y_i^*)_{i=1}^n \|_{r, |w|}. \end{aligned}$$

Thus $v \circ P \circ u$ is positive weakly (q, r) -dominated and

$$d_{w, (q, r)}^{m+}(v \circ P \circ u) \leq \|v\| d_{w, (q, r)}^{m+}(P) \|u\|^m.$$

□

The pair $(\mathcal{P}_{w,(q,r)}^+, d_{w,(q,r)}^{m+})$ defines a positive Banach polynomial ideal. The proof follows directly from the previous proposition and the inclusion (2.8), while the remaining details are straightforward. We now turn to the characterization of positive weakly (q, r) -dominated polynomials through a Pietsch domination theorem.

Theorem 2.9 (Pietsch Domination Theorem). *Let $m \in \mathbb{N}^*$. Let $1 \leq p, q, r \leq \infty$ with $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. Let E and F be Banach lattices. The following statements are equivalent:*

- 1) *The polynomial $P \in \mathcal{P}(^m E; F)$ is positive weakly (q, r) -dominated.*
- 2) *There is a constant $C > 0$ and Borel probability measures μ on $B_{\mathcal{P}^r(mE)}^+$ and η on $B_{F^{**}}^+$ such that*

$$|\langle P(x), y^* \rangle| \leq C \left(\int_{B_{\mathcal{P}^r(mE)}^+} \phi(|x|)^q d\mu \right)^{\frac{1}{q}} \left(\int_{B_{F^{**}}^+} \langle |y^*|, y^{**} \rangle^r d\eta \right)^{\frac{1}{r}} \quad (2.9)$$

for all $(x, y^*) \in E \times F^*$. Therefore, we have

$$d_{w,(q,r)}^{m+}(P) = \inf\{C > 0 : C \text{ satisfies (2.9)}\}.$$

- 3) *There are a constant $C > 0$ and Borel probability measures μ on $B_{\mathcal{P}^r(mE)}^+$ and η on $B_{F^{**}}^+$ such that*

$$|\langle P(x), y^* \rangle| \leq C \left(\int_{B_{\mathcal{P}^r(mE)}^+} \phi(x)^q d\mu \right)^{\frac{1}{q}} \left(\int_{B_{F^{**}}^+} \langle y^*, y^{**} \rangle^r d\eta \right)^{\frac{1}{r}} \quad (2.10)$$

for all $(x, y^*) \in E^+ \times F^{*+}$. Therefore, we have

$$d_{w,(q,r)}^{m+}(P) = \inf\{C > 0 : C \text{ satisfies (2.10)}\}.$$

Proof. 1) $\iff$ 2) : We choose the parameters as specified in [13, Theorem 4.6]

$$\begin{cases} S : \mathcal{P}(^m E; F) \times (E \times F^*) \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^+ : \\ S(P, (x, y^*), \lambda_1, \lambda_2) = |\lambda_2| |\langle P(x), y^* \rangle| \\ R_1 : B_{\mathcal{P}^r(mE)}^+ \times (E \times F^*) \times \mathbb{R} \rightarrow \mathbb{R}^+ : R_1(\phi, (x, y^*), \lambda_1) = \langle |x|, \phi \rangle \\ R_2 : B_{F^{**}}^+ \times (E \times F^*) \times \mathbb{R} \rightarrow \mathbb{R}^+ : R_2(y^{**}, (x, y^*), \lambda_2) = |\lambda_2| \langle |y^*|, y^{**} \rangle. \end{cases}$$

These maps satisfy conditions (1) and (2) from [13, p. 1255], allowing us to conclude that $P : E \rightarrow F$ is positive weakly (q, r) -dominated if and only if

$$\begin{aligned} & \left(\sum_{i=1}^n S(P, (x_i, y_i^*), \lambda_{i,1}, \lambda_{i,2})^p \right)^{\frac{1}{p}} \\ & \leq C \sup_{\phi \in B_{\mathcal{P}^r(mE)}^+} \left(\sum_{i=1}^n R_1(\phi, (x_i, y_i^*), \lambda_{i,1})^q \right)^{\frac{1}{q}} \sup_{y^{**} \in B_{F^{**}}^+} \left(\sum_{i=1}^n R_2(y^{**}, (x_i, y_i^*), \lambda_{i,2})^r \right)^{\frac{1}{r}}, \end{aligned}$$

that is, P is R_1, R_2 - S -abstract (q, r) -summing. As outlined in [13, Theorem 4.6], this implies that P is R_1, R_2 - S -abstract (q, r) -summing if and only if there exist a

positive constant C and probability measures μ on $B_{\mathcal{P}^r(m_E)}^+$ and η on $B_{F^{**}}^+$, such that

$$\begin{aligned} & S(P, (x, y^*), \lambda_1, \lambda_2) \\ & \leq C \left(\int_{B_{\mathcal{P}^r(m_E)}^+} R_1(\phi, (x, y^*), \lambda_1)^q d\mu \right)^{\frac{1}{q}} \left(\int_{B_{F^{**}}^+} R_2(y^{**}, (x, y^*), \lambda_2)^r d\eta \right)^{\frac{1}{r}}. \end{aligned}$$

