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GEOMETRIC CHARACTERIZATIONS AND THE BANACH–STEINHAUS THEOREM FOR MULTILINEAR MULTIVALUED MAPS

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ABSTRACT. We introduce and investigate a new concept of bounded multilinear relations between normed spaces, highlighting some algebraic properties of this new class of multivalued mappings. We characterize the norm of a bounded multilinear relation, thereby, obtaining an important geometric characterization of these mappings. As an application, we prove the uniform boundedness principle in the framework of multilinear relations on Banach spaces.

1. Introduction and notation

The theory of multivalued linear operators (or relations) between normed spaces is a natural extension of linear functional analysis, introduced in 1932 by Neumann [13] and has been developed in various directions. A systematic treatment was given by Arens [3] and by Coddington [6]. This approach turned out to be very successful and a number of concepts have been fruitfully generalized to the multivalued setting by several authors (see [1, 2, 7, 10, 12, 14, 15] and the references therein).

In the present work, we introduce and investigate a new concept of multilinear relations (multivalued) operators between normed spaces. After presenting some algebraic properties of multilinear relations between vector spaces, we characterize the notion of boundedness for these mappings, and we prove the Banach–Steinhaus theorem. To the best of our knowledge, this is the first attempt in this direction. In this direction (but applied to other frameworks), many papers

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have been devoted to fundamental theorems (see [11] for bilinear mappings on asymmetric normed spaces and [5] for linear relations on asymmetric normed spaces)

The article is divided as follows. In section 1, after fixing some notation, we recall the most important results on the theory of multivalued linear operators. In section 2, we present the definition of multilinear relation, and we discuss some algebraic properties of these mappings. The third section is devoted to studying and characterizing the boundedness of multilinear relation defined on normed spaces where we give geometric characterization for this notion. Finally, in section 4, after proving the separate continuity of multilinear relations defined on Banach spaces, we show our main result: Banach–Steinhaus theorem in the framework of multilinear relations on Banach spaces.

The definitions and notations used in the paper are, in general, standard. Let $m \in \mathbb{N}$ and $X_1, \dots, X_m, X, Y$ be vector normed spaces over $\mathbb{K}$ (either $\mathbb{R}$ or $\mathbb{C}$). The space of all continuous m-linear mappings $T : X_1 \times \dots \times X_m \to Y$ will be denoted by $\mathcal{L}(X_1, \dots, X_m; Y)$. It becomes a vector normed space with the natural norm

$$\|T\| = \sup \left\{ \left\|T(x^1, \dots, x^m)\right\| : x^j \in B_{X_j}, j = 1, \dots, m \right\},$$

where B_{X_j} denotes the closed unit ball of $X_j (1 \leq j \leq m)$.

Let F and G be arbitrary nonempty sets. A relation $T : F \rightrightarrows G$ from F into G is a mapping defined on a subspace $\mathcal{D}(T)$ of F, called the domain of T, which takes on values in the collection of nonempty subsets of G. If T maps the points of its domain to singletons, then T is said to be a single-valued mapping, or function. The class of all relations from F to G is denoted by $\mathcal{R}(F, G)$. A relation $T \in \mathcal{R}(F, G)$ is uniquely determined by its graph $Gr(T)$, which is defined by

$$Gr(T) = \{(x, y) \in F \times G : x \in \mathcal{D}(T), y \in Tx\},$$

so that we can identify T with $Gr(T)$. The inverse of T is the relation $T^{-1} : G \rightrightarrows F$ defined by

$$Gr(T^{-1}) = \{(y, x) \in G \times F : (x, y) \in Gr(T)\}.$$

If T^{-1} is single-valued, then T is called injective. The image of $A \subset X$ is given by

$$T(A) = \bigcup \{Tx : x \in A \cap \mathcal{D}(T)\}. \tag{1.1}$$

For a nonempty subsets $B \subset Y$, take

$$T^{-1}(B) = \{x \in \mathcal{D}(T) : B \cap Tx \neq \emptyset\}. \tag{1.2}$$

In particular, for $y \in T(F)$,

$$T^{-1}y = \{x \in \mathcal{D}(T) : y \in Tx\}.$$

Regarding the linear relations, let X and Y be two vector spaces over the field $\mathbb{K}$. The relation $T : X \rightrightarrows Y$ is called linear (or multi-valued linear operator) if

