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DERIVED SHIFT SPACE OF CODED SYSTEMS AND SYNCHRONIZED ENTROPY OF TOTALLY SYNCHRONIZING GENERATED SYSTEMS

MANOUCHEHR SHAHAMAT[1*], OMID YOUSEFIKIA[1]

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Abstract. Our aim is to introduce a class of minimally generated systems generated by some certain synchronizing blocks called totally synchronizing generated system. This has been used to give a new shorter proof for computing synchronized entropy $h_{syn}(X)$. Also, the derived shift space has been characterized as well.

1. Introduction

One of the most studied dynamical systems is a *synchronized system*. Recall that a block m is *synchronizing* if whenever v_1m and v_2m are both blocks of X, then v_1mv_2 is a block of X as well. If an irreducible system has at least one synchronizing block, then it is called a synchronized system and its examples are *sofics*. Synchronized systems, has attracted much attention and extension of them, has been of interest since that notion was introduced [1, p. 36].

In section 3, we introduce the notion of a totally synchronizing generated system. These systems are precisely a tool that enables us to create a bridge between dynamical systems and other mathematical branches. By creating such a connection, we are able to introduce the basic concepts linear independence and dependence from the branch of linear algebra to dynamic systems and enter the topic of applied mathematics. See [11, pp. 43–45]. Thomsen [13, pp. 3564–3575] considered a synchronized component of a general subshift and investigates the approximation of entropy from inside of this synchronized component by some certain *shift of finite types* (SFTs). Indeed, many results in [13, Theorem 3.2,

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* Corresponding author.
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Propositions 4.5 and 4.6] are based on this result. Minimal synchronizing blocks are main tools for our aim. The notion of minimal synchronizing blocks is introduced in section 4 as well as is showed that a sofic space may have infinitely many minimal synchronizing blocks. Thomsen proved that $\lim_{k\to\infty} h(X_k) = h_{syn}(X)$ where

$$h_{syn}(X) := \limsup_{n} \frac{1}{n} \log \left(\text{cardinal}\left\{a \in W_n(X) : mam \in W(X)\right\}\right), \tag{1.1}$$

in which m is an arbitrary synchronizing block in $W(X)$ and X_k's are $SFTs$ approaching X from inside. For the totally synchronizing generated systems, give a simpler way to calculate synchronizing entropy. We cover this in section 5.

2. Background and definitions

This section is devoted to basic definitions for our later work. The notations has been taken from [4, pp. 10–17], [6, pp. 11–23], [3, pp. 139–141] and [9, pp. 2–4] for the relevant concepts. First, we present some elementary concepts from [6, pp. 11–23]. Let $\mathcal{A}$ be a set of nonempty finite symbols called *alphabet*. The full $\mathcal{A}$-shift denoted by $\mathcal{A}^{\mathbb{Z}}$, is the collection of all bi-infinite sequences of symbols in $\mathcal{A}$. Equip $\mathcal{A}$ with discrete topology and $\mathcal{A}^{\mathbb{Z}}$ with product topology. A *block* over $\mathcal{A}$ is a finite sequence of symbols from $\mathcal{A}$. It is convenient to include the sequence of no symbols, called the *empty block* which is denoted by ε. If x is a point in $\mathcal{A}^{\mathbb{Z}}$ and $i \leq j$, then we denote a block of length $j - i + 1$ by $x_{[i,j]} = x_i x_{i+1} \ldots x_j$. If $n \geq 1$, then u^n denotes the concatenation of n copies of u, and put $u^0 = \varepsilon$. The *shift map* σ on the full shift $\mathcal{A}^{\mathbb{Z}}$ maps a point x to the point $y = \sigma(x)$ whose ith coordinate is $y_i = x_{i+1}$. The domain of the function σ is the set $\mathcal{A}^{\mathbb{Z}}$. If $X \subset \mathcal{A}^{\mathbb{Z}}$, then the function σ with the same rule is also defined on the subset of X.

By our topology, σ is a homeomorphism. Let $\mathcal{F}$ be the collection of all forbidden blocks over $\mathcal{A}$. The complement of $\mathcal{F}$ is the set of *admissible blocks* or just blocks in X. For a full shift $\mathcal{A}^{\mathbb{Z}}$, define $X_{\mathcal{F}}$ to be the subset of sequences in $\mathcal{A}^{\mathbb{Z}}$ not containing any block from $\mathcal{F}$. A *shift space* or a *subshift* is a subset X of a full shift $\mathcal{A}^{\mathbb{Z}}$ such that $X = X_{\mathcal{F}}$ for some collection $\mathcal{F}$ of forbidden blocks.

