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THE MULTIPLICATIVE TENSOR PRODUCT OF n-FOLD MATRIX FACTORIZATION OF POLYNOMIALS

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Abstract. The motivation behind this work is the generalization of the construction of the multiplicative tensor product of matrix factorizations. In fact, in this paper, we first construct for each $n \geq 2$, three bifunctorial operations $\widetilde{\otimes}_n$, $\overline{\otimes}_n$, and $\widetilde{\otimes}'_n$, which are such that if X (respectively, Y) is an n-fold matrix factorization of $f \in K[x_1, x_2, \ldots, x_r]$ (respectively, $g \in K[y_1, y_2, \ldots, y_s]$), then both $X\widetilde{\otimes}_nY$ and $X\widetilde{\otimes}'_nY$ are n-fold matrix factorizations of $fg \in K[x, y]$, where $x = x_1, x_2, \ldots, x_r$ and $y = y_1, y_2, \ldots, y_s$. When n is even, $X\widetilde{\otimes}'_nY$ is an n-fold matrix factorization of $fg \in K[x, y]$. We call $\widetilde{\otimes}_n$ the multiplicative tensor product of n-fold matrix factorizations, $\overline{\otimes}_n$ its refined version and $\widetilde{\otimes}'_n$ its variant.

Next, the associativity of these operations is proven. Moreover, we show that for each $n \geq 2$, $\widetilde{\otimes}_n$ can readily be used to construct a *semi-unital semi-monoidal category*. Finally, we briefly compare the operations $\widetilde{\otimes}_n$, $\overline{\otimes}_n$,and $\widetilde{\otimes}'_n$.

1. Introduction

The notion of matrix factorization of an element in a ring has been defined over the years in several papers under different angles depending on what authors wanted to do. Originally, a matrix factorization of an element f in a ring R with unity (see [10, p.15]) is an ordered pair of maps of free R-modules $\phi : F \to G$ and $\psi : G \to F$ such that $\phi\psi = f \cdot 1_G$ and $\psi\phi = f \cdot 1_F$. In their paper [7] published in 2016, Carqueville and Murfet defined a matrix factorization using linear factorizations and $\mathbb{Z}_2$-graded modules (see [7, p.8]). Another (simpler) way

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in which matrix factorizations of a polynomial can be defined is found in Yoshino's paper [21]. Yoshino [21] defined a matrix factorization (of size n) of a power series $f \in K[[x]]$ to be a pair of matrices (P, Q) such that $fI_n = PQ$. We will recall the extension of this definition to the case of n-fold matrix factorizations of a polynomial f where the number of matrix factors of f is $n \geq 2$ (see Definition 3.1). A correlation between these definitions of a matrix factorization of an element in a ring can be found in [13]. In what follows, we will sometimes drop the word "fold" in the phrase "n-fold matrix factorization" and simply talk of "n-matrix factorization".

Matrix factorizations and some of their properties were studied in several papers including [10, 21, 7, 9, 6, 11]. It is essential to study matrix factorizations of polynomials and their properties for many reasons (see the introductions of [11, 14]).

For $n \geq 2$, the category of n-matrix factorizations has been studied in several papers ([21, 8, 2, 15, 4, 3, 5]) with diverse motivations. Reference [20] contains a recent study of this category. The objects of this category are n-matrix factorizations of a polynomial. Arrows in this category are n-tuples of matrices making a certain diagram commutative (see section 3). We shall denote by $MF(R, f)_n$ or $MF(f)_n$ (when there is no risk of confusion) the category of n-matrix factorizations of a polynomial $f \in R = K[x] = K[x_1, \ldots, x_n]$ (see section 3). Some objects of this new category will be used in this work in section 5 where we give an application of $\widetilde{\otimes}_n$ for each $n \geq 2$.

In [11, 12], the multiplicative tensor product of matrix factorizations $\widetilde{\otimes}$ and its refined version were constructed. They are bifunctorial operations, which yield a matrix factorization of the product fg of two power series f and g from their respective matrix factorizations. This is in contrast with the Yoshino tensor product of matrix factorizations $\widehat{\otimes}$, which produces a matrix factorization of the sum $f + g$ of two power series f and g from their respective matrix factorizations. Moreover, $\widetilde{\otimes}$ was used in conjunction with the Yoshino tensor product of matrix factorizations $\widehat{\otimes}$ to reduce the size of matrix factors on the class of summand-reducible polynomials [11]. Also, $\widetilde{\otimes}$ was replaced by its refined version $\overline{\otimes}$ in [12] to improve the results in [11].

In what follows, by the size of a square matrix M, we mean the number of rows of M.

One of the goals of this paper is to generalize the construction of $\widetilde{\otimes}$ to the case of polynomials with n-matrix factors. The case when $n = 3$ was successfully treated in [14].

Our first main result is stated as follows.

Theorem 1.1 (Theorem A). (1) *Let X be an n-matrix factorization of $f \in K[x]$ of size p and let Y be an n-matrix factorization of $g \in K[y]$ of size m. Then, there is a tensor product $\widetilde{\otimes}_n$ of n-matrix factorizations that produces an n-matrix factorization $X\widetilde{\otimes}_n Y$ of the product $fg \in K[x_1, \ldots, x_r, y_1, \ldots, y_s]$, which is of size $2pm$. Moreover, $\widetilde{\otimes}_n$ is called the multiplicative tensor product of n-matrix factorizations.*

(2) *The multiplicative tensor product of n-matrix factorizations* $(-)\widetilde{\otimes}_n(-) : MF(K[x], f)_n \times MF(K[y], g)_n \to MF(K[x, y], fg)_n$ *is a bifunctor.*

Our second main result is the construction of a refined version $\overline{\otimes}$ of $\widetilde{\otimes}_n$ and is stated as follows.

Theorem 1.2 (Theorem B). (1) *Let X be an n-matrix factorization of $f \in K[x]$ of size p and let Y be an n-matrix factorization of $g \in K[y]$ of size m. Then, there is a tensor product $\overline{\otimes}_n$ of n-matrix factorizations that produces an n-matrix factorization $X\overline{\otimes}_nY$ of the product $fg \in K[x_1, \dots, x_r, y_1, \dots, y_s]$, which is of size pm. Moreover, $\overline{\otimes}_n$ is called the refined multiplicative tensor product of n-matrix factorizations.*

(2) *The refined multiplicative tensor product of n-matrix factorizations* $(-)\overline{\otimes}_n(-) : MF(K[x], f)_n \times MF(K[y], g)_n \to MF(K[x, y], fg)_n$ *is a bifunctor.*

In addition, $\overline{\otimes}_n$ is the refined version of $\widetilde{\otimes}_n$. It generalizes $\overline{\otimes}$, which was used to prove results in [12]. From Definitions 4.3 and 4.5, we see that the operation $\widetilde{\otimes}_n$ can be obtained from $\overline{\otimes}_n$ by applying the direct sum of matrices, which has the effect of simply doubling the sizes of matrix factors.

Another bifunctorial operation hereunder denoted by $\widetilde{\otimes}'_n$ that has the same effect as $\widetilde{\otimes}_n$ when n is even, will be defined in this work. It will be referred to as the variant of $\widetilde{\otimes}_n$.

Thus, another result of this work is the following theorem.

Theorem 1.3 (Theorem C). (1) *Let X be an n-matrix factorization of $f \in K[x]$ of size p and let Y be an n-matrix factorization of $g \in K[y]$ of size m. Then for any even integer $n \geq 2$, there is a tensor product $\widetilde{\otimes}'_n$ of n-matrix factorizations that produces an n-matrix factorization $X\widetilde{\otimes}'_nY$ of the product $fg \in K[x_1, \dots, x_r, y_1, \dots, y_s]$, which is of size $2pm$. Moreover, $\widetilde{\otimes}'_n$ is called the variant of $\widetilde{\otimes}_n$.*

(2) *The variant of the multiplicative tensor product of n-matrix factorizations* $(-)\widetilde{\otimes}'_n(-) : MF(K[x], f)_n \times MF(K[y], g)_n \to MF(K[x, y], fg)_n$ *is a bifunctor.*

Since the proofs of the foregoing three results are very similar, we will prove only Theorem 1.1 in this work (see Theorem 4.11). To prove Theorems 1.2 and 1.3, only little adaptations are needed from the proof of Theorem 1.1 and so it is not necessary to also present their proofs here. Moreover, we will show that the bifunctorial operations $\widetilde{\otimes}_n$, $\overline{\otimes}_n$ and $\widetilde{\otimes}'_n$ are associative.

In addition in this work, $\widetilde{\otimes}_n$ will be used to construct a *semi-unital semi-monoidal category* for each $n \geq 2$ (see section 5).
The concept of *semi-unital semi-monoidal category* is very important. In fact in [1], the author used the so called Takahashi's tensor-like product $\boxtimes$ of semi-modules over an associative semi-ring A [19], to introduce notions of semi-unital semi-rings and semi-counital semi-corings. However, these could not be realized as monoids (comonoids) in the category ${}_AS_A$ of (A, A)-bisemi-modules, where A is

a semi-algebra. This is mainly due to the fact that the category $({}_AS_A, \boxtimes, A)$ is not monoidal in general. Motivated by the desire to fix this problem, the author of [1] introduced and investigated a notion of *semi-unital semi-monoidal categories* with prototype $({}_AS_A, \boxtimes, A)$ and investigated semi-monoids (semi-comonoids) in such categories as well as their categories of semi-modules (semi-comodules). He realized that although the base semi-algebra A is not a unit in ${}_AS_A$, A nevertheless has the properties of what he called a *semi-unit*. This motivated the introduction of a more generalized notion of monads (comonads) in arbitrary categories (for more on this, see [1]). It is worth noting that the notion of *semi-unital semi-monoidal category* is a generalization of the classical notion of monoidal category (see [1, Remark 3.3]). From the preceding lines, it is obvious that the importance of this concept cannot be overemphasized as it led to the generalization of some concepts like monads, which have far-reaching applications in computer science. Unfortunately, it is still extremely scarce in the literature as apart from the example given in [14], we found only one example of it (see [1, Theorem 5.11]).

In this paper, we generalize the construction given in [14]. Actually, we use our newly defined tensor product $\widetilde{\otimes}_n$ together with some special objects from the category $MF(1)_n$ of n-matrix factorizations of the constant polynomial *1* to construct a *semi-unital semi-monoidal category* for each $n \geq 2$. In fact, we will define the concept of *one-step connected category* and prove that for each $n \geq 2$, there is a *one-step connected subcategory* of $(MF(1)_n, \widetilde{\otimes}_n)$ that is a *semi-unital semi-monoidal category* (see Theorem 5.5).

Thus, another main result of this paper is stated as follows.

Theorem 1.4 (Theorem D). *For each $n \geq 2$, there is a one-step connected subcategory of $(MF(1)_n, \widetilde{\otimes}_n)$ that is a semi-unital semi-monoidal category.*

Furthermore, the operations $\widetilde{\otimes}_n$, $\overline{\otimes}_n$, and $\widetilde{\otimes}'_n$ will be compared. As a result, we shall prove the following result.

Lemma 1.5 (Lemma A). *For each even integer $n \geq 2$, there is a one-step connected subcategory of $(MF(1)_n, \widetilde{\otimes}'_n)$ that is semi-monoidal.*

This paper is organized as follows: In the next section, we recall the definitions of some needful classes of categories. In section 3, we recall the definition of the n-matrix factorization of a polynomial and proceed to the presentation of the construction of the category of n-matrix factorizations of a polynomial for the case $n \geq 2$. In section 4, the multiplicative tensor product of n-matrix factorizations, its refined version and its variant are constructed and proved to be bifunctors. Their associativity is also proven in this section. An application of $\widetilde{\otimes}_n$ is provided for in section 5. Finally in the last section, the operations $\widetilde{\otimes}_n$, $\overline{\otimes}_n$, and $\widetilde{\otimes}'_n$ are compared.

2. Some needful definitions

This section gathers some needful definitions like that of *semi-monoidal categories* and *monoidal categories*, which are well known in the literature (see [18, 16, 1]) together with the definition of the notion of a *semi-unital semi-monoidal category* (see Definition 2.3).

Definition 2.1. A **semi-monoidal category** $\mathcal{M} =< \mathcal{M}, \Box, \alpha >$ is a category $\mathcal{M}$, a bifunctor $\Box : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ and a natural transformation α, satisfying the following condition:
• α is a natural isomorphism with components $\alpha_{a,b,c} : (a\Box b)\Box c \to a\Box(b\Box c)$ such that the following pentagonal diagram

$$\begin{tikzcd}
a\Box(b\Box(c\Box d)) \arrow[rr, "1_a\Box\alpha"] \arrow[d, "\alpha"'] & & a\Box((b\Box c)\Box d) \arrow[d, "\alpha"] \\
(a\Box b)\Box(c\Box d) \arrow[dr, "\alpha"'] & & (a\Box(b\Box c))\Box d \arrow[dl, "\alpha\Box 1_d"] \\
& ((a\Box b)\Box c)\Box d &
\end{tikzcd}$$

commutes for all $a, b, c, d \in \mathcal{M}$. ($\alpha$ is also called associator (see [7, p.11])).

Definition 2.2. A **monoidal category** $\mathcal{M} =< \mathcal{M}, \Box, e, \alpha, \lambda, \rho >$ is a semi-monoidal category $\mathcal{M} =< \mathcal{M}, \Box, \alpha >$ with an object $e \in \mathcal{M}$, and two natural isomorphisms $\lambda\, and\, \rho$, such that
• λ and ρ are natural (see [7, p.10], λ and ρ are also called left and right unit actions or unitors).
$\lambda_a : e\Box a \cong a$, $\rho_a : a\Box e \cong a$, for all objects $a \in \mathcal{M}$. Moreover, the triangular diagram

$$\begin{tikzcd}
a\Box(e\Box b) \arrow[rr, "\alpha"] \arrow[dr, "1_a\Box\lambda_b"'] & & (a\Box e)\Box b \arrow[dl, "\rho_a\Box 1_b"] \\
& a\Box b &
\end{tikzcd}$$

commutes for all $a, b \in \mathcal{M}$ and $\lambda_e = \rho_e : e\Box e \to e$.