$$T(\alpha x_1 + \beta x_2) = \alpha T(x_1) + \beta T(x_2),$$

for all nonzero $\alpha, \beta \in \mathbb{K}$ and $x_1, x_2 \in \mathcal{D}(T)$. The class of all linear relations from X to Y is denoted by $\mathcal{LR}(X, Y)$. The multivalued part of T is defined by

$$T(0) := \{y : (0, y) \in Gr(T)\}.$$

Moreover, [7, Proposition I.2.8] gives a formula that characterizes the elements $y \in Tx$ for all $x \in D(T)$,

$$Tx = y + T(0). \tag{1.3}$$

By [7, Proposition I.3.1], if $B \subset Y$ with $B \neq \emptyset$ and $A \subset X$ then

$$T\left(T^{-1}(B)\right) = B \cap R(T) + T(0) \tag{1.4}$$

and

$$T^{-1}(T(A)) = A \cap \mathcal{D}(T) + T^{-1}(0). \tag{1.5}$$

All the other relevant terminology and preliminaries, which we will use, are given in corresponding sections. For the theory of multivalued linear operators we refer to the books [7, 2].

2. Multilinear relations

Definition 2.1. Let $m \in \mathbb{N}$ and $X_j (j = 1, \ldots, m)$, Y be vector spaces over $\mathbb{K}$. The relation $T : X_1 \times \cdots \times X_m \rightrightarrows Y$ is called multilinear (or m-linear) if it is linear separately in each coordinate, that is, if the mappings

$$T_j : \mathcal{D}(T_j) \subset X_j \rightrightarrows Y, \quad \text{where } T_j(x^j) = T(x^1, \ldots, x^j, \ldots, x^m) \tag{2.1}$$

are linear relations for each set of fixed $x^k \in X_k, k \neq j$. The collection of all m-linear relations is denoted by $\mathcal{LR}(X_1, \ldots, X_m; Y)$. If T maps the points of $\mathcal{D}(T)$ to singletons, then T is said to be a single-valued multilinear relation or multilinear mapping.

As a consequence of [7, Proposition I.2.3], we obtain the following characterization of the multilinear relations.

Theorem 2.2. *$T : X_1 \times \cdots \times X_m \rightrightarrows Y$ is a multilinear relation if and only if $Gr(T_j)$ is a subspace of $X_j \times Y$ for all $j = 1, \ldots, m$.*

It is well known that, if T is a multilinear mapping from $X_1 \times \cdots \times X_m$ to Y, then the set of all vectors in Y of the form $T(x^1, \ldots, x^m)$, $x^j \in X_j (j = 1, \ldots, m)$ is not in general a linear subspace of Y (see [9, Section 1.1]). Let $V(T)$ be the linear subspace of Y spanned by the set $T(X_1 \times \cdots \times X_m)$.

In what follows, if T is a multilinear relation, then we prove that the subset $T(0, \ldots, 0)$ is a linear subspace of Y (to simplify the notation, let us write 0 instead of $(0, \ldots, 0)$). We generalize in this way the result for the linear case that can be found in [7]. For the proof we need the following results.

Proposition 2.3. *Let $T : X_1 \times \cdots \times X_m \rightrightarrows Y$ be a multilinear relation. Then for every $(x^1, \ldots, x^m) \in \mathcal{D}(T)$ we have*

$$T(x^1, \ldots, x^m) - T(x^1, \ldots, x^m) = T(0). \tag{2.2}$$

Proof. For all $(x^1,\ldots,x^m)\in\mathcal{D}(T)$, we have

$$\begin{aligned} T(0) &= T(x^1-x^1,\ldots,x^m-x^m) \\ &= T(x^1,x^2-x^2,\ldots,x^m-x^m)-T(x^1,x^2-x^2,\ldots,x^m-x^m). \end{aligned}$$

By repeating this argument m times, we get

$$T(0)=2^{m-1}\left(T(x^1,\ldots,x^m)-T(x^1,\ldots,x^m)\right).$$

Then (2.2) follows. □

Corollary 2.4. *Let $T\in\mathcal{LR}(X_1,\ldots,X_m;Y)$. Then $0\in T(0)$ and*

$$T(x^1,\ldots,x^m)+T(0)=T(x^1,\ldots,x^m), \tag{2.3}$$

for all $(x^1,\ldots,x^m)\in\mathcal{D}(T)$.

