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$\mathscr{F}$-MEAN PROXIMALITY FOR AMENABLE SEMIGROUPS ACTIONS

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Abstract. In this paper, we introduce the notions of proximality and regionally proximal relations via a Furstenberg family with respect to the semigroup S. We also explore the relationship between $\mathscr{F}$-equicontinuity and $\tau\mathscr{F}$-equicontinuity. Furthermore, we show that if $(X, \{T_s\}_{s\in S})$ is $\mathscr{F}$-mean equicontinuous, then $P_{\mathscr{F}}(X, \{T_s\}_{s\in S})$ and $Q_{\mathscr{F}}(X, \{T_s\}_{s\in S})$ coincide for amenable semigroup actions. We prove that if a countable, discrete, infinite, and commutative semigroup action is $\mathscr{F}$-mean sensitive, and there exists an $\mathscr{F}$-mean proximal pair consisting of a transitive point and a periodic point, then the system is $\mathscr{F}$-mean Li–Yorke chaotic.

1. Introduction and preliminaries

1.1. **Dynamical system.** A dynamical system is a pair $(X, \{T_s\}_{s\in S})$, where S is a countable discrete semigroup, (X, d) is a compact metric space, and $T_s : X \to X$ is a continuous map for every $s \in S$, such that for all $s, t \in S$, we have $T_s \circ T_t = T_{st}$. If $(X, \{T_s\}_{s\in S})$ is a dynamical system, then the semigroup S is said to act on X, and we refer to this as an action of S on X (or an S-action on X). We say that S acts on X via $\{T_s\}_{s\in S}$. A $\mathbb{Z}^+$-action topological dynamical system (X, T) means that X is a compact metric space (referred to as the phase space) with a metric d, and T is a continuous function from X into X. A point $x \in X$ is called transitive if its orbit, $\mathrm{orb}(x, S) = \{T_s(x) : s \in S\}$ is dense in X.

For each pair of nonempty open subsets $U, V \subset X$, we define $N(U, V) = \{s \in S : U \cap T_s^{-1}(V) \neq \emptyset\}$, and $J(U, V) = \{s \in S : T_s(U) \subset V\}$.

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A dynamical system $(X, \{T_s\}_{s\in S})$ is said to be transitive if $N(U, V) \neq \emptyset$ for every pair of nonempty open sets $U, V \subset X$.

The system $(X, \{T_s\}_{s\in S})$ is called minimal if every point $x \in X$ is transitive.

We say that $(X, \{T_s\}_{s\in S})$ is equicontinuous if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$d(x, y) < \delta \quad \Rightarrow \quad d(T_s x, T_s y) < \varepsilon, \quad \text{for all } s \in S.$$

The concept of sensitivity for $\mathbb{Z}_+$-actions was first introduced in [2] and is commonly used in the study of dynamical systems. A dynamical system $(X, \{T_s\}_{s\in S})$ is called sensitive if there exists $\delta > 0$ such that for every $x \in X$ and every neighborhood U of x, there exist $y \in U$ and $s \in S$ such that $d(T_s x, T_s y) > \delta$. (see [3, 13]). A pair $(x, y) \in X \times X$ is called proximal if $\inf_{s\in S} d(T_s x, T_s y) = 0$. We denote the set of all proximal pairs by $P(X, \{T_s\}_{s\in S})$. A dynamical system $(X, \{T_s\}_{s\in S})$ is called proximal if every pair of points in X is proximal; that is, $P(X, \{T_s\}_{s\in S}) = X \times X$. If $\inf_{s\in S} d(T_s x, T_s y) > 0$, then the pair (x, y) is said to be distal. A dynamical system $(X, \{T_s\}_{s\in S})$ is distal if every pair of distinct points in X is distal. A pair $(x, y) \in X \times X$ is called regionally proximal if for every $\varepsilon > 0$, there exist $x', y' \in X$ and $s \in S$ such that

$$d(x, x') < \varepsilon, \quad d(y, y') < \varepsilon, \quad \text{and} \quad d(T_s x', T_s y') < \varepsilon.$$

We denote the set of all regionally proximal pairs by $Q(X, \{T_s\}_{s\in S})$, or simply Q. Clearly, $Q(X, \{T_s\}_{s\in S}) \supseteq P(X, \{T_s\}_{s\in S})$ (see [3, 12]).

1.2. **Basic concepts of amenable semigroups.** Let S be a discrete semigroup. A left invariant mean for S on $\ell^\infty(S)$ is a positive linear functional $m : \ell^\infty(S) \to \mathbb{R}$ satisfying the following conditions:

(a) $m(1_S) = 1$.
(b) $m(f \circ \lambda_s) = m(f)$, for all $f \in \ell^\infty(S)$ and $s \in S$, where $\lambda_s(x) = sx$ for $x \in S$.

A right invariant mean is defined analogously. The semigroup S is called left amenable if a left invariant mean exists, and amenable if both left and right invariant means exist. In this paper, we consider discrete countable amenable cancellative semigroups. It is well known that such semigroups admit Følner sequences [4]. A (left) Følner sequence is a family $\{F_n\}_{n=1}^\infty$ of finite nonempty subsets of a countable semigroup S such that

$$\lim_{n\to\infty} \frac{|sF_n \triangle F_n|}{|F_n|} = 0,$$

for all $s \in S$, where $sF_n = \{sf : f \in F_n\}$, $\triangle$ denotes the symmetric difference, and $|\cdot|$ denotes cardinality. A right Følner sequence is defined analogously. A sequence in S is called a Følner sequence if it is both a left and right Følner sequence. For countable left cancellative semigroups, the existence of a left Følner sequence is equivalent to left amenability [4]. Throughout this paper, we consider only left Følner sequences and will routinely omit the adjective “left”.

