Khayyam J. Math. 12 (2026), no. 2, 215-228
DOI:10.22067/kjm.2026.565510.3788

Khayyam Journal of Mathematics
emis.de/journals/KJM
kjm-math.org

DOMINION CLOSURE IN VARIETIES OF COMMUTATIVE POSEMIGROUPS

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Communicated by B. Mashayekhy

ABSTRACT. We establish sufficient conditions for certain commutative varieties of posemigroups to be closed under dominions. Furthermore, we extend a key result from the class of commutative posemigroups to more generalized classes, demonstrating that the dominions of these posemigroups remain within the same varieties. This partial generalization broadens the understanding of dominions in the context of commutative posemigroups and their algebraic structure.

1. Introduction

The study of epimorphisms in semigroup theory has attracted considerable attention for several decades. Unlike many familiar algebraic categories, epimorphisms of semigroups need not be surjective. This phenomenon naturally led to the introduction of *dominions* by Isbell [14], which provide a powerful tool for characterizing epimorphisms and studying their algebraic properties. Since then, dominion theory has become one of the principal techniques for investigating epimorphisms, identity preservation, and structural properties of semigroups and their varieties.

A central theme in dominion theory is the determination of classes of semigroups for which every epimorphism is surjective. This problem is closely related to the notions of *closed* and *saturated* semigroups and varieties. A variety is called closed if every member is closed within the variety. Closed varieties occupy a fundamental position in the theory of epimorphisms, since they characterize situations in which identities are preserved under the operation of taking dominions.

Date: Received: 20 February 2026; Revised: 14 August 2026; Accepted: 17 August 2026.
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2020 *Mathematics Subject Classification.* Primary: 06505; Secondary: 20M07, 20M14.
Key words and phrases. Posemigroup, domain, epimorphism, zigzag equation.

Consequently, the characterization of closed and saturated varieties has become one of the major open problems in semigroup theory.

One of the earliest and most influential results in this direction was obtained by Isbell [14], who proved that commutativity is preserved under the operation of taking dominions. In other words, the dominion of a commutative semigroup is itself commutative. This result initiated a systematic study of identity preservation under epimorphisms. The corresponding theorem for ordered semigroups was later established by Ahanger and Shah [4], who proved that the dominion of a commutative posemigroup is likewise commutative.

Subsequent research has focused on determining broader classes of semigroups and posemigroups for which identities remain invariant under dominions. Khan [15] demonstrated that no nontrivial permutation identity, apart from commutativity, is preserved under dominions. As a consequence, nontrivial permutative varieties distinct from the commutative variety fail to be closed under dominions. It is worth noting, however, that it remains an open question whether every subvariety of the variety of commutative semigroups and posemigroups is closed under dominions. Motivated by the above discussion, it is natural to investigate which subvarieties of the variety of commutative semigroups and posemigroups are closed under dominions. In this direction, Ahanger and Shah [17] established sufficient conditions ensuring that certain commutative varieties of semigroups are closed under dominions. More recently, Abbas, Ashraf, and Khan [1] partially extended a result of Isbell by showing that the dominion of several generalized classes of commutative semigroups again belongs to the same classes.

More recently, considerable progress has been made in identifying new closed and saturated varieties. Ahanger, Shah and Nabi studied dominions in varieties of bands and obtained several classes of closed band varieties [3, 2]. These results demonstrate that suitable algebraic identities frequently guarantee the preservation of structural properties under dominions.

Parallel developments have taken place in the theory of posemigroups. Permutative varieties of posemigroups were investigated in [7], where their algebraic structure was studied in detail. Subsequently, saturated varieties of posemigroups were characterized in [6], providing new examples of varieties that are closed under dominions in the ordered setting. More recently, identities preserved under epimorphisms of permutative posemigroups were investigated in [8], further strengthening the connection between dominions, epimorphisms, and identity preservation in ordered algebraic structures.

Despite these significant advances, relatively little is known about closure properties of subvarieties of the variety of commutative posemigroups. Although several sufficient conditions for closure have been established, a complete characterization of all subvarieties of commutative semigroups and commutative posemigroups that are closed under dominions remains open. This problem provides the principal motivation for the present work.

In the present paper, we address this problem in the framework of posemigroups by establishing sufficient conditions under which certain varieties of posemigroups are closed under dominions. Moreover, we extend a result of Ahanger and Shah

[17] from the class of commutative posemigroups to broader classes of commutative posemigroups by proving that the dominion of such posemigroups again lies within the same classes. Nevertheless, a complete characterization of all subvarieties of the variety of commutative semigroups and posemigroups that are closed under dominions remains an open problem.

2. Preliminaries

A semigroup S is called a *partially ordered semigroup*, or a *posemigroup*, if it is equipped with a compatible partial order relation $\leq$, that is,

$$\text{for all } s_1, s_2, s' \in S, \qquad s_1 \leq s_2 \implies s's_1 \leq s's_2 \text{ and } s_1s' \leq s_2s'.$$

Typical examples of posemigroups are $(\mathbb{N}_0, +)$ and $(\mathbb{N}, \cdot)$, each equipped with the usual order. Let S and T be posemigroups. A mapping $\psi : S \to T$ is called a *posemigroup morphism* if it preserves the semigroup operation and is order-preserving; that is, $s \leq s'$ implies $\psi(s) \leq \psi(s')$. Such a morphism ψ is said to be an *epimorphism* if, for any posemigroup morphisms $\alpha_1, \alpha_2 : T \to W$, the equality $\alpha_1 \circ \psi = \alpha_2 \circ \psi$ forces $\alpha_1 = \alpha_2$. While every surjective morphism is an epimorphism in any category, the converse does not hold in the category of posemigroups. In particular, the canonical embedding $(\mathbb{N}_0, +) \to (\mathbb{Z}, +)$, where both posemigroups are endowed with the natural order, is an epimorphism that is not surjective.

