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ON THE nX-COMPLEMENTARY GENERATION OF THE PROJECTIVE SYMPLECTIC GROUP $PSp(4,5)$

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ABSTRACT. This paper classifies all the nontrivial conjugacy classes of the finite simple symplectic group $PSp(4,5)$ whether they are complementary generators or not. A group G is said to be nX-complementary generated if given an arbitrary nonidentity element x in G, then there exists an element y in nX such that $G = \langle x, y\rangle$. The group $PSp(4,5)$ is nX-complementary generated for $n = 5, 12, 13, 15, 20, 30$, with $nX \notin \{5A, 5B, 5C, 5D\}$. The proof combines the structure constant method from the character table, detailed analysis of the group's maximal subgroups, and applications of Scott's theorem and other generation criteria. Computational support from GAP is used extensively to calculate structure constants and verify generation.

1. Introduction

The study of generation properties of finite groups has significant implications across group theory and representation theory. In particular, understanding how finite simple groups can be generated by specific types of elements provides insights into their structure and has applications to computational group theory, the genus of simple groups [34], and projective representations [27]. A finite group G is said to be (lX, mY, nZ)-*generated* if it is generated by two elements x and y, such that the orders of x, y, and xy are l, m and n, respectively [28]. Here $[x] = lX$, $[y] = mY$ and $[z] = nZ$, where $[x]$ is the conjugacy class of lX in G containing elements of order l. The same applies to $[y]$ and $[z]$. In this case, G is also a quotient group of the triangular group $T(l, m, n)$ and, by the definition

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of the triangular group, G is also a $(\sigma(l), \sigma(m), \sigma(n))$-generated group for any $\sigma \in S_3$. Therefore, we assume that $l \leq m \leq n$.

A fundamental question in this area concerns generation by a conjugacy class. This leads to the following key definition.

Definition 1.1. Let G be a finite simple group and let nX be a nontrivial conjugacy class of G, $n \in \mathbb{Z}^+$. Then G is said to be nX-complementary generated if given an arbitrary nonidentity element $x \in G$, then there exists an element $y \in nX$ such that $G = \langle x, y \rangle$.

The motivation of studying the nX-complementary generation of a finite simple group comes from a conjecture by Brenner–Guralnick–Wiegold [18] that every finite simple group can be generated by an arbitrary nontrivial element together with another suitable element.

In a series of papers [2, 21, 26, 23, 25, 24, 28, 29], the nX-complementary generations of the sporadic simple groups Th, Co_1, J_1, J_2, J_3, HS, McL, Co_3, Co_2 and F_{22} have been investigated. The authors together with others established all the nX-complementary generations of the Chevalley groups $G_2(3)$ and $G_2(4)$ and also the Tits group ${}^2\mathrm{F}_4(2)'$ (see [9, 10, 8]). We showed that the group $G_2(3)$ is nX-complementary generated if and only if $n \geq 6$ and $nX \notin \{6A, 6B\}$ while $G_2(4)$ is nX-complementary generated if and only if $n \geq 5$ and $nX \notin \{5A, 5B\}$. The first author, together with Seretlo, also established in [17] the nX-complementary generation of the group $PSL(3,5)$.

The following proposition gives a criterion for G to be nX-complementary generated or not.

Proposition 1.2. *(see [22]) A finite nonabelian group G is nX-complementary generated if and only if for each conjugacy class pY of G, where p is prime, there exists a conjugacy class $t_{pY}Z$, depending on pY, such that G is $(pY, nX, t_{pY}Z)$-generated. Moreover, if G is a finite simple group, then G is not $2X$-complementary generated for any conjugacy class of involutions.*

In this paper, we focus on the projective symplectic group $PSp(4,5)$, applying the structure constant method combined with results on generation of simple groups to establish the nX-complementary generators of $PSp(4,5)$, where nX is a nontrivial conjugacy class of elements of order n as in the atlas [20]. We follow the same notation used in [7, 5, 6, 13, 12, 11, 16, 15, 14]. The main result on the nX-complementary generators of $PSp(4,5)$ reads as follows.

Theorem 1.3. *Let $G = PSp(4,5)$ and let nX be a nontrivial conjugacy class of G. Then G is nX-complementary generated for $n = 5$ and $n \geq 12$, except when $nX \in \{5A, 5B, 5C, 5D\}$.*

Proof. The proof follows from a sequence of propositions in section 3. $\square$

The proof of the above theorem proceeds through a detailed analysis of structure constants, maximal subgroups, and applications of generation criteria, supported by computational verification using GAP [32]. The paper is organized as follows: Section 2 reviews the necessary preliminaries on structure constants, generation criteria, and related tools. Section 3 contains the detailed classification for $PSp(4,5)$, presented through a sequence of propositions that collectively prove our main theorem.

2. Preliminaries

This section outlines the theoretical framework and technical tools used in our investigation of the (lX, mY, nZ)-generations, particularly the nX-complementary generations. We begin with the fundamental concepts of structure constants and generation by conjugacy classes.

2.1. **Structure constants and generation criteria.** Let G be a finite group and let $C_1, C_2, \ldots, C_k$ (not necessarily distinct), $k \geq 3$, be conjugacy classes of G with $g_1, g_2, \ldots, g_k$ being representatives for these classes, respectively.

For a fixed representative $g_k \in C_k$ and for $g_i \in C_i,\ 1 \leq i \leq k-1$, denote by $\Delta_G = \Delta_G(C_1, \ldots, C_k)$ the number of distinct $(k-1)$-tuples $(g_1, \ldots, g_{k-1})$ such that $g_1 g_2 \cdots g_{k-1} = g_k$. This number is known as *class algebra constant* or *structure constant*. With $\mathrm{Irr}(G) = \{\chi_1, \chi_2, \ldots, \chi_r\}$ being the set of complex irreducible characters of G, the number Δ_G is easily calculated from the character table of G through the formula

$$\Delta_G(C_1, \ldots, C_k) = \frac{\prod_{i=1}^{k-1} |C_i|}{|G|} \sum_{i=1}^{r} \frac{\chi_i(g_1)\chi_i(g_2)\cdots\chi_i(g_{k-1})\overline{\chi_i(g_k)}}{(\chi_i(1_G))^{k-2}}. \tag{2.1}$$

For generation analysis, we define the more restrictive quantity $\Delta_G^*(C_1, \ldots, C_k)$ as the number of $(k-1)$-tuples $(g_1, \ldots, g_{k-1}) \in C_1 \times \cdots \times C_{k-1}$ satisfying both

$$g_1 g_2 \cdots g_{k-1} = g_k \quad \text{and} \quad \langle g_1, g_2, \ldots, g_{k-1} \rangle = G. \tag{2.2}$$

Definition 2.1. If $\Delta_G^*(C_1, C_2, \ldots, C_k) > 0$, then the group G is said to be $(C_1, C_2, \ldots, C_k)$**-generated**.

Also, if $H \leq G$ is any subgroup containing the fixed element $g_k \in C_k$, then we let $\Sigma(H) = \Sigma_H(C_1, \ldots, C_k)$ be the total number of distinct $(k-1)$-tuples $(g_1, g_2, \ldots, g_{k-1})$ such that $g_1 g_2 \cdots g_{k-1} = g_k$ and $\langle g_1, g_2, \ldots, g_{k-1} \rangle \leq H$. The value of $\Sigma_H(C_1, C_2, \ldots, C_k)$ can be obtained as a sum of the structure constants $\Delta_H(c_1, c_2, \ldots, c_k)$ of H-conjugacy classes $c_1, c_2, \ldots, c_k$ such that $c_i \subseteq H \bigcap C_i$.

Theorem 2.2. *Let G be a finite group and let H be a subgroup of G containing a fixed element g such that $\gcd(o(g), [N_G(H){:}H]) = 1$. Then the number $h(g, H)$ of conjugates of H containing g is $\chi_H(g)$, where $\chi_H(g)$ is the permutation character of G with action on the conjugates of H. In particular*

$$h(g, H) = \sum_{i=1}^{m} \frac{|C_G(g)|}{|C_{N_G(H)}(x_i)|}, \tag{2.3}$$

where $x_1, x_2, \ldots, x_m$ are representatives of the $N_G(H)$-conjugacy classes fused to the G-class of g.