Let $W_n(X)$ denote the set of all admissible n-blocks. The *language* of X is the collection $W(X) = \cup_n W_n(X)$. A shift space X is *irreducible* if for every ordered pair of blocks u, $v \in W(X)$ there is a block $w \in W(X)$ so that $uwv \in W(X)$. It is *mixing* if for every ordered pair u, $v \in W(X)$, there is N such that for each $n \geq N$ there is a block $w \in W_n(X)$ such that $uwv \in W(X)$. A shift space X is called an SFT (*shift of finite type*) if there is a finite set $\mathcal{F}$ of forbidden blocks such that $X = X_{\mathcal{F}}$. A shift of *sofic* is the image of an SFT by a factor code (an onto sliding block code). Every SFT is sofic [6, Theorem 3.1.5], but the converse is not true. For instance, if $\mathcal{F} = \{10^{2n+1}1 : n \in \mathbb{N} \cup \{0\}\}$, then $X_{\mathcal{F}}$ is called the even shift which is not SFT but it is sofic [6, p. 67].

Let G be a graph with edge set $\mathcal{E} = \mathcal{E}(G)$ and the set of vertices $\mathcal{V} = \mathcal{V}(G)$. The *edge shift* X_G is the shift space over the alphabet $\mathcal{A} = \mathcal{E}$ defined by

$$X_G = \{\xi = (\xi_i)_{i\in\mathbb{Z}} \in \mathcal{E}^{\mathbb{Z}} : t(\xi_i) = i(\xi_{i+1})\}.$$

Each edge e initiates at a vertex denoted by $i(e)$ and terminates at a vertex $t(e)$.

A labeled graph is a pair $\mathcal{G} = (G, \mathcal{L})$, where G is a graph with edge set $\mathcal{E}$, and the labeling $\mathcal{L} : \mathcal{E}(G) \to \mathcal{A}$ assigns to each edge e of G a label $\mathcal{L}(e)$ from the finite alphabet $\mathcal{A}$. For a path $\pi = \pi_0 \dots \pi_k$, $\mathcal{L}(\pi) = \mathcal{L}(\pi_0) \dots \mathcal{L}(\pi_k)$ is the label of π.

Let $\mathcal{L}_\infty(\xi)$ be the sequence of bi-infinite labels of a bi-infinite path ξ in G and set

$$X_{\mathcal{G}} := \{\mathcal{L}_\infty(\xi) : \ \xi \in X_G\} = \mathcal{L}_\infty(X_G).$$

We say $\mathcal{G}$ is a *presentation* or *cover* for $X = \overline{X_{\mathcal{G}}}$. In particular, X is sofic if and only if $X = X_{\mathcal{G}}$ for a finite graph G [6, Proposition 3.2.10]. A labeled graph $\mathcal{G} = (G, \mathcal{L})$ is *right-resolving* if for each vertex I of G the edges starting at I carry different labels.

In this part, we collect some information from [5]. Let X be a shift space and let $w \in W(X)$. The follower set $F(w)$ of w is defined by $F(w) = \{v \in W(X) : wv \in W(X)\}$. Let $x \in X$. Then, $x_+ = (x_i)_{i \in Z^+}$ (resp., $x_- = (x_i)_{i \leq 0}$) is called right (resp., left) infinite X-ray.

For a left infinite X-ray, say x_-, its follower set is $w_+(x_-) = \{x_+ \in X^+ : x_- x_+ \in X\}$.

Consider the collection of all follower sets $w_+(x_-)$ as the set of vertices of a graph. There is an edge from I_1 to I_2 labeled a if and only if there is an X-ray x_- such that $x_- a$ is an X-ray and $I_1 = w_+(x_-), I_2 = w_+(x_- a)$.

This labeled graph is called the *Krieger graph* for X. A block $m \in W(X)$ is *synchronizing* if whenever um and mv are in $W(X)$, we have $umv \in W(X)$. An irreducible shift space X is *synchronized system* if it has a synchronizing block, or equivalently, if and only if it admit a countable generating graph G such that $\mathcal{L}_\infty(X_G)$ is residual in X [3, Theorem 1.1]. If X is a synchronized system with synchronizing m, then the irreducible component of the Krieger graph containing the vertex $w_+(m)$ is denoted by X_0^+ and is called the *Fischer cover* of X. If for some $m \in W(X)$ there is a unique vertex I such that $m \in F_-(m)$, then m is called a *magic* block for the Fischer cover.

3. Minimal generator

A shift space which is the closure of the set of sequences obtained by freely concatenating the blocks in a list of countable blocks, called the set of generators, is a *coded system*. Equivalently, a coded system is a shift space that can be presented by an irreducible countable labeled graph. For more information, see reference [4, pp. 111–123].

Definition 3.1. Let X be a coded system generated by G. Then, G is called *minimal generator* (resp., *weak minimal generator*), whenever $u \in G$, then $u \notin W(Z)$, (resp., $X \neq Z$) where $Z = \overline{< G \setminus \{u\} >}$. Such X is called *minimally* (resp., *weak minimally*) system.