Proof. For an element $y\in T(0)$, by (2.2) we get $0=y-y\in T(0)-T(0)=T(0)$. Also, we have

$$T(x^1,\ldots,x^m)+T(0)=2T(x^1,\ldots,x^m)-T(x^1,\ldots,x^m)=T(x^1,\ldots,x^m).$$

□

Proposition 2.5. *Let $(x^1,\ldots,x^m)\in\mathcal{D}(T)$. Then $y\in T(x^1,\ldots,x^m)$ if and only if $T(x^1,\ldots,x^m)=y+T(0)$. In particular, $T(x^1,\ldots,x^m)=T(0)$ if and only if $0\in T(x^1,\ldots,x^m)$.*

Proof. Suppose that $y\in T(x^1,\ldots,x^m)$. By (2.3) we obtain

$$y+T(0)\subset T(x^1,\ldots,x^m)+T(0)=T(x^1,\ldots,x^m).$$

For the reverse inclusion, let $z\in T(x^1,\ldots,x^m)$. According to (2.2) we get

$$z=y+(z-y)\in y+T(0).$$

For the second implication, we have

$$y=y+0\in y+T(0)=T(x^1,\ldots,x^m),$$

concluding the proof. □

Corollary 2.6. *Let T be a multilinear relation. Then $T(0)$ is a linear subspace of Y.*

Proof. Let $z,z'\in T(0)$ and let $\alpha\in\mathbb{K}$. By the previous proposition, we obtain $T(0)=\alpha z+z'+T(0)$. Thus, $\alpha z+z'\in T(0)$. □

3. Bounded multilinear relations

We now study the boundedness of multilinear relations. Let $X_1,\ldots,X_m,Y$ be normed spaces over $\mathbb{K}$. We shall introduce concepts of the norm of a multilinear relation $T\in\mathcal{LR}(X_1,\ldots,X_m;Y)$ at the point $x=(x^1,\ldots,x^m)\in\mathcal{D}(T)$ and the norm of T. We use the natural quotient map $Q:Y\longrightarrow Y/\overline{T(0)}$ for defining the norm of multilinear relation. Before giving the definition, we need the following result.

Proposition 3.1. *Let $T\in\mathcal{LR}(X_1,\ldots,X_m;Y)$. Then $Q\circ T$ is a single-valued map.*

Proof. Let $x = (x^1, \dots, x^m) \in X_1 \times \cdots \times X_m$ and $y_1, y_2 \in Q\left(T(x^1, \dots, x^m)\right)$. Then there exists $a_1, a_2 \in T(x^1, \dots, x^m)$ such that $y_1 = Q(a_1)$ and $y_2 = Q(a_2)$. By using Proposition 2.5, we get

$$Q(a_1) + Q\left(T(0)\right) = Q(a_2) + Q\left(T(0)\right).$$

This implies that

$$y_1 - y_2 \in Q\left(T(0) - T(0)\right) \subset Q\left(\overline{T(0)}\right) = 0,$$

that is $y_1 = y_2$. □

Definition 3.2. Let $T : X_1 \times \cdots \times X_m \rightrightarrows Y$ be a multilinear relation between normed spaces. We define

$$\left\|T(x^1, \dots, x^m)\right\| := \left\|Q \circ T(x^1, \dots, x^m)\right\|, \tag{3.1}$$

for all $(x^1, \dots, x^m) \in \mathcal{D}(T)$ and $y \in T(x^1, \dots, x^m)$. Also, the quantity $\|T\|$, which is called the norm of T, is defined by

$$\|T\| := \|Q \circ T\|. \tag{3.2}$$

The multilinear relation T is said to be bounded if $\mathcal{D}(T) = X_1 \times \cdots \times X_m$ and $\|T\| < \infty$. We denote by $\mathcal{BR}(X_1, \dots, X_m; Y)$ the set of all bounded multilinear relation from $X_1 \times \cdots \times X_m$ to Y.

The next proposition follows immediately from Proposition 2.5, as in the linear case.

Proposition 3.3. *Let $T : X_1 \times \cdots \times X_m \rightrightarrows Y$ be multilinear relation. Then for every $(x^1, \dots, x^m) \in \mathcal{D}(T)$ we have*

(i) $\|T(x^1, \dots, x^m)\| = d(y, T(0))$, for all $y \in T(x^1, \dots, x^m)$;

(ii) $\|T(x^1, \dots, x^m)\| = d(T(x^1, \dots, x^m), T(0)) = d(0, T(x^1, \dots, x^m))$.

From the above definition we easily get a new formula of $\|T\|$.