The notion of dominions for arbitrary algebras was first introduced by Isbell [14]. Later it was extensively used in the category of semigroups to investigate certain problems related to epimorphisms. In the setting of semigroups, Isbell's celebrated zigzag theorem plays a central role and has proved to be an effective technique for studying problems concerning epimorphisms (see [10, 12]). A more systematic investigation of epimorphisms and dominions in posemigroups was later undertaken by Sohail and Tart [18], who established a pomonoid analogue of Isbell's zigzag theorem. More recently, Ahanger, Shah, and Khan [6] extended several classical results of Khan [16] and Higgins [11] to the framework of posemigroups by employing zigzag techniques. Further developments in the study of epimorphisms and dominion-related properties of permutative posemigroups can be found in [5, 8].

Let U be a subposemigroup of a posemigroup S and let $d \in S$. We say that d is *dominated by* U if, for every posemigroup R and for any pair of posemigroup morphisms $f, g : S \to R$, the implication

$$f(u) = g(u) \text{ for all } u \in U \Longrightarrow f(d) = g(d)$$

holds. The *dominion* of U in S, denoted by $\widehat{Dom(U, S)}$, is the set of all elements of S dominated by U. Clearly, $U \subseteq \widehat{Dom(U, S)} \subseteq S$. If $\widehat{Dom(U, S)} = U$, then U is said to be *closed* in S. Moreover, U is called *absolutely closed* if $\widehat{Dom(U, S)} = U$ for every posemigroup S in which U occurs as a subposemigroup. For a subposemigroup U of a posemigroup S, let $Dom(U, S)$ denote the semigroup dominion of U in S. It is immediate that $Dom(U, S) \subseteq \widehat{Dom(U, S)}$. Hence, if U is closed in S as a posemigroup, then it is also closed in S when considered as a semigroup.

A nonempty class $\mathcal{V}$ of posemigroups is called a *variety* if it is closed under morphic images, subposemigroups, and direct (Cartesian) products, where the product is equipped with componentwise multiplication and order. A *posemigroup identity* is an inequality of the form

$$v_1 \leq v_2, \tag{2.1}$$

where v_1 and v_2 are words over the alphabet of a posemigroup S. A posemigroup S is said to *satisfy* the identity (2.1) if, for every substitution of the variables occurring in $v_1 \leq v_2$ by elements of S, the inequality holds in S.

A class $\mathcal{V}$ of posemigroups is said to *admit* an identity of the form (2.1) if every member of $\mathcal{V}$ satisfies it. Moreover, $\mathcal{V}$ is called an *equational class* if there exists a (finite or infinite) set of posemigroup identities such that $\mathcal{V}$ consists precisely of all posemigroups satisfying these identities. By a Birkhoff-type theorem for ordered algebras due to Bloom [9], a class of posemigroups is a variety if and only if it is an equational class. If a variety $\mathcal{V}$ of posemigroups is defined by a posemigroup identity of the form (2.1), we denote it by $\mathcal{V} = [\, v_1 \leq v_2 \,]$.

An identity (2.1) is said to be *preserved* under posemigroup dominions if, whenever a posemigroup U satisfies (2.1), the same identity is also satisfied by $\widehat{\mathrm{Dom}(U,S)}$. A variety $\mathcal{V}$ of posemigroups is called *closed under dominions* if $U \in \mathcal{V}$ implies $\widehat{\mathrm{Dom}(U,S)} \in \mathcal{V}$. In the commutative (or permutative) setting considered in this paper, every posemigroup identity of the form (2.1) forces the reverse inequality $v_2 \leq v_1$, and hence reduces to an identity of equality. Therefore, the subvarieties under investigation can be described equivalently by identities expressed as equalities.

The following characterization of dominions in posemigroups is due to Sohail and Tart [18] called the Zigzag Theorem for posemigroups and will frequently be used in what follows.

Theorem 2.1. *(see [18, Theorem 5]) Let U be a subposemigroup of a posemigroup S. Then we have $d \in \widehat{Dom}(U,S)$ if and only if $d \in U$ or*

$$\begin{aligned} d &\leq x_1 u_0, & u_0 &\leq u_1 y_1, & & \\ x_i u_{2i-1} &\leq s_{i+1} u_{2i}, & u_{2i} y_i &\leq u_{2i+1} y_{i+1}, & & 1 \leq i \leq m-1, \\ x_m u_{2m-1} &\leq u_{2m}, & u_{2m} y_m &\leq d, & & \end{aligned} \tag{2.2}$$

$$\begin{aligned} v_0 &\leq s_1 v_1, & d &\leq v_0 t_1, & & \\ s_j v_{2j} &\leq s_{j+1} v_{2j+1}, & v_{2j-1} t_j &\leq v_{2j} t_{j+1}, & & 1 \leq j \leq m'-1, \\ s_{m'} v_{2m'} &\leq d, & v_{2m'-1} t_{m'} &\leq v_{2m'}, & & \end{aligned} \tag{2.3}$$

where $u_0, v_0, \ldots u_{2m}, v_{2m'} \in U$ and $x_1, y_1, \ldots, x_m, y_m, s_{1'}, t_{1'}, \ldots, s_{m'}, t_{m'} \in S$.