Here, $< G >$ means the set of all concatenations of the elements of G. Clearly, any minimal system is weakly minimal.

Example 3.2. (1) Let X be the Dyke system. Set

$$G_1 := \{(),\ (()),\ [()],\ ((())),\ [(())],\ ([()]),\ [[()]],\ \dots\},$$

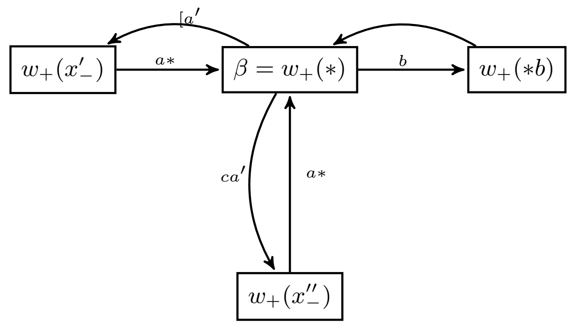

Figure 1. A subgraph of Fischer cover X_0^+, there are two finite paths initialing at $w_+(*)$ and terminating at $w_+(x'_-a*)$ and $w_+(x''_-a*)$, respectively. So u is not a strong synchronizing block

$$G_2 := \{[], ([]), [[]], [[[]]], ([[]]), [([])], (([])), \ldots\}.$$

Then, $G = G_1 \cup G_2$ is a weak minimal generator for the subshift $Z = \overline{<G>}$.

(2) Let $\emptyset \neq S \subseteq \mathbb{N}$. Then, $G := \{10^n 1 : n \in S\}$ is a minimal generator for subshift $X := \overline{<G>}$.

Definition 3.3. Let X be a synchronized system. Call a block m *strong synchronizing* for X if whenever there are two finite paths e, e' in Fischer cover X_0^+ labeled m, then $e = e'$.

An irreducible shift space with a strong synchronizing block is called *strong synchronized*. Any strong synchronized system is a synchronized since any strong synchronizing block is a synchronizing block. Every strong synchronized system has a weak minimal generator [10, Proposition 3.4].

Let X be a strong synchronized system and let $S_t(X)$ (resp., $S(X)$) denote the set of all strong synchronizing (resp., synchronizing) blocks for X.

The next example shows that there are weak minimally systems that are not strong synchronized.

Example 3.4. (1) Pick $S \subseteq \mathbb{N} \cup \{0\}$ such that $0 \in S$. Set $G := \{10^n : n \in S\}$ and claim that G is a weak minimal generator for S-gap shift $X(S)$.

Let $n \in S$. Since G is a generator, we are allowed to arrange its elements in any order, which results in a valid block for the space generated by G. Since $10^n, 1 \in S$, Thus

$$(10^n 1)^\infty = \ldots 10^n 110^n 110^n 1 \ldots \in \overline{<G>}.$$

On the other hand, it is clear that $(10^n 1)^\infty \notin \overline{<G \setminus \{10^n\}>}$ and so $\overline{<G>} \neq \overline{<G \setminus \{10^n\}>}$. Also, $\overline{<G>} = X(S)$ is trivial and we are done.

(2) Let D be the Dyke system. Add a new symbol $*$ (which will be a synchronizing block) to the set of four brackets and let X be the subshift that consists of all bi-infinite sequences of these five symbols such that

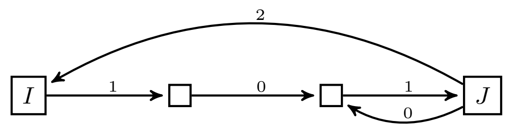

FIGURE 2. The graph H for the cover of a synchronized system such that G_m is not a minimal generator for X_H. $2 \in S_t(X)$ but $012 \notin S_t(X)$.

any finite subblock that does not contain a $*$ obeys the standard brackets rules [8, Theorem 3.4].

We claim that X is not strong synchronized system. Let u be a strong synchronizing block of X. So it is a synchronizing block and $* \subseteq u$.

We can write $u = a * b$ where $a, b \in W(X)$. Since $*$ is a synchronizing block, there is a unique vertex β of Fischer cover X_0^+ with this properties $* \in F_-(\beta)$ and $w_+(\beta) = w_+(*)$ [14, p. 306].

Now, $b \in w_+(*) = w_+(\beta)$, because of $*b \in W(X)$. Hence there is a finite path π_b labeled b in X_0^+ initiating at $w_+(*)$ and terminating at $w_+(*b)$. See Figure 1.