Proposition 3.4. *For all $T \in \mathcal{BR}(X_1, \dots, X_m; Y)$ we have*

$$\|T\| = \sup\left\{\left\|T(x^1, \dots, x^m)\right\|,\ x \in B_{X_1} \times \dots \times B_{X_m}\right\}. \tag{3.3}$$

The above results open the door to give a geometric characterization of $\|T\|$.

Theorem 3.5. *Let $T : X_1 \times \cdots \times X_m \rightrightarrows Y$ be a multilinear relation between normed spaces $X_1, \dots, X_m, Y$. The following statements are equivalent:*

(i) $\|T\| < \infty$.

(ii) There is a number $\lambda > 0$ such that

$$T\left(B_{\mathcal{D}(T)}\right) \subset \lambda B_{V(T)} + T(0), \tag{3.4}$$

where $V(T) = span\left\{T(X_1 \times \cdots \times X_m)\right\}$, the linear subspace of Y spanned by the set $T(X_1 \times \cdots \times X_m)$.

Proof. $(i) \Longrightarrow (ii)$ Take $y \in T\left(B_{\mathcal{D}(T)}\right)$. Then there is $(x^1, \dots, x^m) \in \mathcal{D}(T)$ such that $\|x^j\| \leq 1 (j = 1, \dots, m)$ and $y \in T(x^1, \dots, x^m) \subset T(X_1 \times \cdots \times X_m)$. By using Proposition 3.3, we have

$$\left\|T(x^1, \dots, x^m)\right\| = d(y, T(0)) = \inf_{k \in T(0)} \|y - k\| < \infty.$$

For all $\varepsilon > 0$, choose $k \in T(0) \subset T(X_1 \times \cdots \times X_m)$ such that it satisfies

$$\|y-k\| \le \left\|T(x^1,\ldots,x^m)\right\| + \varepsilon \le \|T\| + \varepsilon.$$

Since $y-k \in V(T)$, it follows from $\|y-k\| \le \|T\|+\varepsilon$ that $y-k \in (\|T\|+\varepsilon)B_{V(T)}$, which means that $y \in \lambda B_{V(T)} + T(0)$ with $\lambda = \|T\| + \varepsilon$.

$(ii) \Longrightarrow (i)$ Suppose that (3.4) holds for a given $\lambda > 0$. Let $(x^1,\ldots,x^m) \in B_{\mathcal{D}(T)}$ and $y \in T(x^1,\ldots,x^m)$. By the hypothesis there exist $y_1 \in B_{V(T)}$ and $k_1 \in T(0)$ such that $y = \lambda y_1 + k_1$. Thus,

$$\begin{aligned} \left\|T(x^1,\ldots,x^m)\right\| &= \inf_{k\in T(0)} \|y-k\| \\ &\le \|y-k_1\| = \lambda \|y_1\| \le \lambda. \end{aligned}$$

Consequently, by (3.3), we get $\|T\| \le \lambda$. □

Now, we extend to multilinear relations the concept of continuity and presenting the relationship between this concept and the boundedness.

Definition 3.6. A multilinear relation $T : X_1 \times \cdots \times X_m \rightrightarrows Y$ between normed spaces is said to be continuous if $T^{-1}(O)$ is an open set in $\mathcal{D}(T)$ for every open set O in $V(T)$.

We do not know if the boundedness of a multilinear relation implies continuity. In the following proposition, we prove that the reverse implication is true.

Proposition 3.7. *Every continuous multilinear relation $T : X_1 \times \cdots \times X_m \rightrightarrows Y$ is bounded.*

Proof. Suppose that T is continuous. Then $T^{-1}(B_{V(T)})$ is an open subset of $X_1 \times \cdots \times X_m$. In addition $(0,\ldots,0) \in T^{-1}(B_{V(T)})$ since $T(0) \cap B_{V(T)} \neq \emptyset$. So, there is $\alpha > 0$ such that $\alpha B_{\mathcal{D}(T)} \subset T^{-1}(B_{V(T)})$. Then, $\alpha T\left(B_{\mathcal{D}(T)}\right) \subset T\left(T^{-1}(B_{V(T)})\right)$. By using (1.1), (1.2), and Proposition 2.5, we get

$$\begin{aligned} T(T^{-1}(B_{V(T)})) &= \bigcup \left\{T(x^1,\ldots,x^m) : T(x^1,\ldots,x^m) \cap B_{V(T)} \neq \emptyset\right\} \\ &\subset \bigcup \left\{y + T(0) : y \in B_{V(T)}\right\} \\ &= B_{V(T)} + T(0). \end{aligned}$$