Definition 2.3. [1] Let $(\mathcal{M}, \Box, \alpha)$ be a semi-monoidal category with natural isomorphism $\alpha_{X,Y,Z} : (X\Box Y)\Box Z \to X\Box(Y\Box Z)$ for all $X, Y, Z \in \mathcal{M}$. Let $\mathbb{I}$ stand for the identity endofunctor on any given category. We say that $\mathbf{I} \in \mathcal{M}$ is a **semi-unit** if the following conditions hold:

(1) There is a natural transformation $\omega : \mathbb{I} \to (\mathbf{I}\Box -)$;
(2) There exists an isomorphism of functors $\mathbf{I}\Box - \cong -\Box\mathbf{I}$, that is, there is a natural isomorphism $l_X : \mathbf{I}\Box X \cong X\Box\mathbf{I}$ in $\mathcal{M}$ with inverse q_X, for each object X of $\mathcal{M}$, such that $l_{\mathbf{I}} = q_{\mathbf{I}}$ and the following diagrams are commutative for all $X, Y \in \mathcal{M}$:

$$\begin{tikzcd}
(\mathbf{I}\Box a)\Box b \arrow[r, "\alpha_{\mathbf{I},a,b}"] \arrow[d, "l_a\Box b"'] & \mathbf{I}\Box(a\Box b) \arrow[r, "l_{a\Box b}"] & (a\Box b)\Box\mathbf{I} \arrow[d, "\alpha_{a,b,\mathbf{I}}"] \\
(a\Box\mathbf{I})\Box b \arrow[r, "\alpha_{a,\mathbf{I},b}"] & a\Box(\mathbf{I}\Box b) \arrow[r, "a\Box l_b"] & a\Box(b\Box\mathbf{I})
\end{tikzcd} \tag{2.1}$$

$$\begin{tikzcd} a\Box b \arrow[rr, "\omega_a\Box b"] \arrow[dr, "\omega_{a\Box b}"'] & & (\mathbf{I}\Box a)\Box b \arrow[dl, "\cong"] \\ & \mathbf{I}\Box(a\Box b) & \end{tikzcd} \tag{2.2}$$

$$\begin{tikzcd} a\Box b \arrow[rr, "a\Box\omega_b"] \arrow[dr, "\omega_{a\Box b}"'] & & a\Box(\mathbf{I}\Box b) \arrow[dl, "\cong"] \\ & \mathbf{I}\Box(a\Box b) & \end{tikzcd} \tag{2.3}$$

A **semi-unital semi-monoidal category** is a semi-monoidal category with a semi-unit.

As hinted in [1, 16, Remark 3.3], since every unit is a semi-unit, the notion of semi-unital semi-monoidal categories generalizes the classical notion of monoidal categories.

3. The category of n-matrix factorizations of $f \in R = K[x]$

In this section, the definition of an n-matrix factorization is first of all recalled. Next, we recall the construction of the category of n-matrix factorizations (see [20]) of $f \in R = K[x]$, where $x = x_1, \ldots, x_r$. In fact, it is a generalization of the definition of the category of matrix factorizations of $f \in R$ for the case where f has only two matrix factors (see [21, section 1]) to the case where f has n matrix factors with $n \geq 2$.

The following definition generalizes the definition of matrix factorizations given in the first section of [21]. It is also found in [17] in a different form (see [17, Definition 1.1]).

Definition 3.1. An $m \times m$ n**-matrix factorization** of a polynomial $f \in R = K[x]$ is an n-tuple of $m \times m$ matrices $(A_1, A_2, \ldots, A_n)$ such that $A_1A_2\cdots A_n = fI_m$, where I_m is the $m \times m$ identity matrix and the coefficients of each matrix A_i, $i \in \{1, 2, \ldots, n\}$, is taken from R.

The category of n-matrix factorizations of a polynomial $f \in K[x]$, where $x = x_1, \ldots, x_r$, which we denote by $MF(R, f)_n$ or $MF(f)_n$ (when there is no risk of confusion) is defined as follows:
- The objects are the n-matrix factorizations of f.
- Given two n-matrix factorizations of f, $(\theta_1, \theta_2, \ldots, \theta_n)$ and $(\psi_1, \psi_2, \ldots, \psi_n)$, respectively, of sizes n_1 and n_2, a morphism from $(\theta_1, \theta_2, \ldots, \theta_n)$ to $(\psi_1, \psi_2, \ldots, \psi_n)$ is an n-tuple of matrices $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ each of size $n_2 \times n_1$, which makes the

following diagram commutative:

$$\begin{tikzcd}
K[x]^{n_1} \arrow[r, "\theta_n"] \arrow[d, "\alpha_1"'] & K[x]^{n_1} \arrow[r, "\theta_{n-1}"] \arrow[d, "\alpha_n"'] & K[x]^{n_1} \arrow[r, "\theta_{n-2}"] \arrow[d, "\alpha_{n-1}"] & K[x]^{n_1} \arrow[r, dashed] \arrow[d, "\alpha_{n-2}"] & K[x]^{n_1} \arrow[r, "\theta_1"] \arrow[d, "\alpha_2"'] & K[x]^{n_1} \arrow[d, "\alpha_1"] \\
K[x]^{n_2} \arrow[r, "\psi_n"] & K[x]^{n_2} \arrow[r, "\psi_{n-1}"] & K[x]^{n_2} \arrow[r, "\psi_{n-2}"] & K[x]^{n_2} \arrow[r, dashed] & K[x]^{n_2} \arrow[r, "\psi_1"] & K[x]^{n_2}
\end{tikzcd}$$

That is,

$$\begin{cases} \alpha_1\theta_1 = \psi_1\alpha_2, \\ \alpha_i\theta_i = \psi_i\alpha_{i+1}, \quad 2 \leq i \leq n-1, \\ \alpha_n\theta_n = \psi_n\alpha_1. \end{cases}$$

• Given three n-matrix factorizations of f: $(\theta_1, \theta_2, \ldots, \theta_n)$, $(\psi_1, \psi_2, \ldots, \psi_n)$ and $(\tau_1, \tau_2, \ldots, \tau_n)$, respectively, of sizes n_1, n_2 and n_3, the composition

$(\beta_1, \ldots, \beta_n) \circ (\alpha_1, \ldots, \alpha_n) : (\theta_1, \ldots, \theta_n) \xrightarrow{(\alpha_1,\ldots,\alpha_n)} (\psi_1, \ldots, \psi_n) \xrightarrow{(\beta_1,\ldots,\beta_n)} (\tau_1, \ldots, \tau_n)$ is the n-tuple of matrices $(\beta_1\alpha_1, \ldots, \beta_n\alpha_n)$.

• It is easy to see that *associativity of composition of maps of n-matrix factorizations of f* is a consequence of the fact that matrix multiplication is associative.

• For any $p \times p$ n-matrix factorization $(\theta_1, \theta_2, \ldots, \theta_n)$ of f, there is a map $1_{(\theta_1,\theta_2,\ldots,\theta_n)} :$ $(\theta_1, \theta_2, \ldots, \theta_n) \to (\theta_1, \theta_2, \ldots, \theta_n)$, which is actually the n-tuple of identity $p \times p$ matrices $(I_p, I_p, \ldots, I_p)$.

• It is also clear that composing any map of n-matrix factorizations of f with $1_{(\theta_1,\theta_2,\ldots,\theta_n)}$ from the left or the right (whenever the composition is possible) leaves the given map unchanged. This ends the definition of the category of n-matrix factorizations of $f \in R = K[x]$.

4. The multiplicative tensor product of n-matrix factorizations, its refined version and its variant

In this section, we construct the multiplicative tensor product of n-matrix factorizations, its refined version and its variant, respectively, denoted by $\widetilde{\otimes}_n$, $\overline{\otimes}_n$ and $\widetilde{\otimes}'_n$. Moreover, the bifunctoriality of $\widetilde{\otimes}_n$ is proved (see subsection 4.2). The proof of the bifunctoriality of $\widetilde{\otimes}'_n$ and $\overline{\otimes}_n$ are omitted because they are similar to that of $\widetilde{\otimes}_n$. Finally, we show that these operations are associative.

4.1. Some definitions and lemmas.

Definition 4.1. Let $X = (\theta_1, \theta_2, \ldots, \theta_n)$ be an n-matrix factorization of $f \in K[x]$ of size p and let $X' = (\theta'_1, \theta'_2, \ldots, \theta'_n)$ be an n-matrix factorization of $g \in K[y]$ of size m. Then, for $i \in \{1, 2, \ldots, n\}$, θ_i and θ'_i can be considered as matrices over $L = K[x, y]$. The **multiplicative tensor product of n-matrix factorizations** $X \widetilde{\otimes}_n X'$ is given by

$$\begin{aligned} X \widetilde{\otimes}_n X' &= (\theta_1, \theta_2, \ldots, \theta_n) \widetilde{\otimes}_n (\theta'_1, \theta'_2, \ldots, \theta'_n) \\ &= ((\theta_1 \otimes \theta'_1) \oplus (\theta_1 \otimes \theta'_1), (\theta_2 \otimes \theta'_2) \oplus (\theta_2 \otimes \theta'_2), \ldots, (\theta_n \otimes \theta'_n) \oplus (\theta_n \otimes \theta'_n)) \\ &= (\begin{bmatrix} \theta_1 \otimes \theta'_1 & 0 \\ 0 & \theta_1 \otimes \theta'_1 \end{bmatrix}, \begin{bmatrix} \theta_2 \otimes \theta'_2 & 0 \\ 0 & \theta_2 \otimes \theta'_2 \end{bmatrix}, \ldots, \begin{bmatrix} \theta_n \otimes \theta'_n & 0 \\ 0 & \theta_n \otimes \theta'_n \end{bmatrix}), \end{aligned}$$

where each component is an endomorphism on $L^p \otimes_L L^m$.

Lemma 4.2. *Let $X = (\theta_1, \theta_2, \ldots, \theta_n)$ be an n-matrix factorization of $f \in K[x]$ of size p and let $X' = (\theta'_1, \theta'_2, \ldots, \theta'_n)$ be an n-matrix factorization of $g \in K[y]$ of size m. Then, $X\widetilde{\otimes}_n X'$ is an object of $MF(K[x,y], fg)_n$ of size $2pm$.*

Proof. We have

$$
\begin{aligned}
&\begin{bmatrix} \theta_1 \otimes \theta'_1 & 0 \\ 0 & \theta_1 \otimes \theta'_1 \end{bmatrix} \begin{bmatrix} \theta_2 \otimes \theta'_2 & 0 \\ 0 & \theta_2 \otimes \theta'_2 \end{bmatrix} \cdots \begin{bmatrix} \theta_n \otimes \theta'_n & 0 \\ 0 & \theta_n \otimes \theta'_n \end{bmatrix} \\
&\quad = \begin{bmatrix} theta_1\theta_2 \cdots \theta_n \otimes \theta'_1\theta'_2 \cdots \theta'_n & 0 \\ 0 & \theta_1\theta_2 \cdots \theta_n \otimes \theta'_1\theta'_2 \cdots \theta'_n \end{bmatrix} \\
&\quad = \begin{bmatrix} f1_p \otimes g1_m & 0 \\ 0 & f1_p \otimes g1_m \end{bmatrix} \\
&\quad = fg \begin{bmatrix} 1_p \otimes 1_m & 0 \\ 0 & 1_p \otimes 1_m \end{bmatrix} \\
&\quad = fg \cdot 1_{2pm}.
\end{aligned}
$$

The second equality is due to the fact that $\theta_1\theta_2 \cdots \theta_n = f1_p$ and $\theta'_1\theta'_2 \cdots \theta'_n = g1_m$. So, $X\widetilde{\otimes}X'$ is an object of $MF(fg)_n$ of size $2pm$ as claimed. □

For any even integer $n \geq 2$, we define variant of $\widetilde{\otimes}_n$.

Definition 4.3. Let $n \geq 2$ be an even integer. Let $X = (\theta_1, \theta_2, \ldots, \theta_n)$ be an n-matrix factorization of $f \in K[x]$ of size p and let $X' = (\theta'_1, \theta'_2, \ldots, \theta'_n)$ be an n-matrix factorization of $g \in K[y]$ of size m. Thus, for $i \in \{1, 2, \ldots, n\}$, θ_i and θ'_i can be considered as matrices over $L = K[x,y]$. The **variant of the multiplicative tensor product of n-matrix factorizations** $X\widetilde{\otimes}'_n X'$ is given by

$$
\begin{aligned}
X\widetilde{\otimes}'_n X' &= (\theta_1, \theta_2, \ldots, \theta_n)\widetilde{\otimes}'_n(\theta'_1, \theta'_2, \ldots, \theta'_n) \\
&= \left(\begin{bmatrix} 0 & \theta_1 \otimes \theta'_1 \\ \theta_1 \otimes \theta'_1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \theta_2 \otimes \theta'_2 \\ \theta_2 \otimes \theta'_2 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & \theta_n \otimes \theta'_n \\ \theta_n \otimes \theta'_n & 0 \end{bmatrix} \right),
\end{aligned}
$$

where each component is an endomorphism on $L^p \otimes_L L^m$.

Nota Bene. Each time we will talk about $\widetilde{\otimes}'_n$ (i.e., the variant of $\widetilde{\otimes}_n$), it will be assumed (sometimes without mentioning) that n is even because this variant makes sense only when n is even. That is, it helps obtain a matrix factorization of two polynomials only when n is even. When n is odd, one cannot obtain a matrix factorization of the given polynomials as we now illustrate for $n = 3$:

Keeping the notations of Definition 4.3, we have

$$
\begin{aligned}
X\widetilde{\otimes}'_3 X' &= (\theta_1, \theta_2, \theta_3)\widetilde{\otimes}'_3(\theta'_1, \theta'_2, \theta'_3) \\
&= \left(\begin{bmatrix} 0 & \theta_1 \otimes \theta'_1 \\ \theta_1 \otimes \theta'_1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \theta_2 \otimes \theta'_2 \\ \theta_2 \otimes \theta'_2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \theta_3 \otimes \theta'_3 \\ \theta_3 \otimes \theta'_3 & 0 \end{bmatrix} \right).
\end{aligned}
$$

Now,

$$\begin{bmatrix} 0 & \theta_1 \otimes \theta'_1 \\ \theta_1 \otimes \theta'_1 & 0 \end{bmatrix} \begin{bmatrix} 0 & \theta_2 \otimes \theta'_2 \\ \theta_2 \otimes \theta'_2 & 0 \end{bmatrix} \begin{bmatrix} 0 & \theta_3 \otimes \theta'_3 \\ \theta_3 \otimes \theta'_3 & 0 \end{bmatrix}$$
$$= (\begin{bmatrix} \theta_1\theta_2 \otimes \theta'_1\theta'_2 & 0 \\ 0 & \theta_1\theta_2 \otimes \theta'_1\theta'_2 \end{bmatrix} \begin{bmatrix} 0 & \theta_3 \otimes \theta'_3 \\ \theta_3 \otimes \theta'_3 & 0 \end{bmatrix})$$
$$= \begin{bmatrix} 0 & \theta_1\theta_2\theta_3 \otimes \theta'_1\theta'_2\theta'_3 \\ \theta_1\theta_2\theta_3 \otimes \theta'_1\theta'_2\theta'_3 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & fgI_{pm} \\ fgI_{pm} & 0 \end{bmatrix} \neq fgI_{2pm}.$$

So $X\widetilde{\otimes}'_3 X' \neq fgI_{2pm}$, that is for $n = 3$, we do not obtain a matrix factorization of fg as claimed above. Similarly, one can observe that the construction of the operation $\widetilde{\otimes}'_n$ for any fixed odd $n \geq 3$ fails to yield a matrix factorization of fg.