Pick $a' \in W(X)$ such that for each $x_- \in X^-$, $x_-a'a \in X^-$ and set $x'_- := \cdots]]][a'$, $x''_- := \cdots]]](a'$. Since $w_+(x'_-a*) = w_+(*) = w_+(x''_-a*)$ and $w_+(x'_-) \neq w_+(x''_-)$, so there are two distinct finite paths π'_{a*} and π_{a*} in the Krieger graph of X labeled $a*$ such that $i(\pi_{a*}) = w_+(x'_-)$, $i(\pi'_{a*}) = w_+(x''_-)$ and $t(\pi_{a*}) = t(\pi'_{a*}) = w_+(*)$. Indeed $w_+(*[a') = w_+(x'_-)$ and $w_+(*(a') = w_+(x''_-)$. So there are two finite paths initialing at $w_+(*)$ and terminating at $w_+(x'_-a*)$ and $w_+(x''_-a*)$, respectively. Hence

$$\{w_+(x'_-a*),\, w_+(x''_-a*)\} \subseteq \mathcal{V}(X_0^+).$$

So there are two finite paths labeled $a * b = u$ in X_0^+. This shows that u is not a strong synchronizing block and we are done.

It is easy to see that $G_* = \{*u : * \notin u \in W(X)\}$ is a weak minimal generator for X.

Note that if $m \in S(X)$ and $m \subseteq u$, then $u \in S(X)$. Nevertheless, it is not true when $m \in S_t(X)$. This can be seen by the fact that in Figure 2, $2 \in S_t(X)$ but $012 \notin S_t(X)$.

Let G be a minimal generator for a subshift X. Set G_{ts} denote the set of all $v \in G$ such that for all $u \in G$, there are not non empty blocks a, b such that $vu = avb$ or $uv = avb$. All the elements of the set G_{ts} are synchronizing [11, Theorem 3.5].

Let G be a minimal generator for a subshift X with $G = G_{ts}$. Then, G is called a *totally synchronizing generator*. Such X is called *totally synchronizing generated system*.

These systems are precisely a tool that enables us to show that for such subshifts X, each $x \in X$ can be written uniquely as $x = \ldots v_{-1}v_0v_1v_2\ldots$, where

$\{\ldots, v_{-1}, v_0, v_1, v_2, \ldots\} \in G$. In fact, these systems are precisely a tool that enables us to create a bridge between dynamical systems and other mathematical branches. By creating such a connection, we are able to introduce the basic concepts linear independence and dependence from the branch of linear algebra to dynamic systems and enter the topic of applied mathematics.

4. Synchronized components

In this section, we introduce the notion of minimal synchronizing blocks and exploit them to identify synchronized components.

Definition 4.1. A block $m \in S(X)$ is *minimal synchronizing block*, whenever $w \subsetneq m$, then w is not synchronizing. If a shift space X has finitely many minimal synchronizing blocks and $S(X) \neq \emptyset$, then we say that X is a *FmSyn* system; otherwise, it is called an *ImSyn* system.

Example 4.2.
(1) The block 1 is minimal synchronizing for any S-gap shift $X(S)$ and no other minimal synchronizing blocks exist which means that this system is an FmSyn even when it is not sofic. See [10, Theorem 3.13] for criteria on S to have nonsofic $X(S)$.
(2) Let G be the graph as in Figure 3 and $X = X_G = R(X)$. Then all blocks in $A = \{2, 101, 10^3 1, 10^5 1, \ldots\}$ are synchronizing blocks.
To this end, we first prove that the block 2 is synchronizing. Since the terminal vertex of all edges with label 2 is vertex I_0, then if $u2, 2v \in W(X)$, this means that there exists a path in the graph with label $u2$ and ending at i_0 and a path in the graph with label $2v$ and starting at I_0. The concatenation of these two paths introduces a path in the graph with label $u2v$. The remaining cases are proven in a similar fashion.
However, blocks in
$$\{1, 0, 100, 1000, \ldots\} \cup \{01, 001, 0001, \ldots\}$$
are not. To prove this part, it suffices to show that block $1o^n$ is not synchronizing. The remaining cases will be proven similarly. Let $n' \in \mathbb{N}$ such that $n + n'$ is even. It is clear that $10^n, 0^{n'}1 \in W(X)$ but $10^{n+n'}1 \notin W(X)$.
Therefore, no blocks in A has a synchronizing subblock and so X is an ImSyn.

The next proposition tells us that a sofic space may have infinitely many minimal synchronizing block. First let the collection of all follower sets in X is denoted by $\mathcal{C}_X$.

Proposition 4.3. *A sofic space may have infinitely many minimal synchronizing block.*

Proof. Consider sets of words $V = \{1^{2n} : n \in \mathbb{N}\}$ and $W = \{1^{2n+1} : n \in \mathbb{N}\}$. Also, consider the subshift X formed as the limit of concatenations of blocks of the form $a0v_1 0v_2 0v_3 \ldots 0v_n$ where v_i belongs in V and n is any positive integer and $a0w_1 0w_2 0w_3 \ldots 0w_m$, where w_i's belong to W and m is any positive integer (in particular, notice the special symbol a at the beginning of all the blocks).