Let $A \subseteq S$ and let $\mathcal{F} = \{F_n\}_{n=1}^{\infty}$ be a Følner sequence. The upper density of A with respect to $\mathcal{F}$ is defined by

$$\overline{D}_{\mathcal{F}}(A) = \limsup_{n\to\infty} \frac{|A \cap F_n|}{|F_n|}.$$

Similarly, the lower density of A is defined by

$$\underline{D}_{\mathcal{F}}(A) = \liminf_{n\to\infty} \frac{|A \cap F_n|}{|F_n|}.$$

Suppose that $\mathscr{B}$ is a collection of infinite subsets of S. For each $a \in [0,1)$, define

$$\overline{D}(a^+) = \{A \in \mathscr{B} : \ \overline{D}(A) > a\}.$$

For $A \subseteq S$, the Banach density of A is defined by

$$BD(A) = \sup\left\{\overline{D}_{\mathcal{F}}(A) : \ \mathcal{F} \text{ is a Følner sequence on } S\right\}.$$

Again, assuming $\mathscr{B}$ is a collection of infinite subsets of S, and for $a \in [0,1)$, we set $BD(a^+) = \{A \in \mathscr{B} : \ BD(A) > a\}$. (For more details, see [6, 7, 10]).

The authors in [11] first introduced the important concept of mean equicontinuity and mean sensitivity for $\mathbb{Z}^+$-actions. Let $(X, \{T_s\}_{s\in S})$ be a topological dynamical system and $\mathcal{F} = \{F_n\}_{n=1}^{\infty}$ a left Følner sequence in the semigroup S. The system $(X, \{T_s\}_{s\in S})$ is said to be mean equicontinuous with respect to $\mathcal{F}$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $x, y \in X$ satisfy $d(x,y) < \delta$, we have

$$\limsup_{n\to\infty} \frac{1}{|F_n|} \sum_{s\in F_n} d(T_s x, T_s y) < \varepsilon.$$

A point $x \in X$ is called a mean equicontinuous point (with respect to $\mathcal{F}$) if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $y \in X$ with $d(x,y) < \delta$, we have

$$\limsup_{n\to\infty} \frac{1}{|F_n|} \sum_{s\in F_n} d(T_s x, T_s y) < \varepsilon.$$

A transitive system is said to be almost mean equicontinuous if it has at least one mean equicontinuous point (see [5, 10]).

Yan and Zeng [18] introduced the notions of mean sensitivity, mean proximality, and mean Li–Yorke chaos for actions of amenable groups. The system $(X, \{T_s\}_{s\in S})$ is said to be mean sensitive (with respect to $\mathcal{F}$) if there exists $\delta > 0$ such that for every $x \in X$ and every neighborhood U of x, there exists $y \in U$ satisfying

$$\limsup_{n\to\infty} \frac{1}{|F_n|} \sum_{s\in F_n} d(T_s x, T_s y) > \delta.$$

A pair $(x,y) \in X \times X$ is called a mean proximal pair for $\mathcal{F}$ if

$$\liminf_{n\to\infty} \frac{1}{|F_n|} \sum_{s\in F_n} d(T_s x, T_s y) = 0.$$

Let MP denote the set of all mean proximal pairs in $(X, \{T_s\}_{s\in S})$. The system is called mean proximal if every pair $(x, y) \in X \times X$ is mean proximal. Let $\eta > 0$. A pair $(x, y) \in X \times X$ is called a mean Li–Yorke pair with modulus η for $\mathcal{F}$ if

$$\liminf_{n\to\infty} \frac{1}{|F_n|} \sum_{s\in F_n} d(T_s x, T_s y) = 0, \quad \text{and} \quad \limsup_{n\to\infty} \frac{1}{|F_n|} \sum_{s\in F_n} d(T_s x, T_s y) \geq \eta.$$

Let $MLY_\eta(X, \{T_s\}_{s\in S})$ denote the set of all mean Li–Yorke pairs with modulus η. A subset $K \subseteq X$ is called a mean Li–Yorke set with modulus $\eta > 0$ if every pair of distinct points $(x, y) \in K \times K$ is a mean Li–Yorke pair, that is,

$$K \times K \setminus \Delta_X \subseteq MLY_\eta(X, \{T_s\}_{s\in S}),$$

where $\Delta_X = \{(x, x) \in X \times X : x \in X\}$. The system $(X, \{T_s\}_{s\in S})$ is said to be mean Li–Yorke chaotic if it contains an uncountable mean Li–Yorke set (see [18] for further information).

1.3. **Furstenberg family.** In this section, we discuss some definitions of dynamical systems via Furstenberg families. Recently, researchers have investigated fundamental concepts in topological dynamical systems —such as transitivity, mixing, equicontinuity, sensitivity, and proximality— via Furstenberg families. Ju et al. [8] introduced a new type of $\mathscr{F}$-equicontinuity, which is characterized as the negation of the existence of $\kappa\mathscr{F}$-sensitive pairs almost everywhere. They also established an analogue of the Auslander–Yorke dichotomy theorem, showing that a transitive system either possesses $\mathscr{F}$-sensitive pairs almost everywhere or is almost $\kappa\mathscr{F}$-equicontinuous of the new type.

According to [16, Theorem 6], there exist two distinct definitions of $\mathscr{F}$-sensitivity for a system (X, T), which are generally not equivalent. In [9], these two notions were referred to as $\mathscr{F}$-orbit sensitivity and $\mathscr{F}$-diameter sensitivity, respectively. Nonetheless, the existence of $\mathscr{F}$-sensitive pairs almost everywhere is equivalent under both definitions. Song [14] introduced the concept of the proximal relation via Furstenberg families and studied $\mathscr{F}$-proximity, $\mathscr{F}$-ductility, $\mathscr{F}$-almost ductility, and $\mathscr{F}$-semi-distality for $\mathbb{Z}^+$-actions. Later, Ri, Ju, and Kim [15] investigated $\mathscr{F}$-equicontinuity using a generalized regionally proximal relation defined via Furstenberg families. They proved that a system (X, T) is $\mathscr{F}$-equicontinuous if and only if it has no nontrivial $\kappa\mathscr{F}$-regionally proximal pairs.