Consequently,

$$|\langle P(x), y^* \rangle| \leq C \left(\int_{B_{\mathcal{P}^r(m_E)}^+} \langle |x|, \phi \rangle^q d\mu \right)^{\frac{1}{q}} \left(\int_{B_{F^{**}}^+} \langle |y^*|, y^{**} \rangle^r d\eta \right)^{\frac{1}{r}}.$$

The implications $2) \implies 3)$ and $3) \implies 1)$ are straightforward to prove. $\square$

3. TENSORIAL REPRESENTATION

Tensorial representation plays a fundamental role in the study of Banach spaces. For the projective tensor product, it is well known that the space of multilinear operators $\mathcal{L}(X_1, \dots, X_m; Y)$ coincides with the dual of $X_1 \widehat{\otimes}_\pi \dots \widehat{\otimes}_\pi X_m \widehat{\otimes}_\pi Y^*$. Similarly, replacing the projective norm with the injective norm ε yields a coincidence with the space of integral multilinear operators. For this reason, tensorial representation has become a standard objective in the study of new classes of operators. It is therefore natural, when introducing a new class, to seek an appropriate tensor norm that produces an analogous identification. In the present section, we define a tensor norm on the algebraic tensor product $E_1 \otimes \dots \otimes E_m \otimes F^*$ and show that its topological dual is isometric to the space of positive weakly (q, r) -dominated multilinear operators. This tensorial approach provides a natural framework to identify such operators. In the polynomial case, we similarly define a tensor norm on $(\widehat{\otimes}_{s, |\pi|}^m E) \otimes F$ and show that its topological dual is isometric to the space of positive weakly (q, r) -dominated polynomials.

3.1. Multilinear case. Fix $m \in \mathbb{N}^*$. Let $E_1, \dots, E_m, F$ be Banach lattices. Let $1 \leq p, q, r \leq \infty$ such that $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$ and $u \in E_1 \otimes \dots \otimes E_m \otimes F$. For simplify, we denote it by

$$\widehat{E}_{|\pi|} = E_1 \widehat{\otimes}_{|\pi|} \dots \widehat{\otimes}_{|\pi|} E_m.$$

Consider

$$\mu_{(q,r)}^{m+}(u) = \inf \left\{ \|(\lambda_i)\|_{\ell_{p^*}^n} \| (x_i^1 \otimes \dots \otimes x_i^m) \|_{\ell_{q, |w|}^n(\widehat{E}_{|\pi|})} \| (y_i) \|_{\ell_{r, |w|}^n(F)} \right\},$$

where the infimum is taken over all general representations of u of the form

$$u = \sum_{i=1}^n \lambda_i x_i^1 \otimes \dots \otimes x_i^m \otimes y_i, \quad (3.1)$$

with $(x_i^j)_{i=1}^n \subset E_j$, $(y_i)_{i=1}^n \subset F$, $(\lambda_i)_{i=1}^n \subset \mathbb{R}$, $(1 \leq j \leq m)$ and $n \in \mathbb{N}^*$.

Lemma 3.1. *Let $u \in E_1 \otimes \dots \otimes E_m \otimes F$ of the form (3.1). The following properties are equivalent:*

1) $u = 0$.

- 2) $\sum_{i=1}^n \lambda_i x_1^*(x_i^1) \cdots x_m^*(x_i^m) y^*(y_i) = 0$ for every $x_j^* \subset E_j^{*+}$, $y^* \in F^{*+}$, $1 \leq j \leq m$.
- 3) $\sum_{i=1}^n \lambda_i \varphi(x_i^1, \dots, x_i^m) y^*(y_i) = 0$ for every $\varphi \in \mathcal{L}^+(E_1, \dots, E_m)$ and $y^* \in F^{*+}$
- 4) $\sum_{i=1}^n \lambda_i \varphi(x_i^1, \dots, x_i^m) y^*(y_i) = 0$ for every $\varphi \in \mathcal{L}^r(E_1, \dots, E_m)$ and $y^* \in F^{*+}$.

Proof. 1) $\Leftrightarrow$ 2) : The first implication is straightforward by [16, Proposition 1.2]. For the second, assume that 2) holds. Let $x_j^* \in E_j^*$ for $1 \leq j \leq m$ and $y^* \in F^*$. Then

$$\begin{aligned} & \sum_{i=1}^n x_1^*(x_i^1) \cdots x_m^*(x_i^m) y^*(y_i) \\ &= \sum_{i=1}^n (x_1^{*+} - x_1^{*-}) (x_i^1) \cdots (x_m^{*+} - x_m^{*-}) (x_i^m) (y^{*+} - y^{*-}) (y_i). \end{aligned}$$

Expanding the products yields a finite sum of terms of the form

$$\sum_{i=1}^n x_1^{*\epsilon_1}(x_i^1) \cdots x_m^{*\epsilon_m}(x_i^m) y^{*\epsilon}(y_i),$$

where $\epsilon_1, \dots, \epsilon_m \in \{+, -\}$ and $\epsilon \in \{+, -\}$. Each of these sums corresponds to positive functionals and, by assumption 2), they all vanish. Therefore,

$$\sum_{i=1}^n \lambda_i x_1^*(x_i^1) \cdots x_m^*(x_i^m) y^*(y_i) = 0.$$

Hence $u = 0$.