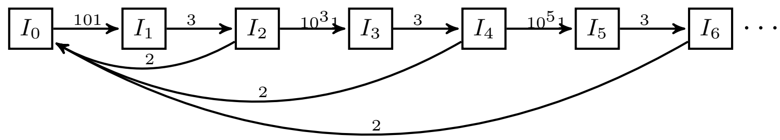

FIGURE 3. Graph G for the cover of an ImSyn, all blocks in $A = \{2, 101, 10^31, 10^51, \ldots\}$ are synchronizing blocks and no blocks in A has a synchronizing subblock.

The block 0 is not synchronizing, because 0110 belongs in the language of X (0110 appears for example in the block $a0v0v$, where $v = 11$ belongs to V) and 010 belongs in the language of X (010 appears for example, in the block $a0w0w$ where $w = 1$ belongs to W), but 010110 does not belong in the language of X, because the separator symbol "a" must appear between even and odd blocks of the ones. Note that "a" is a symbol which is different from 0 and 1.

It is now sufficient to prove that X is a sofic. According to [6, Theorem 3.2.10], it is enough to show that X has a finite number of follower sets. Since

$$\mathcal{C}_X = \{w_+(a), w_+(0), w_+(1)\},$$

as a result, the proof is complete.

$\square$

5. COMPUTING SYNCHRONIZED ENTROPY FOR A TOTALLY SYNCHRONIZING GENERATED SYSTEMS

Let X be a shift space. Set $R(X) = \overline{\mathrm{Per}X}$, and let $S(X)$ denote the set of synchronizing blocks for $R(X)$. For $s, t \in S(X)$, we write $s \sim t$ when there are blocks $u, v \in W(R(X))$ such that sut, $tvs \in W(R(X))$. Then, $\sim$ is an equivalence relation in $S(X)$.

Note that $s \sim t$ if only if there is $x \in R(X)$ such that s, $t \subseteq x$. Consider an element $\alpha \in S(X)/\sim$.

Let $x \in X$ and $p \leq s$ for integers p and s. Set the *gap* between two blocks $x_{[p,q]}$ and $x_{[s,t]}$ to be 0 when $s \leq q$ and $s - q$ otherwise. Denote this gap by $\mathrm{gap}(x_{[p,q]}, x_{[s,t]})$.

Definition 5.1. Let $p < s$, let $q \leq t$, and let $u = x_{[p,q]}, v = x_{[s,t]} \in \alpha \in S(X)/\sim$ be two minimal synchronizing blocks. If the only minimal synchronizing blocks in $x_{[p,t]}$ are u and v, then u and v are called the *consecutive minimal pairs of* α *in* x.

We recall that α is an equivalence class whose elements are synchronizing blocks. See [9, p. 5] for details. Let $x \in X_{(\alpha,0)}$, by (5.1),

$$\{\mathrm{gap}(u, v) : u, v \text{ are the consecutive minimal pairs of } \alpha \text{ in } x\}$$

is bounded and we denote the maximum by $\mathrm{maxgap}(x, \alpha)$.

Let $X_{(\alpha,0)}$ denote the set of elements $x \in R(X)$ as follows.

(1) For all $i \in \mathbb{Z}$ there are u_i, $v_i \in \alpha$ such that $u_i \subseteq x_{(-\infty, i]}$, $v_i \subseteq x_{[i, +\infty)}$.
(2) There is $M > 0$ such that if u, v are the consecutive minimal pairs of α in x, then $\text{gap}(u, v) \leq M$.
(3) $A = \{w \subset x :\ \text{w is a minimal synchronizing}\} \cap \alpha$ is finite.

In particular, if $x \in R(X)$ is periodic point with a subblock in α, then $x \in X_{(\alpha, 0)}$.

Thomsen [13, pp. 5–12] proved that $X_{(\alpha, 0)}$ is the set of elements $x \in R(X)$ for which

$$\sup_{i \in \mathbb{Z}} \left\{ \inf \left\{ (j - i) \geq 0 :\ \exists w \in \alpha,\, w \subseteq x_{[i, j]} \right\} \right\} \tag{5.1}$$

is finite. Here, it follows from the convention that $\inf \emptyset = \infty$. As demonstrated in my paper [9], the two definitions are indeed equivalent.