Throughout this paper, S denotes a countable discrete semigroup, and $P(S)$ is the power set of S. A collection $\mathscr{F} \subset P(S)$ is a Furstenberg family if $F_1 \subset F_2$ and $F_1 \in \mathscr{F}$ imply $F_2 \in \mathscr{F}$. Given a Furstenberg family $\mathscr{F}$, define its dual family $\kappa\mathscr{F}$ as follows:

$$\kappa\mathscr{F} = \{F \in P(S) : S \setminus F \notin \mathscr{F}\} = \{F \in P(S) : \forall F' \in \mathscr{F},\ F \cap F' \neq \emptyset\}.$$

It is easy to see that $\mathscr{F}$ is a Furstenberg family if and only if $\kappa\mathscr{F}$ is so, and $\kappa(\kappa\mathscr{F}) = \mathscr{F}$. Given two Furstenberg families $\mathscr{F}_1$ and $\mathscr{F}_2$, define

$$\mathscr{F}_1 \cdot \mathscr{F}_2 = \{F_1 \cap F_2 : F_1 \in \mathscr{F}_1,\ F_2 \in \mathscr{F}_2\}.$$

A Furstenberg family $\mathscr{F}$ is called a filter if it satisfies $\mathscr{F} \cdot \mathscr{F} \subset \mathscr{F}$, and it is said to have the Ramsey property if $F_1 \cup F_2 \in \mathscr{F}$ implies $F_1 \in \mathscr{F}$ or $F_2 \in \mathscr{F}$. It is easy to check that a Furstenberg family $\mathscr{F}$ has the Ramsey property if and only if $\kappa\mathscr{F}$ is a filter (see [1] for further information).

For $s \in S$ and $F \subseteq S$, define $s^{-1}F = \{t \in S : st \in F\}$. A Furstenberg family $\mathscr{F}$ is said to be translation invariant if for every $F \in \mathscr{F}$ and $s \in S$, we have $s^{-1}F \in \mathscr{F}$. For a Furstenberg family $\mathscr{F}$, define

$$\tau\mathscr{F} = \left\{ F \in P(S) : \bigcap_{j=1}^{n} (Fs_j^{-1}) \in \mathscr{F} \text{ for every finite set } \{s_1, \ldots, s_n\} \subset S \right\}.$$

The family $\mathscr{F}$ is thick if $\tau\mathscr{F} = \mathscr{F}$. It is easy to see that $\tau\mathscr{F}$ is the largest thick family contained in $\mathscr{F}$. Clearly, if $\mathscr{F}$ is a filter, then $\tau\mathscr{F}$ is also a filter (see [14]). Let $(X, \{T_s\}_{s\in S})$ be a topological dynamical system. In this paper, we denote by $(X \times X, \{T_s \times T_s\}_{s\in S})$ the induced topological dynamical system on $X \times X$, where the action is defined by

$$(T_s \times T_s)(x, y) = (T_s x, T_s y), \quad s \in S.$$

For simplicity, we write this system as $(X \times X, S \times S)$. For a given point $x \in X$, a positive number $\varepsilon > 0$, and a subset $A \subset X$, define

$$N_S(x, A) = \{s \in S : T_s x \in A\}.$$

The ε-neighborhood of the diagonal in $X \times X$ is defined by

$$\Delta_\varepsilon = \{(x, y) \in X \times X : d(x, y) < \varepsilon\}.$$

Since the metric $d : X \times X \to [0, \infty)$ is continuous, we have $\Delta_\varepsilon = d^{-1}([0, \varepsilon))$, which is an open subset of $X \times X$. Moreover, $\overline{\Delta}_\varepsilon = \{(x, y) \in X \times X : d(x, y) \leq \varepsilon\}$. For any pair $(x, y) \in X \times X$ and any subset $A \subset X \times X$, define

$$N_{S\times S}((x, y), A) = \{s \in S : (T_s x, T_s y) \in A\}.$$

Therefore,

$$\begin{aligned} N_{S\times S}((x, y), \Delta_\varepsilon) &= \{s \in S : (T_s x, T_s y) \in \Delta_\varepsilon\} \\ &= \{s \in S : d(T_s x, T_s y) < \varepsilon\}. \end{aligned}$$

Ju et al. [8] introduced the notion of $\mathscr{F}$-equicontinuity for topological dynamical systems (X, T); we generalize this definition to amenable semigroup actions. The definition is as follows.

Definition 1.1. Let $(X, \{T_s\}_{s\in S})$ be a topological dynamical system, and let $\mathscr{F}$ be a Furstenberg family. For $\varepsilon > 0$, define

$$\begin{aligned} &Eq_\varepsilon^{\mathscr{F}} = \{x \in X : \exists \delta > 0 \text{ such that } \forall y, z \in B(x, \delta),\ N_{S\times S}((y, z), \Delta_\varepsilon) \in \mathscr{F}\}, \\ &Eq_{\mathscr{F}} = \bigcap_{\varepsilon > 0} Eq_\varepsilon^{\mathscr{F}}. \end{aligned}$$

A point $x \in X$ is called an $\mathscr{F}$-equicontinuous point if $x \in Eq_{\mathscr{F}}$. That is, for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $y, z \in B(x, \delta)$ we have

$$N_{S\times S}((y, z), \Delta_\varepsilon) \in \mathscr{F}.$$

The system $(X, \{T_s\}_{s\in S})$ is said to be $\mathscr{F}$-equicontinuous if $Eq_{\mathscr{F}} = X$. A transitive system is said to be almost $\mathscr{F}$-equicontinuous if $Eq_{\mathscr{F}} \neq \emptyset$.

In [16, 17], the authors introduced the concept of $\mathscr{F}$-sensitivity under conditions (a) and (b) for $\mathbb{Z}^+$-actions. We generalize this definition to actions of amenable semigroups as follows:

Definition 1.2. Let $(X, \{T_s\}_{s\in S})$ be a topological dynamical system and let $\mathscr{F}$ be a Furstenberg family.

(a) A pair $(x, y) \in X \times X$ is called $(\mathscr{F}, \varepsilon)$-sensitive if

$$N_{S\times S}((x, y), X \times X \setminus \overline{\Delta}_\varepsilon) \in \mathscr{F}.$$

Denote the set of all $(\mathscr{F}, \varepsilon)$-sensitive pairs by $(\mathscr{F}, \varepsilon)$.