Lemma 4.4. *Let $X = (\theta_1, \theta_2, \dots, \theta_n)$ be an n-matrix factorization of $f \in K[x]$ of size p and let $X' = (\theta'_1, \theta'_2, \dots, \theta'_n)$ be an n-matrix factorization of $g \in K[y]$ of size m. Then, $X\widetilde{\otimes}'_n X'$ is an object of $MF(K[x, y], fg)_n$ of size $2pm$.*

Proof. This proof is similar to that of Lemma 4.4 and so is omitted. □

Definition 4.5. Let $X = (\theta_1, \theta_2, \dots, \theta_n)$ be an n-matrix factorization of $f \in K[x]$ of size p and let $X' = (\theta'_1, \theta'_2, \dots, \theta'_n)$ be an n-matrix factorization of $g \in K[y]$ of size m. Thus, for $i \in \{1, 2, \dots, n\}$, θ_i and θ'_i can be considered as matrices over $L = K(x, y)$. The **refined multiplicative tensor product of n-matrix factorizations** $X\overline{\otimes}_n X'$ is given by

$X\overline{\otimes}_n X' = (\theta_1, \theta_2, \dots, \theta_n)\overline{\otimes}_n(\theta'_1, \theta'_2, \dots, \theta'_n) = ([\theta_1 \otimes \theta'_1], [\theta_2 \otimes \theta'_2], \dots, [\theta_n \otimes \theta'_n])$,

where each component is an endomorphism on $L^p \otimes_L L^m$.

Lemma 4.6. *Let $X = (\theta_1, \theta_2, \dots, \theta_n)$ be an n-matrix factorization of $f \in K[x]$ of size p and let $X' = (\theta'_1, \theta'_2, \dots, \theta'_n)$ be an n-matrix factorization of $g \in K[y]$ of size m. Then, $X\overline{\otimes} X'$ is an object of $MF(K[x, y], fg)_n$ of size pm.*

Proof. This proof is similar to that of Lemma 4.4 and so is omitted. □

4.2. **Functoriality of $\widetilde{\otimes}_n$, $\widetilde{\otimes}'_n$ and $\overline{\otimes}_n$.** This subsection is entirely devoted to the discussion of the bifunctoriality of the operation $\widetilde{\otimes}_n$. As earlier mentioned, the proof of the bifunctoriality of $\widetilde{\otimes}'_n$ and $\overline{\otimes}_n$ are completely analogous to the one we present for $\widetilde{\otimes}_n$ and so they are omitted.

Setting the stage:

Let $X_f = (\theta_1, \theta_2, \dots, \theta_n)$, $X'_f = (\theta'_1, \theta'_2, \dots, \theta'_n)$ and $X''_f = (\theta''_1, \theta''_2, \dots, \theta''_n)$ be objects of $MF(K[x], f)_n$, respectively, of sizes p, p' and p''. Let $X_g = (\tau_1, \tau_2, \dots, \tau_n)$, $X'_g = (\tau'_1, \tau'_2, \dots, \tau'_n)$ and $X''_g = (\tau''_1, \tau''_2, \dots, \tau''_n)$ be objects of $MF(K[y], g)_n$, respectively, of sizes m, m' and m''. Before stating one of the main results of this paper (see Theorem 4.11), we explicitly state some definitions (Definitions 4.7, 4.8, and 4.9) that will be alluded to in the proofs of Theorem 5.5 and of Lemma 5.4. We keep the notations given at the beginning of this section 4.2.

Definition 4.7. For a morphism

$\Phi_f = (\alpha_1, \dots, \alpha_n) : X_f = (\theta_1, \dots, \theta_n) \to X'_f = (\theta'_1, \dots, \theta'_n)$ in $MF(K[x], f)_n$ and for any $m \times m$ n-matrix factorization $X_g = (\tau_1, \dots, \tau_n)$ in $MF(K[y], g)_n$, we define $\Phi_f\widetilde{\otimes}X_g$

by

$$(\begin{bmatrix}\alpha_1\otimes 1_m & 0\\ 0 & \alpha_1\otimes 1_m\end{bmatrix},\begin{bmatrix}\alpha_2\otimes 1_m & 0\\ 0 & \alpha_2\otimes 1_m\end{bmatrix},\dots,\begin{bmatrix}\alpha_n\otimes 1_m & 0\\ 0 & \alpha_n\otimes 1_m\end{bmatrix}).$$

Definition 4.8. For a morphism
$\Phi_g = (\beta_1,\dots,\beta_n) : X_g = (\tau_1,\dots,\tau_n) \to X'_g = (\tau'_1,\dots,\tau'_n)$ in $MF(K[y], g)_n$ and for any $p\times p$ n-matrix factorization $X_f = (\theta_1,\dots,\theta_n)$ in $MF(K[x], f)$, we define $X_f\widetilde{\otimes}\Phi_g$ by

$$(\begin{bmatrix}1_p\otimes\beta_1 & 0\\ 0 & 1_p\otimes\beta_1\end{bmatrix},\begin{bmatrix}1_p\otimes\beta_2 & 0\\ 0 & 1_p\otimes\beta_2\end{bmatrix},\dots,\begin{bmatrix}1_p\otimes\beta_n & 0\\ 0 & 1_p\otimes\beta_n\end{bmatrix}).$$

Definition 4.9. For morphisms $\Phi_f = (\alpha_1,\dots,\alpha_n) : X_f = (\theta_1,\dots,\theta_n) \to X'_f = (\theta'_1,\dots,\theta'_n)$ and $\Phi_g = (\beta_1,\dots,\beta_n) : X_g = (\tau_1,\dots,\tau_n) \to X'_g = (\tau'_1,\dots,\tau'_n)$, respectively, in $MF(K[x], f)_n$ and $MF(K[y], g)_n$,
we define $\Phi_f\widetilde{\otimes}_n\Phi_g : X_f\widetilde{\otimes}_nX_g \to X'_f\widetilde{\otimes}_nX'_g$
that is, $(\theta_1,\theta_2,\dots,\theta_n)\widetilde{\otimes}_n(\tau_1,\tau_2,\dots,\tau_n) \to (\theta'_1,\theta'_2,\dots,\theta'_n)\widetilde{\otimes}_n(\tau'_1,\tau'_2,\dots,\tau'_n)$
by

$$(\begin{bmatrix}\alpha_1\otimes\beta_1 & 0\\ 0 & \alpha_1\otimes\beta_1\end{bmatrix},\begin{bmatrix}\alpha_2\otimes\beta_2 & 0\\ 0 & \alpha_2\otimes\beta_2\end{bmatrix},\dots,\begin{bmatrix}\alpha_n\otimes\beta_n & 0\\ 0 & \alpha_n\otimes\beta_n\end{bmatrix}).$$

Lemma 4.10. *$\Phi_f\widetilde{\otimes}_n\Phi_g$: $X_f\widetilde{\otimes}_nX_g \to X'_f\widetilde{\otimes}_nX'_g$ is an arrow in $MF(K[x,y], fg)_n$.*

Proof. For a better visibility and readability, in the following diagram for $i\in\{1,2,\dots,n\}$, instead of writing $(\alpha_i\otimes\beta_i)\oplus(\alpha_i\otimes\beta_i)$ and $(\theta_i\otimes\tau_i)\oplus(\theta_i\otimes\tau_i)$, we simply write $\alpha_i\otimes\beta_i$ and $\theta_i\otimes\tau_i$.

Likewise, instead of $(\alpha'_i\otimes\beta'_i)\oplus(\alpha'_i\otimes\beta'_i)$ and $(\theta'_i\otimes\tau'_i)\oplus(\theta'_i\otimes\tau'_i)$, we simply write $\alpha'_i\otimes\beta'_i$ and $\theta'_i\otimes\tau'_i$. This is done only for the diagram, the discussion after it continues normally without any abbreviations.
We need to show that the following diagram commutes.

$$\begin{tikzcd}
K[x,y]^q \arrow[r, "\theta_n\otimes\tau_n"'] \arrow[d, "s_1"'] & K[x,y]^q \arrow[r, "\theta_{n-1}\otimes\tau_{n-1}"'] \arrow[d, "s_n"'] & K[x,y]^q \arrow[r, "\theta_{n-2}\otimes\tau_{n-2}"'] \arrow[d, "s_{n-1}"'] & K[x,y]^q \arrow[r, dashed] \arrow[d, "s_{n-2}"'] \arrow[r, phantom, dashed] & K[x,y]^q \arrow[r, "\theta_1\otimes\tau_1"'] \arrow[d, "s_2"'] & K[x,y]^q \arrow[d, "s_1"] \\
K[x,y]^{q'} \arrow[r, "\theta'_n\otimes\tau'_n"'] & K[x,y]^{q'} \arrow[r, "\theta'_{n-1}\otimes\tau'_{n-1}"'] & K[x,y]^{q'} \arrow[r, "\theta'_{n-2}\otimes\tau'_{n-2}"'] & K[x,y]^{q'} \arrow[r, dashed] & K[x,y]^{q'} \arrow[r, "\theta'_1\otimes\tau'_1"'] & K[x,y]^{q'}
\end{tikzcd}$$

that is, all the squares in the foregoing diagram commute where $q = 2pm$, $q' = 2p'm'$ and here, $s_i = \alpha_i\otimes\beta_i$.
• The commutativity of each of these squares from the right to the left is expressed by the following equalities:

$$\begin{cases}
\begin{bmatrix}\alpha_1\otimes\beta_1 & 0\\ 0 & \alpha_1\otimes\beta_1\end{bmatrix}\begin{bmatrix}\theta_1\otimes\tau_1 & 0\\ 0 & \theta_1\otimes\tau_1\end{bmatrix} = \begin{bmatrix}\theta'_1\otimes\tau'_1 & 0\\ 0 & \theta'_1\otimes\tau'_1\end{bmatrix}\begin{bmatrix}s_2 & 0\\ 0 & s_2\end{bmatrix},\\
\begin{bmatrix}\alpha_i\otimes\beta_i & 0\\ 0 & \alpha_i\otimes\beta_i\end{bmatrix}\begin{bmatrix}\theta_i\otimes\tau_i & 0\\ 0 & \theta_i\otimes\tau_i\end{bmatrix} = \begin{bmatrix}\theta'_i\otimes\tau'_i & 0\\ 0 & \theta'_i\otimes\tau'_i\end{bmatrix}\begin{bmatrix}s_{i+1} & 0\\ 0 & s_{i+1}\end{bmatrix},\\
\begin{bmatrix}\alpha_n\otimes\beta_n & 0\\ 0 & \alpha_n\otimes\beta_n\end{bmatrix}\begin{bmatrix}\theta_n\otimes\tau_n & 0\\ 0 & \theta_n\otimes\tau_n\end{bmatrix} = \begin{bmatrix}\theta'_n\otimes\tau'_n & 0\\ 0 & \theta'_n\otimes\tau'_n\end{bmatrix}\begin{bmatrix}s_1 & 0\\ 0 & s_1\end{bmatrix},
\end{cases}$$

(where $2\le i\le n-1$) and $s_{i+1} = \alpha_{i+1}\otimes\beta_{i+1}$.

That is,

$$\begin{cases} (\alpha_1 \otimes \beta_1)(\theta_1 \otimes \tau_1) = (\theta'_1 \otimes \tau'_1)(\alpha_2 \otimes \beta_2), \\ (\alpha_i \otimes \beta_i)(\theta_i \otimes \tau_i) = (\theta'_i \otimes \tau'_i)(\alpha_{i+1} \otimes \beta_{i+1}), \quad 2 \leq i \leq n-1, \\ (\alpha_n \otimes \beta_n)(\theta_n \otimes \tau_n) = (\theta'_n \otimes \tau'_n)(\alpha_1 \otimes \beta_1). \end{cases}$$

That is, all we need to show is the set of equalities:

$$\alpha_1\theta_1 \otimes \beta_1\tau_1 = \theta'_1\alpha_2 \otimes \tau'_1\beta_2, \tag{4.1}$$

$$\alpha_i\theta_i \otimes \beta_i\tau_i = \theta'_i\alpha_{i+1} \otimes \tau'_i\beta_{i+1}, \qquad 2 \leq i \leq n-1, \tag{4.2}$$

$$\alpha_n\theta_n \otimes \beta_n\tau_n = \theta'_n\alpha_1 \otimes \tau'_n\beta_1. \tag{4.3}$$

Now, by hypothesis,
$\Phi_f = (\alpha_1, \alpha_2, \ldots, \alpha_n) : X_f = (\theta_1, \theta_2, \ldots, \theta_n) \to X'_f = (\theta'_1, \theta'_2, \ldots, \theta'_n)$ and
$\Phi_g = (\beta_1, \beta_2, \ldots, \beta_n) : X_g = (\tau_1, \tau_2, \ldots, \tau_n) \to X'_g = (\tau'_1, \tau'_2, \ldots, \tau'_n)$ are morphisms, meaning that the following diagrams commute:

$$\begin{tikzcd} K[x]^p \arrow[r, "\theta_n"] \arrow[d, "\alpha_1"'] & K[x]^p \arrow[r, "\theta_{n-1}"] \arrow[d, "\alpha_n"'] & K[x]^p \arrow[r, "\theta_{n-2}"] \arrow[d, "\alpha_{n-1}"'] & K[x]^p \arrow[r, dashed] \arrow[d, "\alpha_{n-2}"'] \arrow[dr, phantom, "\dashrightarrow"] & K[x]^p \arrow[r, "\theta_1"] \arrow[d, "\alpha_2"'] & K[x]^p \arrow[d, "\alpha_1"] \\ K[x]^{p'} \arrow[r, "\theta'_n"] & K[x]^{p'} \arrow[r, "\theta'_{n-1}"] & K[x]^{p'} \arrow[r, "\theta'_{n-2}"] & K[x]^{p'} \arrow[r, dashed] & K[x]^{p'} \arrow[r, "\theta'_1"] & K[x]^{p'} \end{tikzcd}$$

and

$$\begin{tikzcd} K[y]^m \arrow[r, "\tau_n"] \arrow[d, "\beta_1"'] & K[y]^m \arrow[r, "\tau_{n-1}"] \arrow[d, "\beta_n"'] & K[y]^m \arrow[r, "\tau_{n-2}"] \arrow[d, "\beta_{n-1}"'] & K[y]^m \arrow[r, dashed] \arrow[d, "\beta_{n-2}"'] \arrow[dr, phantom, "\dashrightarrow"] & K[y]^m \arrow[r, "\tau_1"] \arrow[d, "\beta_2"'] & K[y]^m \arrow[d, "\beta_1"] \\ K[y]^{m'} \arrow[r, "\tau'_n"] & K[y]^{m'} \arrow[r, "\tau'_{n-1}"] & K[y]^{m'} \arrow[r, "\tau'_{n-2}"] & K[y]^{m'} \arrow[r, dashed] & K[y]^{m'} \arrow[r, "\tau'_1"] & K[y]^{m'} \end{tikzcd}$$

That is,

$$\alpha_1\theta_1 = \theta'_1\alpha_2, \tag{4.4}$$

$$\alpha_i\theta_i = \theta'_i\alpha_{i+1}, \qquad 2 \leq i \leq n-1, \tag{4.5}$$

$$\alpha_n\theta_n = \theta'_n\alpha_1, \tag{4.6}$$

and

$$\beta_1\tau_1 = \tau'_1\beta_2, \tag{4.7}$$

$$\beta_i\tau_i = \tau'_i\beta_{i+1}, \qquad 2 \leq i \leq n-1, \tag{4.8}$$

$$\beta_n\tau_n = \tau'_n\beta_1. \tag{4.9}$$

Now, considering (4.4) and (4.7), we immediately see that equality (4.1) holds. Similarly, (4.5) and (4.8) yield (4.2). Finally, (4.6) and (4.9) yield (4.3).