Proof. See for example Ganief and Moori [22, 26, 23]. □

The above number $h(g,H)$ is useful in giving a lower bound for $\Delta^*_G(C_1,\ldots,C_k)$, namely, $\Delta^*_G(C_1,C_2,\ldots,C_k) \geq \Theta_G(C_1,C_2,\ldots,C_k)$, where

$$\Theta_G(C_1,\ldots,C_k) = \Delta_G(C_1,\ldots,C_k) - \sum h(g_k,H)\Sigma_H(C_1,\ldots,C_k), \qquad (2.4)$$

where g_k is a representative of the class C_k and the sum is taken over all the representatives H of G-conjugacy classes of maximal subgroups containing elements of all the classes $C_1,C_2,\ldots,C_k$.

If $\Theta_G > 0$, then certainly G is $(C_1,C_2,\ldots,C_k)$-generated. In the case where $C_1 = C_2 = \cdots = C_{k-1} = C$, then G can be generated by $k-1$ elements suitably chosen from C and hence $rank(G{:}C) \leq k-1$.

2.2. **Generation and nongeneration results.** We now state several key results that facilitate the analysis of group generation by conjugacy classes.

Theorem 2.3 and Lemma 2.4 are in many cases useful in determining whether G is nX-complementary generated, while Lemma 2.6 is in some cases useful in establishing nongeneration for finite groups.

Theorem 2.3 (see, Ali and Moori [1] or [4])**.** *Let G be a $(2X,sY,tZ)$-generated simple group; then G is $(sY,sY,(tZ)^2)$-generated.*

Lemma 2.4 (see [22])**.** *If G is sY-complementary generated and $(rX)^n = sY$, then G is rX-complementary generated.*

Proof. Let rX and sY be nontrivial conjugacy classes of G such that $(rX)^n = sY$ for some integer n. If G is not rX-complementary generated, then there exists an element x of prime order such that $\langle x,y\rangle < G$ for all $y \in rX$. Since $x, y^n \in \langle x,y\rangle$, it follows that $\langle x,y^n\rangle \leq \langle x,y\rangle < G$ for all $y^n \in sY$. Thus the results follows by method of contrapositive. □

Lemma 2.5 (see Ali and Moori [1] or Conder, Wilson, and Woldar [19])**.** *Let G be a finite simple group such that G is (lX,mY,nZ)-generated. Then G is $(\underbrace{lX,\ldots,lX}_{m-times},(nZ)^m)$-generated.*

Proof. Since G is (lX,mY,nZ)-generated group, it follows that there exist $x \in lX$ and $y \in mY$ such that $xy \in nZ$ and $\langle x,y\rangle = G$. Let $N := \left\langle x, x^y, \ldots, x^{y^{m-1}}\right\rangle$. Then $N \trianglelefteq G$. Since G is simple group and N is nontrivial subgroup we obtain that $N = G$. Furthermore, we have

$$\begin{aligned} xx^yx^{y^2}x^{y^{m-1}} &= x(yxy^{-1})(y^2xy^{-2})\cdots(y^{m-1}xy^{1-m}) \\ &= (xy)^m \in (nZ)^m. \end{aligned}$$

Since $x^{y^i} \in lX$ for all i, the result follows. □

The following two results are in some cases useful in establishing nongeneration for finite groups.

Lemma 2.6 (e.g., see Ali and Moori [1] or Conder, Wilson, and Woldar [19])**.** *Let G be a finite centerless group. If $\Delta^*_G(C_1,C_2,\ldots,C_k) < |C_G(g_k)|$, $g_k \in C_k$, then $\Delta^*_G(C_1,C_2,\ldots,C_k) = 0$ and therefore G is not $(C_1,C_2,\ldots,C_k)$-generated.*

Proof. Here we prove the contrapositive of the statement; that is, if $\Delta^*_G(C_1, \ldots, C_k) > 0$, then $\Delta^*_G(C_1, \ldots, C_k) \geq |C_G(g_k)|$, for a fixed $g_k \in C_k$. So let us assume that $\Delta^*_G(C_1, C_2, \ldots, C_k) > 0$. Thus, there exists at least one $(k-1)$-tuple $(g_1, g_2, \ldots, g_{k-1}) \in C_1 \times C_2 \times \cdots \times C_{k-1}$ satisfying (2.2). Let $x \in C_G(g_k)$. Then we obtain

$$x(g_1g_2\cdots g_{k-1})x^{-1} = (xg_1x^{-1})(xg_2x^{-1})\cdots(xg_{k-1}x^{-1}) = (xg_kx^{-1}) = g_k.$$

Thus, the $(k-1)$-tuple $(xg_1x^{-1}, xg_2x^{-1}, \ldots, xg_{k-1}x^{-1})$ will generate G. Moreover, if x_1 and x_2 are distinct elements of $C_G(g_k)$, then the $(k-1)$-tuples
$(x_1g_1x_1^{-1}, \ldots, x_1g_{k-1}x_1^{-1})$ and $(x_2g_1x_2^{-1}, x_2g_2x_2^{-1}, \ldots, x_2g_{k-1}x_2^{-1})$
are also distinct since G is centerless. Thus we have at least $|C_G(g_k)|$ $(k-1)$-tuples $(g_1, g_2, \ldots, g_{k-1})$ generating G. Hence $\Delta^*_G(C_1, C_2, \ldots, C_k) \geq |C_G(g_k)|$. □

The following result is due to Ree [30].

Theorem 2.7. *Let G be a transitive permutation group generated by permutations $g_1, g_2, \ldots, g_s$ acting on a set of n elements such that $g_1g_2\cdots\ g_s = 1_G$. If the generator g_i has exactly c_i cycles for $1 \leq i \leq s$, then $\sum_{i=1}^{s} c_i \leq (s-2)n + 2$.*

Proof. See for example Ali and Moori [1]. □

The following result is due to Scott ([19] and [31]).

Theorem 2.8 (Scott's Theorem)**.** *Let $g_1, g_2, \ldots, g_s$ be elements generating a group G such that $g_1g_2\cdots g_s = 1_G$ and $\mathbb{V}$ be an irreducible module for G with $\dim \mathbb{V} = n \geq 2$. Let $C_{\mathbb{V}}(g_i)$ denote the fixed point space of $\langle g_i \rangle$ on $\mathbb{V}$ and let d_i be the codimension of $C_{\mathbb{V}}(g_i)$ in $\mathbb{V}$. Then $\sum_{i=1}^{s} d_i \geq 2n$.*

With χ being the ordinary irreducible character afforded by the irreducible module $\mathbb{V}$ and $\mathbf{1}_{\langle g_i \rangle}$ being the trivial character of the cyclic group $\langle g_i \rangle$, the codimension d_i of $C_{\mathbb{V}}(g_i)$ in $\mathbb{V}$ can be computed using the following formula ([22]):

$$\begin{aligned} d_i &= \dim(\mathbb{V}) - \dim(C_{\mathbb{V}}(g_i)) = \dim(\mathbb{V}) - \left\langle \chi\downarrow^G_{\langle g_i \rangle}, \mathbf{1}_{\langle g_i \rangle} \right\rangle \\ &= \chi(1_G) - \frac{1}{|\langle g_i \rangle|} \sum_{j=0}^{o(g_i)-1} \chi(g_i^j). \end{aligned} \tag{2.5}$$

These results provide the theoretical foundation for our analysis of $PSp(4,5)$ in section 3.

3. The nX-complementary generations of $PSp(4,5)$

In this section, we apply the results, discussed in section 2, to the group $PSp(4,5)$. We establish the nX-complementary generators of the group $PSp(4,5)$, for all its nontrivial conjugacy classes of elements.

The group $Sp(4,5)$, is defined to be the set of 4×4 matrices over the finite field $\mathbb{F}_5$ preserving a nondegenerate symplectic form. It has a center consisting of $\{I_4, -I_4\}$. The factor group $Sp(4,5)/\{I_4, -I_4\}$, which we denote by $PSp(4,5)$, is a simple group of order $4680000 = 2^6 \times 3^2 \times 5^4 \times 13$. According to the atlas of finite groups [20], $PSp(4,5)$ has precisely 34 conjugacy classes of its elements. Our objective is to find which of the 33 nontrivial classes are the complementary generators. The atlas lists 8 conjugacy classes of maximal subgroups for $PSp(4,5)$. For easy reference, we list these maximal subgroups, along with their orders and indices in G in Table 1.