(b) The system $(X, \{T_s\}_{s\in S})$ has $\mathscr{F}$-sensitive pairs almost everywhere if there exists $\varepsilon > 0$ such that for every nonempty open subset $U \subset X$ there exist $x, y \in U$ such that (x, y) is $(\mathscr{F}, \varepsilon)$-sensitive. Lian, Huang, and Li [12] showed that, when mean equicontinuity is defined with respect to a Følner sequence, the proximal and regionally proximal relations coincide. This result reveals a structural phenomenon associated with mean equicontinuity. Motivated by this observation, we investigate the corresponding phenomenon in the framework of Furstenberg families and amenable semigroup actions, and we show that $\mathscr{F}$-mean equicontinuity also leads to the equality $P_{\mathscr{F}} = Q_{\mathscr{F}}$ in this setting. Yan and Zeng [18] proved that, for amenable group actions defined with respect to a Følner sequence, mean sensitivity together with the existence of a mean proximal pair formed by a transitive point and a periodic point implies mean Li–Yorke chaos. In the present paper, we establish an analogue of this phenomenon within the context of Furstenberg families for amenable semigroup actions.

2. Main results

In [14, 15], the authors proved some properties of proximality and regional proximality for $\mathbb{Z}^+$-actions. In this section, we prove these properties for amenable semigroup actions via Furstenberg families. First, we generalize the notions of $\mathscr{F}$-proximality, $\mathscr{F}$-Li–Yorke pairs, and $\mathscr{F}$-regional proximality to the action of a countable amenable semigroup S. In addition, we investigate the connection between $\mathscr{F}$-equicontinuity and $\tau\mathscr{F}$-equicontinuity. In particular, we establish that if $(X, \{T_s\}_{s\in S})$ is $\mathscr{F}$-mean equicontinuous, then

$$P_{\mathscr{F}}(X, \{T_s\}_{s\in S}) = Q_{\mathscr{F}}(X, \{T_s\}_{s\in S}).$$

We also demonstrate that if a countable, discrete, and infinite commutative semigroup action is $\mathscr{F}$-mean sensitive and contains an $\mathscr{F}$-mean proximal pair formed by a transitive point and a periodic point, then the system exhibits $\mathscr{F}$-mean Li–Yorke chaos.

Definition 2.1. Let $(X, \{T_s\}_{s\in S})$ be a topological dynamical system and let $\mathscr{F}$ be a Furstenberg family.

(a) A pair $(x, y) \in X \times X$ is called $\mathscr{F}$-proximal if (x, y) $\mathscr{F}$-adheres; that is, for every $\varepsilon > 0$,

$$N_{S\times S}((x, y), \Delta_\varepsilon) \in \mathscr{F}.$$

We denote the set of all $\mathscr{F}$-proximal pairs by $P_{\mathscr{F}}(X, \{T_s\}_{s\in S})$, or simply $P_{\mathscr{F}}$.

(b) A pair $(x, y) \in X \times X$ is said to be $\mathscr{F}$-regionally proximal if for every $\varepsilon > 0$, there exists a set $F \in \mathscr{F}$ such that for every $s \in F$, there exist points $x', y' \in X$ satisfying

$$d(x, x') < \varepsilon, \qquad d(y, y') < \varepsilon, \qquad d(T_s x', T_s y') < \varepsilon.$$

The set of all such pairs is denoted by $Q_{\mathscr{F}}(X, \{T_s\}_{s\in S})$, or simply $Q_{\mathscr{F}}$.

(c) A pair $(x, y) \in X \times X$ is called an $\mathscr{F}$-Li–Yorke pair if it is $\kappa\mathscr{F}$-proximal but not $\mathscr{F}$-proximal; that is, $(x, y) \in P_{\kappa\mathscr{F}} \setminus P_{\mathscr{F}}$. The collection of all such pairs is denoted by $LY_{\mathscr{F}}$.

It is clear that $P_{\mathscr{F}}(X, \{T_s\}_{s\in S}) \subset Q_{\mathscr{F}}(X, \{T_s\}_{s\in S})$. A topological dynamical system $(X, \{T_s\}_{s\in S})$ is said to be $\mathscr{F}$-distal if $P_{\kappa\mathscr{F}} = \Delta_X$. In Proposition 2.2, we prove that $P_{\mathscr{F}}\left(X, \{T_s\}_{s\in S}\right) = P_{\tau\mathscr{F}}\left(X, \{T_s\}_{s\in S}\right)$, and in Proposition 2.3, we show that an analogous statement holds for $\mathscr{F}$-regionally proximal pairs; that is,

$$Q_{\mathscr{F}}\left(X, \{T_s\}_{s\in S}\right) = Q_{\tau\mathscr{F}}\left(X, \{T_s\}_{s\in S}\right).$$

Proposition 2.2. *Let $(X, \{T_s\}_{s\in S})$ be a topological dynamical system and let $\mathscr{F}$ be a Furstenberg family. Then $P_{\mathscr{F}} = P_{\tau\mathscr{F}}$.*

Proof. Since $\tau\mathscr{F} \subset \mathscr{F}$, we immediately have $P_{\tau\mathscr{F}} \subset P_{\mathscr{F}}$. To prove the reverse inclusion, let $(x, y) \in P_{\mathscr{F}}$ and fix $\varepsilon > 0$. Define

$$F = N_{S\times S}((x, y), \Delta_\varepsilon).$$

Since X is compact and each T_s is continuous, there exists $0 < \delta < \varepsilon$ and a finite set $\{s_1, \dots, s_m\} \subset S$ such that

$$d(z, w) < \delta \;\Rightarrow\; d(T_{s_j} z, T_{s_j} w) < \varepsilon \quad \text{for all } j = 1, \dots, m.$$

Let

$$F' = N_{S\times S}((x, y), \Delta_\delta).$$

Since $(x, y) \in P_{\mathscr{F}}$, we have $F' \in \mathscr{F}$. Moreover,

$$F' \subset \{t \in S : \{ts_1, \dots, ts_m\} \subset F\} \subset \bigcap_{j=1}^{m} F s_j^{-1}.$$

Thus $\bigcap_{j=1}^m F s_j^{-1} \in \mathscr{F}$, meaning $F \in \tau\mathscr{F}$. Therefore $(x, y) \in P_{\tau\mathscr{F}}$. □