So, $\Phi_f \widetilde{\otimes}_n \Phi_g$ is a morphism in $MF(K[x, y], fg)_n$.

$\square$

We can now state the following result.

Theorem 4.11. (1) *Let X be an n-matrix factorization of $f \in K[x]$ of size p and let Y be an n-matrix factorization of $g \in K[y]$ of size m. Then, there is a tensor* *product* $\widetilde{\otimes}_n$ *of* n*-matrix factorizations which produces an* n*-matrix factorization* $X\widetilde{\otimes}_n Y$ *of the product* $fg \in K[x_1, \ldots, x_r, y_1, \ldots, y_s]$ *which is of size* $2pm$. $\widetilde{\otimes}_n$ *is called the multiplicative tensor product of* n*-matrix factorizations.*

(2) *The multiplicative tensor product of* n*-matrix factorizations* $(-)\widetilde{\otimes}_n(-) : MF(K[x], f)_n \times MF(K[y], g)_n \to MF(K[x, y], fg)_n$ *is a bifunctor.*

Proof. (1) This is exactly what we proved above in subsection 4.1.

(2) We show that $\widetilde{\otimes}_n$ is a bifunctor.
In order to ease our computations, let us write $F = (-)\widetilde{\otimes}_n(-)$. We show that F is a bifunctor. Let Φ_f and Φ_g be as defined in Definition 4.9. Moreover, let $\Phi'_f = (\alpha'_1, \alpha'_2, \ldots, \alpha'_n) : X'_f = (\theta'_1, \theta'_2, \ldots, \theta'_n) \to X''_f = (\theta''_1, \theta''_2, \ldots, \theta''_n)$ and $\Phi'_g = (\beta'_1, \beta'_2, \ldots, \beta'_n) : X'_g = (\tau'_1, \tau'_2, \ldots, \tau'_n) \to X''_g = (\tau''_1, \tau''_2, \ldots, \tau''_n)$, respectively, in $MF(K[x], f)_n$ and $MF(K[y], g)_n$. We have

$$(-)\widetilde{\otimes}_n(-) : MF(f)_n \times MF(g)_n \longrightarrow MF(fg)_n$$

$$\begin{tikzcd}
(X_f, \arrow[d, "\Phi_f"'] & X_g) \arrow[d, "\Phi_g"'] \arrow[rr] & & X_f\widetilde{\otimes}_n X_g \arrow[d, "\Phi_f\widetilde{\otimes}_n\Phi_g := (\alpha_1\otimes\beta_1\oplus\alpha_1\otimes\beta_1, \ldots, \alpha_n\otimes\beta_n\oplus\alpha_n\otimes\beta_n)"] \\
(X'_f, \arrow[d, "\Phi'_f"'] & X'_g) \arrow[d, "\Phi'_g"'] \arrow[rr] & & X'_f\widetilde{\otimes}_n X'_g \arrow[d, "\Phi'_f\widetilde{\otimes}_n\Phi'_g := (\alpha'_1\otimes\beta'_1\oplus\alpha'_1\otimes\beta'_1, \ldots, \alpha'_n\otimes\beta'_n\oplus\alpha'_n\otimes\beta'_n)"] \\
(X''_f, & X''_g) \arrow[rr] & & X''_f\widetilde{\otimes}_n X''_g
\end{tikzcd}$$

We showed in Lemma 4.10 that $\Phi_f\widetilde{\otimes}_n\Phi_g := (\alpha_1 \otimes \beta_1 \oplus \alpha_1 \otimes \beta_1, \ldots, \alpha_n \otimes \beta_n \oplus \alpha_n \otimes \beta_n)$ is a morphism in $MF(K[x, y], fg)_n$.
Similarly, $\Phi'_f\widetilde{\otimes}_n\Phi'_g = (\alpha'_1 \otimes \beta'_1 \oplus \alpha'_1 \otimes \beta'_1, \ldots, \alpha'_n \otimes \beta'_n \oplus \alpha'_n \otimes \beta'_n)$ is a morphism in $MF(K[x, y], fg)_n$.

It now remains to show the composition and the identity axioms.

Identity Axiom:
We show that $F(id_{(X_f,X_g)}) = id_{F(X_f,X_g)}$.
Now, $F(id_{(X_f,X_g)}) = F(id_{X_f}, id_{X_g}) := id_{X_f}\widetilde{\otimes}_n id_{X_g} : X_f\widetilde{\otimes}_n X_g \to X_f\widetilde{\otimes}_n X_g$.
By Definition 4.9, $id_{X_f}\widetilde{\otimes}_n id_{X_g}$ is the n-tuple of matrices

$$(\begin{bmatrix} I_p \otimes I_m & 0 \\ 0 & I_p \otimes I_m \end{bmatrix}, \begin{bmatrix} I_p \otimes I_m & 0 \\ 0 & I_p \otimes I_m \end{bmatrix}, \ldots, \begin{bmatrix} I_p \otimes I_m & 0 \\ 0 & I_p \otimes I_m \end{bmatrix}) \tag{4.10}$$

Next, we compute $id_{F(X_f,X_g)} = id_{X_f\widetilde{\otimes}_n X_g} : X_f\widetilde{\otimes}_n X_g \to X_f\widetilde{\otimes}_n X_g$. By the definition of a morphism in the category $MF(fg)_n$, we know that

$$id_{X_f\widetilde{\otimes}_n X_g} := (\begin{bmatrix} I_{pm} & 0 \\ 0 & I_{pm} \end{bmatrix}, \begin{bmatrix} I_{pm} & 0 \\ 0 & I_{pm} \end{bmatrix}, \ldots, \begin{bmatrix} I_{pm} & 0 \\ 0 & I_{pm} \end{bmatrix}) \tag{4.11}$$

Since $I_p \otimes I_m = I_{pm}$, we see that (4.10) and (4.11) are the same, therefore $F(id_{(X_f,X_g)}) = id_{F(X_f,X_g)}$ as desired.

Composition Axiom:
Consider the situation:

$$X_f \xrightarrow{\Phi_f} X'_f \xrightarrow{\Phi'_f} X''_f$$

$$X_g \xrightarrow{\Phi_g} X'_g \xrightarrow{\Phi'_g} X''_g$$

$$X_f\widetilde{\otimes}_n X_g \xrightarrow{F(\Phi_f,\Phi_g)} X'_f\widetilde{\otimes}_n X'_g \xrightarrow{F(\Phi'_f,\Phi'_g)} X''_f\widetilde{\otimes}_n X''_g$$

We need to show $F(\Phi'_f \circ \Phi_f, \Phi'_g \circ \Phi_g) = F(\Phi'_f, \Phi'_g) \circ F(\Phi_f, \Phi_g)$.
Now, $\Phi'_f \circ \Phi_f = (\alpha'_1\alpha_1, \alpha'_2\alpha_2, \ldots, \alpha'_n\alpha_n)$ and $\Phi'_g \circ \Phi_g = (\beta'_1\beta_1, \beta'_2\beta_2, \ldots, \beta'_n\beta_n)$.

Thanks to Definition 4.9, we obtain

$$\begin{aligned}&(\Phi'_f \circ \Phi_f)\widetilde{\otimes}_n(\Phi'_g \circ \Phi_g)\\ &= (\begin{bmatrix}\alpha'_1\alpha_1 \otimes \beta'_1\beta_1 & 0\\ 0 & \alpha'_1\alpha_1 \otimes \beta'_1\beta_1\end{bmatrix}, \ldots, \begin{bmatrix}\alpha'_n\alpha_n \otimes \beta'_n\beta_n & 0\\ 0 & \alpha'_n\alpha_n \otimes \beta'_n\beta_n\end{bmatrix}).\end{aligned} \tag{4.12}$$

Next,

$$\begin{aligned}&(\Phi'_f\widetilde{\otimes}_n\Phi'_g) \circ (\Phi_f\widetilde{\otimes}_n\Phi_g)\\ &= (P'_1, \ldots, P'_n) \circ (P_1, \ldots, P_n)\\ &= (\begin{bmatrix}\alpha'_1\alpha_1 \otimes \beta'_1\beta_1 & 0\\ 0 & \alpha'_1\alpha_1 \otimes \beta'_1\beta_1\end{bmatrix}, \ldots, \begin{bmatrix}\alpha'_n\alpha_n \otimes \beta'_n\beta_n & 0\\ 0 & \alpha'_n\alpha_n \otimes \beta'_n\beta_n\end{bmatrix}),\end{aligned} \tag{4.13}$$

where for $i \in \{1, 2, \ldots, n\}$, $P'_i = \begin{bmatrix}\alpha'_i \otimes \beta'_i & 0\\ 0 & \alpha'_i \otimes \beta'_i\end{bmatrix}$ and
$P_i = \begin{bmatrix}\alpha_i \otimes \beta_i & 0\\ 0 & \alpha_i \otimes \beta_i\end{bmatrix}$

From (4.12) and (4.13), we see that $F(\Phi'_f \circ \Phi_f, \Phi'_g \circ \Phi_g) = F(\Phi'_f, \Phi'_g) \circ F(\Phi_f, \Phi_g)$. Thus, $(-)\widetilde{\otimes}_n(-)$ is a bifunctor.

□

Remark 4.12. Note that counterparts of Definitions 4.7, 4.8, 4.9 and Lemmas 4.4, 4.10 can be formulated for $\widetilde{\otimes}'_n$. Therefore, we can also state Theorem 4.14 below which is the counterpart of Theorem 4.11 for $\widetilde{\otimes}'_n$. Its proof is similar to the proof of Theorem 4.11 and so will be omitted.

For the sake of illustration, we state the counterpart of Definition 4.9 for $\widetilde{\otimes}'_n$. We will use it in section 6 of this work when comparing $\widetilde{\otimes}_n$ and $\widetilde{\otimes}'_n$.

Definition 4.13. For morphisms $\Phi_f = (\alpha_1, \alpha_2, \ldots, \alpha_n) : X_f = (\theta_1, \theta_2, \ldots, \theta_n) \to X'_f = (\theta'_1, \theta'_2, \ldots, \theta'_n)$ and $\Phi_g = (\beta_1, \beta_2, \ldots, \beta_n) : X_g = (\tau_1, \tau_2, \ldots, \tau_n) \to X'_g = (\tau'_1, \tau'_2, \ldots, \tau'_n)$, respectively, in $MF(K[x], f)_n$ and $MF(K[y], g)_n$,
we define $\Phi_f\widetilde{\otimes}'_n\Phi_g : X_f\widetilde{\otimes}'_n X_g \to X'_f\widetilde{\otimes}'_n X'_g$,
that is, $(\theta_1, \theta_2, \ldots, \theta_n)\widetilde{\otimes}'_n(\tau_1, \tau_2, \ldots, \tau_n) \to (\theta'_1, \theta'_2, \ldots, \theta'_n)\widetilde{\otimes}'_n(\tau'_1, \tau'_2, \ldots, \tau'_n)$
by

$$(\begin{bmatrix}0 & \alpha_1 \otimes \beta_1\\ \alpha_1 \otimes \beta_1 & 0\end{bmatrix}, \begin{bmatrix}0 & \alpha_2 \otimes \beta_2\\ \alpha_2 \otimes \beta_2 & 0\end{bmatrix}, \ldots, \begin{bmatrix}0 & \alpha_n \otimes \beta_n\\ \alpha_n \otimes \beta_n & 0\end{bmatrix}).$$

Here is the counterpart of Theorem 4.11 for $\widetilde{\otimes}'_n$:

Theorem 4.14. (1) *Let X be an n-matrix factorization of $f \in K[x]$ of size p and let Y be an n-matrix factorization of $g \in K[y]$ of size m. Then for any even integer $n \geq 2$, there is a tensor product $\widetilde{\otimes}'_n$ of n-matrix factorizations which produces an n-matrix factorization $X\widetilde{\otimes}'_n Y$ of the product $fg \in K[x_1, \ldots, x_r, y_1, \ldots, y_s]$ which is of size $2pm$. $\widetilde{\otimes}'_n$ is referred to as the variant of the multiplicative tensor product of n-matrix factorizations.*

(2) *The variant of the multiplicative tensor product of n-matrix factorizations $(-)\widetilde{\otimes}'_n(-) : MF(K[x], f)_n \times MF(K[y], g)_n \to MF(K[x, y], fg)_n$ is a bifunctor.*

Here is the counterpart of Theorem 4.11 for $\overline{\otimes}_n$.

Theorem 4.15. (1) *Let X be an n-matrix factorization of $f \in K[x]$ of size p and let Y be an n-matrix factorization of $g \in K[y]$ of size m. Then, there is a tensor product $\overline{\otimes}_n$ of n-matrix factorizations which produces an n-matrix factorization $X\overline{\otimes}_n Y$ of the product $fg \in K[x_1, \ldots, x_r, y_1, \ldots, y_s]$ which is of size pm. $\overline{\otimes}_n$ is called the refined multiplicative tensor product of n-matrix factorizations.*

(2) *The refined multiplicative tensor product of n-matrix factorizations $(-)\overline{\otimes}_n(-) : MF(K[x], f)_n \times MF(K[y], g)_n \to MF(K[x, y], fg)_n$ is a bifunctor.*

4.3. **Associativity of $\widetilde{\otimes}_n$, $\widetilde{\otimes}'_n$ and $\overline{\otimes}_n$.** Here, we show that $\widetilde{\otimes}_n$,its refined version and its variant are associative for $n \geq 2$. Let $X = (\theta_1, \theta_2, \ldots, \theta_n)$ be an $l \times l$ n-matrix factorization of $f \in K[x]$. We also let $X' = (\theta'_1, \theta'_2, \ldots, \theta'_n)$ (resp. $X'' = (\theta''_1, \theta''_2, \ldots, \theta''_n)$) denote a $(p \times p)$ (resp. $(m \times m)$) n-matrix factorization of $g \in K[y]$ (resp. of $h \in K[z] := K[z_1, \ldots, z_t]$).