TABLE 1. Maximal subgroups of $PSp(4,5)$

Maximal Subgroup	Order	Index
$M_1 = 5_+^{1+2}{:}4A_5$	$30000 = 2^4 \times 3 \times 5^4$	156
$M_2 = 5^3{:}(2 \times A_5)^{\cdot}2$	$30000 = 2^4 \times 3 \times 5^4$	156
$M_3 = L_2(25){:}2_2$	$15600 = 2^4 \times 3 \times 5^2 \times 13$	300
$M_4 = 2^{\cdot}(A_5 \times A_5) \cdot 2$	$14400 = 2^6 \times 3^2 \times 5^2$	325
$M_5 = 2^4{:}A_5$	$960 = 2^6 \times 3 \times 5$	4875
$M_6 = S_3 \times S_5$	$720 = 2^4 \times 3^2 \times 5$	6500
$M_7 = (2^2 \times A_5){:}2$	$480 = 2^5 \times 3 \times 5$	9750
$M_8 = A_6$	$360 = 2^3 \times 3^2 \times 5$	13000

In this section, we let $G = PSp(4,5)$. By the electronic atlas of Wilson [33], G has two permutation generators g_1 and g_2 on 156 points. These generators can be taken as follows:

$$
\begin{aligned}
g_1 =\,&(1,2)(3,5)(4,7)(6,10)(8,12)(9,13)(11,16)(14,20)(15,21)(17,23)(19,25)\\
&(22,29)(24,31)(26,33)(27,34)(28,36)(30,39)(32,41)(35,45)(37,47)(38,48)\\
&(40,42)(44,53)(46,56)(49,58)(50,59)(51,61)(52,63)(54,65)(55,66)(57,69)\\
&(62,74)(64,76)(70,82)(71,83)(72,73)(75,88)(77,91)(78,92)(79,94)(80,96)\\
&(84,100)(85,101)(86,102)(89,105)(90,106)(93,110)(95,107)(97,103)(98,112)\\
&(99,114)(104,118)(108,122)(109,124)(111,127)(113,116)(115,130)(117,121)\\
&(119,133)(125,137)(126,138)(128,140)(131,142)(132,143)(134,145)(136,147)\\
&(139,150)(141,152)(144,154)(146,149)(151,153)(155,156);\\
g_2 =\,&(1,3,6)(2,4,8)(5,9,14)(7,11,17)(10,15,22)(12,18,24)(13,19,26)(20,27,35)\\
&(21,28,37)(23,30,40)(25,32,42)(29,38,49)(31,36,46)(33,43,52)(34,44,54)\\
&(39,50,60)(41,51,62)(45,55,67)(47,57,70)(53,64,77)(56,68,80)(58,71,84)\\
&(59,72,85)(61,73,86)(63,75,89)(65,78,93)(66,79,95)(69,81,97)(74,87,103)\\
&(76,90,107)(82,98,113)(83,99,115)(88,104,119)(91,108,123)(92,109,125)\\
&(94,111,128)(96,106,121)(100,116,102)(101,117,131)(105,120,134)\\
&(110,126,139)(112,129,141)(114,130,138)(118,132,144)(122,135,146)\\
&(124,136,148)(133,137,149)(140,151,145)(142,153,143)(150,155,154);
\end{aligned}
$$

with $o(g_1) = 2,\ o(g_2) = 3$ and $o(g_1 g_2) = 13$.

Using (2.3) on GAP [32], we calculated the values of $h(g, M_i)$, where g is a representative of a nontrivial conjugacy class of $PSp(4,5)$ and over all the maximal subgroups M_i of $PSp(4,5)$. We list these values in Table 2.

TABLE 2. The values $h(g, M_i)$, $1 \le i \le 8$, for nonidentity classes and maximal subgroups of $PSp(4,5)$

	M_1	M_2	M_3	M_4	M_5	M_6	M_7	M_8
$1A$	156	156	300	325	4875	6500	9750	13000
$2A$	36	12	60	61	75	260	510	0
$2B$	8	12	12	17	35	60	66	60
$3A$	6	0	15	10	30	11	15	40
$3B$	0	6	0	1	0	20	0	40
$4A$	12	8	0	1	15	0	30	0
$4B$	2	0	4	3	3	4	4	6
$5A$	6	31	0	25	0	0	0	0
$5B$	6	31	0	25	0	0	0	0
$5C$	1	6	15	15	0	25	0	0
$5D$	11	6	10	10	0	0	25	0
$5E$	1	1	0	0	5	0	0	5
$5F$	1	1	0	0	5	0	0	5
$6A$	0	6	0	1	0	20	0	0
$6B$	0	0	3	4	6	5	3	0
$6C$	2	0	3	2	2	3	3	0
$10A$	6	7	0	1	0	0	0	0
$10B$	6	7	0	1	0	0	0	0
$10C$	1	2	0	6	0	0	5	0
$10D$	1	2	0	6	0	0	5	0
$10E$	1	2	5	1	0	5	0	0
$10F$	3	2	2	2	0	0	1	0
$12A$	2	0	1	0	0	1	1	0
$12B$	0	2	0	1	0	0	0	0
$13A$	0	0	1	0	0	0	0	0
$13B$	0	0	1	0	0	0	0	0
$13C$	0	0	1	0	0	0	0	0
$15A$	0	1	0	1	0	0	0	0
$15B$	0	1	0	1	0	0	0	0
$15C$	1	0	0	0	0	1	0	0
$20A$	2	3	0	1	0	0	0	0
$20B$	2	3	0	1	0	0	0	0
$30A$	0	1	0	1	0	0	0	0
$30B$	0	1	0	1	0	0	0	0

Remark 3.1. It has been mentioned in Proposition 1.2 that a finite simple group G cannot be $2X$-complementary generated; for if it is $2X$-complementary generated, then there exists a conjugacy class nZ of G such that G is $(2Y, 2X, nZ)$-generated group. We know that two involutions generate a dihedral group, which is not simple group. Therefore, if G is a simple group, then it is not $2X$-complementary generated for any conjugacy class $2X$ of involutions of G. Hence, the investigation of the nX-complementary generation in simple groups will be when $n \ge 3$.

Proposition 3.2. *Let $X \in \{A, B\}$. The group G is not $3X$-complementary generated.*

Proof. Due to the condition $\frac{1}{l}+\frac{1}{m}+\frac{1}{t}<1$, we accomplish the result by dealing with the case $(2A,3X,tZ)$ for $t\geq 10$. Hence, we consider the set

$$T=\{10A,10B,10C,10D,10E,10F,12A,12B,\\ 13A,13B,13C,15A,15B,15C,20A,20B,30A,30B\}$$

and obtain the result by determining the values of $\Delta^{*}(2A,3X,tZ)$ for all conjugacy classes $tZ\in T$.

Case $(2A,3A,tZ)$**:** Computations with the GAP [32] yield

$$\Delta_G^{*}(2A,3A,tZ)=\Delta_G(2A,3A,tZ)=0$$

for all $tZ\in S=\{10A,10B,10C,10D,12A,12B,15A,15B,\ 15C,20A,20B,30A,30B\}$. From [3, Proposition 6] we know that G is not $(2A,3A,tZ)$-generated for $tZ\in R=\{13A,13B,13C\}$. Lastly, we obtained $\Delta_G(2A,3A,10E)=10<50=|C_G(10E)|$ and $\Delta_G(2A,3A,10F)=5<20=|C_G(10F)|$.

By Lemma 2.6, we must have $\Delta^{*}_{PSp(4,5)}(2A,3A,tZ)=0$ for $tZ\in U=\{10E,10F\}$.

Since we find $\Delta^{*}_{PSp(4,5)}(2A,3A,tZ)=0$ for every conjugacy class $tZ\in T=S\cup R\cup U$, it follows that G is not $(2A,3A,tZ)$-generated for every $tZ\in T$. We conclude, by Proposition 1.2, that G is not $3A$-complementary generated.

Case $(2A,3B,tZ)$**:** In this case, we find $\Delta_G^{*}(2A,3B,tZ)=\Delta_G(2A,3B,tZ)=0$ for all $tZ\in V=\{10A,10B,10C,10D,10E,10F,12B,15A,15B,15C,20A,20B\}$. Again, from [3, Proposition 6], we know that G is not $(2A,3B,tZ)$-generated for $tZ\in R=\{13A,13B,13C\}$. Finally, we obtain $\Delta_G(2A,3B,12B)=4<12=|C_G(12B)|$ and $\Delta_G(2A,3B,30X)=6<30=|C_G(30X)|$, $X\in\{A,B\}$. By Lemma 2.6, we must have $\Delta_G^{*}(2A,3B,tZ)=0$ for $tZ\in W=\{12B,30A,30B\}$.