2) $\Leftrightarrow$ 3) : Assume 2). By the same reasoning as above, we obtain

$$\sum_{i=1}^n \lambda_i x_1^*(x_i^1) \cdots x_m^*(x_i^m) y^*(y_i) = 0$$

for all $x_j^* \subset E_j^*$, $y^* \in F^*$. From [16, Proposition 1.2], this implies $u = 0$, and consequently,

$$\langle u, \psi \rangle = 0 \text{ for every } (m+1)\text{-linear form } \psi.$$

In particular, this holds for $\varphi \in \mathcal{L}^+(E_1, \dots, E_m)$ and $y^* \in F^{*+}$, so that

$$\sum_{i=1}^n \lambda_i \varphi(x_i^1, \dots, x_i^m) y^*(y_i) = 0.$$

Conversely, if 3) holds, then take $x_j^* \subset E_j^{*+}$, $y^* \in F^{*+}$. We have $x_1^* \otimes \cdots \otimes x_m^* \in \mathcal{L}^+(E_1, \dots, E_m)$. Hence,

$$\sum_{i=1}^n \lambda_i x_1^* \otimes \cdots \otimes x_m^* (x_i^1, \dots, x_i^m) y^*(y_i) = \sum_{i=1}^n \lambda_i x_1^*(x_i^1) \cdots x_m^*(x_i^m) y^*(y_i) = 0.$$

3) $\implies$ 4) : Let $\varphi \in \mathcal{L}^r(E_1, \dots, E_m)$, there exist $\varphi_1, \varphi_2 \in \mathcal{L}^+(E_1, \dots, E_m)$ such that

$$\varphi = \varphi_1 - \varphi_2.$$

Let $y^* \in F^{*+}$. Then

$$\begin{aligned} & \sum_{i=1}^n \lambda_i \varphi(x_i^1, \dots, x_i^m) y^*(y_i) \\ &= \sum_{i=1}^n \lambda_i \varphi_1(x_i^1, \dots, x_i^m) y^*(y_i) - \sum_{i=1}^n \lambda_i \varphi_2(x_i^1, \dots, x_i^m) y^*(y_i) = 0. \end{aligned}$$

4) $\implies$ 3) : This is immediate, since $\mathcal{L}^+(E_1, \dots, E_m) \subset \mathcal{L}^r(E_1, \dots, E_m)$. $\square$

The following proposition can be proved easily.

Proposition 3.2. *Let $1 \leq p, q, r \leq \infty$ such that $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$ and $m \in \mathbb{N}^*$. Then $\mu_{(q,r)}^{m+}$ is a tensor norm on $E_1 \otimes \dots \otimes E_m \otimes F$.*

Proof. It is clear that for any element $u \in E_1 \otimes \dots \otimes E_m \otimes F$ of the form (3.1) and any scalar α , we have

$$\mu_{(q,r)}^{m+}(u) \geq 0 \text{ and } \mu_{(q,r)}^{m+}(\alpha u) = |\alpha| \mu_{(q,r)}^{m+}(u).$$

Let $\varphi \in B_{\mathcal{L}^r(E_1, \dots, E_m)}^+$ and $y^* \in B_{F^{*+}}$. Then,

$$\begin{aligned} & |\langle u, \varphi \otimes y^* \rangle| \\ &= \left| \sum_{i=1}^n \lambda_i \varphi(x_i^1, \dots, x_i^m) y^*(y_i) \right| \\ &\leq \sum_{i=1}^n |\lambda_i| \varphi(|x_i^1|, \dots, |x_i^m|) y^*(|y_i|) \quad (\text{by the Hölder inequality}) \\ &\leq \|(\lambda_i)\|_{\ell_{p^*}^n} \left(\sum_{i=1}^n \varphi(|x_i^1|, \dots, |x_i^m|)^q \right)^{\frac{1}{q}} \left(\sum_{i=1}^n y^*(|y_i|)^r \right)^{\frac{1}{r}} \\ &\leq \|(\lambda_i)\|_{\ell_{p^*}^n} \sup_{\varphi \in B_{\mathcal{L}^r(E_1, \dots, E_m)}^+} \left(\sum_{i=1}^n \varphi(|x_i^1|, \dots, |x_i^m|)^q \right)^{\frac{1}{q}} \sup_{y^* \in B_{F^{*+}}} \left(\sum_{i=1}^n y^*(|y_i|)^r \right)^{\frac{1}{r}}. \end{aligned}$$

Since

$$\sup_{\varphi \in B_{\mathcal{L}^r(E_1, \dots, E_m)}^+} \left(\sum_{i=1}^n \varphi(|x_i^1|, \dots, |x_i^m|)^q \right)^{\frac{1}{q}} = \|(x_i^1 \otimes \dots \otimes x_i^m)\|_{\ell_{q,|w|}^n(\widehat{E}_{|\pi|})},$$

we obtain

$$|\langle u, \varphi \otimes y^* \rangle| \leq \|(\lambda_i)\|_{\ell_{p^*}^n} \|(x_i^1 \otimes \dots \otimes x_i^m)\|_{\ell_{q,|w|}^n(\widehat{E}_{|\pi|})} \|y^*\|_{\ell_{r,|w|}^n(F)}.$$