The systems of inequalities described above are called *zigzag inequalities* in S over U with value d. The first family of inequalities in (2.2) yields the chain

$$d \leq x_1 u_0 \leq x_1 u_1 y_1 \leq x_2 u_2 y_1 \leq \cdots \leq x_m u_{2m-1} y_m \leq u_{2m} y_m \leq d.$$

Consequently, all terms in this chain coincide, and hence

$$d = x_1u_0 = x_1u_1y_1 = x_2u_2y_1 = \cdots = x_mu_{2m-1}y_m = u_{2m}y_m. \tag{2.4}$$

In an analogous manner, the second family of inequalities in (2.3) gives rise to

$$d \le v_0t_1 \le s_1v_1t_1 \le s_1v_2t_2 \le \cdots \le s_{m'}v_{2m'-1}t_{m'} \le s_{m'}v_{2m'} \le d,$$

and therefore all expressions are equal, that is,

$$d = v_0t_1 = s_1v_1t_1 = s_1v_2t_2 = \cdots = s_{m'}v_{2m'-1}t_{m'} = s_{m'}v_{2m'}. \tag{2.5}$$

Theorem 2.2. *(see [4, Theorem 2.3]) If U is a commutative subposemigroup of a posemigroup S, then $\widehat{\mathrm{Dom}}(U,S)$ is also a commutative posemigroup.*

3. Subvarieties of commutative posemigroups closed under dominions

In this section, we study dominions of commutative posemigroups. Our goal is to identify conditions under which the dominion of a commutative posemigroup within any containing posemigroup again satisfies the same identities as the original posemigroup.

Throughout this section, U denotes a commutative posemigroup and S is any posemigroup containing U as a subposemigroup. Using the method of minimal zigzag inequalities, we show that each element of $\widehat{\mathrm{Dom}}(U,S)$ satisfies the same defining identities as elements of U, thereby establishing closure under dominions.

Theorem 3.1. *For each $n > 1$, the subvariety*

$$\mathcal{V}_n = [\, xy = x^ny = xy^n \,]$$

of the variety of commutative posemigroups is closed under dominions.

Proof. Let $\mathcal{V}_n$ be the given variety and let $U \in \mathcal{V}_n$. Suppose that S is a posemigroup containing U as a subposemigroup. We show that $\widehat{\mathrm{Dom}}(U,S) \in \mathcal{V}_n$ by verifying that it satisfies the defining identity of $\mathcal{V}_n$. Let $x, y \in \widehat{\mathrm{Dom}}(U,S)$. If $x, y \in U$, there is nothing to prove. Hence, it suffices to consider the remaining cases.

Case I. Suppose that $x = d \in \widehat{\mathrm{Dom}}(U,S) \setminus U$ and $y = u \in U$. Let (2.2) and (2.3) be the zigzag inequalities for d. Then

$$\begin{aligned} (a) du &= s_{m'}v_{2m'}u && \text{(by zigzag equations (2.5))}\\ &= s_{m'}v_{2m'}u^n && \text{(since U is in $\mathcal{V}$)}\\ &= du^n. \end{aligned}$$

To prove part (b) of Case I, namely $du = d^nu$, we first establish the following lemma.

Lemma 3.2. *Let U satisfy the identity $uv = u^nv = uv^n$, $n > 1$, and let $d \in \widehat{\mathrm{Dom}}(U,S) \setminus U$ admit a pair of zigzag inequalities of types (2.2) and (2.3) of respective lengths (m, m'). Then*

$$v_0^{n-1}v_2^{n-1}\cdots v_{2m'}^{n-1}u = d^{n-1}u.$$

Proof. Let $d \in \widehat{\mathrm{Dom}(U,S)} \setminus U$ admit zigzag inequalities of types (2.2) and (2.3). Using the identity $uv = u^n v = uv^n$ with $n > 1$, we obtain

$$\begin{aligned}
& v_0^{n-1} v_2^{n-1} \cdots v_{2m'}^{n-1} u \\
& \geq v_0^{n-1} v_2^{n-1} \cdots v_{2m'-2}^{n-1} v_{2m'-1} t_{m'} v_{2m'}^{n-2} u && \text{(by zigzag inequalities (2.3))} \\
& = v_0^{n-1} v_2^{n-1} \cdots v_{2m'-2}^{n-2} v_{2m'-1} v_{2m'-2} t_{m'} v_{2m'}^{n-2} u && \text{(since } U \text{ is commutative)} \\
& \geq v_0^{n-1} v_2^{n-1} \cdots v_{2m'-2}^{n-2} v_{2m'-1} v_{2m'-3} t_{m'-1} v_{2m'}^{n-2} u && \text{(by zigzag inequalities (2.3))} \\
& \vdots \\
& = v_0^{n-1} v_2^{n-2} \cdots v_{2m'-2}^{n-2} v_{2m'-1} v_{2m'-3} \cdots v_1 t_1 v_{2m'}^{n-2} u \\
& = v_0^{n-2} v_2^{n-2} \cdots v_{2m'-2}^{n-2} v_{2m'-1} v_{2m'-3} \cdots v_1 v_0 t_1 v_{2m'}^{n-2} u \\
& && \text{(since } U \text{ is commutative)} \\
& \geq v_0^{n-2} v_2^{n-2} \cdots v_{2m'-2}^{n-2} v_{2m'-1} v_{2m'-3} \cdots v_1 d v_{2m'}^{n-2} u \\
& && \text{(by zigzag inequalities (2.3))} \\
& = v_0^{n-2} v_2^{n-2} \cdots v_{2m'-2}^{n-2} v_{2m'-1} v_{2m'-3} \cdots v_1 v_{2m'}^{n-2} d u && \text{(by Theorem 2.2)}
\end{aligned}$$