Let X be a shift space. The *entropy* of X is defined by

$$h(X) = \lim_{n \to \infty} \frac{1}{n} \log |W_n(X)|.$$

A shift space X is *almost sofic* if there are sofic shifts $X_n \subseteq X$ such that

$$\lim_{n \to \infty} h(X_n) = h(X).$$

6. Characterization of ∂X and the role of minimally generated totally systems

This section provides the rigorous characterization of the derived shift space ∂X specifically for the **minimally generated totally systems** (a novel class introduced in this manuscript).

The derived shift space ∂X is central to understanding the nonsynchronizing behavior embedded within the system X.

For general synchronized systems, the entropy decomposition

$$h(X) = \max\{h_{\text{syn}}(X), h(\partial(X))\} \tag{6.1}$$

(see Thomsen [12, Proposition 4.5]) demonstrates that $h(\partial X)$ quantifies the entropy contribution from sequences lacking the structure necessary for synchronization.

We introduce the minimally generated totally system class, defined by the minimal generator G and the property of unique generator decomposition.

This class plays a foundational role in describing the structure of the shift space X, analogous to the role of basis vectors in linear algebra.

Just as each vector is uniquely expressed as a linear combination of basis vectors, every word in the language $W(X)$ of our system can be uniquely decomposed into a concatenation of words from G. This strong constraint allows for a precise characterization of ∂X.

These systems, ∂X, are defined by sequences $x \in R(X)$ whose blocks do not admit a synchronizing subblock:

$$\partial X := \{x \in R(X) : w \subseteq x \Rightarrow w\ \notin\ S(X)\}. \tag{6.2}$$

IF $v_0 \in S_t(X)$, then $v_0uv_0 \in W(X)$ if and only if u is a finite concatenation of elements in G, since there is exactly one path labeled v_0 in $\mathcal{H}_G = X_0^+$. See Lemma 6.1 for further details Furthermore, the equivalence $v_0uv_0 \in W(X) \Leftrightarrow u$ is a G-concatenation is rigorously proven.

This result directly links the existence of a word in the synchronized space $W(X)$ to the unique decomposition property of u, solidifying the necessary and sufficient conditions that dictate whether a sequence belongs to ∂X or not.

For any synchronized system X, the *synchronized entropy* h_{syn} of X is defined by

$$h_{\text{syn}}(X) = \limsup_n \frac{1}{n} \log\left(\text{cardinal}\{a \in W_n(Y) :\ mam \in W(X)\}\right), \tag{6.3}$$

where $m \in W(X)$ is an arbitrary synchronizing block.

Lemma 6.1. *Let G be a minimal generator for the coded system X and let $v_0 \in G_{ts}$. Then,*

$$h_{syn}(X) = \limsup_n \frac{1}{n} \log |\{(v_1, \dots, v_k) \in G^k : \Sigma_{i=1}^k |v_i| = n\}|.$$

Proof. First, we prove that if $v \in G_{ts}$ and av, $vb \in W(X)$, then a and b are terminal segment and initial segment of a finite concatenation of elements in G, respectively.

There is $\{v_1, v_2, \dots, v_n\} \subseteq G$, since $av \in W(X)$ in which $av \subseteq v_1v_2 \dots v_n$ as well as

$$v_1v_2 \dots v_n = v_1'v_1''v_2 \dots v_j \dots v_{n-2}v_{n-1}'v_{n-1}''v_n'v_n'',$$

where $v_i = v_i'v_i''$ for $i = 1, n-1, n$, $v = v_{n-1}''v_n'$ and $a = v_1''v_2 \dots v_{n-2}v_{n-1}'$.

On the other hand, $v \subseteq v_{n-1}v_n$, then $v_n = v$ or $v_{n-1} = v$. Suppose now $v_n = v$; then $v_{n-1}v = v_{n-1}v_n = v_{n-1}'vv_n''$ and therefore $v_{n-1}' = \varepsilon$ or $v_n'' = \varepsilon$. If $v_{n-1}' = \varepsilon$, then $a = v_1''v_2 \dots v_{n-2}$.

Now, let $v_{n-1}' \neq \varepsilon$. Then, v_n'' must be an empty block and so $v_n' = v_n = v = v_{n-1}''v_n'$. Hence $v_{n-1}'' = \varepsilon$. Thus $v_{n-1} = v_{n-1}'$ and consequently $a = v_1''v_2 \dots v_{n-1}$. Similar reasoning works for $v_{n-1} = v$.

If $vb \in W(X)$, then by the use of the same routine as it was seen in the before case, we can show that there is $\{u_1, u_2, \dots, u_{n'}\} \subseteq G$ such that $b = u_2 \dots u_{n'-1}u_{n'}'$ where $u_{n'} = u_{n'}'u_{n'}''$.