Thus, $T\left(B_{\mathcal{D}(T)}\right) \subset \alpha^{-1}B_{V(T)}+T(0)$, and the result follows from Theorem 3.5. □

4. Applications

4.1. **Separate boundedness of multilinear relation.** A multilinear mapping $T : X_1\times\cdots\times X_m \longrightarrow Y$ between normed spaces is said to be separately continuous if it is continuous with respect to each variable separately, while the other variable is fixed. Obviously, the continuity implies the separately continuity, but the reverse implication is true if $X_1,\ldots,X_m$ are Banach spaces (see [8, p. 8]. In the following, we see that this result is true in the multivalued framework but under name of "separately bounded multilinear relation" because the continuity and the boundedness of multilinear relation are not equivalent as mentioned above.

Definition 4.1. A multilinear relation $T : X_1 \times \cdots \times X_m \rightrightarrows Y$ between normed spaces is said to be separately bounded if it is bounded with respect to each variable while the other variable is fixed, that is, if the restricted mappings

$$T_j : X_j \rightrightarrows Y, \quad T_j(x^j) = T(x^1, \ldots, x^j, \ldots, x^m)$$

are bounded linear relations for each set of fixed $x^k \in X_k, k \neq j$.

To get the separate boundedness of multilinear relations we need the multivalued version of the Banach–Steinhaus theorem for normed linear relations, which can be found in [7].

Theorem 4.2. *Let X be a Banach space and let Y be a normed vector space. Suppose that $\mathcal{F}$ is a collection of bounded linear relation from X to Y such that $\sup\limits_{T\in\mathcal{F}} \|T(x)\| < \infty$ for every $x \in X$. Then $\sup\limits_{T\in\mathcal{F}} \|T\| < \infty$.*

Theorem 4.3. *Let $X_1, \ldots, X_m$ be Banach spaces and let Y be normed space. Then the multilinear relation $T : X_1, \ldots, X_m \rightrightarrows Y$ is separately bounded if and only if T is bounded.*

Proof. The part "if" is obvious. We prove the "only if" part by induction on $m \in \mathbb{N}$. For $m = 1$, there is nothing to prove. Suppose that the statement is true for $m - 1$. Given $(x^1, \ldots, x^m) \in X_1 \times \cdots \times X_m$, define the mappings $T_{x^1,\ldots,x^{m-1}} : X_m \rightrightarrows Y$ and $T_{x^m} : X_1 \times \cdots \times X_{m-1} \rightrightarrows Y$ by

$$T_{x^m}(x^1, \ldots, x^{m-1}) = T_{x^1,\ldots,x^{m-1}}(x^m) := T(x^1, \ldots, x^m).$$

By the assumption it is clear that the linear relation $T_{x^1,\ldots,x^{m-1}}$ is bounded and the $(m-1)$-linear relation T_{x^m} is separately bounded. Then T_{x^m} is bounded by the inductive hypothesis. Now, consider the family

$$\mathcal{F} = \left\{T_{x^1,\ldots,x^{m-1}} : \left\|x^j\right\| \leq 1,\ j = 1, \ldots, m-1\right\}.$$

Using the formula (3.3), the boundedness of T_{x^m} implies that

$$\sup_{T_{x^1,\ldots,x^{m-1}}\in\mathcal{F}} \|T_{x^1,\ldots,x^{m-1}}(x^m)\| = \sup_{\|x^j\|\leq 1, j=1,\ldots,m-1} \left\|T_{x^m}(x^1, \ldots, x^{m-1})\right\| < \infty,$$

for every $x^m \in X_m$. So by the previous theorem, we have

$$\sup_{T_{x^1,\ldots,x^{m-1}}\in\mathcal{F}} \left(\sup_{\|x^m\|\leq 1} \|T_{x^1,\ldots,x^{m-1}}(x^m)\| \right) < \infty.$$