Since $\Delta_G^{*}(2A,3B,tZ)=0$ for every conjugacy class $tZ\in T=V\cup R\cup W$, it follows that G is not $(2A,3B,tZ)$-generated for every $tZ\in T$. Hence, G is not $3B$-complementary generated. □

Proposition 3.3. *The group G is not $4X$-complementary generated, for $X\in\{A,B\}$.*

Proof. In this case, let $T=\{tZ|\,tZ \text{is a conjugacy class of} G,\ t\geq 5\}$.

Case $(2A,4A,tZ)$**:** Computations via GAP reveal the structure constants

$$\Delta_G(2A,4A,tZ)=0$$

for any $tZ\in T$ except $tZ=12A,20A,20B$. For these three cases, we find

$$\Delta_G(2A,4A,tZ)\ =\ \begin{cases}6<12=|C_G(12A)|, & tZ=12A,\\ 4<20=|C_G(20Z)|, & tZ=20Z \text{ for } Z=\{A,B\}.\end{cases}$$

Thus, $\Delta_G^{*}(2A,4A,tZ)=0$ for all $tZ\in T$. Hence G is not $4A$-complementary generated.

Case $(2A,4B,tZ)$**:** In this case, we partition T into a disjoint union $T=S\cup U\cup V$, where

$$S=\{5A,5B,5D,6A,10A,10B,10C,10D,10F,15C\},\\ U=\{5C,6B,6C,10E,15A,15B,30A,30B\},$$

and

$$V=\{12A,12B,13A,13B,13C,20A,20B\}.$$

Computations with GAP reveal $\Delta_G^{*}(2A,4B,tZ)=\Delta_G(2A,4B,tZ)=0$ for $tZ\in S$. For tZ in U, we have

$$\begin{aligned}\Delta_G(2A,4B,5C) &= 125<750=|C_G(5C)|,\\ \Delta_G(2A,4B,6B) &= 18<36=|C_G(6B)|,\end{aligned}$$

$$\begin{aligned}
\Delta_G(2A,4B,6C) &= 12 < 24 = |C_G(6C)|,\\
\Delta_G(2A,4B,10E) &= 25 < 50 = |C_G(10E)|,\\
\Delta_G(2A,4B,15X) &= 15 < 30 = |C_G(5X)|,\ X \in \{A,B\},\\
\Delta_G(2A,4B,30X) &= 15 < 30 = |C_G(30X)|,\ X \in \{A,B\}.
\end{aligned}$$

Application of Lemma 2.6 renders $\Delta^*_G(2A,4B,tZ) = 0$ for all tZ in U.

For $tZ \in V$ we have

(i) $\Delta_G(2A,4B,12A) = 27$ and by Table 2, only the maximal subgroups M_2, M_3, M_6, and M_7 have classes that fuse into classes $2A$, $4B$, and $12A$ of G. However, the intersections $M_2 \cap M_3$, $M_3 \cap M_6$, and $M_3 \cap M_7$ have no classes that fuse into the class $12A$ of G. So, since $\Sigma(M_3) = 27$ and $h(g,M_3) = 1$, $g \in 12A$, choosing a fixed element z in $M_3 \cap 12A$ such that $xy = z$ for $x \in 2A$ and $y \in 4B$, yields

$$\Delta^*_G(2A,4B,12A) \leq \Delta_G(2A,4B,12A) - 1\cdot\Sigma(M_3) = 27 - 1(27) = 0.$$

(ii) $\Delta_G(2A,4B,12B) = 12$ and only the maximal subgroup M_4 make a contribution with $\Sigma(M_4) = 12$ and $h(g,M_4) = 1$, $g \in 12B$. Thus,

$$\Delta^*_G(2A,4B,12B) \leq \Delta_G(2A,4B,12B) - 1\cdot\Sigma(M_4) = 12 - 1(12) = 0.$$

(iii) $\Delta_G(2A,4B,13X) = 13$ for $X \in \{A,B,C\}$ and only the maximal subgroup M_3 make a contribution with $\Sigma^*(M_3) = 13$ and $h(g,M_3) = 1$, $g \in 13X$. Hence, for $X \in \{A,B,C\}$,

$$\Delta^*_G(2A,4B,13X) = \Delta_G(2A,4B,13X) - 1\cdot\Sigma^*(M_3) = 13 - 1(13) = 0.$$

(iv) $\Delta_G(2A,4B,20X) = 20$ for $X \in \{A,B\}$. By Table 2, only the maximal subgroups M_2, and M_4 have classes that fuse into classes $2A$, $4B$, and $20X$ of G. We find $\Sigma^*(M_2) = 10$ and $h(g,M_2) = 2$, $g \in 20X$ while $\Sigma^*(M_4) = 20$ and $h(g,M_4) = 1$, $g \in 20X$. Furthermore, the intersection $M_2 \cap M_4$ does not have classes that fuse into the class $20X$ of G for $X \in \{A,B\}$. Now, choosing a fixed element z in $M_2 \cap 20X$ such that $xy = z$ for $x \in 2A$ and $y \in 4B$, yields

$$\Delta^*_G(2A,4B,20X) \leq \Delta_G(2A,4B,20X) - 2\cdot\Sigma^*(M_2) = 20 - 2(10) = 0.$$

Results in (i) to (iv) show that $\Delta^*_G(2A,4B,tZ) = 0$ for all tZ in V. Finally,

$$\Delta^*_G(2A,4B,tZ) = 0$$

for all $tZ \in T = S \cup U \cup V$. Hence, G is not $4B$-complementary generated. □

For Propositions 3.4–3.11, we let

$$T = \{tZ |\, tZ \text{is a conjugacy class of}\, G,\ t \geq 4\}$$

to satisfy the condition $\frac{1}{l} + \frac{1}{m} + \frac{1}{t} < 1$.

Proposition 3.4. *Let $X \in \{A,B\}$. The group G is not $5X$-complementary generated.*

Proof. Computations with GAP reveal that $\Delta_G(2A,5X,tZ) = 0$ for all $tZ \in T \setminus \{10Y,10W\}$, where $(Y,W) = (B,D)$ when $X = A$ and $(Y,W) = (A,C)$ when $X = B$. Now,

$$\begin{aligned}
&\Delta_G(2A,5X,tZ)\\
&= \begin{cases} 1 < 600 = |C_G(10Y)|, & Y = B \,\text{when}\, X = A \,\text{and}\, Y = A \,\text{when}\, X = B,\\ 2 < 100 = |C_G(10W)|, & W = D \,\text{when}\, X = A \,\text{and}\, W = C \,\text{when}\, X = B. \end{cases}
\end{aligned}$$

Thus, $\Delta^*_G(2A,5X,tZ) = 0$ for all $tZ \in T$. Hence G is not $5X$-complementary generated for $X \in \{A,B\}$. □

Proposition 3.5. *The group G is not $5C$-complementary generated.*

Proof. Calculations show that $\Delta^*_G(2A,5C,tZ) = \Delta_G(2A,5C,tZ) = 0$ for all $tZ \in T \setminus \{4B, 6B, 10C, 10D, 10E\}$. For the remaining classes it is found that

$$\begin{aligned}
\Delta_G(2A,5C,4B) &= 4 < 24 = |C_G(4B)|,\\
\Delta_G(2A,5C,6B) &= 6 < 36 = |C_G(6B)|,\\
\Delta_G(2A,5C,10C) &= 2 < 100 = |C_G(10C)|,\\
\Delta_G(2A,5C,10D) &= 2 < 100 = |C_G(10D)|,\\
\Delta_G(2A,5C,10E) &= 3 < 50 = |C_G(10E)|.
\end{aligned}$$

We conclude, by Lemma 2.6, that $\Delta^*_G(2A,5C,tZ) = 0$ for all

$$tZ \in \{4B, 6B, 10C, 10D, 10E\}.$$

Ultimately, $\Delta^*_G(2A,5C,tZ) = 0$ for all $tZ \in T$ showing that G is not $5C$-complementary generated. □

Proposition 3.6. *The group G is not $5D$-complementary generated.*

Proof. For classes $tZ \in \{6C, 10A, 10B, 10C, 10D, 10E, 10F\} \subset T$, we have

$$\begin{aligned}
\Delta_G(2A,5D,6C) &= 6 < 24 = |C_G(6C)|,\\
\Delta_G(2A,5D,10X) &= 24 < 600 = |C_G(10X)|, X \in \{A,B\},\\
\Delta_G(2A,5D,10X) &= 1 < 100 = |C_G(10X)|, X \in \{C,D\},\\
\Delta_G(2A,5D,10E) &= 2 < 50 = |C_G(10E)|,\\
\Delta_G(2A,5D,10F) &= 5 < 20 = |C_G(10F)|.
\end{aligned}$$