Proposition 2.3. *Let $(X, \{T_s\}_{s\in S})$ be a dynamical system and let $\mathscr{F}$ be a Furstenberg family. Then a pair $(x, y) \in X \times X$ is $\mathscr{F}$-regionally proximal if and only if for any $\varepsilon > 0$ and any neighborhoods U of x and V of y,*

$$N_{S\times S}(U \times V, \Delta_\varepsilon) \in \mathscr{F}.$$

Proof. Suppose that (x, y) is $\mathscr{F}$-regionally proximal. Fix $\varepsilon > 0$ and neighborhoods U of x and V of y. Choose $0 < \varepsilon' < \varepsilon$ such that $U_d(x, \varepsilon') \subset U$ and $U_d(y, \varepsilon') \subset V$. Since (x, y) is $\mathscr{F}$-regionally proximal, there exists $F \in \mathscr{F}$ such that for every $s \in F$ there exist $x', y' \in X$ with

$$d(x, x') < \varepsilon', \quad d(y, y') < \varepsilon', \quad d(T_s x', T_s y') < \varepsilon'.$$

Thus

$$F \subset N_{S\times S}(U_d(x, \varepsilon') \times U_d(y, \varepsilon'), \Delta_{\varepsilon'}) \subset N_{S\times S}(U \times V, \Delta_\varepsilon),$$

and therefore $N_{S\times S}(U \times V, \Delta_\varepsilon) \in \mathscr{F}$. Conversely, suppose that for any $\varepsilon > 0$ and any neighborhoods U of x and V of y, we have $N_{S\times S}(U \times V, \Delta_\varepsilon) \in \mathscr{F}$. Set $U = U_d(x, \varepsilon)$ and $V = U_d(y, \varepsilon)$. Then

$$F := N_{S\times S}(U \times V, \Delta_\varepsilon) \in \mathscr{F}.$$

For each $s \in F$ there exist $x' \in U$ and $y' \in V$ with $d(T_s x', T_s y') < \varepsilon$, showing that (x, y) is $\mathscr{F}$-regionally proximal. □

Proposition 2.4. *Let $(X, \{T_s\}_{s\in S})$ be a dynamical system and let $\mathscr{F}$ be a Furstenberg family. Then $Q_{\mathscr{F}} = Q_{\tau\mathscr{F}}$.*

Proof. Since $\tau\mathscr{F} \subset \mathscr{F}$, we have $Q_{\tau\mathscr{F}} \subset Q_{\mathscr{F}}$. Let $(x, y) \in Q_{\mathscr{F}}$. Fix $\varepsilon > 0$ and neighborhoods U of x and V of y. By Proposition 2.3,

$$F := N_{S\times S}(U \times V, \Delta_\varepsilon) \in \mathscr{F}.$$

Let $s_1, \ldots, s_n \in S$. By continuity, there exists $0 < \delta < \varepsilon$ such that $d(z, w) < \delta$ implies $d(T_{s_j} z, T_{s_j} w) < \varepsilon$ for all j.

Set

$$F' := N_{S\times S}(U \times V, \Delta_\delta).$$

Then $F' \in \mathscr{F}$ by Proposition 2.3. Moreover,

$$F' s_j^{-1} \subset F \quad \text{for each } j,$$

so

$$F' \subset \bigcap_{j=1}^{n} F s_j^{-1},$$

which is in $\mathscr{F}$. Hence $F \in \tau\mathscr{F}$, and by Proposition 2.3, $(x, y) \in Q_{\tau\mathscr{F}}$. □

The next lemma establishes the equivalence between $\mathscr{F}$-equicontinuity and $\tau\mathscr{F}$-equicontinuity.

Lemma 2.5. *Let $(X, \{T_s\}_{s\in S})$ be a dynamical system and let $\mathscr{F}$ be a Furstenberg family. Then $(X, \{T_s\}_{s\in S})$ is $\mathscr{F}$-equicontinuous if and only if it is $\tau\mathscr{F}$-equicontinuous.*

Proof. Clearly, $\tau\mathscr{F}$-equicontinuity implies $\mathscr{F}$-equicontinuity. Thus, it remains to prove the converse. Assume that $(X, \{T_s\}_{s\in S})$ is $\mathscr{F}$-equicontinuous. Then for every $x \in X$ and every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$J(U_d(x, \delta) \times U_d(x, \delta), \Delta_\varepsilon) \in \mathscr{F}.$$

Let

$$F := J(U_d(x, \delta) \times U_d(x, \delta), \Delta_\varepsilon) \in \mathscr{F}.$$

Now, fix any finite set $\{s_1, \ldots, s_n\} \subset S$. By continuity there exists $0 < \delta_1 < \delta$ such that for all $y, z \in B(x, \delta_1)$ and for all j,

$$d(T_{s_j} y, T_{s_j} z) < \delta.$$

Define

$$F' := J(U_d(x, \delta_1) \times U_d(x, \delta_1), \Delta_\varepsilon).$$

Then $F' \subset F$, so $F' \in \mathscr{F}$. Moreover, for each j,

$$F' s_j^{-1} \subset F,$$

and therefore,

$$F' \subset \bigcap_{j=1}^{n} F s_j^{-1} \in \mathscr{F}.$$

This implies that $F \in \tau\mathscr{F}$. Hence $(X, \{T_s\}_{s\in S})$ is $\tau\mathscr{F}$-equicontinuous. $\square$