Proposition 4.16. *(Associativity)*
There is an identity:

(1) $(X\widetilde{\otimes}_n X')\widetilde{\otimes}_n X'' = X\widetilde{\otimes}_n(X'\widetilde{\otimes}_n X'')$ *in* $MF(fgh)_n$.
(2) $(X\widetilde{\otimes}'_n X')\widetilde{\otimes}'_n X'' = X\widetilde{\otimes}'_n(X'\widetilde{\otimes}'_n X'')$ *in* $MF(fgh)_n$.
(3) $(X\overline{\otimes}_n X')\overline{\otimes}_n X'' = X\overline{\otimes}_n(X'\overline{\otimes}'_n X'')$ *in* $MF(fgh)_n$*, when n is even.*

Proof. The desired identities follow from the fact that the standard tensor product for matrices is associative. □

5. An application: A semi-unital semi-monoidal subcategory of $(MF(1)_n, \widetilde{\otimes}_n)$, for each $n \geq 2$

Here, the concept of *one-step connected category* is recalled (see [14, Definition 5.1]) and it is proved that for each $n \geq 2$, there is a one-step connected subcategory $(\mathcal{T}, \widetilde{\otimes}_n)$ of $(MF(1)_n, \widetilde{\otimes}_n)$ which is a *semi-unital semi-monoidal category.* Our interest in this is that the concept of *semi-unital semi-monoidal category* generalizes the classical concept of monoidal category and it led to the generalization of other classical concepts like that of monads in category theory (see [1]) which have great applications in computer science. Unfortunately, apart from the example given in [14] which is being generalized in this paper, we found only one example of a *semi-unital semi-monoidal category* in the literature (see [1, Theorem 5.11]) and its construction required a considerable amount of somewhat intricate set-up. Indeed, the example (see Theorem 5.5) we give here requires an easy-to-understand set-up. In fact, we will use some particular n-matrix factors of the constant polynomial *1* and the multiplicative tensor product of n-matrix factorizations to construct our example.

Before we start giving our application, we need the following definition: a $(1,0)$-matrix is a matrix with entries from the set $\{0,1\}$. The terminology $(1,0)$-matrix is preferred here to that of $(0,1)$-matrix because some authors use the terminology $(0,1)$-matrix to refer to what we call here $(1,0)$-matrix with some additional conditions.
It is worth recalling that a permutation matrix is a square matrix obtained from the same size identity matrix by a permutation of rows.

Notation 5.1. $0_{p,q}$ will denote the zero matrix of size $p \times q$ whenever there would be a risk of confusion on the size of the zero matrix under consideration. Otherwise, we will simply write 0.

Remark 5.2. At this juncture, it is good to recall (see Definition 3.1) that objects of $MF(1)_n$ are of the form $(A_{1r}, A_{2r}, , A_{nr})$, where $A_{1r}, A_{2r}, \dots, A_{nr}$ are $r \times r$ matrices, $r \in \mathbb{N} - \{0\}$ and $A_{1r}A_{2r}\cdots A_{nr} = 1.I_r$.
Some particular objects of $MF(1)_n$ are crucial for this work. They are of the form e^r, where $r \in \mathbb{N} - \{0\}$. They are characterized as follows: $e^1 = e = (1, 1, \dots, 1)$, where *1* appears n times,

$$\begin{aligned}
e^2 &= e\widetilde{\otimes}_n e = \left(\begin{pmatrix} 1\otimes 1 & 0 \\ 0 & 1\otimes 1 \end{pmatrix}, \begin{pmatrix} 1\otimes 1 & 0 \\ 0 & 1\otimes 1 \end{pmatrix}, \dots, \begin{pmatrix} 1\otimes 1 & 0 \\ 0 & 1\otimes 1 \end{pmatrix} \right) \\
&= (I_2, I_2, \dots, I_2), \\
e^3 &= e\widetilde{\otimes}_n e^2 \\
&= (1, 1, \dots, 1)\widetilde{\otimes}_n \left(\begin{pmatrix} 1\otimes 1 & 0 \\ 0 & 1\otimes 1 \end{pmatrix}, \begin{pmatrix} 1\otimes 1 & 0 \\ 0 & 1\otimes 1 \end{pmatrix}, \dots, \begin{pmatrix} 1\otimes 1 & 0 \\ 0 & 1\otimes 1 \end{pmatrix} \right) \\
&= (Q, Q, \dots, Q),
\end{aligned}$$

where $Q = \begin{pmatrix} 1\otimes \begin{pmatrix} 1\otimes 1 & 0 \\ 0 & 1\otimes 1 \end{pmatrix} & 0 \\ 0 & 1\otimes \begin{pmatrix} 1\otimes 1 & 0 \\ 0 & 1\otimes 1 \end{pmatrix} \end{pmatrix} = I_4.$

So, $e^3 = (I_4, I_4, \dots, I_4)$. In general, $e^r = (I_{2^{r-1}}, I_{2^{r-1}}, \dots, I_{2^{r-1}})$, where I_{r-1} appears n times.

Section 3 shows that for $m, p \in \mathbb{N} - \{0\}$, a morphism $e^m \to e^p$ in $MF(1)_n$ is an n-tuple of matrices $(\alpha_1, \alpha_2, \dots, \alpha_n)$ each of size $2^{p-1} \times 2^{m-1}$, which makes the following diagram commutative:

$$\begin{tikzcd}
K[x]^{2^{m-1}} \arrow[r, "I_{2^{m-1}}"] \arrow[d, "\alpha_1"'] & K[x]^{2^{m-1}} \arrow[r, "I_{2^{m-1}}"] \arrow[d, "\alpha_n"'] & K[x]^{2^{m-1}} \arrow[r, "I_{2^{m-1}}"] \arrow[d, "\alpha_{n-1}"] & K[x]^{2^{m-1}} \arrow[r, dashed] \arrow[d, "\alpha_{n-2}"'] & K[x]^{2^{m-1}} \arrow[r, "I_{2^{m-1}}"] \arrow[d, "\alpha_2"'] & K[x]^{2^{m-1}} \arrow[d, "\alpha_1"] \\
K[x]^{2^{p-1}} \arrow[r, "I_{2^{p-1}}"] & K[x]^{2^{p-1}} \arrow[r, "I_{2^{p-1}}"] & K[x]^{2^{p-1}} \arrow[r, "I_{2^{p-1}}"] & K[x]^{2^{p-1}} \arrow[r, dashed] & K[x]^{2^{p-1}} \arrow[r, "I_{2^{p-1}}"] & K[x]^{2^{p-1}}
\end{tikzcd} \tag{5.1}$$

That is,

$$\begin{cases} \alpha_1 I_{2^{m-1}} = I_{2^{p-1}}\alpha_2, \\ \alpha_i I_{2^{m-1}} = I_{2^{p-1}}\alpha_{i+1}, \quad 2 \leq i \leq n-1, \\ \alpha_n I_{2^{m-1}} = I_{2^{p-1}}\alpha_1. \end{cases} \tag{5.2}$$

Thus from (5.2), a morphism $e^m \to e^p$ in $MF(1)_n$ is an n-tuple $(\alpha_1, \alpha_2, \dots, \alpha_n)$ of matrices with $\alpha_1 = \alpha_2 = \cdots = \alpha_n$. Notice that this does not impose any restrictions on the entries of α_i, $1 \leq i \leq n$. The entries could be anything provided we have the equalities $\alpha_1 = \alpha_2 = \cdots = \alpha_n$. Indeed, we will impose some restrictions on the entries of these matrices to obtain a morphism between two objects of the subcategory we will construct in Lemma 5.4 below.

Definition 5.3. [14] A category is said to be a **one-step connected category** if for every two objects of the category, there exists a nonzero morphism between them.

The following lemma was already proved in [14] for the case $n = 3$. Indeed in this section, we are actually proving that the example we constructed in [14] has counterparts for each $n \in \mathbb{N} - \{0, 1, 3\}$.

Lemma 5.4. *There is a one-step connected subcategory of $MF(1)_n$ which is a semi-monoidal category.*

Proof. A one-step connected subcategory of $MF(1)_n$ will be extracted from $MF(1)_n$ and it will be proved that it is a semi-monoidal category. $\mathcal{T}$ is the name we give to this subcategory in the sequel.
• Objects of $\mathcal{T}$ are of the form $e^r = (I_{2^{r-1}}, I_{2^{r-1}}, \dots, I_{2^{r-1}})$, $r \in \mathbb{N} - \{0\}$.
(It is worth remarking here that the choice of these objects is motivated by the proof of the commutativity of the diagrams under the proof of Theorem 5.5 which is one of our main results).
• Definition of a morphism in $\mathcal{T}$.
Arrows of $\mathcal{T}$ are defined as in $MF(1)_n$, but with some restrictions. An arrow $e^m \to e^p$ in $\mathcal{T}$ is an n-tuple of matrices $(\alpha_1, \alpha_2, \dots, \alpha_n)$ such that $\alpha_1 = \alpha_2 = \cdots = \alpha_n$ is a $(1,0)$-matrix of size $2^{p-1} \times 2^{m-1}$ fulfilling the requirements below:

$$\alpha_1 = \cdots = \alpha_n = \begin{cases} (I_{2^{p-1}}, 0_{2^{p-1}, 2^{m-1} - 2^{p-1}}) & \text{if } m > p, \\ \begin{pmatrix} I_{2^{m-1}} \\ 0_{2^{p-1} - 2^{m-1}, 2^{m-1}} \end{pmatrix} & \text{if } m < p, \\ Z & \text{if } m = p, \end{cases} \tag{5.3}$$

where Z is a $2^{m-1} \times 2^{m-1}$ permutation matrix. Intuitively, $\alpha_1 = \alpha_2 = \cdots = \alpha_n$ is a $(1,0)$-matrix *with at most one nonzero entry in each row and each column*. This restriction will ensure that the composition of two morphisms in $\mathcal{T}$ is again a morphism in $\mathcal{T}$.
From diagram (5.1) above, it follows that for each $i \in \{1, 2, \dots, n\}$, α_i should be of size $2^{p-1} \times 2^{m-1}$. It is evident from diagram (5.1) that we have a morphism from e^m to e^p for the above values of $\alpha_1 = \alpha_2 = \cdots = \alpha_n$.

► $\mathcal{T}$ is a subcategory of $MF(1)_n$ which is one-step connected.
For every pair of morphisms ζ and ζ' in $hom(\mathcal{T})$, the composite $\zeta \circ \zeta'$ is in $hom(\mathcal{T})$ whenever it is defined. In fact, ζ and ζ' by the definition are n-tuples of $(1,0)$-matrices such that in each matrix; each column and each row has at most one nonzero entry. Therefore, the composition of such matrices will yield another $(1,0)$-matrix in which each column and each row would have at most one nonzero entry, whence we will still have a morphism of $\mathcal{T}$.

In addition, $\mathcal{T}$ is a one-step connected category since between any two objects of $\mathcal{T}$ say e^m and e^p, there exists a nonzero morphism as can be seen from the above case distinction (5.3).

▶ Next, we prove that $(\mathcal{T}, \widetilde{\otimes}_n)$ is a semi-monoidal category (see Definition 2.1):
Considering Definition 2.1, it is necessary to first show that α is a natural isomorphism:
- To see that $\widetilde{\otimes}_n : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ is a bifunctor, use Theorem 4.11 in which $MF(1)_n$ is restricted to $\mathcal{T}$, f and g are replaced by the constant polynomial *1*, and let $x = y$.
- There is a natural isomorphism α from the functor $((-)\widetilde{\otimes}_n(-))\widetilde{\otimes}_n(-) : \mathcal{T} \times \mathcal{T} \times \mathcal{T} \to (\mathcal{T} \times \mathcal{T}) \times \mathcal{T}$ to the functor $(-)\widetilde{\otimes}_n((-)\widetilde{\otimes}_n(-)) : \mathcal{T} \times \mathcal{T} \times \mathcal{T} \to \mathcal{T} \times (\mathcal{T} \times \mathcal{T})$ with components $\alpha_{a,b,c} : (a\widetilde{\otimes}_n b)\widetilde{\otimes}_n c \to a\widetilde{\otimes}_n(b\widetilde{\otimes}_n c)$, where $a, b\, and\, c$ are matrix factorizations of the constant polynomial *1* in $\mathcal{T}$.
Let $a, b, c, a', b', and\, c'$ be objects of $\mathcal{T}$. Let $f : a \to a'$, $g : b \to b'$ and $h : c \to c'$ be maps in $\mathcal{T}$. We show that the following diagram commutes:

$$\begin{tikzcd}
(a\widetilde{\otimes}_n b)\widetilde{\otimes}_n c \arrow[r, "\alpha_{a,b,c}"] \arrow[d, "(f\widetilde{\otimes}_n g)\widetilde{\otimes}_n h"'] & a\widetilde{\otimes}_n(b\widetilde{\otimes}_n c) \arrow[d, "f\widetilde{\otimes}_n(g\widetilde{\otimes}_n h)"'] \\
(a'\widetilde{\otimes}_n b')\widetilde{\otimes}_n c' \arrow[r, "\alpha_{a',b',c'}"'] & a'\widetilde{\otimes}_n(b'\widetilde{\otimes}_n c')
\end{tikzcd}$$

that is,

$$f\widetilde{\otimes}_n(g\widetilde{\otimes}_n h) \circ \alpha_{a,b,c} = \alpha_{a',b',c'} \circ (f\widetilde{\otimes}_n g)\widetilde{\otimes}_n h \tag{5.4}$$

In fact, the matrices representing $\alpha_{a,b,c}$ and $\alpha_{a',b',c'}$ are identity matrices. Besides, the tensor product of maps is associative. Thus, (5.4) holds. That is α is a natural transformation. Moreover, for all a, b and c, it is true that $\alpha_{a,b,c}$ is an equality and so, it is an isomorphism.
Hence, α is a natural isomorphism.

Next, we show that the pentagonal diagram of Definition 2.1 commutes for all objects a, b, c, d of $MF(1)_n$:

$$\begin{tikzcd}
a\widetilde{\otimes}_n(b\widetilde{\otimes}_n(c\widetilde{\otimes}_n d)) \arrow[rr, "1_a\widetilde{\otimes}_n\alpha"] \arrow[d, "\alpha"'] & & a\widetilde{\otimes}_n((b\widetilde{\otimes}_n c)\widetilde{\otimes}_n d) \arrow[d, "\alpha"] \\
(a\widetilde{\otimes}_n b)\widetilde{\otimes}_n(c\widetilde{\otimes}_n d) \arrow[dr, "\alpha"'] & & (a\widetilde{\otimes}_n(b\widetilde{\otimes}_n c))\widetilde{\otimes}_n d \arrow[dl, "\alpha\widetilde{\otimes}_n 1_d"] \\
& ((a\widetilde{\otimes}_n b)\widetilde{\otimes}_n c)\widetilde{\otimes}_n d &
\end{tikzcd}$$

This diagram commutes because all the maps linking the vertices of the pentagon are identity maps. Indeed, we know that α the associator is an identity map. Furthermore, since the n-tuple of matrices making up α are identity matrices, it follows from Definition 4.9 of the multiplicative tensor product of n-matrix factorizations that $1_a\widetilde{\otimes}_n\alpha$ and $\alpha\widetilde{\otimes}_n 1_d$ are also identity maps.