We conclude, by applying Lemma 2.6, that $\Delta^*_G(2A,5D,tZ) = 0$ for all

$$tZ \in \{6C, 10A, 10B, 10C, 10D, 10E, 10F\}.$$

For the remaining classes tZ in $T \setminus \{6C, 10A, 10B, 10C, 10D, 10E, 10F\}$ we have $\Delta^*_G(2A,5D,tZ) = \Delta_G(2A,5D,tz) = 0$. This shows that $\Delta^*_G(2A,5D,tZ) = 0, \forall\, tZ \in T$ and thus, G is not $5D$-complementary generated. □

Proposition 3.7. *The group G is $5X$-complementary generated for $X \in \{E,F\}$.*

Proof. In [3, Propositions 9(ii), 10(ii), 16 and 19] state that G is generated by the triples $(2A,5X,13A)$, $(2B,5X,13A)$, $(3A,5X,13A)$, $(3B,5X,13A)$, $(5Y,5X,13A)$ for $Y \in \{A,B,C,D,E,F\}$ and $X \in \{E,F\}$. By [3, Proposition 20], G is $(5X,13Z,13A)$-generated for $X \in \{E,F\}$ and $Z \in \{A,B,C\}$. Hence the triple $(13Z,5X,13A)$ also generates G. We see here that G is $(pY,5X,13A)$-generated for all $pY \in P$ and $X \in \{E,F\}$. Consequently, application of Proposition 1.2 infers that the classes $5E$ and $5F$ are complementary generators of G. □

Proposition 3.8. *The group G is not $6X$-complementary generated for $X \in \{A,B,C\}$.*

Proof. **Case** $(2A,6A,tZ)$**:** For classes $tZ \in \{6C, 15A, 15B, 15C\} \subset T$, we have

$$\begin{aligned}
\Delta_G(2A,6A,6C) &= 6 < 24 = |C_G(6C)|,\\
\Delta_G(2A,6A,15X) &= 6 < 30 = |C_G(15X)|, X \in \{A,B\},\\
\Delta_G(2A,6A,15C) &= 5 < 15 = |C_G(15C)|.
\end{aligned}$$

We conclude, by applying Lemma 2.6, that $\Delta^*_G(2A,6A,tZ) = 0$ for all

$$tZ \in \{6C, 15A, 15B, 15C\}.$$

For the remaining classes tZ in $T \setminus \{6C, 15A, 15B, 15C\}$ we have $\Delta_G^*(2A, 6A, tZ) = \Delta_G(2A, 6A, tz) = 0$. This shows that $\Delta_G^*(2A, 6A, tZ) = 0$ for all $tZ \in T$ and thus, G is not $6A$-complementary generated.

Case $(2A, 6B, tZ)$**:** Let R and S be disjoint subsets of T such that

$$R = \{4B, 5C, 6B, 6C, 10C, 10D, 20A, 20B, 30A, 30B\}, \quad S = \{13A, 13B, 13C, 12A, 12B\}.$$

(i) For classes tZ in $T \setminus (R \cup S)$, computations show that

$$\Delta_G^*(2A, 6B, tZ) = \Delta_G(2A, 6B, tZ) = 0.$$

(ii) For classes tZ in R, the following cases occur:

$$\begin{aligned}
\Delta_G(2A, 6B, 4B) &= 12 < 24 = |C_G(4B)|, \\
\Delta_G(2A, 6B, 5C) &= 125 < 750 = |C_G(5C)|, \\
\Delta_G(2A, 6B, 6B) &= 18 < 36 = |C_G(6B)|, \\
\Delta_G(2A, 6B, 6C) &= 2 < 24 = |C_G(6C)|, \\
\Delta_G(2A, 6B, 10X) &= 25 < 100 = |C_G(10X)|, \ X \in \{C, D\}, \\
\Delta_G(2A, 6B, 20X) &= 10 < 20 = |C_G(20X)|, \ X \in \{A, B\}, \\
\Delta_G(2A, 6B, 30X) &= 15 < 30 = |C_G(30X)|, \ X \in \{A, B\}.
\end{aligned}$$

(iii) For classes in S, we first deal with $12X$, $X \in \{A, B\}$. Calculations give

$$\Delta_G(2A, 6B, 12X) = 12$$

for both $X = A$ and $X = B$. When $X = A$, classes with the required fusions give $\Sigma^*(M_3) = 12$, $\Sigma(M_6) = 0$, $\Sigma(M_7) = 0$. When $X = B$, only one maximal subgroup is involved, namely M_4, with $\Sigma(M_4) = 12$. With the corresponding values of $h(g, M)$, the following calculations arise:

$$\begin{aligned}
&\Delta_G^*(2A, 6B, 12X) \\
&\le \begin{cases} \Delta_G(2A, 6B, 12A) - 1 \cdot \Sigma^*(M_3) - 1 \cdot \Sigma(M_6) - 1 \cdot \Sigma(M_7) & \text{when } X = A, \\ \Delta_G(2A, 6B, 12B) - 1 \cdot \Sigma(M_4) & \text{when } X = B, \end{cases} \\
&= \begin{cases} 12 - 1(12) - 1(0) - 1(0) = 0 & \text{when } X = A, \\ 12 - 1(12) = 0 & \text{when } X = B. \end{cases}
\end{aligned}$$

We see that $\Delta_G^*(2A, 6B, 12X) = 0$ for $X \in \{A, B\}$.

Dealing with $13X$ for $X \in \{A, B, C\}$, we obtain $\Delta_G(2A, 6B, 13X) = 13$ and there is only one maximal subgroup involved in the calculations, namely M_3, with $\Sigma^*(M_3) = 13$. Since $h(g, M_3) = 1$ for $g \in 13X$, it follows that $\Delta_G^*(2A, 6B, 13X) = \Delta_G(2A, 6B, 13X) - 1 \cdot \Sigma^*(M_3) = 13 - 13 = 0$. Therefore, none of the triples $(2A, 6B, tZ)$, $tZ \in T$ generate G, hence G is not $6B$-complementary generated.

Case $(2A, 6C, tZ)$**:** Consider sets

$$R = \{4A, 5A, 5B, 5C, 10A, 10B, 10C, 10D, 30A, 30B\},$$
$$S = \{4B, 5D, 6A, 6B, 6C, 10E, 12A, 15A, 15B, 20A, 20B\},$$

and

$$U = \{5E, 5F, 10F, 12B, 13A, 13B, 13C, 15C\}.$$

Clearly, $R \cup S \cup U = T$. For all classes tZ in R, calculations show that $\Delta_G^*(2A, 6C, tZ) = \Delta_G(2A, 6C, tZ) = 0$. For tZ in S we have

$$\begin{aligned}
\Delta_G(2A, 6C, 4B) &= 12 < 24 = |C_G(4B)|, \\
\Delta_G(2A, 6C, 5D) &= 125 < 500 = |C_G(5D)|,
\end{aligned}$$

$$\begin{aligned}
\Delta_G(2A,6C,6A) &= 60 < 360 = |C_G(6A)|,\\
\Delta_G(2A,6C,6B) &= 3 < 36 = |C_G(6B)|,\\
\Delta_G(2A,6C,6C) &= 12 < 24 = |C_G(6C)|,\\
\Delta_G(2A,6C,10E) &= 25 < 50 = |C_G(10E)|,\\
\Delta_G(2A,6C,12A) &= 2 < 12 = |C_G(12A)|,\\
\Delta_G(2A,6C,15X) &= 15 < 30 = |C_G(15X)|,\ X \in \{A,B\},\\
\Delta_G(2A,6C,20X) &= 10 < 20 = |C_G(20X)|,\ X \in \{A,B\}.
\end{aligned}$$

We see that $\Delta_G(2A,6C,tZ) < |C_G(tZ)|$, based on Lemma 2.6, we conclude that $\Delta^*_G(2A,6C,tZ) = 0$ for all tZ in S.