Continuing the same process, we get

$$\begin{aligned}
& v_0^{n-1} v_2^{n-1} \cdots v_{2m'}^{n-1} u \\
& = v_0 v_2 \cdots v_{2m'-2} v_{2m'-1}^{n-2} v_{2m'-3}^{n-2} \cdots v_1^{n-2} v_{2m'} d^{n-2} u \\
& \geq v_0 v_2 \cdots v_{2m'-2} v_{2m'-1}^{n-2} v_{2m'-3}^{n-2} \cdots v_1^{n-2} v_{2m'-1} t_{m'} d^{n-2} u \\
& && \text{(by zigzag inequalities (2.3))} \\
& = v_0 v_2 \cdots v_{2m'-1}^{n-1} v_{2m'-3}^{n-2} \cdots v_1^{n-2} v_{2m'-2} t_{m'} d^{n-2} u && \text{(since } U \text{ is commutative)} \\
& \vdots \\
& = v_0 v_{2m'-1}^{n-1} v_{2m'-3}^{n-1} \cdots v_1^{n-2} v_2 t_2 d^{n-2} u \\
& \geq v_0 v_{2m'-1}^{n-1} v_{2m'-3}^{n-1} \cdots v_1^{n-2} v_1 t_1 d^{n-2} u && \text{(by zigzag inequalities (2.3))} \\
& = v_{2m'-1}^{n-1} v_{2m'-3}^{n-1} \cdots v_1^{n-1} v_0 t_1 d^{n-2} u && \text{(since } U \text{ is commutative)} \\
& \geq v_{2m'-1}^{n-1} v_{2m'-3}^{n-1} \cdots v_1^{n-1} d^{n-1} u && \text{(by zigzag inequalities (2.3))} \\
& = d^{n-1} v_{2m'-1}^{n-1} v_{2m'-3}^{n-1} \cdots v_1^{n-1} u && \text{(by Theorem 2.2)} \\
& \geq s_{m'} v_{2m'} d^{n-2} v_{2m'-1}^{n-1} v_{2m'-3}^{n-1} \cdots v_1^{n-1} u && \text{(by zigzag inequalities (2.3))} \\
& = s_{m'} d^{n-2} v_{2m'-1}^{n-1} v_{2m'-3}^{n-1} \cdots v_1^{n-1} u v_{2m'} && \text{(by Theorem 2.2)} \\
& \geq s_{m'} d^{n-2} v_{2m'-1}^{n-1} v_{2m'-3}^{n-1} \cdots v_1^{n-1} u v_{2m'-1} t_{m'} && \text{(by zigzag inequalities (2.3))} \\
& = s_{m'} d^{n-2} v_{2m'-1}^{n} v_{2m'-3}^{n-1} \cdots v_1^{n-1} u t_{m'} \\
& = s_{m'} d^{n-2} v_{2m'-1} v_{2m'-3}^{n-1} \cdots v_1^{n-1} u t_{m'} && \text{(since } U \text{ is in } \mathcal{V}) \\
& = s_{m'} v_{2m'-1} d^{n-2} v_{2m'-3}^{n-1} \cdots v_1^{n-1} u t_{m'} && \text{(by Theorem 2.2)} \\
& \geq s_{m'-1} v_{2m'-2} d^{n-2} v_{2m'-3}^{n-1} \cdots v_1^{n-1} u t_{m'} && \text{(by zigzag inequalities (2.3))} \\
& \vdots
\end{aligned}$$

$$
\begin{aligned}
&\geq s_1 v_2 d^{n-2} v_1^{n-1} u t_2 \\
&= s_1 d^{n-2} v_1^{n-1} u v_2 t_2 && \text{(by Theorem 2.2)} \\
&\geq s_1 d^{n-2} v_1^{n-1} u v_1 t_1 && \text{(by zigzag inequalities (2.3))} \\
&= s_1 d^{n-2} v_1^{n} u t_1 \\
&= s_1 d^{n-2} v_1 u t_1 && \text{(since } U \text{ is in } \mathcal{V}) \\
&= s_1 v_1 d^{n-2} u t_1 && \text{(by Theorem 2.2)} \\
&\geq v_0 d^{n-2} u t_1 && \text{(by zigzag inequalities (2.3))} \\
&= d^{n-2} u v_0 t_1 && \text{(by Theorem 2.2)} \\
&\geq d^{n-2} u d && \text{(by zigzag inequalities (2.3))} \\
&= d^{n-1} u \qquad . && \text{(by Theorem 2.2)}
\end{aligned}
$$

□

Using the preceding lemma, we now complete the proof of part (b) as follows.