So, $\mathcal{T}$ is a semi-monoidal category. □

Theorem 5.5. *For a fixed $n \geq 2$, there is a one-step connected subcategory of $MF(1)_n$ which is a semi-unital semi-monoidal category.*

Proof. Lemma 5.4 already shows that there is a one-step connected subcategory of $MF(1)_n$ which is a semi-monoidal category. We called it $\mathcal{T}$. Therefore, to prove the theorem, it will be enough to find a *semi-unit* (see Definition 2.3) in the semi-monoidal

category $\mathcal{T}$.
►**Claim:** $e = (1, 1, \dots, 1)$ is a *semi-unit* in $\mathcal{T}$.
According to the first point of Definition 2.3, we need to find a natural transformation $\gamma : (-) = G \to F = e\widetilde{\otimes}_n(-)$, where G is the identity endofunctor on $\mathcal{T}$ and F is an endofunctor on $\mathcal{T}$, such that $F(a) = e\widetilde{\otimes}_n a$. Components of γ are $\gamma_a : a \to e\widetilde{\otimes}_n a$, where $a = e^p = (\phi_1, \phi_2, \dots, \phi_n)$ is an object of $\mathcal{T}$ of size n_1, that is $\phi_1 = \phi_2 = \cdots = \phi_n = I_{n_1}$ with $n_1 = 2^{p-1}$. So, $\phi_i = I_{2^{p-1}}$ for $i \in \{1, 2, \dots, n\}$. We have
$e\widetilde{\otimes}_n a = \left(\begin{bmatrix} I_{2^{p-1}} & 0 \\ 0 & I_{2^{p-1}} \end{bmatrix}, \begin{bmatrix} I_{2^{p-1}} & 0 \\ 0 & I_{2^{p-1}} \end{bmatrix}, \dots, \begin{bmatrix} I_{2^{p-1}} & 0 \\ 0 & I_{2^{p-1}} \end{bmatrix} \right) = (I_{2^p}, I_{2^p}, \dots, I_{2^p})$ is of size $2^p = 2n_1$.
The family of morphisms γ should fulfill the following two conditions:

(1) For each $a \, in \, Ob(\mathcal{T})$, γ_a should be a morphism in $\mathcal{T}$.
Owing to the manner in which arrows are defined in $\mathcal{T}$ (see the proof of Lemma 5.4), we define $\gamma_a = (\delta_1, \delta_2, \dots, \delta_n) = ((I_{n_1}, 0)^t, (I_{n_1}, 0)^t, \dots, (I_{n_1}, 0)^t)$, where t is the operation of taking the transpose, 0 is the zero $n_1 \times n_1$ matrix. It follows that γ_a is an arrow in $\mathcal{T}$.

(2) Naturality of γ:
Let $b = (\phi'_1, \phi'_1, \dots, \phi'_n)$ be a matrix factorization in $\mathcal{T}$ of size n_2 and let $\mu = (\mu_1, \mu_2, \dots, \mu_n) : a \to b$ be a map of matrix factorizations. It is easy to see that for $i \in \{1, 2, \dots, n\}$, μ_i is of size $n_2 \times n_1$. The following diagram commutes:

$$\begin{tikzcd} a \arrow[r, "\gamma_a"] \arrow[d, "\mu"'] & e\widetilde{\otimes}_n a \arrow[d, "e\widetilde{\otimes}_n \mu"'] \\ b \arrow[r, "\gamma_b"'] & e\widetilde{\otimes}_n b \end{tikzcd}$$

that is,

$$e\widetilde{\otimes}_n \mu \circ \gamma_a = \gamma_b \circ \mu \tag{5.5}$$

We know that $e\widetilde{\otimes}_n a$ is of size $2n_1$ since a is of size n_1.
We also know that $\gamma_b = [(I_{n_2}, 0)^t, (I_{n_2}, 0)^t, \dots, (I_{n_2}, 0)^t]$. Now, by definition of composition of two morphisms in $\mathcal{T}$, the right hand side of equality (5.5) becomes

$$\begin{aligned} &\gamma_b \circ \mu \\ &= [(I_{n_2}, 0)^t, (I_{n_2}, 0)^t, \dots, (I_{n_2}, 0)^t] \circ (\mu_1, \mu_2, \dots, \mu_n) \\ &= [(I_{n_2}, 0)^t \mu_1, (I_{n_2}, 0)^t \mu_2, \dots, (I_{n_2}, 0)^t \mu_n] \\ &= [(\mu_1, 0)^t, (\mu_2, 0)^t, \dots, (\mu_n, 0)^t], \end{aligned} \tag{5.6}$$

where 0 in $[(\mu_1, 0)^t, (\mu_2, 0)^t, \dots, (\mu_n, 0)^t]$ is the $n_2 \times n_1$ zero matrix.
As for the left hand side of (5.5), first recall that

$$\gamma_a = [(I_{n_1}, 0)^t, (I_{n_1}, 0)^t, \dots, (I_{n_1}, 0)^t],$$

(where 0 is the zero $n_1 \times n_1$ matrix) and by Definition 4.8 of the multiplicative tensor product of n-matrix factorizations, we know that

$$
\begin{aligned}
&e\widetilde{\otimes}\mu \\
&= (1,1,\dots,1)\widetilde{\otimes}(\mu_1,\mu_2,\dots,\mu_n) \\
&= (\begin{bmatrix} 1\otimes\mu_1 & 0 \\ 0 & 1\otimes\mu_1 \end{bmatrix}, \begin{bmatrix} 1\otimes\mu_2 & 0 \\ 0 & 1\otimes\mu_2 \end{bmatrix}, \dots, \begin{bmatrix} 1\otimes\mu_n & 0 \\ 0 & 1\otimes\mu_n \end{bmatrix}) \\
&= (\begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_1 \end{bmatrix}, \begin{bmatrix} \mu_2 & 0 \\ 0 & \mu_2 \end{bmatrix}, \dots, \begin{bmatrix} \mu_n & 0 \\ 0 & \mu_n \end{bmatrix}).
\end{aligned}
$$

So,

$$
\begin{aligned}
&e\widetilde{\otimes}\mu \circ \gamma_a \\
&= (\begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_1 \end{bmatrix}, \begin{bmatrix} \mu_2 & 0 \\ 0 & \mu_1 \end{bmatrix}, \dots, \begin{bmatrix} \mu_n & 0 \\ 0 & \mu_n \end{bmatrix}) \circ [(I_{n_1},0)^t, (I_{n_1},0)^t, \dots, (I_{n_1},0)^t] \\
&= (\begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_1 \end{bmatrix}(I_{n_1},0)^t, \begin{bmatrix} \mu_2 & 0 \\ 0 & \mu_2 \end{bmatrix}(I_{n_1},0)^t, \dots, \begin{bmatrix} \mu_n & 0 \\ 0 & \mu_n \end{bmatrix}(I_{n_1},0)^t)) \\
&= [(\mu_1,0)^t, (\mu_2,0)^t], \dots, (\mu_n,0)^t].
\end{aligned}
\tag{5.7}
$$

From (5.6) and (5.7), we see that equality (5.5) holds.

Hence γ is a natural transformation.

It is now time to verify that the requirements stated in the second point of Definition 2.3 are satisfied. So, the next step towards proving that $e = (1,1,\dots,1)$ is a semi-unit in $\mathcal{T}$ is to prove that there is an isomorphism of functors $e\widetilde{\otimes}_n(-) \cong (-)\widetilde{\otimes}_n e$, (i.e., there is a natural isomorphism) $l_a : e\widetilde{\otimes}_n(a) \cong (a)\widetilde{\otimes}_n e$ with inverse q_a, for each object a of $\mathcal{T}$ such that $l_e = q_e$ and the following diagrams are commutative for all objects a and b of $\mathcal{T}$:

$$
\begin{tikzcd}
(e\widetilde{\otimes}_n a)\widetilde{\otimes}_n b \arrow[r, "\alpha_{e,a,b}"] \arrow[d, "l_a\widetilde{\otimes}_n b"'] & e\widetilde{\otimes}_n(a\widetilde{\otimes}_n b) \arrow[r, "l_{a\widetilde{\otimes}_n b}"] & (a\widetilde{\otimes}_n b)\widetilde{\otimes}_n e \arrow[d, "\alpha_{a,b,e}"] \\
(a\widetilde{\otimes}_n e)\widetilde{\otimes}_n b \arrow[r, "\alpha_{a,e,b}"] & a\widetilde{\otimes}_n(e\widetilde{\otimes}_n b) \arrow[r, "a\widetilde{\otimes}_n l_b"] & a\widetilde{\otimes}_n(b\widetilde{\otimes}_n e)
\end{tikzcd}
\tag{5.8}
$$

$$
\begin{tikzcd}
a\widetilde{\otimes}_n b \arrow[rr, "\gamma_a\widetilde{\otimes}_n b"] \arrow[dr, "\gamma_{a\widetilde{\otimes}_n b}"'] & & (e\widetilde{\otimes}_n a)\widetilde{\otimes}_n b \arrow[dl, "\cong"] \\
& e\widetilde{\otimes}_n(a\widetilde{\otimes}_n b) &
\end{tikzcd}
\tag{5.9}
$$

$$
\begin{tikzcd}
a\widetilde{\otimes}_n b \arrow[rr, "a\widetilde{\otimes}_n \gamma_b"] \arrow[dr, "\gamma_{a\widetilde{\otimes}_n b}"'] & & a\widetilde{\otimes}_n(e\widetilde{\otimes}_n b) \arrow[dl, "\cong"] \\
& e\widetilde{\otimes}_n(a\widetilde{\otimes}_n b) &
\end{tikzcd}
\tag{5.10}
$$

Before we define l_a, observe that $e\widetilde{\otimes}_n(a) = (a)\widetilde{\otimes}_n e$.

♣ We define $l_a : e\widetilde{\otimes}_n(a) \to (a)\widetilde{\otimes}_n e$ to be the n-tuple of matrices $(I_{2n_1}, I_{2n_1}, \dots, I_{2n_1}) = (I_{2^p}, I_{2^p}, \dots, I_{2^p})$, where $a = e^p$ is of size n_1. Clearly, l_a is a morphism in $\mathcal{T}$.

♣ Naturality of l:
Let $b = (\phi'_1, \phi'_1, \dots, \phi'_n)$ be a matrix factorization of size n_2 and let $\mu' = (\mu'_1, \mu'_2, \dots, \mu'_n) : a \to b$ be a map of matrix factorizations. Clearly, for each $i \in \{1, 2, \dots, n\}$, μ_i is of size $n_2 \times n_1$. The following diagram commutes:

$$\begin{tikzcd} e\widetilde{\otimes}_n a \arrow[r, "l_a"] \arrow[d, "e\widetilde{\otimes}_n\mu'"'] & a\widetilde{\otimes}_n e \arrow[d, "\mu'\widetilde{\otimes}_n e"'] \\ e\widetilde{\otimes}_n b \arrow[r, "l_b"'] & b\widetilde{\otimes}_n e \end{tikzcd}$$

that is,

$$\mu'\widetilde{\otimes}_n e \circ l_a = l_b \circ e\widetilde{\otimes}_n \mu'. \tag{5.11}$$

Since l_a and l_b are just n-tuples of identity matrices, it suffices to show that $\mu'\widetilde{\otimes}_n e = e\widetilde{\otimes}_n \mu'$.
By Definition 4.8 of the multiplicative tensor product of n-matrix factorizations, we know that

$$\begin{aligned}
&e\widetilde{\otimes}_n\mu' \\
&= (1, 1, \dots, 1)\widetilde{\otimes}_n(\mu'_1, \mu'_2, \dots, \mu'_n) \\
&= (\begin{bmatrix} 1\otimes\mu'_1 & 0 \\ 0 & 1\otimes\mu'_1 \end{bmatrix}, \begin{bmatrix} 1\otimes\mu'_2 & 0 \\ 0 & 1\otimes\mu'_2 \end{bmatrix}, \dots, \begin{bmatrix} 1\otimes\mu'_n & 0 \\ 0 & 1\otimes\mu'_n \end{bmatrix}) \\
&= (\begin{bmatrix} \mu'_1 & 0 \\ 0 & \mu'_1 \end{bmatrix}, \begin{bmatrix} \mu'_2 & 0 \\ 0 & \mu'_2 \end{bmatrix}, \dots, \begin{bmatrix} \mu'_n & 0 \\ 0 & \mu'_n \end{bmatrix}).
\end{aligned}$$

We also have by Definition 4.7 of the multiplicative tensor product of n-matrix factorizations, that

$$\begin{aligned}
&\mu'\widetilde{\otimes}_n e \\
&= (\mu'_1, \mu'_2, \dots, \mu'_n)\widetilde{\otimes}_n(1, 1, \dots, 1) \\
&= (\begin{bmatrix} \mu'_1\otimes 1 & 0 \\ 0 & \mu'_1\otimes 1 \end{bmatrix}, \begin{bmatrix} \mu'_2\otimes 1 & 0 \\ 0 & \mu'_2\otimes 1 \end{bmatrix}, \dots, \begin{bmatrix} \mu'_n\otimes 1 & 0 \\ 0 & \mu'_n\otimes 1 \end{bmatrix}) \\
&= (\begin{bmatrix} \mu'_1 & 0 \\ 0 & \mu'_1 \end{bmatrix}, \begin{bmatrix} \mu'_2 & 0 \\ 0 & \mu'_2 \end{bmatrix}, \dots \begin{bmatrix} \mu'_n & 0 \\ 0 & \mu'_n \end{bmatrix}).
\end{aligned}$$

So $e\widetilde{\otimes}_n\mu' = \mu'\widetilde{\otimes}_n e$, hence l is a natural transformation.

♣ $l_a : e\widetilde{\otimes}_n(a) \cong (a)\widetilde{\otimes}_n e$ is a natural isomorphism. In fact, note that $e\widetilde{\otimes}_n(a) = (a)\widetilde{\otimes}_n e$ and define the inverse of l denoted q by $q_a : (a)\widetilde{\otimes}_n e = e\widetilde{\otimes}_n(a)$ for all objects a of $\mathcal{T}$. It follows that $l_e = q_e$.

Finally, for the second point of Definition 2.3 to be satisfied we need to check the commutativity of the diagrams (5.8), (5.9) and (5.10).

For diagram (5.8), it would commute if

$$\alpha_{a,b,e} \circ l_{a\widetilde{\otimes}_n b} \circ \alpha_{e,a,b} = a\widetilde{\otimes}_n l_b \circ \alpha_{a,e,b} \circ l_a\widetilde{\otimes}_n b. \tag{5.12}$$

We show that all the maps involved in equality (5.12) are identities.

For all objects x, y, z *in* $Ob(\mathcal{T})$, we have by the definitions of $\alpha_{x,y,z}$ and l_x that they are identity maps. We now show that the other maps involved in diagram (5.8) are also identity maps.