Now, assume that $v \in G_{ts}$ and av, $vb \in W(X)$. Then, it follows that $av = v_1''v_2 \dots v_nv$ and $vb = vu_2 \dots u_{n'-1}u_{n'}'$.
Thus

$$avb = v_1''v_2 \dots v_nvu_2 \dots u_{n'-1}u_{n'}' \subseteq v_1v_2 \dots v_nvu_2 \dots u_{n'}.$$

Finally, $avb \in W(X)$ is obtained. Moreover,

$$h_{\text{syn}}(X) = \limsup_n \frac{1}{n} \log |\{u \in W_n(X) :\ v_0uv_0 \in W(X)\}|,$$

since $v_0 \in S_t(X)$. Also, $v_0uv_0 \in W(X)$ if and only if u is a finite concatenation of elements in G, since there is exactly one path labeled v_0 in $\mathcal{H}_G = X_0^+$, so we are done. $\square$

We recall that the elements of $S(X)/\sim$ are equivalence classes, since $\sim$ is an equivalence relation on $S(X)$. For every $v \in G_{ts} \subset S(X)$, the equivalence class containing v is denoted by the symbol $[v]$. Also, we recall that, a point $x \in X$ is periodic if $\sigma^n(x) = x$ for some $n \in \mathbb{N}$.

We consider Per$X$ to be the set of all periodic points of the shift space X. The minimal period of a periodic point $x \in$ Per$X$ is denoted by period(x). Thus period(X) is the greatest common divisor of period$(x) : x \in$ Per$X$.

It is important to note that Thompson [12] established the existence of a subsequence. This subsequence allows us to replace the limsup by an actual lim thereby justifying the use of a true limit rather than only a limsup.

Proposition 6.2. *Let G be a minimal generator for the coded system X and let $v \in G_{ts}$. Then, $h_{syn}(X)$ is equal to*

$$\lim_{n\to\infty} \frac{1}{n period(X_{([v],0)})} \log |\{(v_1, \dots, v_k) \in G^k : \Sigma_{i=1}^k |v_i| = n period(\mathcal{L}(X_{([v],0)}))\}|.$$

Proof. Since $v \in S(X)$ (see [7, Theorem 5.2]), so X is a synchronized system.

Let $a \in W(X)$. There are blocks $b_1, b_2 \in W(X)$ such that $vb_1ab_2v \in W(X)$. Consequently, $(vb_1ab_2)^\infty \in R(X)$ as well as $a \in W(R(X))$. Therefore we conclude that $X = R(X)$. From the fact all elements of $S(X)$ are equivalent then $S(X)/\sim = \{[v]\}$.

Note that the periodic element $(vb_1ab_2)^\infty \in X_{([v],0)}$, since $a \in W(X)$ was an arbitrary. Thus by [13, Lemma 3.5], $\overline{X_{([v],0)}} = X$, and

$$h_{\text{syn}}(X) = \lim_{n\to\infty} \frac{1}{n\text{period}(X_{([v],0)})} \log |\{u \in W_{n\text{period}(X_{([v],0)})}(X) : vuv \in W(X)\}|, \tag{6.4}$$

by [13, Proposition 3.2]. However,

$$\{u \in W_N(X) : vuv \in W(X)\} = \{(v_1, \dots, v_k) \in G^k : \Sigma_{i=1}^k |v_i| = N\},$$

where $N = n\text{period}(X_{([v],0)})$, by Lemma 6.1. □

An implication of the above proposition is the fact that if X is mixing synchronized, then by [13, Lemma 3.6], period$(X_{([v],0)}) = 1$. Therefore,

$$h_{\text{syn}}(X) = \lim_{n\to\infty} \frac{1}{n} \log |\{(v_1, \dots, v_k) \in G^k : \Sigma_{i=1}^k |v_i| = n\}|.$$

7. Computing derived shift space

Let $\partial(X)$ be as in (6.2), and denote the set of synchronizing blocks for $R(\partial^k(X))$ by $S(\partial^k(X))$.

Define the *depth* of X to be Depth$(X) = \sup\{k \in \mathbb{N} : \partial^k(X) \neq \emptyset\}$. If $\alpha \in S(\partial^k(X))/\sim$, then $(\partial^k(X))_{(\alpha,0)}$ is denoted by $X_{(\alpha,k)}$. Recall that when X is a sofic shift with nonwandering part $R(X)$, we can consider the shift space

$$\partial X = \{x \in R(X) : x \text{ consists of no blocks that are synchronizing for } R(X)\},$$

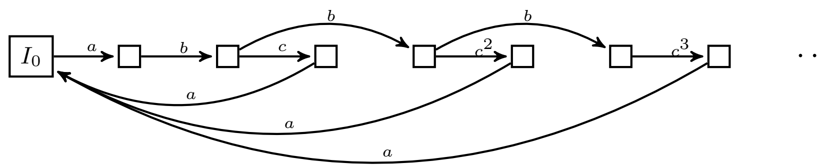

FIGURE 4. The graph $\mathcal{G}_{u_i \hookrightarrow a_i}$; merged graph $\mathcal{H}_G$ from $\mathcal{G}_{u_i \hookrightarrow a_i}$ have two irreducible components $\{b^\infty\}$ and $\{c^\infty\}$ at level 0 for $\partial(\overline{< G >})$.