Li, Tu, and Ye, [11] introduced the notions of mean equicontinuity and mean sensitivity for $\mathbb{Z}^+$-actions. Subsequently, Lian, Huang, and Li [12] and Yan and Zeng [18] extended these notions to amenable group actions. Inspired by [11, 18, 12], we introduce the concepts of $\mathscr{F}$-mean equicontinuity and $\mathscr{F}$-mean sensitivity for amenable semigroup actions. For this purpose, we assume that $\{F_n\}$ is a Følner sequence in the semigroup S, that $\mathscr{F}$ is a Furstenberg family, and that $F \in \mathscr{F}$. In the case where $F_n \cap F = \emptyset$, we adopt the convention

$$\sum_{s\in\emptyset} d(T_s x, T_s y) = 0.$$

Definition 2.6. Let $(X, (T_s)_{s\in S})$ be a topological dynamical system, let $\mathscr{F}$ be a Furstenberg family, and let $\mathcal{F} = \{F_n\}_{n=1}^{\infty}$ be a Følner sequence in S. For every $\varepsilon > 0$ and some $F \in \mathscr{F}$, we define

$$MEq_\varepsilon^F = \left\{ \begin{array}{c} x \in X : \exists\delta > 0 \text{ such that } \limsup\limits_{n\to\infty} \dfrac{1}{|F_n \cap F| + 1} \sum\limits_{s\in F_n\cap F} d\left(T_s(y), T_s(z)\right) \\ < \varepsilon \end{array} \right\},$$

for all $y, z \in B(x, \delta)$. We then set

$$MEq_\varepsilon^{\mathscr{F}} = \bigcup_{F\in\mathscr{F}} MEq_\varepsilon^F,$$
$$MEq_{\mathscr{F}} = \bigcap_{\varepsilon>0} MEq_\varepsilon^{\mathscr{F}}.$$

A point $x \in X$ is called an $\mathscr{F}$-mean equicontinuous point if $x \in MEq_{\mathscr{F}}$, that is, for every $\varepsilon > 0$ and every $F \in \mathscr{F}$, there exists $\delta > 0$ such that for all $y, z \in B(x, \delta)$,

$$\limsup_{n\to\infty} \frac{1}{|F_n \cap F| + 1} \sum_{s\in F_n\cap F} d\left(T_s(y), T_s(z)\right) < \varepsilon.$$

If $MEq_{\mathscr{F}} = X$, then the dynamical system $(X, (T_s)_{s\in S})$ is called $\mathscr{F}$-mean equicontinuous. If $MEq_\varepsilon^{\mathscr{F}}$ is an open and dense subset of X for each $\varepsilon > 0$, then the system $(X, (T_s)_{s\in S})$ is said to be almost $\mathscr{F}$-mean equicontinuous.

Definition 2.7. Let $(X, \{T_s\}_{s\in S})$ be a topological dynamical system, let $\mathscr{F}$ be a Furstenberg family, and let $\mathcal{F} = \{F_n\}_{n\in\mathbb{N}}$ be a left Følner sequence in S.

(a) A pair $(x,y) \in X \times X$ is called $(\mathscr{F}, \varepsilon)$-*mean sensitive* if for every $F \in \mathscr{F}$,

$$\limsup_{n\to\infty} \frac{1}{|F_n \cap F| + 1} \sum_{s\in F_n\cap F} d(T_s x, T_s y) > \varepsilon.$$

(b) The system $(X, \{T_s\}_{s\in S})$ is said to have $\mathscr{F}$-*mean sensitive pairs almost everywhere* if there exists $\varepsilon > 0$ such that for every nonempty open subset $U \subset X$, there exist $x, y \in U$ such that (x,y) is $(\mathscr{F}, \varepsilon)$-mean sensitive.

Now, we define the notions of $\mathscr{F}$-*mean proximality* and $\mathscr{F}$-*mean Li–Yorke chaos* for actions of countable amenable semigroups.

Definition 2.8. Let $(X, \{T_s\}_{s\in S})$ be a topological dynamical system, let $\mathscr{F}$ be a Furstenberg family, and let $\mathcal{F} = \{F_n\}_{n\in\mathbb{N}}$ be a left Følner sequence on S.

(a) A pair $(x,y) \in X \times X$ is called an $\mathscr{F}$-*mean proximal pair* if for every $F \in \mathscr{F}$,

$$\liminf_{n\to\infty} \frac{1}{|F_n \cap F| + 1} \sum_{s\in F_n\cap F} d(T_s x, T_s y) = 0.$$

We denote the set of all $\mathscr{F}$-mean proximal pairs by $MP_{\mathscr{F}}$. The system $(X, \{T_s\}_{s\in S})$ is called $\mathscr{F}$-*mean proximal* if every pair $(x,y) \in X \times X$ is $\mathscr{F}$-mean proximal.

(b) A pair $(x,y) \in X \times X$ is called an $\mathscr{F}$-*mean asymptotic pair* if for every $F \in \mathscr{F}$,

$$\lim_{n\to\infty} \frac{1}{|F_n \cap F| + 1} \sum_{s\in F_n\cap F} d(T_s x, T_s y) = 0.$$

The system $(X, \{T_s\}_{s\in S})$ is called $\mathscr{F}$-*mean asymptotic* if every pair $(x,y) \in X \times X$ is $\mathscr{F}$-mean asymptotic.

(c) Let $\eta > 0$. A pair $(x,y) \in X \times X$ is called an $\mathscr{F}$-*mean Li–Yorke pair with modulus* η if for every $F \in \mathscr{F}$,

$$\liminf_{n\to\infty} \frac{1}{|F_n \cap F| + 1} \sum_{s\in F_n\cap F} d(T_s x, T_s y) = 0,$$

and

$$\limsup_{n\to\infty} \frac{1}{|F_n \cap F| + 1} \sum_{s\in F_n\cap F} d(T_s x, T_s y) \geq \eta.$$

We denote the set of all such pairs by $MLY_{\mathscr{F}}$.

(d) A pair $(x,y) \in X \times X$ is called an $\mathscr{F}$-*mean regionally proximal pair* if for every $\varepsilon > 0$, there exists $F \in \mathscr{F}$ such that for every $s \in F$, there exist $x', y' \in X$ with $d(x,x') < \varepsilon$, $d(y,y') < \varepsilon$, and

$$\limsup_{n\to\infty} \frac{1}{|F_n \cap F| + 1} \sum_{s\in F_n\cap F} d(T_s x', T_s y') < \varepsilon.$$

We denote the set of all such pairs by $MQ_{\mathscr{F}}\,(X, \{T_s\}_{s\in S})$ or simply $Q_{M\mathscr{F}}$.

We now turn to the relationship between $\mathscr{F}$-proximality and $\mathscr{F}$-regional proximality under the assumption of $\mathscr{F}$-mean equicontinuity. The following theorem shows that, in this setting, the two notions coincide.