Now, we use Scott's theorem to show that G is not $(2A,6C,tZ)$-generated for $tZ \in \{5E,5F,10F\} \subset U$. The group G has 13-dimensional complex irreducible module $\mathbb{V}$. For any conjugacy class nX, let $d_{nX} = \dim(\mathbb{V}/C_{\mathbb{V}}(nX))$ denote the codimension of the fixed space (in $\mathbb{V}$) of a representative of nX. Using (2.5) together with the power maps associated with the character table of G given in the atlas, with respect to $\mathbb{V}$, we found that $d_{2A} = 4$, $d_{6C} = 10$ and $d_{tZ} = 10$ for $tZ \in \{5E,5F,10F\}$. If G is $(2A,6C,nX)$-generated group for any nontrivial class nX of G in S, then we must have $d_{2A} + d_{6C} + d_{nX} \geq 2 \times 13$. However, we find that $d_{2A} + d_{6C} + d_{tZ} = 4 + 10 + 10 < 26$ for each tZ in $\{5E,5F,10F\}$. Thus G is not $(2A,6C,tZ)$-generated group for any $tZ \in \{5E,5F,10F\}$. Dealing with the remaining classes $12B, 13A, 13B, 13C$, and $15C$ in U, we find the following result: (i) Only the maximal subgroup M_4 has a nonempty intersection with the triple $(2A,6C,12B)$. We obtain $\Delta_G(2A,6C,12B) = 12$ and $\Sigma(M_4) = 12$. Now,

$$\Delta^*_G(2A,6C,12B) \leq \Delta_G(2A,6C,12B) - 1 \cdot \Sigma(M_4) = 12 - 1(12) = 0.$$

(ii) Only the maximal subgroup M_3 has a nonempty intersection with the triple $(2A,6C,13X)$ for $X \in \{A,B,C\}$. We obtain $\Delta_G(2A,6C,13X) = 13$ and $\Sigma(M_3) = 13$. Now,

$$\Delta^*_G(2A,6C,13X) \leq \Delta_G(2A,6C,13X) - 1 \cdot \Sigma(M_3) = 13 - 1(13) = 0.$$

(iii) Only two maximal subgroups, namely, M_2 and M_6 have a nonempty intersection with the triple $(2A,6C,15C)$. Computations give $\Delta_G(2A,6C,15C) = 30$, $\Sigma(M_2) = 30$, and $\Sigma(M_6) = 0$. Now,

$$\Delta^*_G(2A,6C,15C) = \Delta_G(2A,6C,15C) - 1 \cdot \Sigma(M_2) - 1 \cdot \Sigma(M_6) = 30 - 1(30) - 1(0) = 0.$$

We see here that $\Delta^*_G(2A,6C,tZ) = 0$ for $tZ = 12B, 13A, 13B, 13C, 15C$. Thus, $\Delta^*_G(2A,6C,tZ) = 0$ for all $tZ \in T$ and the result follows. □

Proposition 3.9. *The group G is not $10X$-complementary generated for $X \in \{A,B\}$.*

Proof. Let $U = \{5B,5D,5F,10F,15C\}$ and $S = T \setminus U$. Computations with GAP show that $\Delta_G(2A,10A,tZ) = 0$ for any $tZ \in T \setminus U$. Therefore, $\Delta^*_G(2A,10A,tZ) = 0$ for all $tZ \in T \setminus U$. It remains to show that $\Delta_G(2A,10A,tZ) = 0$ for any tZ in U. Computations reveal

$$\begin{aligned}
\Delta_G(2A,10A,5B) &= 25 < 15000 = |C_G(5B)|,\\
\Delta_G(2A,10A,5D) &= 20 < 500 = |C_G(5D)|,\\
\Delta_G(2A,10A,5F) &= 5 < 25 = |C_G(5F)|,\\
\Delta_G(2A,10A,10F) &= 2 < 20 = |C_G(10F)|,\\
\Delta_G(2A,10A,15C) &= 3 < 15 = |C_G(15C)|.
\end{aligned}$$

Then using Lemma 2.6, we deduce that G is not $(2A, 10A, tZ)$-generated group for any conjugacy class tZ in U. It follows that G is not a $10A$-complementary generated group. The proof for $10B$ follows similarly with classes $5A$, $5E$ changing roles with $5B$, $5F$, respectively. □

Proposition 3.10. *The group G is not $10X$-complementary generated for $X \in \{C, D\}$.*

Proof. Computations via GAP yield $\Delta^*_G(2A, 10C, tZ) = \Delta_G(2A, 10C, tZ) = 0$ for any class

$$\begin{aligned} tZ &\in R \\ &= \{4A, 4B, 5A, 5E, 6A, 6C, 10A, 10B, 10D, 10E, 12A, 13A, 13B, 13C, 15B, 20A, 30\} \\ &\subset T. \end{aligned}$$

For the classes $tZ \in T \setminus R$, we establish that

$$\begin{aligned}
\Delta_G(2A, 10C, 5B) &= 300 < 15000 = |C_G(5B)|, \\
\Delta_G(2A, 10C, 5C) &= 15 < 750 = |C_G(5C)|, \\
\Delta_G(2A, 10C, 5D) &= 5 < 500 = |C_G(5D)|, \\
\Delta_G(2A, 10C, 5F) &= 10 < 25 = |C_G(5F)|, \\
\Delta_G(2A, 10C, 6B) &= 9 < 36 = |C_G(6B)|, \\
\Delta_G(2A, 10C, 10C) &= 25 < 100 = |C_G(10C)|, \\
\Delta_G(2A, 10C, 10F) &= 1 < 20 = |C_G(10F)|, \\
\Delta_G(2A, 10C, 12B) &= 6 < 12 = |C_G(12B)|, \\
\Delta_G(2A, 10C, 15A) &= 15 < 30 = |C_G(15A)|, \\
\Delta_G(2A, 10C, 15C) &= 3 < 15 = |C_G(15C)|, \\
\Delta_G(2A, 10C, 20B) &= 10 < 20 = |C_G(20B)|, \\
\Delta_G(2A, 10C, 30A) &= 15 < 30 = |C_G(30A)|.
\end{aligned}$$

Using Lemma 2.6, we conclude that $\Delta^*_G(2A, 10C, tZ) = 0$ for the classes $tZ \in T \setminus R$. Consequently, G is not $(2A, 10C, tZ)$-generated for all $tZ \in T$. Hence G is not $10C$-complementary generated. The proof that G is not $10D$-complementary generated follow the same pattern, but with some classes swapping contributions as shown below: $5A \leftrightarrow 5B$, $5E \leftrightarrow 5F$, $10C \leftrightarrow 10D$, $15A \leftrightarrow 15B$, $20A \leftrightarrow 20B$, $30A \leftrightarrow 30B$. □

Proposition 3.11. *The group G is not $10X$-complementary generated for $X \in \{E, F\}$.*

Proof. We first deal with the case $(2A, 10E, tZ)$. The direct computations reveal $\Delta^*_G(2A, 10E, tZ) = \Delta_G(2A, 10E, tZ) = 0$ for classes tZ in $T \setminus R$, where

$$R = \{4B, 5X, 6C, 10F, 12A, 13Y, 15C\},$$

for $X \in \{C, D, E, F\}$ and $Y \in \{A, B, C\}$. For classes in R, we have

$$\begin{aligned}
\Delta_G(2A, 10E, 4B) &= 12 < 24 = |C_G(4B)|, \\
\Delta_G(2A, 10E, 5C) &= 45 < 750 = |C_G(5C)|, \\
\Delta_G(2A, 10E, 5D) &= 20 < 500 = |C_G(5D)|, \\
\Delta_G(2A, 10E, 5X) &= 10 < 25 = |C_G(5X)|, \ X \in \{E, F\}, \\
\Delta_G(2A, 10E, 6C) &= 12 < 24 = |C_G(6C)|, \\
\Delta_G(2A, 10E, 10F) &= 4 < 20 = |C_G(10F)|, \\
\Delta_G(2A, 10E, 15C) &= 3 < 15 = |C_G(15C)|.
\end{aligned}$$

Then using Lemma 2.6, we deduce that G is not a $(2A, 10E, tZ)$-generated group for any conjugacy class $tZ \in R \setminus \{12A, 13X\}$ for $X \in \{A, B, C\}$. Now, for the triple $(2A, 10E, 12A)$, there are three maximal subgroups with the required fusions giving $\Sigma(M_2) = 0$, $\Sigma^*(M_3) = \Sigma(M_3) = 12$, and $\Sigma(M_6) = 0$. Since $\Delta_G(2A, 10E, 12A) = 12$, it follows that

$$\begin{aligned}\Delta_G^*(2A, 10E, 12A) &= \Delta_G(2A, 10E, 12A) - 1 \cdot \Sigma^*(M_2) - 1 \cdot \Sigma^*(M_3) - 1 \cdot \Sigma^*(M_6) \\ &= 12 - 1(0) - 1(12) - 1(0) = 0.\end{aligned}$$

For the triple $(2A, 10E, 13X)$, $X \in \{A, B, C\}$, there is only one maximal subgroup involved with $\Sigma^*(M_3) = \Sigma(M_3) = 13$. Calculations result in

$$\Delta_G^*(2A, 10E, 13X) = \Delta_G(2A, 10E, 13X) - 1 \cdot \Sigma^*(M_3) = 13 - 1(13) = 0,$$

for $X \in \{A, B, C\}$. This shows that G is not generated by $(2A, 10E, tZ)$, $tZ \in \{12A, 13A, 13B, 13C\} \subset R$. Consequently, the group G is not generated by the triple $(2A, 10E, tZ)$ for any $tZ \in T$. Therefore, G is not a $10E$-complementary generated group.