Since a is of size n_1, we have $l_a = (I_{2n_1}, I_{2n_1}, \ldots, I_{2n_1})$. Let $b = (\phi'_1, \phi'_2, \ldots, \phi'_n)$ be of size n_2.
That is, $\phi'_1 = \phi'_2 = \cdots = \phi'_n = I_{n_2}$. By Definition 4.7,

$$\begin{aligned}
&l_a \widetilde{\otimes}_n b \\
&= (I_{2n_1}, I_{2n_1}, \ldots, I_{2n_1}) \widetilde{\otimes}_n (\phi'_1, \phi'_2, \ldots, \phi'_n) \\
&= (\begin{bmatrix} I_{2n_1} \otimes I_{n_2} & 0 \\ 0 & I_{2n_1} \otimes I_{n_2} \end{bmatrix}, \begin{bmatrix} I_{2n_1} \otimes I_{n_2} & 0 \\ 0 & I_{2n_1} \otimes I_{n_2} \end{bmatrix}, \ldots, \begin{bmatrix} I_{2n_1} \otimes I_{n_2} & 0 \\ 0 & I_{2n_1} \otimes I_{n_2} \end{bmatrix}) \\
&= (\begin{bmatrix} I_{2n_1 n_2} & 0 \\ 0 & I_{2n_1 n_2} \end{bmatrix}, \begin{bmatrix} I_{2n_1 n_2} & 0 \\ 0 & I_{2n_1 n_2} \end{bmatrix}, \ldots, \begin{bmatrix} I_{2n_1 n_2} & 0 \\ 0 & I_{2n_1 n_2} \end{bmatrix}) \\
&= (I_{4n_1 n_2}, I_{4n_1 n_2}, \ldots, I_{4n_1 n_2}). \qquad (5.13)
\end{aligned}$$

Hence $l_a \widetilde{\otimes}_n b$ is an identity map as expected.

Similarly using Definition 4.8, we prove that $a \widetilde{\otimes}_n l_b$ is an identity map.
Let $a = (\phi_1, \phi_2, \ldots, \phi_n)$ be of size n_1 (that is, $\phi_i = I_{n_1}$ for all $i \in \{1, 2, \ldots, n\}$) and b be as above. Then,

$$\begin{aligned}
&a \widetilde{\otimes}_n l_b \\
&= (\phi_1, \phi_2, \ldots, \phi_n) \widetilde{\otimes}_n (I_{2n_2}, I_{2n_2}, \ldots, I_{2n_2}) \\
&= (\begin{bmatrix} I_{n_1} \otimes I_{2n_2} & 0 \\ 0 & I_{n_1} \otimes I_{2n_2} \end{bmatrix}, \begin{bmatrix} I_{n_1} \otimes I_{2n_2} & 0 \\ 0 & I_{n_1} \otimes I_{2n_2} \end{bmatrix}, \ldots, \begin{bmatrix} I_{n_1} \otimes I_{2n_2} & 0 \\ 0 & I_{n_1} \otimes I_{2n_2} \end{bmatrix}) \\
&= (\begin{bmatrix} I_{2n_1 n_2} & 0 \\ 0 & I_{2n_1 n_2} \end{bmatrix}, \begin{bmatrix} I_{2n_1 n_2} & 0 \\ 0 & I_{2n_1 n_2} \end{bmatrix}, \ldots, \begin{bmatrix} I_{2n_1 n_2} & 0 \\ 0 & I_{2n_1 n_2} \end{bmatrix}) \\
&= (I_{4n_1 n_2}, I_{4n_1 n_2}, \ldots, I_{4n_1 n_2}). \qquad (5.14)
\end{aligned}$$

Equations (5.13) and (5.14) show that $a \widetilde{\otimes}_n l_b = l_a \widetilde{\otimes}_n b$. Observe that all the other maps involved in diagram (5.8) are equal to $(I_{4n_1 n_2}, I_{4n_1 n_2}, \ldots, I_{4n_1 n_2})$. So diagram (5.8) is commutative.

Next, we show that diagram (5.9) commutes. To this end, we need to find an isomorphism $\zeta : (e \widetilde{\otimes}_n a) \widetilde{\otimes}_n b \to e \widetilde{\otimes}_n (a \widetilde{\otimes}_n b)$, such that $\zeta \circ \gamma_a \widetilde{\otimes}_n b = \gamma_{a \widetilde{\otimes}_n b}$.
Now, we know that $\gamma_a = [(I_{n_1}, 0)^t, (I_{n_1}, 0)^t, \ldots, (I_{n_1}, 0)^t]$, where 0 is the $n_1 \times n_1$ zero matrix and $b = (\phi'_1, \phi'_2, \ldots, \phi'_n)$ is of size n_2, that is $\phi'_i = I_{n_2}$ for all $i \in \{1, 2, \ldots, n\}$. Hence by Definition 4.7

$$\begin{aligned}
&\gamma_a \widetilde{\otimes}_n b \\
&= [(I_{n_1}, 0)^t, (I_{n_1}, 0)^t, \ldots, (I_{n_1}, 0)^t] \widetilde{\otimes}_n (I_{n_2}, I_{n_2}, \ldots, I_{n_2}) \\
&= (M, M, \ldots, M) \\
&= (Q, Q, \ldots, Q), \qquad (5.15)
\end{aligned}$$

where

$$M = \begin{bmatrix} (I_{n_1}, 0_{n_1, n_1})^t \otimes I_{n_2} & 0_{2n_1 n_2, n_1 n_2} \\ 0_{2n_1 n_2, n_1 n_2} & (I_{n_1}, 0_{n_1, n_1})^t \otimes I_{n_2} \end{bmatrix}$$

and

$$Q = \begin{bmatrix} (I_{n_1 n_2}, 0_{n_1 n_2, n_1 n_2})^t & 0_{2n_1 n_2, n_1 n_2} \\ 0_{2n_1 n_2, n_1 n_2} & (I_{n_1 n_2}, 0_{n_1 n_2, n_1 n_2})^t \end{bmatrix}.$$

Next, since $a\widetilde{\otimes}_n b$ is an object of size $2n_1n_2$, we obtain from the way γ is defined that

$$\gamma_{a\widetilde{\otimes}_n b} = ((I_{2n_1n_2}, 0_{2n_1n_2,2n_1n_2})^t, (I_{2n_1n_2}, 0_{2n_1n_2,2n_1n_2})^t, \ldots, (I_{2n_1n_2}, 0_{2n_1n_2,2n_1n_2})^t). \tag{5.16}$$

From (5.15) and (5.16), we see that $\gamma_{a\widetilde{\otimes}_n b}$ and $\gamma_a\widetilde{\otimes}_n b$ are both $(1,0)$-matrices with the same number of rows and columns. Moreover, they have the same number of *1s* and each of these *1s* is the only nonzero entry in its row and in its column. More precisely, the matrix from $\gamma_{a\widetilde{\otimes}_n b}$ is (row-)permutation equivalent to the matrix in $\gamma_a\widetilde{\otimes}_n b$. That is the rows have simply been interchanged.
This suggests that for the equality $\zeta \circ \gamma_a\widetilde{\otimes}_n b = \gamma_{a\widetilde{\otimes}_n b}$ to hold, the n-tuple of matrices constituting ζ must be permutation matrices.

Hence, there exists a $4n_1n_2 \times 4n_1n_2$ permutation matrix P such that

$$P\begin{bmatrix} (I_{n_1n_2}, 0_{n_1n_2,n_1n_2})^t & 0_{2n_1n_2,n_1n_2} \\ 0_{2n_1n_2,n_1n_2} & (I_{n_1n_2}, 0_{n_1n_2,n_1n_2})^t \end{bmatrix} = (I_{2n_1n_2}, 0_{2n_1n_2,2n_1n_2})^t,$$

where P is a permutation matrix, P is invertible and its inverse is P^t.

Now that we know P exists and is invertible, we need to verify that the n-tuple of matrices $(P, P, \ldots, P)$ is a map from $(e\widetilde{\otimes}_n a)\widetilde{\otimes}_n b$ to $e\widetilde{\otimes}_n(a\widetilde{\otimes}_n b)$ in $\mathcal{T}$ and also that the n-tuple of matrices $(P^{-1}, P^{-1}, \ldots, P^{-1})$ is a map from $e\widetilde{\otimes}_n(a\widetilde{\otimes}_n b)$ to $(e\widetilde{\otimes}_n a)\widetilde{\otimes}_n b$.

We do it for $(P, P, \ldots, P)$, the case of $(P^{-1}, P^{-1}, \ldots, P^{-1})$ is completely similar.

Once more, a and b are objects of $\mathcal{T}$. So, we can let $a = e^p$ and $b = e^m$. Observe that

$$\begin{aligned} e\widetilde{\otimes}_n(a\widetilde{\otimes}_n b) = (e\widetilde{\otimes}_n a)\widetilde{\otimes}_n b = (e\widetilde{\otimes}_n e^p)\widetilde{\otimes}_n e^m = e^{1+p+m} \\ = (I_{2^{1+p+m-1}}, I_{2^{1+p+m-1}}, \ldots, I_{2^{1+p+m-1}}) = (I_{2^{p+m}}, I_{2^{p+m}}, \ldots, I_{2^{p+m}}). \end{aligned}$$

Let $r = 2^{p+m}$. All we need check now to conclude that $(P, P, \ldots, P)$ is a map in $\mathcal{T}$ is that the following diagram commutes:

$$\begin{tikzcd} K[x]^r \arrow[r, "I_r"] \arrow[d, "P"'] & K[x]^r \arrow[r, "I_r"] \arrow[d, "P"'] & K[x]^r \arrow[r, "I_r"] \arrow[d, "P"] & K[x]^r \arrow[r, dashed] \arrow[d, "P"'] \arrow[r, phantom, shift right=5ex, "\dashrightarrow"] & K[x]^r \arrow[r, "I_r"] \arrow[d, "P"'] & K[x]^r \arrow[d, "P"] \\ K[x]^r \arrow[r, "I_r"] & K[x]^r \arrow[r, "I_r"] & K[x]^r \arrow[r, "I_r"] & K[x]^r \arrow[r, dashed] & K[x]^r \arrow[r, "I_r"] & K[x]^r \end{tikzcd} \tag{5.17}$$

Now, this diagram clearly commutes, so we can take
$\zeta := (P, P, \ldots, P)$ and $\zeta^{-1} := (P^{-1}, P^{-1}, \ldots, P^{-1}) = (P^t, P^t, \ldots, P^t)$. Therefore there exists an isomorphism namely ζ such that diagram (5.9) commutes.
A small remark: The foregoing proof for the commutativity of diagram (5.9) helps understand the motivation behind the choice of the objects of $\mathcal{T}$.
In fact, if objects are chosen arbitrarily the diagram (5.17) above may not commute. Furthermore, though diagram (5.17) commutes even if P is replaced by any matrix, what we need is a matrix that will make diagram (5.9) commute and that matrix should also be invertible because we need an isomorphism in diagram (5.9).

The commutativity of diagram (5.10) is proved in a manner similar to the proof given for the commutativity of diagram (5.9). So e is a semi-unit in $(\mathcal{T}, \widetilde{\otimes}_n)$.

Conclusion: $(\mathcal{T}, \widetilde{\otimes}_n)$ is a one-step connected semi-unital semi-monoidal subcategory of $MF(1)_n$. □

The foregoing proof works well for $\mathcal{T}$ since the objects of $\mathcal{T}$ are carefully chosen so that an n-tuple of matrices that constitutes an object in $\mathcal{T}$ is an n-tuple of identity matrices thanks to which diagrams will be commutative. Indeed, diagrams (5.9) and (5.10) in Definition 2.3, commute when a and b are objects in $\mathcal{T}$, that is, of the form e^r for some $r \in \mathbb{N} - \{0\}$. Indeed, for arbitrary values of a and b, it is possible to verify that the diagrams (5.9) and (5.10) do not always commute. This entails that $(MF(1)_n, \widetilde{\otimes}_n)$ is not a semi-unital semi-monoidal category though it is clearly a semi-monoidal category.

6. Brief comparison of the operations $\widetilde{\otimes}_n$, $\overline{\otimes}_n$ and $\widetilde{\otimes}'_n$

In this section, we briefly compare the bifunctorial operations $\widetilde{\otimes}_n$, $\overline{\otimes}_n$ and $\widetilde{\otimes}'_n$.

6.1. $\widetilde{\otimes}_n$ **vs** $\widetilde{\otimes}'_n$**.** The operation $\widetilde{\otimes}'_n$ can only be used to produce a matrix factorization of the product of two polynomials from their respective matrix factorizations when n is even. For all integers $n \geq 2$, we were able to prove Lemma 5.4 using $\widetilde{\otimes}_n$. In fact, we were able to extract a one-step connected subcategory of $(MF(1)_n, \widetilde{\otimes}_n)$ which is a semi-monoidal category. We will show that when n is even, one can extract a one-step connected semi-monoidal subcategory of $(MF(1)_n, \widetilde{\otimes}'_n)$ (see Lemma 6.1).

Lemma 6.1. *There is a one-step connected subcategory of* $(MF(1)_n, \widetilde{\otimes}'_n)$*, which is semi-monoidal.*

Proof. We will construct a one-step connected subcategory of $(MF(1)_n, \widetilde{\otimes}'_n)$ which is a semi-monoidal category. We call it $\mathcal{C}$. We will generate objects for $\mathcal{C}$ just like we did above for $\mathcal{T}$ but this time around using the operation $\widetilde{\otimes}'_n$.

• Objects of $\mathcal{C}$ are of the form e^r, where $r \in \mathbb{N} - \{0\}$. They are

$$\begin{aligned}
e^1 &= e = (1, \ldots, 1)\\
e^2 &= e\widetilde{\otimes}' e = \left(\begin{pmatrix} 0 & 1\otimes 1 \\ 1\otimes 1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & 1\otimes 1 \\ 1\otimes 1 & 0 \end{pmatrix} \right) = (J_2, \ldots, J_2)\\
e^3 &= e\widetilde{\otimes} e^2 = (1, \ldots, 1)\widetilde{\otimes}' \left(\begin{pmatrix} 0 & 1\otimes 1 \\ 1\otimes 1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & 1\otimes 1 \\ 1\otimes 1 & 0 \end{pmatrix} \right) = (J_4, \ldots, J_4),
\end{aligned}$$

where $J_4 = \begin{pmatrix} 0 & 1\otimes \begin{pmatrix} 0 & 1\otimes 1 \\ 1\otimes 1 & 0 \end{pmatrix} \\ 1\otimes \begin{pmatrix} 0 & 1\otimes 1 \\ 1\otimes 1 & 0 \end{pmatrix} & 0 \end{pmatrix}$.

In general, $e^r = (J_{2^{r-1}}, \ldots, J_{2^{r-1}})$.