Theorem 2.9. *Let $(X, \{T_s\}_{s\in S})$ be a dynamical system and let $\mathscr{F}$ be a Furstenberg family. If $(X, \{T_s\}_{s\in S})$ is $\mathscr{F}$-mean equicontinuous, then*

$$P_{\mathscr{F}}(X, \{T_s\}_{s\in S}) = Q_{\mathscr{F}}(X, \{T_s\}_{s\in S}).$$

Proof. It is clear that $P_{\mathscr{F}} \subset Q_{\mathscr{F}}$. Thus it suffices to show that $Q_{\mathscr{F}} \subset P_{\mathscr{F}}$. Let $(x, y) \in Q_{\mathscr{F}}$. Since the system is $\mathscr{F}$-mean equicontinuous, for every $\varepsilon > 0$ and every $F \in \mathscr{F}$, there exists $\delta > 0$ such that for all $z, w \in B(x, \delta)$,

$$\limsup_{n\to\infty} \frac{1}{|F_n \cap F| + 1} \sum_{s\in F_n\cap F} d(T_s z, T_s w) < \frac{\varepsilon}{3}.$$

Because $(x, y) \in Q_{\mathscr{F}}$, there exists $F \in \mathscr{F}$ such that for every $g \in F$ there exist $x', y' \in X$ satisfying

$$d(x, x') < \delta, \qquad d(y, y') < \delta, \qquad d(T_g x', T_g y') < \delta.$$

Consequently, we have

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{|F_n \cap F| + 1} \sum_{s\in F_n\cap F} d(T_s x, T_s y) \\
&\leq \limsup_{n\to\infty} \frac{1}{|F_n \cap F| + 1} \sum_{s\in F_n\cap F} \Big(d(T_s x, T_s x') + d(T_s y, T_s y') + d(T_s x', T_s y') \Big) \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we conclude that

$$\limsup_{n\to\infty} \frac{1}{|F_n \cap F| + 1} \sum_{s\in F_n\cap F} d(T_s x, T_s y) = 0,$$

which shows that (x, y) is $\mathscr{F}$-proximal. Hence $Q_{\mathscr{F}} \subset P_{\mathscr{F}}$, which completes the proof. □

Theorem 2.10. *Let S be a countable discrete infinite commutative amenable semigroup, let $\mathscr{F}$ be a translation-invariant Furstenberg family, and let $F \in \mathscr{F}$. Let $\mathcal{F} = \{F_n\}_{n=1}^{\infty}$ be a Følner sequence in S. If $(X, \{T_s\}_{s\in S})$ is $\mathscr{F}$-mean sensitive and there exists an $\mathscr{F}$-mean proximal pair in $X \times X$ consisting of a transitive point and a periodic point, then $(X, \{T_s\}_{s\in S})$ is $\mathscr{F}$-mean Li–Yorke chaotic.*

Proof. By $\mathscr{F}$-mean sensitivity, there exists $\delta > 0$ such that for every $x \in X$ and every neighborhood U of x, there is $y \in U$ with

$$\limsup_{n\to\infty} \frac{1}{|F_n \cap F| + 1} \sum_{s\in F_n\cap F} d(T_s x, T_s y) > \delta.$$

Fix $\eta = \delta/2 > 0$ and define

$$D_\eta = \left\{ (x, y) \in X \times X : \limsup_{n\to\infty} \frac{1}{|F_n \cap F| + 1} \sum_{s\in F_n\cap F} d(T_s x, T_s y) \geq \eta \right\}.$$

We first show that D_η is a G_δ subset of $X \times X$. For $m, l \in \mathbb{N}$, set

$$E_{m,l} = \bigcup_{n=l}^{\infty} \left\{ (x,y) \in X \times X : \frac{1}{|F_n \cap F| + 1} \sum_{s \in F_n \cap F} d(T_s x, T_s y) > \eta - \frac{1}{m} \right\}.$$

For fixed n, the map

$$(x,y) \mapsto \frac{1}{|F_n \cap F| + 1} \sum_{s \in F_n \cap F} d(T_s x, T_s y)$$

is continuous on $X \times X$, so the set in braces is open; hence each $E_{m,l}$ is open. Moreover,

$$D_\eta = \bigcap_{m=1}^{\infty} \bigcap_{l=1}^{\infty} E_{m,l},$$

so D_η is a G_δ subset of $X \times X$. Next, we show that D_η is dense. Assume to the contrary that it is not dense. Then there exists a nonempty open rectangle $U_1 \times U_2 \subset X \times X$ such that $(U_1 \times U_2) \cap D_\eta = \emptyset$. Since $d(T_s x, T_s y) = d(T_s y, T_s x)$ for all $s \in S$ and $x, y \in X$, the set D_η is symmetric, and therefore also $(U_2 \times U_1) \cap D_\eta = \emptyset$. Let $U := U_1 \cap U_2$, which is a nonempty open subset of X. For any $x, y \in U$, we then have $(x,y) \in U_1 \times U_2$ and hence $(x,y) \notin D_\eta$, implying

$$\limsup_{n \to \infty} \frac{1}{|F_n \cap F| + 1} \sum_{s \in F_n \cap F} d(T_s x, T_s y) < \eta.$$

Now, fix $x \in U$. By $\mathscr{F}$-mean sensitivity, applied to the neighborhood U of x, there exists $y \in U$ such that

$$\limsup_{n \to \infty} \frac{1}{|F_n \cap F| + 1} \sum_{s \in F_n \cap F} d(T_s x, T_s y) > \delta = 2\eta,$$

which contradicts the previous inequality. Hence D_η is dense in $X \times X$. Recall that the set of all $\mathscr{F}$-mean proximal pairs is

$$MP_{\mathscr{F}} = \left\{ (x,y) \in X \times X : \liminf_{n \to \infty} \frac{1}{|F_n \cap F| + 1} \sum_{s \in F_n \cap F} d(T_s x, T_s y) = 0 \right\}.$$