We show that G is not $10F$-complementary generated. From Table 3, we see that $\Delta_G(2A, 10F, tZ) < |C_G(tZ)|$ for every $tZ \in R = \{5D, 10X, 15Y, 20Y, 30Y\} \subset T$, where $X \in \{A, B, C, D, E\}$ and $Y \in \{A, B\}$. It follows by Lemma 2.6 that G is not generated by the triple $(2A, 10F, tZ)$ for any tZ in R.

TABLE 3. Structure constant and centralizer order of tZ in R

tZ	$5D$	$10A/B$	$10C/D$	$10E$	$15A/B$	$20A/B$	$30A/B$
$\Delta_G(2A, 10F, tZ)$	125	60	5	10	15	10	15
$\lvert C_G(tZ)\rvert$	500	600	100	50	30	20	30

We use Scott's theorem to show that G is not $(2A, 10F, tZ)$-generated for $tZ \in S = \{5E, 5F, 6C, 10F\} \subset T$. Consider the 13-dimensional complex irreducible module $\mathbb{V}$ mentioned in the proof of Proposition 3.8. With respect to this module $\mathbb{V}$, we have $d_{2A} = 4$ and $d_{tZ} = 10$ for $tZ \in \{5E, 5F, 6C, 10F\}$. If G is $(2A, 10F, tZ)$-generated group for any class tZ in S, then we must have $d_{2A} + d_{3B} + d_{tZ} \geq 2 \times 13$. However, in this case $d_{2A} + d_{10F} + d_{tZ} = 4 + 10 + 10 < 26$ for each tZ in $\{5E, 5F, 6C, 10F\}$. Thus G is not $(2A, 10F, tZ)$-generated group for any $tZ \in \{5E, 5F, 6C, 10F\}$. Dealing with tZ in $U = \{12X, 13Y, 15C\}$, $X \in \{A, B\}$, and $Y \in \{A, B, C\}$, we find the following result: (i) Three maximal subgroups with classes that fuse into classes $2A$, $10F$, and $12A$ of G are M_2, M_3, and M_7. Computations with the GAP system yield $\Delta_G(2A, 10F, 12A) = 12$, $\Sigma^*(M_2) = \Sigma(M_2) = 0$, $\Sigma^*(M_3) = \Sigma(M_3) = 12$, and $\Sigma^*(M_7) = \Sigma(M_7) = 0$. Now,

$$\begin{aligned}\Delta_G^*(2A, 10F, 12A) &= \Delta_G(2A, 10F, 12A) - 1 \cdot \Sigma^*(M_2) - 1 \cdot \Sigma^*(M_3) - 1 \cdot \Sigma^*(M_7) \\ &= 12 - 1(0) - 1(12) - 1(0) = 0.\end{aligned}$$

(ii) Only two maximal subgroups, namely, M_1 and M_4 have classes that fuse into classes $2A$, $10F$ and $12B$ of G. Computations with the GAP system yield $\Delta_G(2A, 10F, 12B) = 12$, $\Sigma^*(M_1) = \Sigma(M_1) = 0$ and $\Sigma^*(M_4) = \Sigma(M_4) = 6 + 6 = 12$. Now,

$$\begin{aligned}\Delta_G^*(2A, 10F, 12B) &= \Delta_G(2A, 10F, 12B) - 1 \cdot \Sigma^*(M_1) - 1 \cdot \Sigma^*(M_4) \\ &= 12 - 1(0) - 1(12) = 0.\end{aligned}$$

(iii) For the triple $(2A, 10F, 13X)$, $X \in \{A, B, C\}$, there is only one maximal subgroup involved in the computation of $\Delta^*(2A, 10F, 13X)$. We get

$$\Delta^*_G(2A, 10F, 13X) = \Delta_G(2A, 10F, 13X) - 1 \cdot \Sigma^*(M_3) = 13 - 1(13) = 0,$$

for $X \in \{A, B, C\}$. (iv) For the triple $(2A, 10F, 15C)$, there is only one maximal subgroup involved in the computation of $\Delta^*(2A, 10F, 15C)$. We get

$$\Delta^*_G(2A, 10F, 15C) \leq \Delta_G(2A, 10F, 15C) - 1 \cdot \Sigma(M_2) = 30 - 1(30) = 0.$$

The results in (i) to (iv) show that G is not $(2A, 10F, tZ)$-generated for any tZ in U.

Lastly, direct computations via GAP reveal that $\Delta^*_G(2A, 10F, tZ) = \Delta_G(2A, 10F, tZ) = 0$ for all $tZ \in V = T \setminus (R \cup S \cup U)$. Therefore, G is not generated by $(2A, 10F, tz)$, for all $tZ \in T = R \cup S \cup U \cup V$, showing that G is not $10F$-complementary generated. □

Conclusions in Propositions 3.12–3.18 are based on the application of Proposition 1.2. For that, we let

$$P = \{pX \,|\, p \,\text{is prime}, p \,\text{divides}\, |G|\}.$$

Proposition 3.12. *The group G is $12A$-complementary generated.*

Proof. We achieve the result by showing that G is $(pX, 12A, tZ)$-generated for all conjugacy classes $pX \in P$ and some nontrivial conjugacy class tZ of G. We observe from Table 2 that for all $pX \in P$, all the maximal subgroups of G have an empty intersection with the triple $(pX, 12A, 30A)$. Thus, we will have $\Delta^*_G(pX, 12A, 30A) = \Delta_G(pX, 12A, 30A)$. Now, direct computations with the GAP system yield

- $\Delta_G(2A, 12A, 30A) = 30$,
- $\Delta_G(2B, 12A, 30A) = 900$,
- $\Delta_G(3A, 12A, 30A) = 840$,
- $\Delta_G(3B, 12A, 30A) = 1200$,
- $\Delta_G(5A, 12A, 30A) = 60$,
- $\Delta_G(5C, 12A, 30A) = 540$,
- $\Delta_G(5D, 12A, 30A) = 900$,
- $\Delta_G(5E, 12A, 30A) = 18900$,
- $\Delta_G(5F, 12A, 30A) = 12900$.
- $\Delta_G(13X, 12A, 30A) = 30000$, $X \in \{A, B, C\}$.

Since all of the structure constants calculated above are greater than zero except when $pX = 5B$, it follows that $\Delta^*_G(pX, 12A, 30A) > 0$ for all $pX \in P$ except $pX = 5B$. Though $\Delta_G(5B, 12A, 30A) = 0$, we find that $\Delta_G(5B, 12A, 13A) = 26$. Again, from Table 2, we see that no maximal subgroup of G has a nonempty intersection with the triple $(5B, 12A, 13A)$. It follows that $\Delta^*_G(5B, 12A, 13A) = \Delta_G(5B, 12A, 13A) = 26$, showing generation by this triple. Since G is $(5B, 12A,\ 13A)$-generated and $(pX, 12A, 30A)$-generated for all $pX \in P \setminus \{5B\}$ we conclude that G is $12A$-complementary generated. □

Proposition 3.13. *The group G is $12B$-complementary generated.*

Proof. The proof follows the same pattern in Proposition 3.12. We observe from Table 2 that for all $pX \in P$, all the maximal subgroups of G have an empty intersection with the triple $(pX, 12B, 13A)$. Thus, we will have $\Delta^*_G(pX, 12B, 13A) = \Delta_G(pX, 12B, 13A)$. Direct computations with the GAP system yield

- $\Delta_G(2A, 12B, 13A) = 26$,
- $\Delta_G(2B, 12B, 13A) = 806$,
- $\Delta_G(3X, 12B, 13A) = 1092$, $X \in \{A, B\}$,

- $\Delta_G(5X, 12B, 13A) = 26,\ X \in \{A, B\}$,
- $\Delta_G(5C, 12B, 13A) = 520$,
- $\Delta_G(5D, 12B, 13A) = 780$,
- $\Delta_G(5X, 12B, 13A) = 15600,\ X \in \{E, F\}$
- $\Delta_G(13X, 12B, 13A) = 29952\ X \in \{A, B, C\}$.