$$
\begin{aligned}
du &\geq s_{m'} v_{2m'} u && \text{(by zigzag inequalities (2.3))} \\
&= s_{m'} v_{2m'}^{n} u && \text{(since } U \text{ is in } \mathcal{V}) \\
&= s_{m'} v_{2m'}^{n-1} u v_{2m'} && \text{(since } U \text{ is commutative)} \\
&= s_{m'} v_{2m'}^{n-1} u v_{2m'-1} t_{m'} && \text{(by zigzag inequalities (2.3))} \\
&= s_{m'} v_{2m'-1} v_{2m'}^{n-1} u t_{m'} && \text{(since } U \text{ is commutative)} \\
&\geq s_{m'-1} v_{2m'-2} v_{2m'}^{n-1} u t_{m'} && \text{(by zigzag inequalities (2.3))} \\
&= s_{m'-1} v_{2m'-2}^{n} v_{2m'}^{n-1} u t_{m'} && \text{(since } U \text{ is in } \mathcal{V}) \\
&= s_{m'-1} v_{2m'-2}^{n-1} v_{2m'}^{n-1} u v_{2m'-2} t_{m'} && \text{(since } U \text{ is commutative)} \\
&\geq s_{m'-1} v_{2m'-2}^{n-1} v_{2m'}^{n-1} u v_{2m'-3} t_{m'-1} && \text{(by zigzag inequalities (2.3))} \\
&\vdots \\
&= s_1 v_2^{n-1} v_4^{n-1} \cdots v_{2m'}^{n-1} u v_1 t_1 \\
&= s_1 v_1 v_2^{n-1} v_4^{n-1} \cdots v_{2m'}^{n-1} u t_1 && \text{(since } U \text{ is commutative)} \\
&\geq v_0 v_2^{n-1} v_4^{n-1} \cdots v_{2m'}^{n-1} u t_1 && \text{(by zigzag inequalities (2.3))} \\
&= v_0^{n} v_2^{n-1} v_4^{n-1} \cdots v_{2m'}^{n-1} u t_1 && \text{(since } U \text{ is in } \mathcal{V}) \\
&= v_0^{n-1} v_2^{n-1} \cdots v_{2m'}^{n-1} u v_0 t_1 && \text{(since } U \text{ is commutative)} \\
&\geq v_0^{n-1} v_2^{n-1} \cdots v_{2m'}^{n-1} u d && \text{(by zigzag inequalities (2.3))} \\
&= d^{n-1} u d && \text{(by Lemma 3.2)} \\
&= d^{n} u \ . && \text{(by Theorem 2.2)}
\end{aligned}
$$

The reverse inequality $d^n u \leq du$ can be proved analogously by applying the zigzag inequalities (2.2). Hence $du = d^n u$, as required.

Case II: $u \in U$, $d \in \widehat{\mathrm{Dom}(U,S)} \setminus U$. Here we take $x = u$ and $y = d$, reversing the roles of Case I, and proceed analogously.

Case III: Assume that, $z_1, z_2 \in \widehat{\mathrm{Dom}(U,S)}$ and let (2.3) and (2.2) be the zigzag inequalities for z_1. Then

$$\begin{aligned} z_1z_2 &= s_{m'}v_{2m'}z_2 && \text{(by zigzag equations)}\\ &= s_{m'}v_{2m'}z_2^n && \text{(by Case II)}\\ &= z_1z_2^n. \end{aligned}$$

Next, we may let (2.3) and (2.2) be the zigzag inequalities for z_2. Then

$$\begin{aligned} z_1z_2 &= z_1v_0t_1 && \text{(by zigzag equations)}\\ &= z_1^nv_0t_1 && \text{(by Case I)}\\ &= z_1^nz_2. \end{aligned}$$

□

4. Dominions and generalized classes of commutative posemigroups

Isbell [13, Corollary 2.5] proved that the dominion of a commutative semigroup retains commutativity, and this preservation was later extended to posemigroups by Ahanger and Shah [4]. However, this behavior is highly restrictive: Khan [15] showed that apart from commutativity, no nontrivial permutation identity is stable under the dominion construction. These observations naturally raise the question of whether meaningful subclasses of commutative posemigroups still exhibit some form of closure under dominions. In this work, we address this question by identifying certain generalized classes of commutative posemigroups for which a weaker, yet nontrivial, preservation property holds.

Definition 4.1. A nontrivial permutation identity

$$z_1z_2z_3 = z_{\pi(1)}z_{\pi(2)}z_{\pi(3)},$$

where π is a nonidentity permutation of $\{1,2,3\}$, is said to be

(i) *externally commutative (EC)* if $\pi(1) = 3$ and $\pi(3) = 1$;
(ii) *right commutative (RC) [left commutative (LC)]* if $\pi(2) = 3$ and $\pi(3) = 2$ [$\pi(1) = 2$ and $\pi(2) = 1$];
(iii) *cyclic commutative (CC)[dual-cyclic commutative (DCC)]* if $\pi = (1\,2\,3)$ [$\pi = (1\,3\,2)$].

Similarly, a nontrivial permutation identity

$$z_1z_2z_3z_4 = z_{\sigma(1)}z_{\sigma(2)}z_{\sigma(3)}z_{\sigma(4)},$$

where σ is a nonidentity permutation of $\{1,2,3,4\}$, is said to be

(i) *left externally commutative (LEC)* if $\sigma(1) = 3$ and $\sigma(3) = 1$;
(ii) *left cyclic commutative (LCC)* if $\sigma = (1\,2\,3)$;
(iii) *right dual-cyclic commutative (RDCC)* if $\sigma(2) = 4$, $\sigma(3) = 2$, and $\sigma(4) = 3$;
(iv) *left semicommutative (LSC)* if $\sigma(1) = 2$ and $\sigma(2) = 1$.

A semigroup satisfying any nontrivial permutation identity is called a permutative semigroup. A posemigroup is said to be permutative whenever its underlying semigroup is permutative.

Remark 4.2. Throughout this section, we freely use the abbreviations EC, RC, LC, CC, DCC, LEC, LCC, RDCC, and LSC for the corresponding permutation identities and the classes of posemigroups satisfying them.