• For $m, p \in \mathbb{N} - \{0\}$, a morphism $e^m \to e^p$ in $\mathcal{C}$ is an n-tuple of matrices $(\alpha_1, \ldots, \alpha_n)$ each of size $2^{p-1} \times 2^{m-1}$ such that the following diagram commutes:

$$\begin{tikzcd}
K[x]^{2^{m-1}} \arrow[r, "J_{2^{m-1}}"] \arrow[d, "\alpha_1"'] & K[x]^{2^{m-1}} \arrow[r, "J_{2^{m-1}}"] \arrow[d, "\alpha_n"'] & K[x]^{2^{m-1}} \arrow[r, "J_{2^{m-1}}"] \arrow[d, "\alpha_{n-1}"] & K[x]^{2^{m-1}} \arrow[r, dashed] \arrow[d, "\alpha_{n-2}"'] & K[x]^{2^{m-1}} \arrow[r, "J_{2^{m-1}}"] \arrow[d, "\alpha_2"'] & K[x]^{2^{m-1}} \arrow[d, "\alpha_1"] \\
K[x]^{2^{p-1}} \arrow[r, "J_{2^{p-1}}"] & K[x]^{2^{p-1}} \arrow[r, "J_{2^{p-1}}"] & K[x]^{2^{p-1}} \arrow[r, "J_{2^{p-1}}"] & K[x]^{2^{p-1}} \arrow[r, dashed] & K[x]^{2^{p-1}} \arrow[r, "J_{2^{p-1}}"] & K[x]^{2^{p-1}}
\end{tikzcd}$$

That is,

$$\begin{cases} \alpha_1 J_{2^{m-1}} = J_{2^{p-1}}\alpha_2, \\ \alpha_i J_{2^{m-1}} = J_{2^{p-1}}\alpha_{i+1}, \quad 2 \le i \le n-1, \\ \alpha_n J_{2^{m-1}} = J_{2^{p-1}}\alpha_1. \end{cases} \tag{6.1}$$

Thus from (6.1), we see that a morphism $e^m \to e^p$ in $\mathcal{C}$ can be defined in several ways. First of all observe that if $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$, then above diagram commutes. This diagram can also commute for some nonzero values of α_i, for each $i \in \{1, \dots, n\}$. In fact, we notice that given that the matrices $J_{2^{m-1}}$ *and* $J_{2^{p-1}}$ are permutation matrices, one can define the n-tuple of matrices $(\alpha_1, \alpha_2, \dots, \alpha_n)$ to be made up of equal matrices in which all entries are equal. Indeed, if we do so, the composition of two morphisms in $\mathcal{C}$ will no longer be in $\mathcal{C}$ and therefore it would not be a subcategory.

To ensure that the composition of two morphisms in $\mathcal{C}$ is again in $\mathcal{C}$, we found a way of defining these morphisms. we will present it and use it to show that $(\mathcal{C}, \widetilde{\otimes}'_n)$ is a one-step connected semi-monoidal subcategory of $MF(1)_n$.

Definition of morphisms in $\mathcal{C}$: We will define morphisms $(e^m \to e^p)$ in $\mathcal{C}$ distinguishing the case $m = p$ from the case $m \neq p$.

In both *case 1* and *case 2* below, one also has the following possible morphism $(\alpha_1, \alpha_2, \dots, \alpha_n) = (0, 0, \dots, 0)$, where each of the zeros in the n-tuple is a $2^{p-1} \times 2^{m-1}$ matrix.

Case 1: $p = m$.

Take $\alpha_1 = \alpha_2 = \cdots = \alpha_n = I_{2^{m-1}}$.

Case 2: $p \neq m$.

Define $(\alpha_1, \alpha_2, \dots, \alpha_n)$ as follows: For $1 \le r \le n$, α_r is a $2^{p-1} \times 2^{m-1}$ matrix and

$$\alpha_r = \begin{cases} \alpha_1 & \text{if } r \text{ is odd,} \\ \alpha_2 & \text{if } r \text{ is even.} \end{cases}$$

Moreover, α_1 and α_2 are matrices whose entries are all zero except for one entry which is *1* positioned anywhere in the matrix. This requirement should obey the following rule: If α_1 (respectively, α_2) has a *1* at the entry (i, j), that is at the ith row and jth column, then α_2 (respectively, α_1) should have a *1* at the entry $(2^{p-1} - i + 1, 2^{m-1} - j + 1)$. These requirements will ensure that the composition of two morphisms in $\mathcal{C}$ remains in $\mathcal{C}$ for it to be a subcategory.

We give an illustration of this before we proceed.

Let us limit ourselves to $n = 4$ (n must be even since we are dealing with $\widetilde{\otimes}'_n$). Then from the system (5.2) we have above boils down to three equations:

$$\begin{cases} \alpha_1 J_{2^{m-1}} = J_{2^{p-1}}\alpha_2, \\ \alpha_2 J_{2^{m-1}} = J_{2^{p-1}}\alpha_3, \\ \alpha_3 J_{2^{m-1}} = J_{2^{p-1}}\alpha_4. \end{cases}$$

For illustration purposes take $m = 2$ and $p = 3$. Then our equations become

$$\begin{cases} \alpha_1 J_2 = J_4 \alpha_2, \\ \alpha_2 J_2 = J_4 \alpha_3, \\ \alpha_3 J_2 = J_4 \alpha_4. \end{cases}$$

That is,

$$\begin{cases} \alpha_1 J_2 = J_4 \alpha_2, \\ \alpha_2 J_2 = J_4 \alpha_1, \end{cases} \tag{6.2}$$

since $\alpha_3 = \alpha_1$ and $\alpha_4 = \alpha_2$.

In order to specify the $\alpha' s$, we make use of the definition we gave for them. Recall that we have several choices for the α_i $(i = 1, 2)$ we choose to specify first. Indeed, once we have chosen one of the two $\alpha' s$, we have only one possibility for the other one. In this present case, α_i $(i = 1, 2)$ must be a 4×2 matrix due to the sizes of J_4 and J_2. Choose α_1 such that it has a *1* at entry $(1, 1)$. Then our α_2 must have a *1* at entry $(4 - 1 + 1, 2 - 1 + 1) = (4, 2)$. Thus, we have

$$\alpha_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } \alpha_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Our system (6.2) becomes

$$\begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}. \end{cases}$$

This boils down to

$$\begin{cases} \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \end{cases}$$

which is obviously true.

Let us give another illustration with $n = 4$, $m = 3$ and $p = 4$. Hence, proceeding like above, instead of (6.2) we have the following system:

$$\begin{cases} \alpha_1 J_4 = J_8 \alpha_2, \\ \alpha_2 J_4 = J_8 \alpha_1. \end{cases} \tag{6.3}$$

In this present case, α_i $(i = 1, 2)$ must be a 8×4 matrix due to the sizes of J_4 and J_8. Choose α_1 such that it has a *1* at entry $(2, 3)$. Then our α_2 must have a *1* at entry $(8-2+1, 4-3+1) = (7, 2)$. Hence, we have

$$\alpha_1 = \begin{bmatrix} 0&0&0&0\\ 0&0&1&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{bmatrix} \text{ and } \alpha_2 = \begin{bmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&1&0&0\\ 0&0&0&0 \end{bmatrix}.$$

Our system (6.3) becomes

$$\left\{ \begin{array}{l} \begin{bmatrix} 0&0&0&0\\ 0&0&1&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{bmatrix} \begin{bmatrix} 0&0&0&1\\ 0&0&1&0\\ 0&1&0&0\\ 1&0&0&0 \end{bmatrix} = \begin{bmatrix} 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0 \end{bmatrix} \begin{bmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&1&0&0\\ 0&0&0&0 \end{bmatrix}, \\ \begin{bmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&1&0&0\\ 0&0&0&0 \end{bmatrix} \begin{bmatrix} 0&0&0&1\\ 0&0&1&0\\ 0&1&0&0\\ 1&0&0&0 \end{bmatrix} = \begin{bmatrix} 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0 \end{bmatrix} \begin{bmatrix} 0&0&0&0\\ 0&0&1&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{bmatrix}. \end{array} \right.$$

This boils down to

$$\begin{cases} \begin{bmatrix} 0&0&0&0\\ 0&1&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{bmatrix} = \begin{bmatrix} 0&0&0&0\\ 0&1&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{bmatrix}, \\ \begin{bmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&1&0\\ 0&0&0&0 \end{bmatrix} = \begin{bmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&1&0\\ 0&0&0&0 \end{bmatrix}, \end{cases}$$

which is obviously true.

▶ $\mathcal{C}$ is a subcategory of $MF(1)_n$, which is one-step connected.

For every pair of morphisms ζ and ζ' in $hom(\mathcal{C})$, the composite $\zeta \circ \zeta'$ is in $hom(\mathcal{C})$ whenever it is defined. In fact, ζ and ζ' by definition are n-tuples of matrices that are either the zero n-tuple or matrices whose entries are all zeros except for one entry that is one as specified above when defining α_i, $i \in \{1,\dots,n\}$. Therefore, the composition of such matrices will yield another matrix which has only zeros in all its entries or only one nonzero entry which will be *1*. Thus, we will still have a morphism of $\mathcal{C}$.

In addition, $\mathcal{C}$ is a one-step connected category since between any two objects of $\mathcal{C}$ say e^m and e^p, there exists a nonzero morphism as can be seen from the definition of α_i, $i \in \{1,\dots,n\}$.

▶ Next, we prove that $(\mathcal{C}, \widetilde{\otimes}'_n)$ is a semi-monoidal category (see Definition 2.1).

Considering Definition 2.1, it is necessary to first show that α is a natural isomorphism:

- To see that $\widetilde{\otimes}'_n : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a bifunctor, use Theorem 4.11 in which $MF(1)_n$ is restricted to $\mathcal{C}$, f and g are replaced by the constant polynomial *1*, and let $x = y$.
- There is a natural isomorphism α from the functor $((-)\widetilde{\otimes}'_n(-))\widetilde{\otimes}'_n(-) : \mathcal{C} \times \mathcal{C} \times \mathcal{C} \to (\mathcal{C} \times \mathcal{C}) \times \mathcal{C}$ to the functor $(-)\widetilde{\otimes}'_n((-)\widetilde{\otimes}'_n(-)) : \mathcal{C} \times \mathcal{C} \times \mathcal{C} \to \mathcal{C} \times (\mathcal{C} \times \mathcal{C})$ with components $\alpha_{a,b,c} : (a\widetilde{\otimes}'_n b)\widetilde{\otimes}'_n c \to a\widetilde{\otimes}'_n(b\widetilde{\otimes}'_n c)$, where $a, b\, and\, c$ are matrix factorizations of the constant polynomial *1* in $\mathcal{C}$.

Let $a, b, c, a', b', and\, c'$ be objects of $\mathcal{C}$. Let $f : a \to a'$, $g : b \to b'$ and $h : c \to c'$ be maps in $\mathcal{C}$. We show that the following diagram commutes:

$$\begin{tikzcd}
(a\widetilde{\otimes}'_n b)\widetilde{\otimes}'_n c \arrow[r, "\alpha_{a,b,c}"] \arrow[d, "(f\widetilde{\otimes}'_n g)\widetilde{\otimes}'_n h"'] & a\widetilde{\otimes}'_n(b\widetilde{\otimes}'_n c) \arrow[d, "f\widetilde{\otimes}'_n(g\widetilde{\otimes}'_n h)"'] \\
(a'\widetilde{\otimes}'_n b')\widetilde{\otimes}'_n c' \arrow[r, "\alpha_{a',b',c'}"'] & a'\widetilde{\otimes}'_n(b'\widetilde{\otimes}'_n c')
\end{tikzcd}$$

that is,

$$f\widetilde{\otimes}'_n(g\widetilde{\otimes}'_n h)\circ\alpha_{a,b,c} = \alpha_{a',b',c'}\circ(f\widetilde{\otimes}'_n g)\widetilde{\otimes}'_n h \tag{6.4}$$

In fact, the matrices representing $\alpha_{a,b,c}$ and $\alpha_{a',b',c'}$ are identity matrices. Besides, the tensor product of maps is associative. Thus, (6.4) holds. That is α is a natural transformation. Moreover, for all a, b and c, it is true that $\alpha_{a,b,c}$ is an equality and so, it is an isomorphism. Hence, α is a natural isomorphism.

Next, we show that the pentagonal diagram of Definition 2.1 commutes for all objects a, b, c, d of $MF(1)_n$:

$$\begin{tikzcd}
a\widetilde{\otimes}'_n(b\widetilde{\otimes}'_n(c\widetilde{\otimes}'_n d)) \arrow[rr, "1_a\widetilde{\otimes}'_n\alpha"] \arrow[d, "\alpha"'] & & a\widetilde{\otimes}'_n((b\widetilde{\otimes}'_n c)\widetilde{\otimes}'_n d) \arrow[d, "\alpha"] \\
(a\widetilde{\otimes}'_n b)\widetilde{\otimes}'_n(c\widetilde{\otimes}'_n d) \arrow[dr, "\alpha"'] & & (a\widetilde{\otimes}'_n(b\widetilde{\otimes}'_n c))\widetilde{\otimes}'_n d \arrow[dl, "\alpha\widetilde{\otimes}'_n 1_d"] \\
& ((a\widetilde{\otimes}'_n b)\widetilde{\otimes}'_n c)\widetilde{\otimes}'_n d &
\end{tikzcd}$$

This diagram commutes because all the maps linking the vertices of the pentagon are identity maps. Indeed, we know that α the associator is an identity map. Furthermore, since the n-tuple of matrices making up α are identity matrices, it follows from Definition 4.13 of the multiplicative tensor product of n-matrix factorizations that $1_a\widetilde{\otimes}_n\alpha$ and $\alpha\widetilde{\otimes}_n 1_d$ are also identity maps.

So, $\mathcal{C}$ is a semi-monoidal category.

□

6.2. $\widetilde{\otimes}_n$ **vs** $\overline{\otimes}_n$**.** From the definitions of these two operations (see Definitions 4.3 and 4.5), we see that the operation $\widetilde{\otimes}_n$ can be obtained from $\overline{\otimes}_n$ by applying the direct sum of matrices which has the effect of simply doubling the sizes of matrix factors. Moreover,$\widetilde{\otimes}_n$ and $\overline{\otimes}_n$ can both be used to find matrix factorizations of the product of two polynomials from their respective matrix factorizations as seen in Theorems 4.15 and 4.11. Indeed, a striking difference is at the level of the sizes of the matrix factors as with $\widetilde{\otimes}_n$, one obtains matrix factors that are twice as big as the ones obtained when using $\overline{\otimes}_n$. Another observation is that unlike $\overline{\otimes}_n$, the operation $\widetilde{\otimes}_n$ can be used to generate objects of the shape e^r (see section 5) which help one to extract a one-step connected semi-monoidal subcategory of $MF(1)_n$. Finally, for $n = 2$, $\overline{\otimes}_n$ was used in [12] to improve results obtained in [11] using $\widetilde{\otimes}_n$.

6.3. $\widetilde{\otimes}'_n$ **vs** $\overline{\otimes}_n$**.** Clearly for all n, $\overline{\otimes}_n$ can be used to obtain the matrix factorization of two polynomials from their respective matrix factorizations. This is only true of $\widetilde{\otimes}'_n$ when n is even.

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