By taking the infimum over all representations of u , we obtain

$$|\langle u, \varphi \otimes y^* \rangle| \leq \mu_{(q,r)}^{m+}(u).$$

Suppose that $\mu_{(q,r)}^{m+}(u) = 0$; then

$$|\langle u, \varphi \otimes y^* \rangle| = 0.$$

Consequently, if $\varphi \in \mathcal{L}^r(E_1, \dots, E_m)$ and $y^* \in F^{*+}$ with $\varphi \neq 0$ and $y^* \neq 0$, then

$$|\langle u, \varphi \otimes y^* \rangle| = \|\varphi\| \|y^*\| \left| \left\langle u, \frac{\varphi}{\|\varphi\|} \otimes \frac{y^*}{\|y^*\|} \right\rangle \right| = 0,$$

and by Lemma 3.1 we obtain $u = 0$. Let now $u_1, u_2 \in E \otimes F$ of the form (3.1). By the definition of $\mu_{(q,r)}^{m+}$, we find representations

$$\begin{aligned} u_1 &= \sum_{i=1}^{s_1} \lambda_{1,i} x_{1,i}^1 \otimes \cdots \otimes x_{1,i}^m \otimes y_{1,i}, \\ u_2 &= \sum_{i=1}^{s_2} \lambda_{2,i} x_{2,i}^1 \otimes \cdots \otimes x_{2,i}^m \otimes y_{2,i}, \end{aligned}$$

such that

$$\|(\lambda_{1,i})\|_{\ell_{p^*}^{s_1}} \|(x_{1,i}^1 \otimes \cdots \otimes x_{1,i}^m)\|_{\ell_{q,|w|}^{s_1}(\widehat{E}_{|\pi|})} \|(y_{1,i})\|_{\ell_{r,|w|}^{s_1}(F)} \leq \mu_{(q,r)}^{m+}(u) + \varepsilon.$$

Replace $(\lambda_{1,i})$, $(x_{1,i}^1 \otimes \cdots \otimes x_{1,i}^m)$ and $(y_{1,i})$ by an appropriate multiple of them;

$$\begin{aligned} \lambda_{1,i} &= \lambda_{1,i} \frac{\|(y_{1,i})\|_{\ell_{r,|w|}^{s_1}(F)}^{\frac{1}{p^*}} \|(x_{1,i}^1 \otimes \cdots \otimes x_{1,i}^m)\|_{\ell_{q,|w|}^{s_1}(\widehat{E}_{|\pi|})}^{\frac{1}{p^*}}}{\|(\lambda_{1,i})\|_{\ell_{p^*}^{s_1}}^{\frac{1}{p}}}, \\ x_{1,i}^1 \otimes \cdots \otimes x_{1,i}^m &= x_{1,i}^1 \otimes \cdots \otimes x_{1,i}^m \frac{\|(\lambda_{1,i})\|_{\ell_{p^*}^{s_1}}^{\frac{1}{q}} \|(y_{1,i})\|_{\ell_{r,|w|}^{s_1}(F)}^{\frac{1}{q}}}{\|(x_{1,i}^1 \otimes \cdots \otimes x_{1,i}^m)\|_{\ell_{q,|w|}^{s_1}(\widehat{E}_{|\pi|})}^{\frac{1}{q^*}}}, \\ y_{1,i} &= y_{1,i} \frac{\|(\lambda_{1,i})\|_{\ell_{p^*}^{s_1}}^{\frac{1}{r}} \|(x_{1,i}^1 \otimes \cdots \otimes x_{1,i}^m)\|_{\ell_{q,|w|}^{s_1}(\widehat{E}_{|\pi|})}^{\frac{1}{r}}}{\|(y_{1,i})\|_{\ell_{r,|w|}^{s_1}(F)}^{\frac{1}{r^*}}}. \end{aligned}$$

We obtain

$$\begin{aligned} \|(\lambda_{1,i})_{i=1}^{s_1}\|_{\ell_{p^*}^{s_1}} &\leq \left(\mu_{(q,r)}^{m+}(u_1) + \varepsilon \right)^{\frac{1}{p^*}} \\ \|(x_{1,i}^1 \otimes \cdots \otimes x_{1,i}^m)_{i=1}^{s_1}\|_{\ell_{q,|w|}^{s_1}(\widehat{E}_{|\pi|})} &\leq \left(\mu_{(q,r)}^{m+}(u_1) + \varepsilon \right)^{\frac{1}{q}} \\ \|(y_{1,i})_{i=1}^{s_1}\|_{\ell_{r,|w|}^{s_1}(F)} &\leq \left(\mu_{(q,r)}^{m+}(u_1) + \varepsilon \right)^{\frac{1}{r}}. \end{aligned}$$