which is called the derived shift space of X Since ∂X is a shift space we can continue and consider $\partial(\partial X) = \partial^2 X$, $\partial(\partial^2 X) = \partial^3 X$, etc.
We define the *depth* of X to be

$$Depth(X) = \sup\{n \in \mathbb{N} : \ \partial^n X \neq \emptyset\}.$$

The irreducible components at level 0 of $\partial^k X$ are called the irreducible components of X at level k. They are denoted by $X_{(\alpha,\, k)}$ [2, Proposition 2.1.2].

Recall an *internal subblock* of $u = a_1 a_2 \dots a_k$ is a subblock of $a_2 \dots a_{k-1}$, which is denoted by u^0.

Proposition 7.1. *Let X be a coded system generated by G. Then*

$$\partial X = \{\cup_i u_i : \ u_i \subseteq v_{i_1} \dots v_{i_n} \ s.t. \ v_{i_j} \in G, u_i = (u_{i+1})^0 \ and \ u_i \notin S(X)\}.$$

Proof. Set $x := \cup_i u_i$, where $u_i \in W(X)$, $u_i = (u_{i+1})^0$ and $u_i \notin S(X)$. If there is a $m \in S(X)$ such that $m \subseteq x$, then $m \subseteq u_i$ for some $i \in \mathbb{N}$.

In this case, u_i is also a synchronizing block, since any block that contains a synchronizing block itself becomes synchronizing. Consequently, the space X contains no synchronizing blocks and so $x \in \partial X$. Let $x \in \partial X$. Then, for each $i \in \mathbb{N}$, $x_{[-i,\, i]} \in W(X)$ is a block with no synchronizing subblock. Since G is a generator for X, then by the definition, every block is constructed by the generating elements of G. Therefore there are subblocks $v_{i_1} \dots v_{i_n}$ for some $v_{i_j} \in G$ such that $x_{[-i,\, i]} \subseteq v_{i_1} \dots v_{i_n}$. Set $u_i := x_{[-i,\, i]}$; then the conclusion is obtained immediately. □

A *minimal synchronizing* block is a block whose proper subblocks are not synchronizing.

Corollary 7.2. *Assume that all elements of the generator G of a coded system X are minimal synchronizing block. Then,*

$$\partial X = \{\cup_n u_n : \ u_n \subsetneq v \ for\ some\ v \in G \ and \ u_n = (u_{n+1})^0\}.$$

Example 7.3. Let X be a subshift over $\mathcal{A}$ generated by $G := \{u_i := ab^i c^i a : \ i \in \mathbb{N}\}$, where $a \in \mathcal{A}$, $\{b,\, c\} \subseteq W(X)$ and $a \notin bc$. Then,

(1) $G = G_{ts}$.
(2) $W(X) - S(X) = \{b^i c^j : \ i,\, j \geq 0\} \cup \{a\}$ and so $\partial X = \{b^\infty,\, c^\infty,\, b^\infty c^\infty\}$.

Example 7.3 gives a cover of a strong synchronized system of depth 2, that contains one irreducible component X at level 0 and two irreducible components

$\{b^\infty\}$ and $\{c^\infty\}$ at level 1 (Figure 4). One can easily do this for infinitely many irreducible components.

In fact, let $P = \cup P_i$ be the set of all prime numbers such that $|P_i| = \infty$ and $P_i \cap P_j = \emptyset$ for all $i, j \in \mathbb{N}$. Set $G_i := \{2(10^{2i-1}1)^n 2 :\ n \in P_i\}$, where $i \in \mathbb{N}$ and Suppose that X is a subshift generated by $G := \cup_i G_i$. Then, it is not hard to see that $G = G_{ts}$ and $\partial X = \{(10^{2i-1}1)^\infty :\ i \geq 1\} \cup \{0^\infty\}$.

Corollary 7.4. *Suppose that X is a subshift with a minimal generator G and that $v \in G$. Then, $|V_v := \{I \in \mathcal{V}(\mathcal{H}_G) :\ t(\pi_v) = I\}| < \infty$.*

If $w_+(I_0) = w_+(I)$ for all $I \in V_v$, then $v \in G_{ts}$ and so is a strong synchronizing block.

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[1] Institute of Artificial Intelligence and Social and Advanced Technologies, Dez.C., Islamic Azad University, Dezful, Iran.

Email address: `m.shahamat@iau.ac.ir; yousefikia74@iau.ac.ir`