Theorem 4.3. *Let S be an LEC posemigroup and let U be an EC subposemigroup of S. Then $\widehat{Dom}(U,S)$ is an EC posemigroup.*

Proof. Let U be an EC subposemigroup of a LEC posemigroup S. To show that $\widehat{Dom}(U,S)$ is EC, we consider the following cases.
Case (i). If $z_1, z_2, z_3 \in U$, then the required identity holds trivially, since U is EC.
Case (ii). Let $z_1 \in \widehat{\mathrm{Dom}}(U,S) \setminus U$ and $z_2, z_3 \in U$. By Theorem 2.1, the element z_1 admits zigzag inequalities of type (2.2) and (2.3). Now,

$$\begin{aligned}
z_1z_2z_3 &\geq s_{m'}v_{2m'}z_2z_3 && \text{(by zigzag inequalities (2.3))}\\
&= s_{m'}z_3z_2v_{2m'} && \text{(since } U \text{ is EC)}\\
&\geq s_{m'}z_3z_2v_{2m'-1}t_{m'} && \text{(by zigzag inequalities (2.3))}\\
&= s_{m'}v_{2m'-1}z_2z_3t_{m'} && \text{(since } U \text{ is EC)}\\
&\geq s_{m'-1}v_{2m'-2}z_2z_3t_{m'} && \text{(by zigzag inequalities (2.3))}\\
&= s_{m'-1}(z_3z_2v_{2m'-2})t_{m'} && \text{(since } U \text{ is EC)}\\
&\geq s_{m'-1}z_3z_2v_{2m'-3}t_{m'-1} && \text{(by zigzag inequalities (2.3))}\\
&= s_{m'-1}v_{2m'-3}z_2z_3t_{m'-1} && \text{(since } U \text{ is EC)}\\
&\vdots\\
&= (s_1v_1)z_2z_3t_1\\
&\geq v_0z_2z_3t_1 && \text{(by zigzag inequalities (2.3))}\\
&= z_3z_2v_0t_1 && \text{(since } U \text{ is EC)}\\
&\geq z_3z_2z_1.
\end{aligned}$$

A similar argument, using zigzag inequalities (2.2) for z_1, yields $z_1z_2z_3 \leq z_3z_2z_1$. Hence $z_1z_2z_3 = z_3z_2z_1$, as required.
Case (iii). Let $z_1, z_2 \in \widehat{Dom}(U,S) \setminus U$ and $z_3 \in U$. Then, by Theorem 2.1, z_2 has zigzag inequalities of type (2.2) and (2.3). Now, using repeatedly the left external commutativity of S, we compute

$$\begin{aligned}
z_1z_2z_3 &= z_1s_{m'}v_{2m'-1}t_{m'}z_3 && \text{(by zigzag equations (2.5))}\\
&= v_{2m'-1}s_{m'}z_1t_{m'}z_3\\
&= t_{m'}s_{m'}z_1v_{2m'-1}z_3\\
&= t_{m'-1}s_{m'}z_3v_{2m'-1}z_1 && \text{(by Case (ii))}\\
&= v_{2m'-1}s_{m'}z_3t_{m'}z_1\\
&= z_3s_{m'}v_{2m'-1}t_{m'}z_1\\
&= z_3z_2z_1 \qquad . && \text{(by zigzag equations (2.5))}
\end{aligned}$$

Case (iv). Let $z_1, z_2, z_3 \in \widehat{Dom(U, S)} \setminus U$ and (2.2) and (2.3) be the zigzag inequalities for z_3. Now,

$$\begin{aligned}
z_1z_2z_3 &\leq z_1z_2v_0t_1 && \text{(by zigzag inequalities (2.3))}\\
&= v_0z_2z_1t_1 && \text{(by Case (iii))}\\
&\leq s_1v_1z_2z_1t_1 && \text{(by zigzag inequalities (2.3))}\\
&= s_1z_1z_2v_1t_1 && \text{(by Case (iii))}\\
&\leq s_1z_1z_2v_2t_2 && \text{(by zigzag inequalities (2.3))}\\
&= s_1v_2z_2z_1t_2 && \text{(by Case (iii))}\\
&\leq s_2v_3z_2z_1t_2 && \text{(by zigzag inequalities (2.3))}\\
&\vdots\\
&= s_{m'}(v_{2m'-1}z_2z_1)t_{m'}\\
&= s_{m'}z_1z_2v_{2m'-1}t_{m'} && \text{(by Case (iii))}\\
&\leq s_{m'}z_1z_2v_{2m'} && \text{(by zigzag inequalities (2.3))}\\
&= s_{m'}v_{2m'}z_2z_1 && \text{(by Case (iii))}\\
&\leq z_3z_2z_1 \qquad . && \text{(by zigzag inequalities (2.3))}
\end{aligned}$$

Similarly, using zigzag inequalities (2.2), we derive $z_1z_2z_3 \geq z_3z_2z_1$. Consequently, $z_1z_2z_3 = z_3z_2z_1$, completing the proof of the theorem. □

Theorem 4.4. *Let U be a CC subposemigroup of an LCC posemigroup S. Then $\widehat{\mathrm{Dom}(U, S)}$ is a CC posemigroup.*

Proof. Let U be a CC subposemigroup of an LCC posemigroup S. Since the cyclic commutativity already holds within U, it suffices to consider the cases where at least one of the elements lies in $\widehat{\mathrm{Dom}(U, S)} \setminus U$.

Case (i): Assume $z_1 \in \widehat{Dom(U, S)} \setminus U$ and $z_2, z_3 \in U$. Then by Theorem 2.1, z_1 admits zigzag inequalities of type (2.2) and (2.3). Now,

$$\begin{aligned}
z_1z_2z_3 &= s_{m'}(v_{2m'}z_2z_3) && \text{(by zigzag equations (2.5))}\\
&= s_{m'}z_2z_3v_{2m'} && \text{(since U is CC)}\\
&= z_2z_3s_{m'}v_{2m'} && \text{(since S is LCC)}\\
&= z_2z_3z_1 \qquad . && \text{(by zigzag equations (2.5))}
\end{aligned}$$