Similarly for u_2 , we get

$$\begin{aligned}
& \mu_{(q,r)}^{m+}(u_1 + u_2) \\
& \leq \left(\|(\lambda_{1,i})_{i=1}^{s_1}\|_{\ell_{p^*}^{s_1}}^{p^*} + \|(\lambda_{2,i})_{i=1}^{s_2}\|_{\ell_{p^*}^{s_2}}^{p^*} \right)^{\frac{1}{p^*}} \left(\|x_{1,i}^1 \otimes \cdots \otimes x_{1,i}^m\|_{\ell_{q,|w|}^{s_1}(\widehat{E}_{|\pi|})}^{s_1} \right)^q \\
& \quad + \left\| (x_{2,i}^1 \otimes \cdots \otimes x_{2,i}^m)_{i=1}^{s_2} \right\|_{\ell_{q,|w|}^{s_2}(\widehat{E}_{|\pi|})}^q \left(\|y_{1,i}\|_{\ell_{r,|w|}^{s_1}(F)}^{s_1} + \|y_{2,i}\|_{\ell_{r,|w|}^{s_2}(F)}^{s_2} \right)^r \\
& \leq \left(\mu_{(q,r)}^{m+}(u_1) + \mu_{(q,r)}^{m+}(u_2) + 2\varepsilon \right)^{\frac{1}{p^*}} \left(\mu_{(q,r)}^{m+}(u_1) + \mu_{(q,r)}^{m+}(u_2) + 2\varepsilon \right)^{\frac{1}{q}} \\
& \quad \times \left(\mu_{(q,r)}^{m+}(u_1) + \mu_{(q,r)}^{m+}(u_2) + 2\varepsilon \right)^{\frac{1}{r}} \\
& \leq \mu_{(q,r)}^{m+}(u_1) + \mu_{(q,r)}^{m+}(u_2) + 2\varepsilon.
\end{aligned}$$

By letting ε tend to zero, we obtain the triangle inequality for $\mu_{(q,r)}^{m+}$. $\square$

Proposition 3.3. *The norm $\mu_{(q,r)}^{m+}$ is reasonable, that is,*

$$\varepsilon \leq \mu_{(q,r)}^{m+} \leq \pi, \quad (3.2)$$

where ε and π denote the injective and projective norms on $E_1 \otimes \cdots \otimes E_m \otimes F$, respectively.

Proof. Let us prove the right-hand inequality in (3.2). We have

$$\mu_{(q,r)}^{m+}(u) \leq \|(\lambda_i)_{i=1}^n\|_{\ell_{p^*}^n} \left\| (x_i^1 \otimes \cdots \otimes x_i^m)_{i=1}^n \right\|_{\ell_q^n(\widehat{E}_{|\pi|})} \left\| (y_i)_{i=1}^n \right\|_{\ell_r^n(F)}.$$

For each i , we set

$$\begin{aligned}
\lambda_i &= \lambda_i \frac{(|\lambda_i| \|y_i\| \|x_i^1 \otimes \cdots \otimes x_i^m\|)^{\frac{1}{p^*}}}{|\lambda_i|}, \\
x_i^1 \otimes \cdots \otimes x_i^m &= x_i^1 \otimes \cdots \otimes x_i^m \frac{(|\lambda_i| \|y_i\| \|x_i^1 \otimes \cdots \otimes x_i^m\|)^{\frac{1}{q}}}{\|x_i^1 \otimes \cdots \otimes x_i^m\|}, \\
y_i &= y_i \frac{(|\lambda_i| \|y_i\| \|x_i^1 \otimes \cdots \otimes x_i^m\|)^{\frac{1}{r}}}{\|y_i\|}.
\end{aligned}$$

Substituting these expressions into the above inequality and taking the infimum over all representations of u of the form (3.1), we obtain

$$\mu_{(q,r)}^{m+}(u) \leq \sum_{i=1}^n |\lambda_i| \|y_i\| \|x_i^1 \otimes \cdots \otimes x_i^m\| \leq \sum_{i=1}^n |\lambda_i| \|y_i\| \|x_i^1\| \cdots \|x_i^m\|.$$

Hence,

$$\mu_{(q,r)}^{m+}(u) \leq \pi(u).$$

For the left inequality in (3.2), we have

$$\begin{aligned}\varepsilon(u) &= \sup_{\substack{x_j^* \in B_{E_j^*}, y^* \in B_{F^*} \\ 1 \leq j \leq m}} \left\{ \left| \sum_{i=1}^n \lambda_i x_1^*(x_i^1) \cdots x_m^*(x_i^m) y^*(y_i) \right| \right\} \\ &= \sup_{\substack{x_j^* \in B_{E_j^*}, y^* \in B_{F^*} \\ 1 \leq j \leq m}} \left\{ \left| \sum_{i=1}^n \lambda_i x_1^* \otimes \cdots \otimes x_m^*(x_i^1, \dots, x_i^m) y^*(y_i) \right| \right\}.\end{aligned}$$

Since $x_1^* \otimes \cdots \otimes x_m^* \in B_{\mathcal{L}^r(E_1, \dots, E_m)}$, and taking absolute values inside, we get

$$\varepsilon(u) \leq \sup_{\varphi \in B_{\mathcal{L}^r(E_1, \dots, E_m)}, y^* \in B_{F^*}} \left\{ \sum_{i=1}^n |\lambda_i| |\varphi(x_i^1 \otimes \cdots \otimes x_i^m)| |y^*(y_i)| \right\}.$$

Since $|\varphi| \in B_{\mathcal{L}^r(E_1, \dots, E_m)}^+$ and $|y^*| \in B_{F^*}^+$, we further obtain

$$\varepsilon(u) \leq \sup_{\varphi \in B_{\mathcal{L}^r(E_1, \dots, E_m)}^+, y^* \in B_{F^*}^+} \left\{ \sum_{i=1}^n |\lambda_i| \varphi(|x_i^1|, \dots, |x_i^m|) y^*(|y_i|) \right\}.$$