It follows that

$$\begin{aligned} \|T\| &= \sup_{\|x^j\|\leq 1, j=1,\ldots,m} \|T(x^1, \ldots, x^m)\| \\ &= \sup_{\|x^j\|\leq 1, j=1,\ldots,m-1} \left(\sup_{\|x^m\|\leq 1} \|T_{x^1,\ldots,x^{m-1}}(x^m)\| \right) \\ &= \sup_{T_{x^1,\ldots,x^{m-1}}\in\mathcal{F}} \left(\sup_{\|x^m\|\leq 1} \|T_{x^1,\ldots,x^{m-1}}(x^m)\| \right) < \infty, \end{aligned}$$

which proves that T is bounded from $X_1, \ldots, X_m$ to Y. $\square$

4.2. **Multilinear Banach–Steinhaus theorem.** The Banach–Steinhaus theorem firstly presented by Banach and Steinhaus [4], is one of the central principles of functional analysis. Following the original cross's approach of Banach–Steinhaus theorem to the theory of normed multivalued linear operators (see [7]) and as application of our results, we prove the corresponding multilinear version, namely, *multilinear relation Banach–Steinhaus theorem.*

Theorem 4.4. *Let $X_1, \dots, X_m$ be Banach spaces, let Y be a normed space, and let $\mathcal{F}$ be a family of continuous multilinear relations from $X_1 \times \cdots \times X_m$ to Y. Suppose that*

$$\sup_{T \in \mathcal{F}} \left\| T(x^1, \dots, x^m) \right\| < \infty, \tag{4.1}$$

for every $x^j \in X_j, j = 1, \dots, m$. Then $\sup\limits_{T \in \mathcal{F}} \|T\| < \infty$.

Proof. First, note that by Proposition 3.7 we have $\mathcal{F} \subset \mathcal{BR}(X_1, \dots, X_m; Y)$. It is well known that the product space $X_1 \times \cdots \times X_m$, endowed with the norm $p_\infty := \max\left\{ \|\cdot\|_{X_1}, \dots, \|\cdot\|_{X_m} \right\}$, is a Banach space. For each integer $n \in \mathbb{N}$, consider F_n the subset of $X_1 \times \cdots \times X_m$ of all $(x_1, \dots, x_m)$ such that

$$\sup_{T \in \mathcal{F}} \left\| T(x^1, \dots, x^m) \right\| \leq n.$$

Each F_n is closed. Perhaps the simplest way to see this is to note that the single-valued map $\|\cdot\|_Y \circ Q \circ T$ is continuous on $X_1 \times \cdots \times X_m$ with respect to the norm p_∞. In addition, $X_1 \times \cdots \times X_m = \cup_{n \geq 1} F_n$. By the Baire category theorem, there must be some $n_0 \in \mathbb{N}$ such that F_{n_0} contains an open ball $B_{p_\infty}(a, r)$ with $a = (a^1, \dots, a^m) \in X_1 \times \cdots \times X_m$. It is clear that, for every $(t^1, \dots, t^m) \in B_{p_\infty}(0, r)$, we have $(0, t^2, t^3, \dots, t^m) \in B_{p_\infty}(0, r)$. We can compute, for $T \in \mathcal{F}$,

$$\begin{aligned}
&\|T(t^1, a^2 + t^2, \dots, a^m + t^m)\| \\
&\leq \|T(a^1 + t^1, a^2 + t^2, \dots, a^m + t^m)\| + \|T(-a^1, a^2 + t^2, \dots, a^m + t^m)\| \\
&\leq 2n_0.
\end{aligned}$$

Using the same argument and taking in account that both $(t^1, 0, t^3, \dots, t^m)$ and $(0, 0, t^3, \dots, t^m)$ belongs to $B_{p_\infty}(0, r)$, we get

$$\begin{aligned}
&\|T(t^1, t^2, a^3 + t^3, \dots, a^m + t^m)\| \\
&\leq \|T(t^1, a^2 + t^2, \dots, a^m + t^m)\| + \|T(t^1, -a^2, a^3 + t^3, \dots, a^m + t^m)\| \\
&\leq 4n_0.
\end{aligned}$$

By repeating this argument m times, we obtain

$$\left\| T(t^1, \dots, t^m) \right\| \leq 2^m n_0,$$

for every $(t^1, \dots, t^m) \in B_{p_\infty}(0, r)$. On the other hand,

$$\begin{aligned}
\|T\| &= \sup\left\{ \|T(x^1, \dots, x^m)\| : \|x^j\| \leq 1, j = 1, \dots, m \right\} \\
&= \sup\left\{ \left\| T\left(\tfrac{t^1}{r}, \dots, \tfrac{t^m}{r}\right) \right\| : \|t^j\| \leq r, j = 1, \dots, m \right\} \\
&\leq \frac{2^m n_0}{r^m}.
\end{aligned}$$

Since this holds for every $T \in \mathcal{F}$ the result is proved. ∎

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