Case (ii): Let $z_1, z_2 \in \widehat{Dom(U, S)} \setminus U$, let $z_3 \in U$, and let z_2 admit zigzag inequalities of type (2.2) and (2.3). Now,

$$\begin{aligned}
z_1z_2z_3 &= z_1v_0t_1z_3 && \text{(by zigzag equations (2.5))}\\
&= v_0t_1z_1z_3 && \text{(since S is LCC)}\\
&= t_1z_1v_0z_3 && \text{(since S is LCC)}\\
&= t_1v_0z_3z_1 && \text{(by Case (i))}\\
&= v_0z_3t_1z_1 && \text{(since S is LCC)}
\end{aligned}$$

$$
\begin{aligned}
&\leq s_1 v_1 z_3 t_1 z_1 && \text{(by zigzag inequalities (2.3))}\\
&= v_1 z_3 s_1 t_1 z_1 && \text{(since S is LCC)}\\
&= z_3 s_1 v_1 t_1 z_1 && \text{(since S is LCC)}\\
&= z_3 v_0 t_1 z_1 && \text{(by zigzag equations (2.5))}\\
&= v_0 t_1 z_3 z_1 && \text{(since S is LCC)}\\
&= z_2 z_3 z_1.
\end{aligned}
$$

Similarly, applying the zigzag inequalities (2.2), we obtain $z_1z_2z_3 \geq z_2z_3z_1$. Combining this with the previously established inequality yields, $z_1z_2z_3 = z_2z_3z_1$, as required.

Case (iii). Let $z_1, z_2 \in \widehat{Dom(U,S)} \setminus U$ and let $z_3 \in U$. Let z_3 admit zigzag inequalities of type (2.2) and (2.3). Now,

$$
\begin{aligned}
z_1z_2z_3 &= z_1 z_2 s_{m'} v_{2m'} && \text{(by zigzag equations (2.5))}\\
&= z_2 s_{m'} z_1 v_{2m'} && \text{(since S is LCC)}\\
&= s_{m'} z_1 z_2 v_{2m'} && \text{(since S is LCC)}\\
&= s_{m'} z_2 v_{2m'} z_1 && \text{(by Case (ii))}\\
&\geq s_{m'} z_2 v_{2m'-1} t_{m'} z_1 && \text{(by zigzag inequalities (2.3))}\\
&= s_{m'} v_{2m'-1} t_{m'} z_2 z_1 && \text{(since S is LCC)}\\
&= s_{m'} v_{2m'} z_2 z_1 && \text{(by zigzag equations (2.5))}\\
&= v_{2m'} z_2 s_{m'} z_1 && \text{(since S is LCC)}\\
&= z_2 s_{m'} v_{2m'} z_1 && \text{(since S is LCC)}\\
&= z_2 z_3 z_1.
\end{aligned}
$$

A similar argument, using the zigzag inequalities (2.2), yields $z_1z_2z_3 \leq z_2z_3z_1$. Combining this with the previously established reverse inequality, we obtain $z_1z_2z_3 = z_2z_3z_1$, as required. □

Theorem 4.5. *Let U be a DCC subposemigroup of an RDCC posemigroup S. Then $\widehat{Dom(U,S)}$ is a DCC posemigroup.*

Proof. Let U be a DCC subposemigroup of an RDCC posemigroup S. Since the dual-cyclic commutativity already holds within U, it suffices to consider the cases where at least one of the elements lies in $\widehat{\mathrm{Dom}(U,S)} \setminus U$.

Case (i). Assume $z_1 \in \widehat{Dom(U,S)} \setminus U$ and $z_2, z_3 \in U$. Let z_1 admit zigzag inequalities of type (2.2) and (2.3). Now,

$$
\begin{aligned}
z_1z_2z_3 &= v_0 t_1 z_2 z_3 && \text{(by zigzag equations (2.5))}\\
&= v_0 z_3 t_1 z_2 && \text{(since S is RDCC)}\\
&= v_0 z_2 z_3 t_1 && \text{(since S is RDCC)}\\
&= z_3 v_0 z_2 t_1 && \text{(since S is RDCC)}\\
&= z_3 t_1 v_0 z_2 && \text{(since S is RDCC)}\\
&\leq (z_3 t_1 s_1 v_1) z_2 && \text{(by zigzag inequalities (2.3))}
\end{aligned}
$$

$$\begin{aligned} &= z_3 s_1 v_1 t_1 z_2 && \text{(since } S \text{ is RDCC)} \\ &= z_3 z_1 z_2. \end{aligned}$$

A similar argument, using zigzag inequalities of type (2.2) yields $z_1z_2z_3 \geq z_3z_1z_2$. Therefore, $z_1z_2z_3 = z_3z_1z_2$, as required.

Case (ii). Let $z_1, z_2 \in \widehat{Dom(U,S)} \setminus U$ and let $z_3 \in U$. Let (2.2) and (2.3) be the zigzag inequalities for z_2 in S over U. Now,

$$\begin{aligned} z_1z_2z_3 &= z_1 s_{m'} v_{2m'} z_3 && \text{(by zigzag equations (2.5))} \\ &= z_1 z_3 s_{m'} v_{2m'} && \text{(since } S \text{ is RDCC)} \\ &= z_1 v_{2m'} z_3 s_{m'} && \text{(since } S \text{ is RDCC)} \\ &= z_3 z_1 v_{2m'} s_{m'} && \text{(by Case (i))} \\ &\geq z_3 z_1 v_{2m'-1} t_{m'} s_{m'} && \text{(by zigzag inequalities (2.3))} \\ &= z_3 z_1 s_{m'} v_{2m'-1} t_{m'} && \text{(since } S \text{ is RDCC)} \\ &= z_3 z_1 z_2. \end{aligned}$$

A similar argument, using zigzag inequalities of type (2.2) yields, $z_1z_2z_3 \leq z_3z_1z_2$. Therefore, $z_1z_2z_3 = z_3z_1z_2$, as required.