Applying the Hölder's inequality, we have

$$\varepsilon(u) \leq \|(\lambda_i)_{i=1}^n\|_{\ell_{p^*}^n} \sup_{\varphi \in B_{\mathcal{L}^r(E_1, \dots, E_m)}^+} \left(\sum_{i=1}^n \varphi(|x_i^1|, \dots, |x_i^m|)^p \right)^{\frac{1}{p}} \|(y_i)_{i=1}^n\|_{\ell_{r, |w|}^n(F)}.$$

Finally, taking the infimum over all representations of u of the form (3.1) yields

$$\varepsilon(u) \leq \mu_{(q,r)}^{m+}(u).$$

□

We denote by $E_1 \widehat{\otimes}_{\mu_{(q,r)}^{m+}} \cdots \widehat{\otimes}_{\mu_{(q,r)}^{m+}} E_m \widehat{\otimes}_{\mu_{(q,r)}^{m+}} F$ the completed of $E_1 \otimes \cdots \otimes E_m \otimes F$ for the norm $\mu_{(q,r)}^{m+}$. The main result of this section is the following identification.

Proposition 3.4. *Let $1 \leq p, q, r \leq \infty$ such that $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. We have the following isometric identification:*

$$\mathcal{L}_{w, (q,r)}^{m+}(E_1, \dots, E_m; F) = (E_1 \widehat{\otimes}_{\mu_{(q,r)}^{m+}} \cdots \widehat{\otimes}_{\mu_{(q,r)}^{m+}} E_m \widehat{\otimes}_{\mu_{(q,r)}^{m+}} F^*)^*.$$

Proof. Let $T \in \mathcal{L}_{w, (q,r)}^{m+}(E, \dots, E_m; F)$. We define a linear functional on $E_1 \otimes \cdots \otimes E_m \otimes F^*$ by

$$\Psi_T(u) = \sum_{i=1}^n \lambda_i \langle T(x_i^1, \dots, x_i^m), y_i^* \rangle,$$

where $u = \sum_{i=1}^n \lambda_i x_i^1 \otimes \cdots \otimes x_i^m \otimes y_i^*$. Then, by the Hölder's inequality, we have

$$\begin{aligned}|\Psi_T(u)| &= \left| \sum_{i=1}^n \lambda_i \langle T(x_i^1, \dots, x_i^m), y_i^* \rangle \right| \\ &\leq \|(\lambda_i)\|_{\ell_{p^*}^n} \left(\sum_{i=1}^n |\langle T(x_i^1, \dots, x_i^m), y_i^* \rangle|^p \right)^{\frac{1}{p}}.\end{aligned}$$

Since T is positive weakly (q, r) -dominated, we get

$$|\Psi_T(u)| \leq d_{w,(q,r)}^{m+}(T) \|(\lambda_i)\|_{\ell_{p^*}^n} \|(x_i^1 \otimes \cdots \otimes x_i^m)\|_{\ell_{q,|w|}^n(\widehat{E}_{|\pi|})} \|(y_i^*)\|_{\ell_{r,|w|}^n(F^*)}.$$

Hence, as u is arbitrary, Ψ_T is $\mu_{(q,r)}^{m+}$ -continuous on $E_1 \otimes \cdots \otimes E_m \otimes F^*$ and extends continuously to the completed tensor product $E_1 \widehat{\otimes}_{\mu_{(q,r)}^{m+}} \cdots \widehat{\otimes}_{\mu_{(q,r)}^{m+}} E_m \widehat{\otimes}_{\mu_{(q,r)}^{m+}} F^*$ with

$$\|\Psi_T\| \leq d_{w,(q,r)}^{m+}(T).$$

Conversely, let $\Psi \in (E_1 \widehat{\otimes}_{\mu_{(q,r)}^{m+}} \cdots \widehat{\otimes}_{\mu_{(q,r)}^{m+}} E_m \widehat{\otimes}_{\mu_{(q,r)}^{m+}} F^*)^*$. We consider the mapping $B(\Psi)$ defined by

$$B(\Psi)(x^1, \dots, x^m)(y^*) = \Psi(x^1 \otimes \cdots \otimes x^m \otimes y^*).$$

It is clear that $B(\Psi) \in \mathcal{L}(E_1, \dots, E_m; F)$. Let $(x_i^j)_{i=1}^n \subset E_j$, $(j = 1, \dots, m)$ and $y_1^*, \dots, y_n^* \in Y^*$. Then

$$\begin{aligned} & \left| \sum_{i=1}^n \langle B(\Psi)(x_i^1, \dots, x_i^m), y_i^* \rangle \right| \\ &= \left| \left\langle \Psi \left(\sum_{i=1}^n x_i^1 \otimes \cdots \otimes x_i^m \otimes y_i^* \right) \right\rangle \right| \\ &\leq \|\Psi\| \mu_{(q,r)}^{m+} \left(\sum_{i=1}^n x_i^1 \otimes \cdots \otimes x_i^m \otimes y_i^* \right). \end{aligned}$$