We see that all of the structure constants calculated above are greater than zero. It follows that $\Delta^*_G(pX, 12B, 13A) > 0$ for all $pX \in P$. Since G is $(pX, 12B, 13A)$-generated for all $pX \in P$, we conclude that G is $12B$-complementary generated group. □

Proposition 3.14. *The group G is $13Y$-complementary generated for all $Y \in \{A, B, C\}$.*

Proof. The proof follows the same pattern in Proposition 3.12. We observe from Table 2 that for all $pX \in P$, all the maximal subgroups of G have an empty intersection with the triple $(pX, 13Y, 30A)$, $Y \in \{A, B, C\}$. Thus, we will have $\Delta^*_G(pX, 13Y, 30A) = \Delta_G(pX, 13Y, 30A)$. Direct computations with the GAP system yield

- $\Delta_G(2A, 13Y, 30A) = 30$,
- $\Delta_G(2B, 13Y, 30A) = 750$,
- $\Delta_G(3X, 13Y, 30A) = 900,\ X \in \{A, B\}$,
- $\Delta_G(5X, 13Y, 30A) = 30,\ X \in \{A, B\}$,
- $\Delta_G(5C, 13Y, 30A) = 420$,
- $\Delta_G(5D, 13Y, 30A) = 720$,
- $\Delta_G(5X, 13Y, 30A) = 14400,\ X \in \{E, F\}$
- $\Delta_G(13X, 13Y, 30A) = 27000\ X \in \{A, B, C\}$.

Clearly, all of the structure constants calculated are greater than zero. So, we have $\Delta^*_G(pX, 13Y, 30A) > 0$ for all $pX \in P$. Consequently, G is $13Y$-complementary generated for $Y \in \{A, B, C\}$. □

Proposition 3.15. *The group G is $15Y$-complementary generated for $Y \in \{A, B\}$.*

Proof. The proof follows the same pattern in Proposition 3.12. We observe from Table 2 that for all $pX \in P$, all the maximal subgroups of G have an empty intersection with the triple $(pX, 15Y, 15C)$, $Y \in \{A, B\}$. Thus, we will have $\Delta^*_G(pX, 15Y, 15C) = \Delta_G(pX, 15Y, 15C)$. Direct computations with the GAP system yield

- $\Delta_G(2A, 15A, 15C) = 15$,
- $\Delta_G(2B, 15A, 15C) = 360$,
- $\Delta_G(3A, 15A, 15C) = 420$,
- $\Delta_G(3B, 15A, 15C) = 510$,
- $\Delta_G(5A, 15A, 15C) = 0$,
- $\Delta_G(5B, 15A, 15C) = 15$,
- $\Delta_G(5C, 15A, 15C) = 225$,
- $\Delta_G(5D, 15A, 15C) = 405$,
- $\Delta_G(5E, 15A, 15C) = 5190$,
- $\Delta_G(5F, 15A, 15C) = 7740$,
- $\Delta_G(13X, 15A, 15C) = 12300\ X \in \{A, B, C\}$.

The values found above are all greater than zero except when $pX = 5A$, it follows that $\Delta^*_G(pX, 15A, 15C) > 0$ for all $pX \in P$ except $pX = 5A$. Using class $13A$ in place of $15C$ gives $\Delta_G(5A, 15A, 13A) = 13$. Again, from Table 2, we see that no maximal subgroup of G has a nonempty intersection with the triple $(5A, 15A, 13A)$. It follows that $\Delta^*_G(5A, 15A, 13A) = \Delta_G(5A, 15A, 13A) = 13$, showing generation by this triple. Since G is $(5A, 15A, 13A)$-generated and $(pX, 15A, 15C)$-generated for all

$pX \in P \setminus \{5A\}$, we conclude that G is $15A$-complementary generated. Similarly, with $5A$, $5E$ swapping roles with $5B$, $5F$, respectively, we find that G is $15B$-complementary generated. $\square$

Proposition 3.16. *The group G is $15C$-complementary generated.*

Proof. The proof follows the same pattern in Proposition 3.12. We observe from Table 2 that for all $pX \in P$, all the maximal subgroups of G have an empty intersection with the triple $(pX, 15C, 13A)$. Thus, we will have $\Delta^*_G(pX, 15C, 13A) = \Delta_G(pX, 15C, 13A)$. Direct computations with the GAP system yield

- $\Delta_G(2A, 15C, 13A) = 26$,
- $\Delta_G(2B, 15C, 13A) = 650$,
- $\Delta_G(3X, 15C, 13A) = 780,\ X \in \{A, B\}$,
- $\Delta_G(5X, 15C, 13A) = 26,\ X \in \{A, B\}$,
- $\Delta_G(5C, 15C, 13A) = 364$,
- $\Delta_G(5D, 15C, 13A) = 624$,
- $\Delta_G(5X, 15C, 13A) = 12480,\ X \in \{E, F\}$,
- $\Delta_G(13X, 15C, 13A) = 23400\ X \in \{A, B, C\}$.

The values found above are all greater than zero. It follows that $\Delta^*_G(pX, 15C, 13A) > 0$ for all $pX \in P$. Since G is $(pX, 15C, 13A)$-generated for all pX we conclude that G is $15C$-complementary generated. $\square$

Proposition 3.17. *The group G is $20Y$-complementary generated for $Y \in \{A, B\}$.*

Proof. The proof follows the same pattern in Proposition 3.12. We observe from Table 2 that for all $pX \in P$, all the maximal subgroups of G have an empty intersection with the triple $(pX, 20Y, 13A)$, $Y \in \{A, B\}$. Thus, we will have $\Delta^*_G(pX, 20Y, 13A) = \Delta_G(pX, 20Y, 13A)$. Direct computations with the GAP system yield

- $\Delta_G(2A, 20Y, 13A) = 13$,
- $\Delta_G(2B, 20Y, 13A) = 455$,
- $\Delta_G(3X, 20Y, 13A) = 650,\ X \in \{A, B\}$,
- $\Delta_G(5X, 20Y, 13A) = 13,\ X \in \{A, B\}$,
- $\Delta_G(5C, 20Y, 13A) = 312$,
- $\Delta_G(5D, 20Y, 13A) = 442$,
- $\Delta_G(5X, 20Y, 13A) = 9360,\ X \in \{E, F\}$,
- $\Delta_G(13X, 20Y, 13A) = 18200\ X \in \{A, B, C\}$.

The structure constants computed above are all greater than zero. Therefore

$$\Delta^*_G(pX, 20Y, 13A) > 0$$

for all $pX \in P$. Since G is $(pX, 20Y, 13A)$-generated for all pX we conclude that G is $20Y$-complementary generated for $Y \in \{A, B\}$. $\square$

Proposition 3.18. *The group G is $30Y$-complementary generated for $Y \in \{A, B\}$.*

Proof. The proof follows the same pattern in Proposition 3.12. We observe from Table 2 that for all $pX \in P$, all the maximal subgroups of G have an empty intersection with the triple $(pX, 30Y, 13A)$, $Y \in \{A, B\}$. Thus, we will have $\Delta^*_G(pX, 30Y, 13A) = \Delta_G(pX, 30Y, 13A)$. Direct computations with the GAP system yield

- $\Delta_G(2A, 30Y, 13A) = 13$,
- $\Delta_G(2B, 30Y, 13A) = 325$,
- $\Delta_G(3X, 30Y, 13A) = 390,\ X \in \{A, B\}$,
- $\Delta_G(5X, 30Y, 13A) = 13,\ X \in \{A, B\}$,

- $\Delta_G(5C, 30Y, 13A) = 182$,
- $\Delta_G(5D, 30Y, 13A) = 312$,
- $\Delta_G(5X, 30Y, 13A) = 6240$, $X \in \{E, F\}$,
- $\Delta_G(13X, 30Y, 13A) = 11700$ $X \in \{A, B, C\}$.

The structure constants computed above are all greater than zero. Therefore

$$\Delta_G^*(pX, 30Y, 13A) > 0$$

for all $pX \in P$. Since G is $(pX, 30Y, 13A)$-generated for all pX, we conclude that G is $30Y$-complementary generated for $Y \in \{A, B\}$. □

The results established in the Propositions 3.2–3.18 show that the symplectic group $PSp(4,5)$ is nX-complementary generated if and only if $n = 5$, $nX \notin \{5A, 5B, 5C, 5D\}$, and $n \geq 12$. Hence Theorem 1.3 is proved.

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