As usual, we may write

$$MP_{\mathscr{F}} = \bigcap_{m=1}^{\infty} \bigcap_{l=1}^{\infty} \bigcup_{n=l}^{\infty} \left\{ (x,y) \in X \times X : \frac{1}{|F_n \cap F| + 1} \sum_{s \in F_n \cap F} d(T_s x, T_s y) < \frac{1}{m} \right\},$$

so $MP_{\mathscr{F}}$ is a G_δ subset of $X \times X$. By assumption, there exist a periodic point $p \in X$ and a transitive point $q \in X$ such that $(p,q) \in MP_{\mathscr{F}}$. Thus, for every $F \in \mathscr{F}$,

$$\liminf_{n \to \infty} \frac{1}{|F_n \cap F| + 1} \sum_{s \in F_n \cap F} d(T_s p, T_s q) = 0.$$

Let $\mathrm{Stab}_p = \{s \in S : T_s(p) = p\}$ and let $r = [S : \mathrm{Stab}_p] < \infty$ be its index. We may decompose

$$S = s_1 \mathrm{Stab}_p \cup s_2 \mathrm{Stab}_p \cup \cdots \cup s_r \mathrm{Stab}_p,$$

where $s_i^{-1}s_j \notin \mathrm{Stab}_p$ for $i \neq j$. For each $1 \leq j \leq r$, define

$$X_j = \overline{\{T_{s_j s}(q) : s \in \mathrm{Stab}_p\}}.$$

We claim that $MP_{\mathscr{F}}$ is dense in each $X_j \times X_j$. Fix j and let $t_1, t_2 \in \mathrm{Stab}_p$. Then $T_{t_1}p = T_{t_2}p = p$, and for each $s \in S$,

$$d(T_s T_{s_j t_1} q, T_s T_{s_j t_2} q) = d\big(T_{s s_j t_1} q, T_{s s_j t_2} q\big) \leq d\big(T_{s s_j t_1} q, T_{s s_j} p\big) + d\big(T_{s s_j} p, T_{s s_j t_2} q\big).$$

Since p is fixed by all elements of Stab_p, we have

$$T_{s s_j t_k} p = T_{s s_j}(T_{t_k} p) = T_{s s_j} p, \qquad k = 1, 2,$$

and hence each term on the right-hand side can be written in the form $d(T_{s'}p, T_{s'}q)$ for some $s' \in S$. By the amenability of S and the Følner property of $\{F_n\}$, reindexing the sums by finitely many left-translations does not change the limit inferior of the corresponding averages. Consequently, for every $F \in \mathscr{F}$,

$$\liminf_{n\to\infty} \frac{1}{|F_n \cap F| + 1} \sum_{s \in F_n \cap F} d(T_s T_{s_j t_1} q, T_s T_{s_j t_2} q) = 0,$$

so $(T_{s_j t_1} q, T_{s_j t_2} q) \in MP_{\mathscr{F}}$ for all $t_1, t_2 \in \mathrm{Stab}_p$. Therefore

$$\{(T_{s_j t_1} q, T_{s_j t_2} q) : t_1, t_2 \in \mathrm{Stab}_p\} \subset MP_{\mathscr{F}},$$

and taking closures shows that $MP_{\mathscr{F}}$ is dense in $X_j \times X_j$ for each j. Since q is transitive, we have

$$X = \overline{\{T_s(q) : s \in S\}} = \bigcup_{j=1}^{r} X_j.$$

Therefore at least one X_j has nonempty interior; fix such a j and set $A = \mathrm{int}(X_j) \neq \emptyset$. Since $(X, \{T_s\}_{s\in S})$ has no isolated points, A is a perfect compact metric space. Moreover, by the density of D_η in $X \times X$ and the density of $MP_{\mathscr{F}}$ in $X_j \times X_j$, we see that

$$D_\eta \cap (A \times A) \quad \text{and} \quad MP_{\mathscr{F}} \cap (A \times A)$$

are both dense G_δ subsets of $A \times A$. By Mycielski's theorem (together with [18, Remark 3.1]), there exists a Mycielski subset $K \subset A$ such that

$$K \times K \setminus \Delta_X \subset D_\eta \cap MP_{\mathscr{F}},$$

that is, every pair of distinct points in K is simultaneously $\mathscr{F}$-mean proximal and has $\mathscr{F}$-mean separation at least η in the limsup sense. By the definitions given earlier, this means

$$K \times K \setminus \Delta_X \subset MLY_{\mathscr{F},\eta}(X, \{T_s\}_{s\in S}),$$

so K is an uncountable $\mathscr{F}$-mean Li–Yorke set with modulus η, and $(X, \{T_s\}_{s\in S})$ is $\mathscr{F}$-mean Li–Yorke chaotic. □

Lemma 2.11. *Let $\mathscr{F}$ be a translation-invariant Furstenberg family and let $F \in \mathscr{F}$. Let $\mathcal{F} = \{F_n\}_{n=1}^{\infty}$ be a Følner sequence in S. If $(X, \{T_s\}_{s\in S})$ is $\mathscr{F}$-mean asymptotic, then $(X, \{T_s\}_{s\in S})$ is $\mathscr{F}$-mean proximal.*

Proof. This follows directly from the definitions, since for each pair $(x,y)\in X\times X$ and every $F\in\mathscr{F}$, we have

$$\lim_{n\to\infty}\frac{1}{|F_n\cap F|+1}\sum_{s\in F_n\cap F} d(T_sx,T_sy)=0,$$

which immediately implies

$$\liminf_{n\to\infty}\frac{1}{|F_n\cap F|+1}\sum_{s\in F_n\cap F} d(T_sx,T_sy)=0.$$

Thus every $\mathscr{F}$-mean asymptotic pair is $\mathscr{F}$-mean proximal. □

Corollary 2.12. *If $(X,\{T_s\}_{s\in S})$ is $\mathscr{F}$-mean equicontinuous, then $(X,\{T_s\}_{s\in S})$ is $\mathscr{F}$-mean regionally proximal if and only if it is $\mathscr{F}$-mean asymptotic.*

Corollary 2.13. *If $(X,\{T_s\}_{s\in S})$ is an $\mathscr{F}$-mean proximal system, then $(X,\{T_s\}_{s\in S})$ has no $\mathscr{F}$-mean Li–Yorke pairs.*

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