Therefore,

$$\begin{aligned} & \left(\sum_{i=1}^n |\langle B(\Psi)(x_i^1, \dots, x_i^m), y_i^* \rangle|^p \right)^{\frac{1}{p}} \\ &= \sup_{\|(\lambda_i)\|_{\ell_{p^*}^n} \leq 1} \left(\sum_{i=1}^n \lambda_i \langle B(\Psi)(x_i^1, \dots, x_i^m), y_i^* \rangle \right) \\ &= \sup_{\|(\lambda_i)\|_{\ell_{p^*}^n} \leq 1} \left| \Psi \left(\sum_{i=1}^n \lambda_i x_i^1 \otimes \cdots \otimes x_i^m \otimes y_i^* \right) \right| \\ &\leq \sup_{\|(\lambda_i)\|_{\ell_{p^*}^n} \leq 1} \|\Psi\| \|(\lambda_i)\|_{\ell_{p^*}^n} \|(x_i^1 \otimes \cdots \otimes x_i^m)\|_{\ell_{q,|w|}^n(\widehat{E}_{|\pi|})} \|(y_i^*)\|_{\ell_{r,|w|}^n(F^*)} \\ &\leq \|\Psi\| \|(x_i^1 \otimes \cdots \otimes x_i^m)\|_{\ell_{q,|w|}^n(\widehat{E}_{|\pi|})} \|(y_i^*)\|_{\ell_{r,|w|}^n(F^*)}. \end{aligned}$$

Hence, $B(\Psi)$ is positive weakly (q, r) -dominated and

$$d_{w,(q,r)}^{m+}(B(\Psi)) \leq \|\Psi\|.$$

□

3.2. Polynomial case. The polynomial case differs from the multilinear case in essential details; it does not directly follow from the multilinear setting, and the proof steps must be revisited to handle this situation. Let E and F be Banach lattices. Let $1 \leq p, q, r \leq \infty$ such that $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. Consider $u \in \left(\widehat{\otimes}_{s,|\pi|}^m E\right) \otimes F$ of the form

$$u = \sum_{i=1}^n \lambda_i x_i \otimes \overset{(m)}{\cdots} \otimes x_i \otimes y_i, \quad (3.3)$$

where $\lambda_i \in \mathbb{R}, x_i \in E$ and $y_i \in F$ ($1 \leq i \leq n$). This representation of u can be considered a general form, as any other representation can be rewritten in this way. Define

$$\lambda_{(q,r)}^{m+}(u) = \inf \left\{ \|\lambda_i\|_{\ell_{p^*}^n} \sup_{\phi \in B_{\mathcal{P}^r(mE)}^+} \|(\phi(|x_i|))\|_{\ell_q^n} \|(y_i)\|_{\ell_{r,|w|}^n(F)} \right\},$$

where the infimum is taken over all general representations of u of the form (3.3).

Proposition 3.5. *For every u of the form (3.3), we have*

$$\lambda_{(q,r)}^{m+}(u) = \mu_{(q,r)}^{1+}(u).$$

Proof. Let u be of the form (3.3). We have

$$\begin{aligned} & \mu_{(q,r)}^{1+}(u) \\ &= \inf \left\{ \|\lambda_i\|_{\ell_{p^*}^n} \left\| (x_i \otimes \overset{(m)}{\cdots} \otimes x_i) \right\|_{\ell_{q,|w|}^n(\widehat{\otimes}_{s,|\pi|}^m E)} \|(y_i)\|_{\ell_{r,|w|}^n(F)} \right\} \\ &= \inf \left\{ \|\lambda_i\|_{\ell_{p^*}^n} \sup_{\phi \in B_{(\widehat{\otimes}_{s,|\pi|}^m E)^*}^+} \left(\sum_{i=1}^n \phi \left(\left| x_i \otimes \overset{(m)}{\cdots} \otimes x_i \right| \right)^q \right)^{\frac{1}{q}} \|(y_i)\|_{\ell_{r,|w|}^n(F)} \right\} \\ &= \inf \left\{ \|\lambda_i\|_{\ell_{p^*}^n} \sup_{\phi \in B_{(\widehat{\otimes}_{s,|\pi|}^m E)^*}^+} \left(\sum_{i=1}^n \phi \left(|x_i| \otimes \overset{(m)}{\cdots} \otimes |x_i| \right)^q \right)^{\frac{1}{q}} \|(y_i)\|_{\ell_{r,|w|}^n(F)} \right\}. \end{aligned}$$

Since $\mathcal{P}^r(mE) = (\widehat{\otimes}_{s,|\pi|}^m E)^*$, we get

$$\mu_{(q,r)}^{1+}(u) = \inf \left\{ \|\lambda_i\|_{\ell_{p^*}^n} \sup_{\phi \in B_{\mathcal{P}^r(mE)}^+} \|(\phi(|x_i|))\|_{\ell_q^n} \|(y_i)\|_{\ell_{r,|w|}^n(F)} \right\} = \lambda_{(q,r)}^{m+}(u).$$

□

From the above discussion on the tensor norm $\mu_{(q,r)}^{m+}(u)$, and the previous proposition, we obtain the following result.

Corollary 3.6. *Let $1 \leq p, q, r \leq \infty$ such that $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$ and $m \in \mathbb{N}^*$. Then $\lambda_{(q,r)}^{m+}$ is a tensor norm on $(\widehat{\otimes}_{s,|\pi|}^m E) \otimes F$ and we have*

$$\varepsilon \leq \lambda_{(q,r)}^{m+} \leq \pi,$$

where ε and π denote the injective and projective norms on $(\widehat{\otimes}_{s,|\pi|}^m E) \otimes F$, respectively.