Case (iii): Let $z_1, z_2, z_3 \in \widehat{Dom(U,S)} \setminus U$. Let (2.2) and (2.3) be the zigzag inequalities for z_3 in S over U. This case follows by using Case (ii) together with the RDCC property of S. ☐

Theorem 4.6. *Let U be an RC subposemigroup of an RDCC posemigroup S. Then $\widehat{Dom(U,S)}$ is an RC posemigroup.*

Proof. Let U be an RC subposemigroup of an RDCC posemigroup S. Since the right commutativity already holds within U, it suffices to consider the cases where at least one of the elements lies in $\widehat{\mathrm{Dom}(U,S)} \setminus U$.

Case (i). $z_1 \in \widehat{Dom(U,S)} \setminus U$ and $z_2, z_3 \in U$. Then, by Theorem 2.1, z_1 admits zigzag inequalities of type (2.2) and (2.3) in S over U. Now,

$$\begin{aligned} z_1z_2z_3 &= s_{m'} v_{2m'} z_2 z_3 && \text{(by zigzag equations (2.5))} \\ &= s_{m'} v_{2m'} z_3 z_2 && \text{(since } U \text{ is RC)} \\ &= z_1 z_3 z_2, \end{aligned}$$

as required.

Case (ii). $z_1, z_2 \in \widehat{\mathrm{Dom}(U,S)} \setminus U$ and $z_3 \in U$. This case is handled in the same manner as above, using the zigzag inequalities of type (2.2) and (2.3) for z_2, together with the right dual-cyclic commutativity of S.

Case (iii). $z_1, z_2, z_3 \in \widehat{Dom(U,S)} \setminus U$. Let (2.2) and (2.3) be zigzag inequalities for z_3 in S over U. Now,

$$\begin{aligned} z_1z_2z_3 &= z_1 z_2 s_{m'} v_{2m'} && \text{(by zigzag equations (2.5))} \\ &= z_1 v_{2m'} z_2 s_{m'} && \text{(since } S \text{ is RDCC)} \\ &= z_1 z_2 v_{2m'} s_{m'} && \text{(by Case (ii))} \\ &= z_1 s_{m'} z_2 v_{2m'} && \text{(since } S \text{ is RDCC)} \end{aligned}$$

$$\begin{aligned}
&\geq z_1 s_{m'} z_2 v_{2m'-1} t_{m'} && \text{(by zigzag inequalities (2.3))}\\
&= z_1 s_{m'} t_{m'} z_2 v_{2m'-1} && \text{(since } S \text{ is RDCC)}\\
&= z_1 s_{m'} v_{2m'-1} t_{m'} z_2 && \text{(since } S \text{ is RDCC)}\\
&= z_1 z_3 z_2.
\end{aligned}$$

A similar argument, using zigzag inequalities (2.2), yields $z_1 z_2 z_3 \leq z_1 z_3 z_2$. Therefore, $z_1 z_2 z_3 = z_1 z_3 z_2$, as required. □

Theorem 4.7. *Let U be an LC subposemigroup of an LSC posemigroup S. Then $\widehat{Dom(U,S)}$ is an LC posemigroup.*

Proof. Let U be an LC subposemigroup of an LSC posemigroup S. Since the left commutativity already holds within U, it suffices to consider the cases where at least one of the elements lies in $\widehat{\mathrm{Dom}(U,S)} \setminus U$.

Case (i). $z_1 \in \widehat{Dom(U,S)} \setminus U$ and $z_2, z_3 \in U$. Then, by Theorem 2.1, z_1 admits zigzag inequalities of type (2.2) and (2.3) in S over U. Now,

$$\begin{aligned}
z_1 z_2 z_3 &= s_{m'} v_{2m'} z_2 z_3 && \text{(by zigzag equations (2.5))}\\
&= s_{m'} z_2 v_{2m'} z_3 && \text{(since } U \text{ is LC)}\\
&= z_2 s_{m'} v_{2m'} z_3 && \text{(since } S \text{ is LSC)}\\
&= z_2 z_1 z_3,
\end{aligned}$$

as required.

Case (ii). $z_1, z_2 \in \widehat{\mathrm{Dom}(U,S)} \setminus U$ and $z_3 \in U$. This follows from the zigzag inequalities for z_2, the LSC property of S, and Case (i).

Case (iii). $z_1, z_2, z_3 \in \widehat{Dom(U,S)} \setminus U$. Let (2.2) and (2.3) be zigzag inequalities for z_3 in S over U. Now,

$$\begin{aligned}
z_1 z_2 z_3 &= z_1 z_2 v_0 t_1 && \text{(by zigzag equations (2.5))}\\
&= z_2 z_1 v_0 t_1 && \text{(by Case (ii))}\\
&= z_2 z_1 z_3,
\end{aligned}$$

as required. □

5. Conclusion

The results obtained contribute to the ongoing study of epimorphisms and dominions in ordered algebraic structures by identifying new saturated varieties of posemigroups. They also demonstrate that dominion closure can be preserved under a wider range of algebraic identities than was previously known, thereby enriching the theory of posemigroup varieties.

Several interesting questions remain open. It would be worthwhile to investigate whether the sufficient conditions established in this paper are also necessary, to characterize additional varieties of posemigroups that are closed under dominions, and to extend these techniques to noncommutative varieties and other classes of ordered semigroups. Another promising direction is the study of dominion closure and epimorphism preservation in pseudovarieties and profinite posemigroups, where analogous questions remain largely unexplored.

Acknowledgement. The authors wish to thank the anonymous referees for their detailed feedback and thoughtful suggestions, which significantly improved the clarity and quality of this paper.

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