We denote by $(\widehat{\otimes}_{s,|\pi|}^m E) \widehat{\otimes}_{\lambda_{(q,r)}^{m+}} F$ the completed of $(\otimes_{s,|\pi|}^m E) \otimes F$ for the norm $\lambda_{(q,r)}^{m+}$.

Now, the main result of this section is the following identification.

Proposition 3.7. *Let $1 \leq p, q, r \leq \infty$ such that $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. We have the following isometric identification*

$$\mathcal{P}_{w,(q,r)}^{m+}({}^m E; F) = ((\widehat{\otimes}_{s,|\pi|}^m E) \widehat{\otimes}_{\lambda_{(q,r)}^{m+}} F^*)^*.$$

Proof. Let $P \in \mathcal{P}_{w,(q,r)}^{m+}({}^m E; F)$. We define a linear functional on $(\otimes_{s,|\pi|}^m E) \otimes F^*$ by

$$\Psi_P(u) = \sum_{i=1}^n \lambda_i \langle P(x_i), y_i^* \rangle,$$

where $u = \sum_{i=1}^n \lambda_i x_i \otimes \cdots \otimes x_i \otimes y_i^*$. Then, by the Hölder's inequality, we have

$$|\Psi_P(u)| = \left| \sum_{i=1}^n \lambda_i \langle P(x_i), y_i^* \rangle \right| \leq \|(\lambda_i)\|_{\ell_{p^*}^n} \left(\sum_{i=1}^n \langle P(x_i), y_i^* \rangle^p \right)^{\frac{1}{p}}.$$

Since P is positive weakly (q, r) -dominated, we get

$$|\Psi_P(u)| \leq d_{w,(q,r)}^{m+}(P) \|(\lambda_i)\|_{\ell_{p^*}^n} \sup_{\phi \in B_{\mathcal{P}^r({}^m E)}^+} \|(\phi(x_i))\|_{\ell_q^n} \| (y_i^*) \|_{\ell_{r,|w|}^n(F^*)}.$$

Hence, as u is arbitrary, Ψ_P is $\lambda_{(q,r)}^{m+}$ -continuous on $(\otimes_{s,|\pi|}^m E) \otimes F^*$ and extends continuously to the completed tensor product $(\widehat{\otimes}_{s,|\pi|}^m E) \widehat{\otimes}_{\lambda_{(q,r)}^{m+}} F^*$ with

$$\|\Psi_P\| \leq d_{(q,r)}^{m+}(P).$$

Conversely, let $\Psi \in ((\widehat{\otimes}_{s,|\pi|}^m E) \widehat{\otimes}_{\lambda_{(q,r)}^{m+}} F^*)^*$. We consider the mapping $B(\Psi)$ defined by

$$B(\Psi)(x)(y^*) = \Psi \left(x \otimes \cdots \otimes x \otimes y^* \right).$$

It is clear that $B(\Psi) \in \mathcal{P}({}^m E; F)$. Let $x_1, \dots, x_n \in E$, and $y_1^*, \dots, y_n^* \in F^*$, we have

$$\begin{aligned} \left| \sum_{i=1}^n \langle B(\Psi)(x_i), y_i^* \rangle \right| &= \left| \left\langle \Psi \left(\sum_{i=1}^n x_i \otimes \cdots \otimes x_i \otimes y_i^* \right) \right\rangle \right| \\ &\leq \|\Psi\| \lambda_{(q,r)}^{m+} \left(\sum_{i=1}^n x_i \otimes \cdots \otimes x_i \otimes y_i^* \right). \end{aligned}$$

Therefore,

$$\begin{aligned}
& \left(\sum_{i=1}^n |\langle B(\Psi)(x_i), y_i^* \rangle|^p \right)^{\frac{1}{p}} \\
&= \sup_{\|(\lambda_i)\|_{\ell_{p^*}^n} \leq 1} \left| \sum_{i=1}^n \lambda_i \langle B(\Psi)(x_i), y_i^* \rangle \right| \\
&= \sup_{\|(\lambda_i)\|_{\ell_{p^*}^n} \leq 1} \left| \Psi \left(\sum_{i=1}^n \lambda_i x_i \otimes \dots \otimes x_i \otimes y_i^* \right) \right| \\
&\leq \sup_{\|(\lambda_i)\|_{\ell_{p^*}^n} \leq 1} \|\Psi\|_{\ell_{p^*}^n} \sup_{\phi \in B_{\mathcal{P}r(m_E)}^+} \|(\phi(x_i))\|_{\ell_q^n} \| (y_i^*) \|_{\ell_{r,|w|}^n(F^*)} \\
&\leq \|\Psi\| \sup_{\phi \in B_{\mathcal{P}r(m_E)}^+} \|(\phi(x_i))\|_{\ell_q^n} \| (y_i^*) \|_{\ell_{r,|w|}^n(F^*)}.
\end{aligned}$$

Hence, $B(\Psi)$ is positive weakly (q, r) -dominated and

$$d_{w,(q,r)}^{m+}(B(\Psi)) \leq \|\Psi\|.$$

□

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